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Remark on twists of Frobenius algebra and link homology

by

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Abstract. We discuss twists on Frobenius algebras in the context of link homology. In his paper in 2006, Khovanov asserted that a twist of a Frobenius algebra yields an isomorphic chain complex on each link diagram. Although the result has been widely accepted for nearly two decades, a subtle gap in the original proof was found in the induction step of the construction of the isomorphism. Following a discussion with Khovanov, we decided to provide a new proof. Our proof is based on a detailed analysis of configurations of circles in each state.

1. Introduction.

Khovanov [3] defined a link invariant, which is nowadays known as *Khovanov homology*. Later, in [2] and [4], it was suggested that there are some interesting variants of Khovanov homology indexed by Frobenius algebras. Following the common notation and terminology, which will be reviewed in Section 2, we briefly recall Khovanov's construction. For a link diagram D and a commutative Frobenius algebra A, define a commutative cube $V_{D;A}$ by

$$V_{D;A}(S) \coloneqq \bigotimes_{C\in\pi_0(D_S)} A_C$$

for each state S on D, where A_C is a copy of A associated to a circle component of the smoothing D_S. The cube $V_{D;A}$ yields a link homology provided A satisfies a certain set of conditions (see [2] and [4]). On the other hand, if $\theta \in A$ is an invertible element, then one can *twist* the coalgebra structure of A by θ, that is,

$$\varepsilon^\theta(x) \coloneqq \varepsilon(\theta x), \qquad \Delta^\theta(x) \coloneqq \Delta(\theta^{-1}x).$$

We denote by A^θ the resulting Frobenius algebra. In [4], it is asserted that this twisted Frobenius algebra A^θ results in the same link homology as A, and

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this assertion has been widely accepted for nearly two decades. Khovanov however only gave a sketch of the proof. We communicated with Khovanov, and it turned out that the result requires more careful and detailed proof.

We here elaborate the gap in the original construction. In order to construct an isomorphism $V_{D;A^\theta} \cong V_{D;A}$, a key ingredient is a function $\nu : \coprod_S \pi_0(D_S) \to \mathbb{Z}$ such that the following square commutes for each state S and each crossing $c \notin S$:

$$\begin{tikzcd} V_{D;A^\theta}(S) \arrow[r, "\bigotimes_C \theta^{\nu(C)}"] \arrow[d, "\xi_c"'] & V_{D;A}(S) \arrow[d, "\xi_c"] \\ V_{D;A^\theta}(S \cup \{c\}) \arrow[r, "\bigotimes_{C'} \theta^{\nu(C')}"] & V_D(S \cup \{c\}) \end{tikzcd}$$

where ξ_c is the structure morphism of the commutative cubes. Since θ is invertible, it is obvious that the morphisms form an isomorphism of commutative cubes. According to the original proof in [4], it is asserted that one can construct such a function ν on D by induction on the cardinality $|S|$. In each induction step, we need to make a choice of how we distribute the value $\nu(C)$ whenever C splits into two in a new state. The source of the problem is that such a choice cannot be arbitrary.

We can see the problem more explicitly in a concrete example; take D to be the diagram for the right-handed trefoil in Figure 1. In this example,

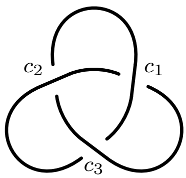


Fig. 1. The right-handed trefoil

there is an unfortunate choice in the induction steps which causes deadlock in later steps. Namely, Figure 2 depicts the list of smoothings D_S for the states S, where the red label attached to each circle C indicates the value $\nu(C)$. It is seen that the function ν cannot extend to $\pi_0(D_{\{1,2,3\}})$ satisfying the condition.

Note that a required function ν on D *does exist* in the example above. For example, by swapping the values of the two circles in $D_{\{1,2\}}$ Figure in 2, one can extend ν to $\pi_0(D_{\{1,2,3\}})$. This choice however seems to be *ad-hoc*, and it is uncertain if a similar technique is available for general diagrams having a similar problem. The observation implies that, if there is an algorithm constructing ν for arbitrary link diagrams, then we would need finer control of a path in the cube through which ν is extended.

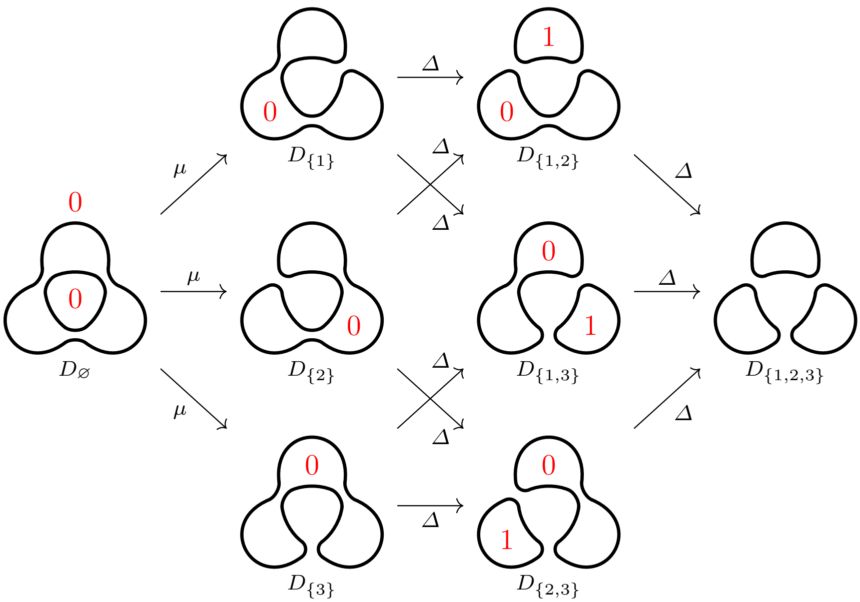
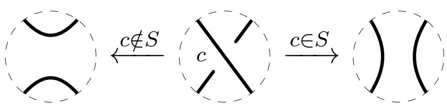


Fig. 2. An "unfortunate choice" of twisting weights

The plan of the paper is as follows. After preparing basic notations and terminology in Section 2, we review the statement of Khovanov's assertion and state our main result in Section 3. The actual construction of the function ν will be done in Section 4.

2. Notation and terminology. Throughout the note, we will denote by $\mathbb{K}$ a fixed commutative ring. Tensor products are always taken over $\mathbb{K}$. In addition, A always denotes a commutative Frobenius algebra over $\mathbb{K}$.

A *link diagram* in this paper is unoriented unless otherwise stated. For a link diagram D, we write $\mathcal{X}(D)$ for the set of crossings of D. A *state* on D is just a subset of $\mathcal{X}(D)$. For each state $S \subset \mathcal{X}(D)$, we construct a 1-dimensional submanifold $D_S \subset \mathbb{R}^2$ by replacing each crossing $c \in \mathcal{X}(D)$ with two disjoint arcs in two different ways according to whether $c \in S$ or not:



Write $\pi_0(D_S)$ for the set of circles in D_S. If (S, c) is a pair of $S \subset \mathcal{X}(D)$ and $c \in \mathcal{X}(D) \setminus S$, then $D_{S \cup \{c\}}$ is obtained from D_S in either of the following ways:

- if c is adjacent to only a single circle in D_S, say $C \in \pi_0(D_S)$, then c *splits* C into two circles in $D_{S\cup\{c\}}$;
- if c is adjacent to two circles in D_S, say $C_1, C_2 \in \pi_0(D_S)$, then c *merges* C_1 and C_2 into a single circle in $D_{S\cup\{c\}}$.

Note that the circles in D_S and $D_{S\cup\{c\}}$ are all in common except for those affected by the state change at c; we will often identify those circles in the following argument.

DEFINITION 1. For a finite set Λ, a *commutative Λ-cube* consists of a family $\{V(S)\}_{S\subset\Lambda}$ of $\mathbb{K}$-modules together with a family of $\mathbb{K}$-linear maps $\xi^S_\lambda : V(S) \to V(S\cup\{\lambda\})$ for each pair of $S \subset \Lambda$ and $\lambda \in \Lambda \setminus S$ which make the diagram below commute:

$$\begin{tikzcd} V(S) \arrow[r, "\xi^S_\lambda"] \arrow[d, "\xi^S_\mu"'] & V(S\cup\{\lambda\}) \arrow[d, "\xi^{S\cup\{\lambda\}}_\mu"] \\ V(S\cup\{\mu\}) \arrow[r, "\xi^{S\cup\{\mu\}}_\lambda"] & V(S\cup\{\lambda,\mu\}) \end{tikzcd} \tag{1}$$

For a commutative Λ-cube $V = \{V(S)\}_{S\subset\Lambda}$, the structure morphism ξ^S_λ is often abbreviated to ξ_λ; hence, the condition (1) is also written $\xi_\lambda\xi_\mu = \xi_\mu\xi_\lambda$.

For a Frobenius algebra A over $\mathbb{K}$, and for a link diagram D, we define a commutative $\mathcal{X}(D)$-cube $V_{D;A}$ by

$$V_{D;A}(S) \coloneqq \bigotimes_{C\in\pi_0(D_S)} A_C,$$

where A_C is a copy of A associated with a circle C in D_S. The structure morphism $\xi_c = \xi^S_c : V_{D;A}(S) \to V_{D;A}(S\cup\{c\})$ consists of

- the comultiplication $\Delta : A_C \to A_{C'_1} \otimes A_{C'_2}$ if c splits C into C'_1 and C'_2;
- the multiplication $\mu : A_{C_1} \otimes A_{C_2} \to A_{C'}$ if c merges C_1 and C_2 into C';
- the identity on A_C otherwise.

Given a commutative Λ-cube $V = \{V(S)\}_S$, one can construct a chain complex $C_V = \{C^i_V\}_i$ by $C^i_V \coloneqq \bigoplus_{|S|=i} V(S)$ with the differential $d = \sum \pm\xi_\lambda$, where the sign is chosen in an appropriate manner (see [3, 1]). In the case of $V = V_{D;A}$ for a link diagram D, we in particular write $C(D;A) \coloneqq C_{V_{D;A}}$. It turns out that the homology $H^i(D;A) \coloneqq H^i(C^*(D;A), d)$ results in a link invariant for some Frobenius algebras A (see [3, 4]).

3. Twist of Frobenius algebra and Khovanov's proposition. We are interested in the comparison of the homology $H^i(D;A)$ for different Frobenius algebras A. In particular, we focus on the following.

DEFINITION 2. For an invertible element $\theta \in A$, the *twist* of A by θ is a Frobenius algebra A^θ which is identical to A as an algebra but with the "twisted" coalgebra structure given by

$$\varepsilon^\theta(x) := \varepsilon(\theta x), \qquad \Delta^\theta(x) := \Delta(\theta^{-1}x).$$

In [4, Proposition 3], the following is asserted.

PROPOSITION 3 ([4, Proposition 3]). *Let A be a Frobenius algebra and $\theta \in A$ an invertible element. Then, for every link diagram D, the commutative $\mathcal{X}(D)$-cubes $V_{D;A}$ and $V_{D;A^\theta}$ are isomorphic. In particular, $C(D;A^\theta) \cong C(D;A)$ as chain complexes.*

In [4], Khovanov only gave a sketch of the proof of Proposition 3, though it has a subtle gap in its induction step of the construction of the isomorphism. The goal of the paper is to give a complete proof for Proposition 3.

We first reduce Proposition 3 to a combinatorial problem.

LEMMA 4. *For an invertible element $\theta \in A$, the following commute for all $p, q \in \mathbb{Z}$:*

$$\begin{tikzcd} A^\theta \arrow[r, "\theta^{p+q-1}"] \arrow[d, "\Delta^\theta"'] & A \arrow[d, "\Delta"] \\ A^\theta \otimes A^\theta \arrow[r, "\theta^p \otimes \theta^q"] & A \otimes A \end{tikzcd} \qquad \begin{tikzcd} A^\theta \otimes A^\theta \arrow[r, "\theta^p \otimes \theta^q"] \arrow[d, "\mu^\theta(=\mu)"'] & A \otimes A \arrow[d, "\mu"] \\ A^\theta \arrow[r, "\theta^{p+q}"] & A \end{tikzcd} \qquad \blacksquare$$

In view of Lemma 4, Proposition 3 is proved by taking the following data.

DEFINITION 5. Let D be a link diagram. A *twisting weight* on D is a map

$$\nu : \coprod_{S \subset \mathcal{X}(D)} \pi_0(D_S) \to \mathbb{Z}$$

satisfying the following conditions:

(a) if $C \in \pi_0(D_S)$ splits into $C'_1, C'_2 \in \pi_0(D_{S\cup\{c\}})$, then $\nu(C) = \nu(C'_1) + \nu(C'_2) - 1$;
(b) if $C_1, C_2 \in \pi_0(D_S)$ merge into $C' \in \pi_0(D_{S\cup\{c\}})$, then $\nu(C_1) + \nu(C_2) = \nu(C')$;
(c) if $C \in \pi_0(D_S)$ is identified with $C' \in \pi_0(D_{S\cup\{c\}})$, then $\nu(C) = \nu(C')$.

We can indeed prove Proposition 3 for a diagram D with a twisting weight ν. Namely, for each state $S \subset \mathcal{X}(D)$, define

$$\widehat{\theta}^\nu : V_{D;A^\theta}(S) = \bigotimes_{C\in\pi_0(D_S)} A^\theta_C \xrightarrow{\bigotimes_C \theta^{\nu(C)}} \bigotimes_{C\in\pi_0(D_S)} A_C = V_{D;A}(S).$$

By virtue of Lemma 4, $\widehat{\theta}^\nu$ commutes with the structure morphisms, so it defines a morphism of commutative $\mathcal{X}(D)$-cubes. Furthermore, since θ is invertible, it is obvious that $\widehat{\theta^{\nu}}$ is in fact an isomorphism. In other words, Proposition 3 follows from the following.

Theorem 6. *Every link diagram D admits a twisting weight.*

Remark 7. As a special case, if the invertible element $\theta \in A$ belongs to the coefficient ring $\mathbb{K}$, then one can prove Proposition 3 without taking a twisting weight. Indeed, in this case, θ freely passes through the tensor products, so the two commutative squares in Lemma 4 reduce to the following:

$$\begin{tikzcd} A^{\theta} \arrow[r, "\theta^k \cdot \mathrm{id}"] \arrow[d, "\Delta^{\theta}"'] & A \arrow[d, "\Delta"] \\ A^{\theta} \otimes A^{\theta} \arrow[r, "\theta^{k+1} \cdot \mathrm{id}"] & A \otimes A \end{tikzcd} \qquad \begin{tikzcd} A^{\theta} \otimes A^{\theta} \arrow[r, "\theta^k \cdot \mathrm{id}"] \arrow[d, "\mu^{\theta} (=\mu)"'] & A \otimes A \arrow[d, "\mu"] \\ A^{\theta} \arrow[r, "\theta^k \cdot \mathrm{id}"] & A \end{tikzcd}$$

Hence, one can construct an isomorphism $V_{D;A^{\theta}}(S) \cong V_{D;A}(S)$ only by counting the number of comultiplications Δ in the path from the initial state $\varnothing$ to S.

4. Construction of twisting weight. In order to construct a twisting weight on a link diagram, it is convenient to describe the structure of the cube $V_{D;A}$ in terms of *cobordisms*. Namely, if S is a state on D, then for each $c \in \mathcal{X}(D) \setminus S$, we regard the structure morphism ξ_c^S as a 2-dimensional cobordism from D_S to $D_{S\cup\{c\}}$ obtained by attaching a saddle around c. It turns out that we still have $\xi_{c_1} \circ \xi_{c_2} = \xi_{c_2} \circ \xi_{c_1}$ as cobordisms; in other words, the family of cobordisms $\{\xi_c\}$ yields a commutative $\mathcal{X}(D)$-cube in the *cobordism category*. In fact, the commutative cube $V_{D;A}$ is obtained from this cube by applying the *TQFT* associated with the Frobenius algebra A.

For a cobordism W, we write

$$\gamma(W) := \frac{|\pi_0(\partial_{\mathrm{out}} W)| - |\pi_0(\partial_{\mathrm{in}} W)| - \chi(W)}{2}.$$

The following is straightforward.

Lemma 8.

(1) $\gamma(W_2 \circ W_1) = \gamma(W_2) + \gamma(W_1)$.
(2) $\gamma(W \amalg W') = \gamma(W) + \gamma(W')$.

Definition 9. Let $W : Y_0 \to Y_1$ be a 2-dimensional cobordism. A pair of functions $\nu_0 : \pi_0(Y_0) \to \mathbb{Z}$ and $\nu_1 : \pi_0(Y_1) \to \mathbb{Z}$ is said to be *compatible with* W if, for each connected component $\Sigma \subset W$, we have

$$\sum_{C \in \pi_0(\partial_{\mathrm{out}} \Sigma)} \nu_1(C) = \sum_{C \in \pi_0(\partial_{\mathrm{in}} \Sigma)} \nu_0(C) + \gamma(\Sigma).$$

Lemma 10. *Let $W_1 : Y_0 \to Y_1$ and $W_2 : Y_1 \to Y_2$ be 2-dimensional cobordisms and $\nu_i : \pi_0(Y_i) \to \mathbb{Z}$.*

(1) *If* (ν_0, ν_1) *and* (ν_1, ν_2) *are compatible with* W_1 *and* W_2 *respectively, then* (ν_0, ν_2) *is compatible with* $W_2 \circ W_1$.
(2) *If* $\pi_0(\partial_{\mathrm{out}} W_1) \to \pi_0(W_1)$ *is bijective, if* (ν_0, ν_1) *is compatible with* W_1, *then* (ν_1, ν_2) *is compatible with* W_2 *if and only if* (ν_0, ν_2) *is compatible with* $W_2 \circ W_1$.

Proof. The part (1) is immediate from Lemma 8. To see (2), notice that, if $\pi_0(\partial_{\mathrm{out}} W_1) \to \pi_0(W_1)$ is bijective, then $\pi_0(W_2) \to \pi_0(W_2 \circ W_1)$ is also bijective. Thus, each connected component Σ of $W_2 \circ W_1$ is of the form $\Sigma^{(2)} \circ (\amalg_i^{(1)} \Sigma_i')$ for connected components $\Sigma^{(2)} \in \pi_0(W_2)$ and $\Sigma_i^{(1)} \in \pi_0(W_1)$. If (ν_0, ν_1) is compatible with W_1, then we obtain

$$\begin{aligned}
\sum_{C \in \pi_0(\partial_{\mathrm{out}} \Sigma)} \nu_2(C) &= \sum_{C \in \pi_0(\partial_{\mathrm{out}} \Sigma^{(2)})} \nu_2(C), \\
\sum_{C \in \pi_0(\partial_{\mathrm{in}} \Sigma)} \nu_0(C) &= \sum_i \sum_{C \in \pi_0(\partial_{\mathrm{in}} \Sigma_i^{(1)})} \nu_0(C) \\
&= \sum_i \sum_{C' \in \pi_0(\partial_{\mathrm{out}} \Sigma_i^{(1)})} \nu_1(C') - \gamma(\Sigma_i^{(1)}) \\
&= \sum_{C' \in \pi_0(\partial_{\mathrm{in}} \Sigma^{(2)})} \nu_1(C') - \gamma\Big(\coprod_i \Sigma_i^{(1)}\Big).
\end{aligned}$$

The result is now obvious. ∎

If Σ is a cobordism with a single saddle, then we have

$$\gamma(\Sigma) = \begin{cases} 1 & \text{if } \Sigma \text{ splits a circle,} \\ 0 & \text{if } \Sigma \text{ merges a pair of circles.} \end{cases}$$

This computation leads to the following.

LEMMA 11. *Let* D *be a link diagram. A set* $\{\nu_S : \pi_0(D_S) \to \mathbb{Z}\}_S$ *of functions forms a twisting weight if and only if each pair of the form* $(\nu_S, \nu_{S \cup \{c\}})$ *for* $c \notin S$ *is compatible with* ξ_c^S.

DEFINITION 12. Let D be a link diagram and S a state.

(1) A pair of crossings c_1 and c_2 of D are said to be *disjoint* in S if there is no circle in D_S adjacent to both of them.
(2) A pair of crossings c_1 and c_2 of D are said to be *parallel* in S if they are adjacent to the same pair of distinct circles in D_S.
(3) A triple of crossings c_1, c_2, and c_3 of D are said to form a *triangle* in S if they are in the position as in Figure 3.

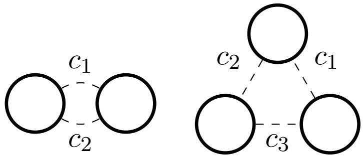


Fig. 3. Parallel crossings and triangle

LEMMA 13. *Let D be a link diagram such that $D_\varnothing$ is connected. Suppose S is a state on D such that, for every $c \in S$, $\xi_c : D_{S\setminus\{c\}} \to D_S$ splits a circle into two. Then S has at least one pair of crossings which are not parallel. Furthermore, if there is no disjoint pair of crossings in S, then S has no triangle.*

Proof. If all of crossings in S are mutually parallel, then the crossings in S are depicted as in Figure 4. Since there are no other crossings, one can

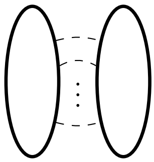


Fig. 4. All parallel crossings

easily see that $D_\varnothing$ is not connected.

We next suppose that S has no disjoint pair of crossings. If S admits a triangle, say c_1, c_2, and c_3 as in Figure 3, then no circle in the picture can have crossings in S in the "interior" of the circle. Hence, the triangle becomes like in Figure 5 in the state $S \setminus \{a, b, c\}$ such that no crossing in $S \setminus \{a, b, c\}$ intersects the red triangle. In particular, $D_\varnothing$ is not connected. ■

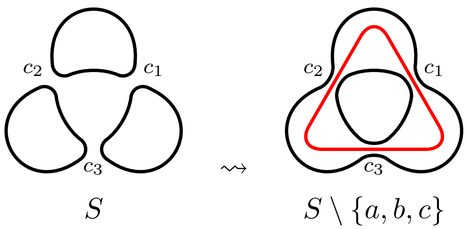


Fig. 5. Triangle in the initial state $\varnothing$

LEMMA 14. *Let S be a state on a link diagram D and $a, b \in S$. Suppose that we are given functions*

$$\nu_{00} : \pi_0(D_{S\setminus\{a,b\}}) \to \mathbb{Z},$$
$$\nu_{10} : \pi_0(D_{S\setminus\{b\}}) \to \mathbb{Z},$$
$$\nu_{01} : \pi_0(D_{S\setminus\{a\}}) \to \mathbb{Z}$$

such that the pairs (ν_{00}, ν_{10}) *and* (ν_{00}, ν_{01}) *are compatible with* $\xi_a^{S\setminus\{a,b\}} : D_{S\setminus\{a,b\}} \to D_{S\setminus\{b\}}$ *and* $\xi_b^{S\setminus\{a,b\}} : D_{S\setminus\{a,b\}} \to D_{S\setminus\{a\}}$ *respectively. For* $C \in \pi_0(D_S)$, *let* $\Sigma_a \in \pi_0(\xi_a^{S\setminus\{a\}})$ *and* $\Sigma_b \in \pi_0(\xi_b^{S\setminus\{b\}})$ *be the connected components containing* C. *If* $\partial_{\mathrm{out}}\Sigma_a = \partial_{\mathrm{out}}\Sigma_b = C$, *then*

$$\sum_{C'\in\pi_0(\partial_{\mathrm{in}}\Sigma_b)} \nu_{10}(C') = \sum_{C'\in\pi_0(\partial_{\mathrm{in}}\Sigma_a)} \nu_{01}(C').$$

Proof. We denote by $\widetilde{\Sigma} \subset \xi_b^{S\setminus\{b\}} \circ \xi_a^{S\setminus\{a,b\}} = \xi_a^{S\setminus\{a\}} \circ \xi_b^{S\setminus\{a,b\}}$ the connected component containing C. We assert that $\partial_{\mathrm{out}}\widetilde{\Sigma} = C$. Indeed, if either of a and b is not adjacent to C in S, then $\widetilde{\Sigma}$ is identified either with Σ_a or Σ_b, so the assertion is obvious. If both a and b are adjacent to C in S, then the possible configurations of these crossings are as in Figure 6. In both cases,

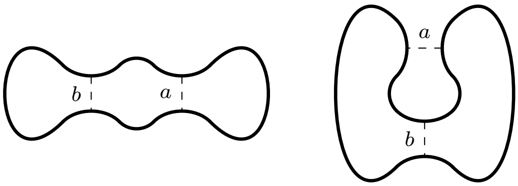


Fig. 6. Configuration of "merging" crossings

the assertion is verified by direct computation. Notice that, by virtue of the assertion, we can write $\widetilde{\Sigma} = \Sigma_b \circ W_a = \Sigma_a \circ W_b$ for $W_a \subset \xi_a^{S\setminus\{a,b\}}$ and $W_b \subset \xi_b^{S\setminus\{a,b\}}$. Since $\gamma(\Sigma_a) = \gamma(\Sigma_b) = 0$, we have $\gamma(W_a) = \gamma(W_b) = \gamma(\widetilde{\Sigma})$. Now, since (ν_{00}, ν_{10}) and (ν_{00}, ν_{01}) are compatible with $\xi_a^{S\setminus\{a,b\}}$ and $\xi_b^{S\setminus\{a,b\}}$ respectively, we have

$$\sum_{C'\in\pi_0(\partial_{\mathrm{in}}\Sigma_b)} \nu_{10}(C') = \sum_{C''\in\pi_0(\partial_{\mathrm{in}}\widetilde{\Sigma})} \nu_{00}(C'') + \gamma(\widetilde{\Sigma}) = \sum_{C'\in\pi_0(\partial_{\mathrm{in}}\Sigma_a)} \nu_{01}(C'),$$

which completes the proof. ∎

Lemma 15. *Let* D *be a link diagram. For every state* S, *exactly one of the following holds:*

(1) *for each circle component* $C \in \pi_0(D_S)$, *there is a crossing* $c \in S$ *such that* $\xi_c^{S\setminus\{c\}} : D_{S\setminus\{c\}} \to D_S$ *contains a connected component* Σ *with* $\partial_{\mathrm{out}}\Sigma = C$;

(2) *there is a circle component* $C_0 \in \pi_0(D_S)$ *such that every crossing* $c \in S$ *is adjacent to* C_0 *and one other circle in* D_S.

Proof. If there is a crossing $c \in S$ such that $\xi_c^{S\setminus\{c\}} : D_{S\setminus\{c\}} \to D_S$ merges circles, then every circle in D_S is a unique target of a connected component of $\xi_c^{S\setminus\{c\}}$. Also, every circle in D_S which is not adjacent to c is the unique target of a connected component of $\xi_c^{S\backslash c} : D_{S\backslash\{c\}} \to D_S$. Therefore, if D_S has a circle which is not a unique target of any connected components of $\xi_c^{S\backslash c}$ for any crossings c, then every such circle must be adjacent to all the crossings. ∎

PROPOSITION 16. *If D is a link diagram such that $D_\varnothing$ is connected, then D admits a twisting weight.*

Proof. In view of Lemma 11, we construct a function $\nu_S : \pi_0(D_S) \to \mathbb{Z}$ for each state S by induction on the cardinality $|S|$ so that $(\nu_S, \nu_{S\cup\{c\}})$ is compatible with the cobordism ξ_c^S for each pair of a state S and a crossing c with $c \notin S$. We set $\nu_\varnothing \equiv 0$. For each crossing c, take $\nu_{\{c\}} : \pi_0(D_{\{c\}}) \to \mathbb{Z}$ arbitrarily so that $(\nu_\varnothing, \nu_{\{c\}})$ is compatible with $\xi_c^\varnothing$. For S with $|S| \geq 2$, we proceed the induction by splitting it into the following two cases in view of Lemma 15.

First, suppose that every circle $C \in \pi_0(D_S)$ admits a crossing $c \in S$ such that $\xi_c^{S\backslash\{c\}}$ contains a connected component Σ_C with $\partial_{\mathrm{out}}\Sigma_C = C$. We define $\nu(C) \in \mathbb{Z}$ by

$$\nu_S(C) := \sum_{C' \in \pi_0(\partial_{\mathrm{in}}\Sigma_C)} \nu_{S\backslash\{c\}}(C'). \tag{2}$$

Note that, by virtue of Lemma 14, $\nu(C)$ does not depend on the choice of c. We show that $(\nu_{S\backslash\{c\}}, \nu_S)$ is compatible with $\xi_c^{S\backslash\{c\}}$ for each $c \in S$. If c merges circles, then the assertion directly follows from the independence mentioned above. Suppose c splits a circle $C' \in \pi_0(D_{S\backslash\{c\}})$ into $C_1, C_2 \in \pi_0(D_S)$. If a connected component $\Sigma \subset \xi_c^{S\backslash\{c\}}$ is not adjacent to c, then Σ is by definition the identity cobordism, and hence $\nu_S(\partial_{\mathrm{out}}\Sigma) = \nu_{S\backslash\{c\}}(\partial_{\mathrm{in}}\Sigma)$ by definition of ν_S. On the other hand, let us denote by Σ_c the connected component of $\xi_c^{S\backslash\{c\}}$ containing the saddle point. If there is a crossing $c' \in S$ disjoint from c, then the circles C', C_1, and C_2 are identified with those in $\pi_0(D_{S\backslash\{c,c'\}})$ and $\pi_0(D_{S\backslash\{c'\}})$ respectively. Similarly, Σ_c is also seen as a connected component of the cobordism $\xi_c^{S\backslash\{c,c'\}}$. The induction hypothesis then implies

$$\begin{aligned} \nu_S(C_1) + \nu_S(C_2) &= \nu_{S\backslash\{c'\}}(C_1) + \nu_{S\backslash\{c'\}}(C_2) \\ &= \nu_{S\backslash\{c,c'\}}(C') + 1 = \nu_{S\backslash\{c\}}(C') + 1. \end{aligned}$$

Hence $(\nu_{S\backslash\{c\}}, \nu_S)$ is compatible with $\xi_c^{S\backslash\{c\}}$ in this case. Also, if there is a crossing $c' \in S$ which merges circles, then it turns out that the cobordism $\xi_{c'}^{S\backslash\{c,c'\}} : D_{S\backslash\{c,c'\}} \to D_{S\backslash\{c\}}$ also merges circles. According to Lemma 10, in this case, $(\nu_{S\backslash\{c\}}, \nu_S)$ is compatible with $\xi_c^{S\backslash\{c\}}$ if and only if $(\nu_{S\backslash\{c,c'\}}, \nu_S)$ is compatible with the composition $\xi_c^{S\backslash\{c\}} \circ \xi_{c'}^{S\backslash\{c,c'\}} = \xi_{c'}^{S\backslash\{c'\}} \circ \xi_c^{S\backslash\{c,c'\}}$. By induction hypothesis, $(\nu_{S\backslash\{c,c'\}}, \nu_{S\backslash\{c'\}})$ is compatible with $\xi_c^{S\backslash\{c,c'\}}$ while

$(\nu_{S\setminus\{c'\}}, \nu_S)$ is compatible with $\xi_{c'}^{S\setminus\{c'\}}$ by definition. Hence the assertion is verified. Suppose now that c admits no disjoint crossings in S and that all the crossings cause splitting of circles. Take crossings $c_i \in S$ for $i = 1, 2$ so that c_i is adjacent only to the circle C_i. By virtue of Lemma 13, the configuration of c, c_1, and c_2 in D_S must be as in Lemma 7. Let $\widetilde{C}_1$ and $\widetilde{C}_2$ be circles as in

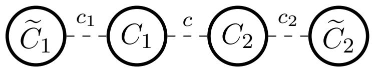


Fig. 7. Linear configuration of three crossings

Figure 7. We also write $\widetilde{C}'_i$ for $i = 1, 2$ for the circle in $D_{S\setminus\{c_i\}}$ which splits into $\widetilde{C}_i$ and C_i. By induction hypothesis, we then have

$$\begin{aligned}
\nu_{S\setminus\{c\}}(\widetilde{C}_1) = \nu_{S\setminus\{c_2\}}(\widetilde{C}_1), \qquad \nu_{S\setminus\{c\}}(\widetilde{C}_2) = \nu_{S\setminus\{c_1\}}(\widetilde{C}_2),\\
\nu_S(C_1) = \nu_{S\setminus\{c_2\}}(C_1), \qquad \nu_S(C_2) = \nu_{S\setminus\{c_1\}}(C_2),\\
\nu_{S\setminus\{c_2\}}(\widetilde{C}_1) + \nu_{S\setminus\{c_2\}}(C_1) = \nu_{S\setminus\{c_1,c_2\}}(\widetilde{C}'_1) + 1,\\
\nu_{S\setminus\{c_1\}}(C_2) + \nu_{S\setminus\{c_1\}}(\widetilde{C}_2) = \nu_{S\setminus\{c_1,c_2\}}(\widetilde{C}'_2) + 1,\\
\nu_{S\setminus\{c_1,c_2\}}(\widetilde{C}'_1) + \nu_{S\setminus\{c_1,c_2\}}(\widetilde{C}'_2) = \nu_{S\setminus\{c,c_1,c_2\}}(\widetilde{C}') + 1,
\end{aligned}$$

where $\widetilde{C}'$ is the circle in $D_{S\setminus\{c,c_1,c_2\}}$ formed by the four circles. In addition, the induction hypothesis also implies that $(\nu_{S\setminus\{c,c_1,c_2\}}, \nu_{S\setminus\{c\}})$ is compatible with the composition of the cobordisms $\xi_{c_1}^{S\setminus\{c,c_1\}} \circ \xi_{c_2}^{S\setminus\{c,c_1,c_2\}}$, so

$$\nu_{S\setminus\{c\}}(\widetilde{C}_1) + \nu_{S\setminus\{c\}}(C') + \nu_{S\setminus\{c\}}(\widetilde{C}_2) = \nu_{S\setminus\{c,c_1,c_2\}}(\widetilde{C}') + 2.$$

Combining the equations above, one obtains

$$\nu_S(C_1) + \nu_S(C_2) = \nu_{S\setminus\{c\}}(C') + 1.$$

This proves that $(\nu_S, \nu_{S\setminus\{c\}})$ is compatible with $\xi_c^{S\setminus\{c\}}$.

Second, suppose that there is a circle C_0 in D_S such that every crossing $c \in S$ is adjacent to C_0 and one other circle. Note that if there is another such circle, then every crossing in S is adjacent to both of the two circles, which is impossible by virtue of Lemma 13. It follows that every circle C other than C_0 admits a crossing c which is not adjacent to C; in this case, we may identify C with the corresponding circle in $\pi_0(D_{S\setminus\{c\}})$. We then define $\nu_S(C)$ by

$$\nu_S(C) := \begin{cases} \gamma(W_S) - \sum_{C\neq C_0} \nu_S(C) & \text{if } C = C_0,\\ \nu_{S\setminus\{c\}}(C) & \text{if } C \neq C_0 \text{ and if } C \text{ is not adjacent to } c, \end{cases}$$

where W_S is the cobordism obtained by composing the chain of cobordisms connecting $D_\varnothing$ to D_S. We verify that $(\nu_{S\setminus\{c\}}, \nu_S)$ is compatible with the cobordism $\xi_s^{S\setminus\{c\}} : D_{S\setminus\{c\}} \to D_S$ for each crossing c. For a connected component $\Sigma \subset \xi_c^{S\setminus\{c\}}$, if Σ is not adjacent to c, then Σ is the identity cobordism such that $\nu_S(\partial_{\mathrm{out}}\Sigma) = \nu_{S\setminus\{c\}}(\partial_{\mathrm{in}}\Sigma)$. Otherwise, Σ is the saddle cobordism from C' to $C_0 \amalg C_1$ for $C' \in \pi_0(D_{S\setminus\{c\}})$ and $C_1 \in \pi_0(D_S)$ with $C_1 \neq C_0$. Thus, we have

$$\begin{aligned}
\nu_S(C_0) + \nu_S(C_1) &= \gamma(W_S) - \sum_{C\in\pi_0(D_S)\setminus\{C_0,C_1\}} \nu_S(C) \\
&= \gamma(\xi_c^{S\setminus\{c\}}) + \gamma(W_{S\setminus\{c\}}) - \sum_{C\in\pi_0(D_{S\setminus\{c\}})\setminus\{C'\}} \nu_{S\setminus\{c\}}(C) \\
&= \gamma(\xi_c^{S\setminus\{c\}}) + \nu_{S\setminus\{c\}}(C') + \sum_{C''\in\pi_0(\partial_{\mathrm{in}}W_{S\setminus\{c\}})} \nu_{\varnothing}(C'') \\
&= \gamma(\xi_c^{S\setminus\{c\}}) + \nu_{S\setminus\{c\}}(C'),
\end{aligned}$$

which completes the proof. ∎

The general case reduces to the case of Proposition 16 by virtue of the following result.

Lemma 17. *Suppose D_+ and D_- are link diagrams which differ from one another only by the type of a single crossing. Then D_+ admits a twisting weight if and only if D_- does.*

Proof. We suppose D_+ admits a twisting weight ν and construct a twisting weight ν' on D_-; the other way is similar. Let c_0 be the crossing whose over- and under-arc are switched in D_+ and in D_-. Choosing an edge e in D_+ adjacent to c_0, we define a function $\phi : \coprod_{S\subset\mathcal{X}(D_-)} \pi_0(D_{-,S}) \to \mathbb{Z}$ by

$$\phi(C) := \begin{cases} 1 & \text{if } C \in \pi_0(D_{-,S}) \text{ with } c_0 \in S \text{ and } e \subset C, \\ 0 & \text{otherwise.} \end{cases}$$

Note that each circle in a resolution of D_- is identified with one in a resolution of D_+. From this point of view, define a function $\nu' : \coprod_{S\subset\mathcal{X}(D_-)} \pi_0(D_{-,S}) \to \mathbb{Z}$ by

$$\nu'(C) := \nu(C) - \phi(C).$$

We show that ν' is indeed a twisting weight. Suppose S is a state on D_- and $c \notin S$. We check the conditions in the following two cases.

Case $c \neq c_0$: First notice that, if a circle C in $D_{-,S}$ is not adjacent to c, and if C' is the corresponding circle in $D_{-,S\cup\{c\}}$, then since the edges of C are exactly those of C', we have

$$\nu'(C) - \nu'(C') = \nu(C) - \phi(C) - \nu(C') + \phi(C') = 0.$$

Hence, we have to verify the condition on the circles adjacent to c. If c splits a circle C in $D_{-,S}$ into C_1' and C_2' in $D_{-,S\cup\{c\}}$, then we have

$$\begin{aligned}\nu'(C) - \nu'(C_1') - \nu'(C_2') + 1 &\\ &= \nu(C) - \phi(C) - \nu(C_1') + \phi(C_1') - \nu(C_2') + \phi(C_2') + 1\\ &= -\phi(C) + \phi(C_1') + \phi(C_2')\end{aligned}$$

since c also splits C into C_1' and C_2' in D_+. Observing that the set of edges of C is exactly the disjoint union of those of C_1' and C_2', one obtains $\nu'(C) = \nu'(C_1') + \nu'(C_2') - 1$. If c merges circles C_1 and C_2 in $D_{-,S}$ into $C' \in D_{-,S\cup\{c\}}$, then

$$\begin{aligned}\nu'(C_1) + \nu'(C_2) - \nu'(C') &\\ &= \nu(C_1) - \phi(C_1) + \nu(C_2) - \phi(C_2) - \nu(C') + \phi(C')\\ &= -\phi(C_1) - \phi(C_2) + \phi(C').\end{aligned}$$

As with the "split" case, the set of edges of C' is exactly the disjoint union of those of C_1 and C_2, so we obtain $\nu'(C_1) + \nu'(C_2) = \nu'(C')$.

Case $c = c_0$*:* As in the preceding case, it is enough to verify the condition on circles adjacent to c_0. Note that if c_0 splits a circle C in $D_{-,S}$ into C_1' and C_2' in $D_{-,S\cup\{c_0\}}$, then it merges C_1' and C_2' into C in the cube of D_+. Furthermore, since the edge e is contained in exactly one of C_1' or C_2', and since $\phi(C) = 0$ by definition, we obtain

$$\begin{aligned}\nu'(C) - \nu'(C_1') - \nu'(C_2') + 1 &\\ &= \nu(C) - \phi(C) - \nu(C_1') + \phi(C_1') - \nu(C_2') + \phi(C_2') + 1\\ &= -\phi(C) - \phi(C_1') - \phi(C_2') + 1 = 0.\end{aligned}$$

The condition is also verified similarly in the case of "merge." ∎

Proof of Main Theorem 6. Let D be an arbitrary link diagram. Notice that, if D is a disjoint union of diagrams $D^{(1)}, \dots, D^{(n)}$, then a twisting weight on D is determined by its restriction to each $D^{(k)}$. Hence, we may assume D is connected without loss of generality. Notice that, in this case, there is a state $S \subset \chi(D)$ for which D_S is connected. Indeed, since D is connected, every adjacent pair of circles in D_S is connected by a crossing $c \in \mathcal{X}(D)$, so they are merged either in the state $S\cup\{c\}$ or in $S\setminus\{c\}$. Iterating the process, one reaches the required state. If S is such a state, then perform crossing change on each crossing $c \in S$. Let us denote by D' the resulting diagram. By construction, the smoothing $D'_\varnothing$ is identified with D_S and hence connected. Therefore, D' admits a twisting weight by Proposition 16. Furthermore, since D and D' are connected by a sequence of crossing change, the twisting weight on D' is transferred to D by virtue of Lemma 17. This completes the proof. ∎

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