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Full mad families of vector spaces and two local Ramsey theories

by

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Abstract. Let E be a vector space over a countable field of dimension $\aleph_0$. Two infinite-dimensional subspaces $V, W \subseteq E$ are *almost disjoint* if $V \cap W$ is finite-dimensional. This paper provides some improvements on results of Smythe (2019) about the definability of maximal almost disjoint families (mad families) of subspaces. We construct a full mad family of block subspaces in ZFC, answering a problem by Smythe in the positive. A variant of this construction shows that there exists a completely separable mad family of block subspaces in ZFC. We also discuss the abstract Mathias forcing introduced by Di Prisco, Mijares and Nieto (2017), and apply it to show that in Solovay's model obtained by the collapse of a Mahlo cardinal, there are no full mad families of subspaces over $\mathbb{F}_2$.

1. Introduction. Let E be a vector space over a countable field $\mathbb{F}$ of dimension $\aleph_0$. Smythe [25] studied *almost disjoint families* of infinite-dimensional subspaces $V, W \subseteq E$, i.e. families of infinite-dimensional subspaces of E whose pairwise intersections are finite-dimensional, and raised the following questions.

Problem 1.1. *Can mad families of subspaces be definable? In particular:*

(1) *Are there analytic mad families of subspaces?*
(2) *Assuming the existence of large cardinals, are there mad families of subspaces in* $\mathbf{L}(\mathbb{R})$*?*

The study of almost disjoint families in various linear-algebraic notions, or applications of the theory of almost disjoint families to linear algebra or functional analysis, predates Smythe's work [25]. Baumgartner–Spinas [2] studied the existence of quadratic spaces, and used almost disjoint families in $[\omega]^\omega$ to show that under the assumption $\mathfrak{b} = \aleph_1$, there exists an

2020 *Mathematics Subject Classification*: Primary 03E15; Secondary 15A03.
Key words and phrases: mad families, vector spaces, Ramsey theory, block subspace, block sequence, topological Ramsey space, definability.
Received 7 June 2025; revised 17 March 2026.
Published online 21 September 2026.

DOI: 10.4064/fm250607-18-3

$\aleph_1$-dimensional $(**)$-space over a class of countable fields. Possible cardinalities of variants of almost disjoint families, including *almost orthogonal families*, of infinite-dimensional subspaces of a separable Hilbert space were also studied in [3, 30]. Kolman [13] also studied almost disjoint families of infinite-dimensional subspaces with no additional topological assumptions (which he called *almost disjoint packings*), but he assumed that almost disjoint families satisfy a property stronger than pairwise almost disjointness (see also [25, footnote 2]).

Horowitz–Shelah [11] proved that there are no analytic mad families of subspaces over $\mathbb{F}_2$, but the first problem remains open for other fields. Smythe remarked in [25] that if there is a supercompact cardinal, then every set in $\mathbf{L}(\mathbb{R})$ is $\mathcal{H}$-strategically Ramsey, so there are no full mad families of subspaces there (the definition of a *full* mad family of subspaces will be introduced in the subsequent paragraph). Strategically Ramsey subsets of $E^{[\infty]}$ were introduced in [19, 20] to study the combinatorics of Banach space theory and dichotomies, and Smythe [24] introduced a localised version of the strategically Ramsey property. To the author's knowledge, the precise large cardinal strength needed for the non-existence of mad families of subspaces in $\mathbf{L}(\mathbb{R})$ is mostly unknown.

Inspired by Mathias' [14] original proof of the non-existence of analytic mad families in $[\omega]^\omega$, Smythe [25] approached this problem using a local Ramsey-theoretic approach, replacing $\mathcal{H}$-Ramsey with $\mathcal{H}$-strategically Ramsey. Given an almost disjoint family $\mathcal{A}$ of vector spaces, he extended Mathias' definition of the "happy family" $\mathfrak{F}_{\mathcal{A}}$ of subsets of ω to a "family" $\mathcal{H}(\mathcal{A})$ of subspaces of E. His approach requires $\mathcal{H}(\mathcal{A})$ to be *full*, a property which closely resembles the pigeonhole property of coideals of $[\omega]^\omega$. Calling a mad family $\mathcal{A}$ *full* if $\mathcal{H}(\mathcal{A})$ is full, Smythe proved the following theorem:

THEOREM 1.2 (Smythe [25]). *There exist no analytic full mad families of subspaces.*

Smythe proved this theorem by studying the combinatorial properties of families of *block subspaces*. Block subspaces are subspaces with a basis of vectors with strictly increasing support (see §2 for a precise definition). Block subspaces are more combinatorially tame, and since every infinite-dimensional subspace contains an infinite-dimensional block subspace, an almost disjoint family of subspaces is maximal with respect to block subspaces iff it is maximal.

While it is clear from Smythe's proof that the fullness property plays an important role in understanding the definability of mad families of subspaces, the extent to which mad families of subspaces satisfy this property remains poorly understood. In particular, Smythe asked the following question:

PROBLEM 1.3 (Smythe [25]). *Does* ZFC *prove the existence of a full mad family of subspaces? Even more, does* ZFC *prove that every mad family of subspaces is full?*

In the same paper, Smythe proved that assuming $\mathfrak{p} = \mathfrak{c}$, there exists a full mad family of block subspaces.

Since every infinite-dimensional subspace contains an infinite-dimensional block subspace, not all generality is lost when restricting our attention to block subspaces. However, several open problems concerning differences between mad families of subspaces and of block subspaces remain. Consider the following two cardinals:

$$\mathfrak{a}^*_{\mathrm{vec},\mathbb{F}} := \min\{|\mathcal{A}| : \mathcal{A} \text{ is an infinite mad family of subspaces over } \mathbb{F}\},$$

$$\mathfrak{a}_{\mathrm{vec},\mathbb{F}} := \min\{|\mathcal{A}| : \mathcal{A} \text{ is an infinite mad family of block subspaces over } \mathbb{F}\}.$$

It is unknown if they are equal in ZFC.

PROBLEM 1.4 (Smythe [25]). *Do we lose any generality when restricting mad families to block subspaces? In particular, is* $\mathfrak{a}^*_{\mathrm{vec},\mathbb{F}} = \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$ *true in* ZFC*?*

The first result in the present paper answers the first part of Problem 1.1 in the positive.

THEOREM 1.5. *There exists a full mad family of block subspaces.*

Our method is inspired by a variant of the construction [15] of a completely separable mad family in $[\omega]^\omega$ assuming $\mathfrak{s} \leq \mathfrak{a}$, provided by Guzmán [9]. Construction of completely separable mad families in $[\omega]^\omega$ has been extensively studied in the past two decades. Furthermore, complete separable mad families in $[\omega]^\omega$ have proved to be a powerful tool for constructing other set-theoretic objects. These include objects of interest in functional analysis, such as a masa in the Calkin algebra which does not lift to a masa of $B(H)$ (see [23]). Shelah [22] attained a breakthrough by showing that a completely separable mad family exists assuming that either $\mathfrak{c} < \aleph_\omega$, or there is no inner model with a measurable cardinal. However, the existence of a completely separable mad family in ZFC alone remains open. We shall show that the natural generalisation of a completely separable mad family to vector spaces (see Definition 13) exists in ZFC.

THEOREM 1.6. *There exists a completely separable mad family of block subspaces.*

Our constructions for Theorems 1.5 and 1.6 utilise various results from the local Ramsey theory of block subspaces developed by Smythe [24, 25], along with a few additional results that we shall prove. The non-requirement of additional set-theoretic hypothesis is a result of the ZFC inequality $\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$ by Smythe [25]. Since $\mathfrak{s} \leq \mathrm{non}(\mathcal{M})$ holds in ZFC, the vector space variant of the inequality $\mathfrak{s} \leq \mathfrak{a}$, i.e. $\mathfrak{s} \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$, is also a ZFC inequality.

Readers may also refer to [17, 18] for more constructions of mad families in $[\omega]^\omega$ involving the cardinal invariant $\mathrm{non}(\mathcal{M})$.

The second part in this paper addresses the existence of full mad families of block subspaces over $\mathbb{F}_2$ in Solovay's model. We utilise the abstract Mathias forcing developed by Di Prisco–Mijares–Nieto [7] – the abstract Mathias forcing was developed to expand the local Ramsey theory of $[\omega]^\omega$ to topological Ramsey spaces (i.e. closed triples $(\mathcal{R}, \leq, r)$ satisfying the axioms **A1**–**A4** introduced by Todorčević [28]).

If E is a countable vector space over $\mathbb{F}_2$, then the pigeonhole principle holds, so $(E^{[\infty]}, \leq, r)$ is a topological Ramsey space. By identifying vectors in E with their supports, this topological Ramsey space coincides with Hindman's space $(\mathbf{FIN}^{[\infty]}, \leq, r)$ studied in [4]. Filters in $\mathbf{FIN}^{[\infty]}$ have also been studied by Blass [4], and are known as *ordered-union ultrafilters*. Additional overlaps between Blass' theory of ordered-union ultrafilters and Smythe's theory of filters in [24, 26] include the notion of a *stable* ordered-union ultrafilter, which in Smythe's terminology is known as a full (p)-filter. This allows us to study the local Ramsey theory of infinite block sequences in E from the following perspectives:

1. Abstract local Ramsey theory by Di Prisco–Mijares–Nieto [7].
2. Local Ramsey theory of block sequences by Smythe [24, 26].
3. The theory of (stable) ordered-union ultrafilters by Blass [4].

In particular, the first two theories proposed versions of (semi)coideals and (ultra)filters in $E^{[\infty]}$, along with definitions of various selectivity properties. In §6 and §7, we shall show that these three theories coincide, so definitions may be used interchangeably and results proven in the individual theories may be freely applied to (semi)coideals and (ultra)filters in $E^{[\infty]}$.

Let $\mathrm{Coll}(\omega, {<}\kappa)$ be the Lévy collapse of the uncountable cardinal κ to ω_1. We may apply the abstract Mathias forcing to obtain the following variant of a celebrated theorem of Mathias [14]:

THEOREM 1.7. *Let κ be a Mahlo cardinal, and let G be a $\mathrm{Coll}(\omega, {<}\kappa)$-generic filter. Suppose that E is a countable vector space over $\mathbb{F}_2$. If $\mathcal{H} \in \mathbf{V}[G]$ is a selective coideal on $E^{[\infty]}$, and $\mathcal{X} \in \mathbf{L}(\mathbb{R})^{\mathbf{V}[G]}$ is a subset of $E^{[\infty]}$, then $\mathcal{X}$ is $\mathcal{H}$-Ramsey.*

A rough sketch of the proof of Theorem 1.7 (in the context of **FIN**) was stated by Blass in [4], and he stated that it "will be shown, in a more general context" in [5], but the published version of [5] does not contain such a proof.

As a consequence, we obtain the following corollary:

COROLLARY 1.8. *Let κ be a Mahlo cardinal, and let G be a $\mathrm{Coll}(\omega, {<}\kappa)$-generic filter. Then there are no full mad families of subspaces over $\mathbb{F}_2$ in $\mathbf{L}(\mathbb{R})^{\mathbf{V}[G]}$.*

Our paper is organised as follows:

(§2) We provide a concise introduction to various definitions and notations for block subspaces of E. We prove that finite intersections of block subspaces are also block subspaces (Lemma 2.1).

(§3) We review the local Ramsey theory of block subspaces developed by Smythe, i.e. combinatorial properties of *semicoideals* of block subspaces (which Smythe calls *families*). We introduce the notion of a *selective* semicoideal of block subspaces, a strengthening of the (p)-property.

(§4) We introduce some definitions and basic results concerning almost disjoint families of subspaces. We show that $\mathfrak{a}^*_{\mathrm{vec},\mathbb{F}}$ is uncountable, addressing a remark by Smythe [25]. We also show that the semicoideal $\mathcal{H}^-(\mathcal{A})$ is selective for any almost disjoint family $\mathcal{A}$ of subspaces – the non-requirement for $\mathcal{A}$ to be maximal is crucial in the proofs of Theorems 1.5 and 1.6.

(§5) We apply various lemmas and results proven in §3 and §4 to provide detailed proofs of Theorems 1.5 and 1.6.

(§6) We provide an overview of the axioms of topological Ramsey spaces **A1**–**A4** introduced by Todorčević, and the abstract local Ramsey theory developed in [7]. We show that given a closed triple $(\mathcal{R}, \leq, r)$ satisfying **A1**–**A4**, if $\mathcal{H} \subseteq \mathcal{R}$ is semiselective then $\mathbb{M}_{\mathcal{H}}$ satisfies the Mathias property and the Prikry property. This was proven in [7], but it was assumed that the almost reduction relation $\leq^*$ is transitive (which is not generally true, as exemplified by the topological Ramsey space of strong subtrees). We provide a proof of this result without this assumption.

(§7) We study the intersection between the local Ramsey theory of block subspaces and abstract local Ramsey theory when considering block subspaces over $\mathbb{F}_2$. We show that various notions and results introduced in §3 and §6 coincide in the setting of block subspaces over $\mathbb{F}_2$ and provide a proof of Theorem 1.7 and Corollary 1.8.

(§8) We conclude the paper with several further remarks and open problems.

2. Vector spaces and block subspaces. We introduce some basic definitions and properties of vector spaces and block subspaces. Throughout this article, only non-zero vectors shall be considered (thus, for instance, for two subspaces V, W, we write $V \cap W = \emptyset$ if their intersection is the trivial subspace).

2.1. Setup and notations. Let E be a vector space over a countable field $\mathbb{F}$, with a countable Hamel basis $(e_n)_{n<\omega}$. For each $x \in E$, we may write $x := \sum_{n<\omega} \lambda_n(x) e_n$, where $\lambda_n(x) \in \mathbb{F}$. This allows us to define the *support* of a vector $x \in E$ by

$$\mathrm{supp}(x) := \{n < \omega : \lambda_n(x) \neq 0\}.$$

Note that $\operatorname{supp}(x)$ is finite for all $x \in E$, as $(e_n)_{n<\omega}$ is a Hamel basis. This gives us a partial order $<$ on E, where for any two vectors $x, y \in E$,

$$x < y \iff \max(\operatorname{supp}(x)) < \min(\operatorname{supp}(y)).$$

A *block sequence* of vectors in E is thus a $<$-increasing sequence. Note that any block sequence of vectors is linearly independent.

Definition 1. An infinite-dimensional subspace $V \subseteq E$ is a *block subspace* if V has a *block basis*, i.e. an infinite block sequence $(x_n)_{n<\omega}$ such that $V = \operatorname{span}\{x_n : n < \omega\}$.

Since such a block basis is unique up to scaling, we may conflate block subspaces of E with infinite block sequences. This allows us to equip $E^{[\infty]}$ (see below) with a Polish topology by considering the first difference metric on infinite block sequences – that is, if $(x_n)_{n<\omega}$ and $(y_n)_{n<\omega}$ are infinite block sequences, then

$$d((x_n)_{n<\omega}, (y_n)_{n<\omega}) = 2^{-\min\{n:\, x_n \neq y_n\}}.$$

Notation 1.

(1) We let $E^{[\infty]}$ (resp. $E^{[<\infty]}$) denote the set of infinite (resp. finite) block sequences of vectors in E.
(2) $E^{[\leq\infty]} := E^{[\infty]} \cup E^{[<\infty]}$.
(3) If $A = (x_n)_{n<\omega} \in E^{[\infty]}$ and $N < \omega$, we write $A/N := (x_n)_{n\geq N}$ [1].
(4) If $A = (x_n)_{n<\omega} \in E^{[\infty]}$ and $n < \omega$, we write $r_n(A) := (x_i)_{i<n}$.
(5) If $a = (x_n)_{n<N}$, $b = (y_m)_{m<M} \in E^{[<\infty]}$, we write $a < b$ if $x_{N-1} < y_0$.
(6) If $A \in E^{[\infty]}$ and $a \in E^{[<\infty]}$, we write $a < A$ if $a < r_1(A)$.
(7) If $A \in E^{[\infty]}$ and $a \in E^{[<\infty]}$, we write $d_A(a) := \min\{N < \omega : a < A/N\}$.
(8) If $A \in E^{[\infty]}$ and $a \in E^{[<\infty]}$, then $A/a := A/d_A(a)$.
(9) If $Y \subseteq E$, we write
$$\langle Y \rangle := \operatorname{span}(Y).$$
(10) If $(x_n)_{n<\alpha} \in E^{[\leq\infty]}$, we write
$$\langle (x_n)_{n<\alpha} \rangle := \langle x_n : n < \alpha \rangle.$$
(11) If $A, B \in E^{[\infty]}$, we write $A \leq B$ iff $\langle A \rangle \subseteq \langle B \rangle$.
(12) If $A, B \in E^{[\infty]}$, we write $A \leq^* B$ iff $A/N \leq B$ for some $N < \omega$.
(13) If $A \in E^{[\infty]}$, we write $E^{[\infty]}\restriction A := \{B \in E^{[\infty]} : B \leq A\}$.
(14) If $A \in E^{[\infty]}$, we write $E^{[<\infty]}\restriction A := \{a \in E^{[<\infty]} : \langle a \rangle \subseteq \langle A \rangle\}$.

Block sequences/subspaces were introduced as they are more combinatorially well-behaved than arbitrary infinite-dimensional subspaces. The focus on block sequences is further justified by the observation that every infinite-dimensional subspace of E contains a block subspace. This will be apparent

[1] This notation differs from that introduced in [26], where A/N was instead defined to be the tail sequence of A consisting of vectors supported above N.

once we have introduced the relevant definitions and notations for almost disjoint families of subspaces.

2.2. Intersection of block subspaces. Given $A, B \in E^{[\infty]}$, if $\langle A\rangle \cap \langle B\rangle$ is infinite-dimensional, then by the remark above it contains a block subspace. The following lemma shows that $\langle A\rangle \cap \langle B\rangle$ is itself a block subspace (2). This observation is central in ensuring that the proofs of Theorems 1.5 and 1.6 do not require additional set-theoretic hypotheses.

LEMMA 2.1. *Let $A, B \in E^{[\infty]}$ be such that $\langle A\rangle \cap \langle B\rangle$ is infinite-dimensional. Then there exists some $C \in E^{[\infty]}$ such that $\langle C\rangle = \langle A\rangle \cap \langle B\rangle$.*

Proof. We write $A = (x_n)_{n<\omega}$ and $B = (y_n)_{n<\omega}$. For each $v \in \langle A\rangle \cap \langle B\rangle$, we let

$$\begin{aligned}\operatorname{supp}_A(v) &:= \{n < \omega : \operatorname{supp}(x_n) \subseteq \operatorname{supp}(v)\},\\ \operatorname{supp}_B(v) &:= \{n < \omega : \operatorname{supp}(y_n) \subseteq \operatorname{supp}(v)\}.\end{aligned}$$

We remark that for all $v, w \in \langle A\rangle \cap \langle B\rangle$,

$$\begin{aligned}\operatorname{supp}(v) \subseteq \operatorname{supp}(w) &\iff \operatorname{supp}_A(v) \subseteq \operatorname{supp}_A(w)\\ &\iff \operatorname{supp}_B(v) \subseteq \operatorname{supp}_B(w).\end{aligned}$$

For an arbitrary $v \in E$ and $k \in \operatorname{supp}(v)$, recall that $\lambda_k(v) \in \mathbb{F}$ denotes the unique coefficient such that $v = u + \lambda_k(v)e_k + w$ for some vectors $u < e_k < w$ (or $u = 0$ or $w = 0$).

CLAIM 1. *Let $v_0, v_1 \in \langle A\rangle \cap \langle B\rangle$; suppose that $\operatorname{supp}_A(v_0) \cap \operatorname{supp}_A(v_1) \neq \emptyset$. Then there exists some $w \in \langle A\rangle \cap \langle B\rangle$ such that $\operatorname{supp}_A(w) = \operatorname{supp}_A(v_0) \cap \operatorname{supp}_A(v_1)$.*

Proof. Let $v_0, v_1 \in \langle A\rangle \cap \langle B\rangle$ be such that $\operatorname{supp}_A(v_0) \cap \operatorname{supp}_A(v_1) \neq \emptyset$. We may write

$$v_0 = \sum_{i\in\operatorname{supp}_A(v_0)} \xi_i^0 x_i = \sum_{j\in\operatorname{supp}_B(v_0)} \mu_j^0 y_j, \tag{2.1}$$

$$v_1 = \sum_{i\in\operatorname{supp}_A(v_1)} \xi_i^1 x_i = \sum_{j\in\operatorname{supp}_B(v_1)} \mu_j^1 y_j, \tag{2.2}$$

where $\xi_i^0, \xi_i^1, \xi_j^0, \xi_j^1$ are all non-zero. We let

$$w := \sum_{i\in\operatorname{supp}_A(v_0)\cap\operatorname{supp}_A(v_1)} \xi_i^0 x_i, \qquad w' := \sum_{j\in\operatorname{supp}_B(v_0)\cap\operatorname{supp}_B(v_1)} \mu_j^0 y_j.$$

It is clear that $\operatorname{supp}_A(w) = \operatorname{supp}_A(v_0) \cap \operatorname{supp}_A(v_1)$. We shall show that $w = w'$. This implies that that $w \in \langle A\rangle \cap \langle B\rangle$, proving the claim.

(2) This lemma corrects footnote 8 of [25], which incorrectly stated that an intersection of block subspaces need not be block.

By the symmetry of the argument, it suffices to show that $\mathrm{supp}(w) \subseteq \mathrm{supp}(w')$, and $\lambda_k(w) = \lambda_k(w')$ for all $k \in \mathrm{supp}(w)$. Let $k \in \mathrm{supp}(w)$, so $k \in \mathrm{supp}(x_i)$ for some $i \in \mathrm{supp}_A(v_0) \cap \mathrm{supp}_A(v_1)$. By the second equalities in (2.1) and (2.2), we know that $k \in \mathrm{supp}(y_j)$ for some $j \in \mathrm{supp}_B(v_0)$, and $k \in \mathrm{supp}(y_{j'})$ for some $j' \in \mathrm{supp}_B(v_1)$. Since B is a block sequence, $k \in \mathrm{supp}(y_j) \cap \mathrm{supp}(y_{j'})$ implies that $j = j'$. Therefore, $j \in \mathrm{supp}_B(v_0) \cap \mathrm{supp}_B(v_1)$, so $k \in \mathrm{supp}(w')$. By (2.1), we can conclude that

$$\lambda_k(w) = \xi_i^0 \lambda_k(x_i) = \lambda_k(v_0) = \mu_j^0 \lambda_k(y_j) = \lambda_k(w'). \ \blacksquare$$

Let $K \subseteq \omega$ be the set of all n such that $n \in \mathrm{supp}_A(v)$ for some $v \in \langle A \rangle \cap \langle B \rangle$. Since $\langle A \rangle \cap \langle B \rangle$ is infinite-dimensional, K is infinite. Define a relation $\sim$ on K such that for all $m, n \in K$,

$$m \sim n \iff \forall v \in \langle A \rangle \cap \langle B \rangle\, [m \in \mathrm{supp}_A(v) \leftrightarrow n \in \mathrm{supp}_A(v)].$$

It is clear from the definition that $\sim$ is an equivalence relation, and that every equivalence class is finite.

Claim 2. *For each $n \in K$, there exists some $v \in \langle A \rangle \cap \langle B \rangle$ such that $\mathrm{supp}_A(v) = [n]_\sim$.*

Proof. It follows from the definition of $\sim$ that

$$[n]_\sim = \bigcap \{\mathrm{supp}_A(u) : u \in \langle A \rangle \cap \langle B \rangle \text{ and } n \in \mathrm{supp}_A(u)\}.$$

However, since $\mathrm{supp}_A(u)$ is finite for each $u \in \langle A \rangle \cap \langle B \rangle$, we may find $u_0, \ldots, u_{m-1} \in \langle A \rangle \cap \langle B \rangle$ such that $[n]_\sim = \bigcap_{i<m} \mathrm{supp}_A(u_i)$. Then the existence of such a v follows from Claim 1. $\blacksquare$

Thus, for each n we let $z_n \in \langle A \rangle \cap \langle B \rangle$ be a fixed vector such that $\mathrm{supp}_A(z_n) = [n]_\sim$.

Claim 3. *Let $m, n, l \in K$ be such that $m < n < l$, and suppose that $m \sim l$. Then $m \sim n \sim l$.*

Proof. Suppose otherwise, so there exist $m < n < l$ in K such that $m \sim l$ but $n \notin [m]_\sim$. We write

$$z_m = \sum_{i \in \mathrm{supp}_A(z_m)} \xi_i x_i = \sum_{j \in \mathrm{supp}_B(z_m)} \mu_j y_j,$$

where ξ_i, μ_j are non-zero. Let $i_0 := \max\{i \in \mathrm{supp}_A(z_m) : i < n\}$ and $i_1 := \min\{i \in \mathrm{supp}_A(z_m) : i > n\}$, and let $k_0 := \max(\mathrm{supp}(x_{i_0}))$ and $k_1 := \min(\mathrm{supp}(x_{i_1}))$. We define

$$w := \sum_{\substack{i \in \mathrm{supp}_A(z_m) \\ \max(\mathrm{supp}(x_i)) \le k_0}} \xi_i x_i.$$

We shall show that $w \in \langle A \rangle \cap \langle B \rangle$. This gives us the required contradiction, as $m \in \mathrm{supp}_A(w)$ but $l \notin \mathrm{supp}_A(w)$.

Observe that for each $j \in \operatorname{supp}_B(z_m)$, either $k_0 < \min(\operatorname{supp}(y_j))$ or $\max(\operatorname{supp}(y_j)) < k_1$ – otherwise, if $j \in \operatorname{supp}_B(z_m)$ is such that $\min(\operatorname{supp}(y_j)) \leq k_0 < k_1 \leq \max(\operatorname{supp}(y_j))$, then by the block sequence property of B, we know that for all $j' \neq j$, either $\max(\operatorname{supp}(y_{j'})) < k_0$ or $k_1 < \min(\operatorname{supp}(y_{j'}))$. Since $k_0 < \min(\operatorname{supp}(x_n)) \leq \max(\operatorname{supp}(x_n)) < k_1$ and $j \notin \operatorname{supp}_B(z_n)$, this implies that

$$\operatorname{supp}(x_n) \cap \bigcup_{j' \in \operatorname{supp}_B(z_n)} \operatorname{supp}(y_{j'}) = \emptyset,$$

which contradicts $\operatorname{supp}(x_n) \subseteq \operatorname{supp}(z_n)$.

Note that for all $k \in \operatorname{supp}(z_m)$, either $k \leq k_0$ or $k_1 \leq k$. Thus, for all $j \in \operatorname{supp}_B(z_m)$, since $\operatorname{supp}(y_j) \subseteq \operatorname{supp}(z_m)$, if $\max(\operatorname{supp}(y_j)) < k_1$ then $\max(\operatorname{supp}(y_j)) \leq k_0$. Similarly, if $k_0 < \min(\operatorname{supp}(y_j))$ then $k_1 \leq \min(\operatorname{supp}(y_j))$. Therefore,

$$z_m = \sum_{\substack{j \in \operatorname{supp}_B(z_m) \\ \max(\operatorname{supp}(y_i)) \leq k_0}} \mu_j y_j + \sum_{\substack{j \in \operatorname{supp}_B(z_m) \\ k_1 \leq \min(\operatorname{supp}(y_i))}} \mu_j y_j.$$

We may now express $\operatorname{supp}(z_m)$ in two different ways:

$$\bigcup_{i \in \operatorname{supp}_A(z_m)} \operatorname{supp}(x_i) = \bigcup_{\substack{i \in \operatorname{supp}_A(z_m) \\ \max(\operatorname{supp}(x_i)) \leq k_0}} \operatorname{supp}(x_i) \cup \bigcup_{\substack{i \in \operatorname{supp}_A(z_m) \\ k_1 \leq \min(\operatorname{supp}(x_i))}} \operatorname{supp}(x_i),$$

$$\bigcup_{j \in \operatorname{supp}_B(z_m)} \operatorname{supp}(y_j) = \bigcup_{\substack{j \in \operatorname{supp}_B(z_m) \\ \max(\operatorname{supp}(y_j)) \leq k_0}} \operatorname{supp}(y_j) \cup \bigcup_{\substack{j \in \operatorname{supp}_B(z_m) \\ k_1 \leq \min(\operatorname{supp}(y_j))}} \operatorname{supp}(y_j).$$

This implies that

$$\bigcup_{\substack{i \in \operatorname{supp}_A(z_m) \\ \max(\operatorname{supp}(x_i)) \leq k_0}} \operatorname{supp}(x_i) = \bigcup_{\substack{j \in \operatorname{supp}_B(z_m) \\ \max(\operatorname{supp}(y_j)) \leq k_0}} \operatorname{supp}(y_j),$$

$$\bigcup_{\substack{i \in \operatorname{supp}_A(z_m) \\ k_1 \leq \min(\operatorname{supp}(x_i))}} \operatorname{supp}(x_i) = \bigcup_{\substack{j \in \operatorname{supp}_B(z_m) \\ k_1 \leq \min(\operatorname{supp}(y_j))}} \operatorname{supp}(y_j).$$

Therefore, if we let

$$w' := \sum_{\substack{j \in \operatorname{supp}_B(z_m) \\ \max(\operatorname{supp}(y_j)) \leq k_0}} \mu_j y_j,$$

then we find that $\operatorname{supp}(w) = \operatorname{supp}(w')$. Furthermore, for any $k \in \operatorname{supp}(w)$, $\lambda_k(w) = \lambda_k(z_m) = \lambda_k(w')$, so $w = w'$. It is clear from the definition that $w \in \langle A \rangle$ and $w' \in \langle B \rangle$, so $w \in \langle A \rangle \cap \langle B \rangle$, as desired. ∎

We now choose an increasing sequence $(n_i)_{i<\omega}$ of representatives of all equivalence classes for $\sim$ (i.e. $\bigcup_{i<\omega} [n_i]_\sim = K$). We let $C := (z_{n_i})_{i<\omega}$. By

Claim 3, C is an infinite block sequence. We may finish the proof of the lemma by proving the following claim:

Claim 4. $\langle C\rangle = \langle A\rangle \cap \langle B\rangle$.

Proof. Since $z_{n_i} \in \langle A\rangle\cap\langle B\rangle$ for all i, we have $\langle C\rangle \subseteq \langle A\rangle\cap\langle B\rangle$. Conversely, let $v \in \langle A\rangle \cap \langle B\rangle$. Let $n_{i_0}, \dots, n_{i_{t-1}}$ be an increasing enumeration of all n_{i_j}'s such that $n_{i_j} \in \mathrm{supp}_A(v)$. Fix any $k_j \in \mathrm{supp}(x_{n_{i_j}}) \subseteq \mathrm{supp}(z_{n_{i_j}})$, and let

$$\xi_j := \frac{\lambda_{k_j}(v)}{\lambda_{k_j}(z_{n_{i_j}})}.$$

Note that $\xi_j \neq 0$ for all j. Now let

$$v' := \sum_{j<t} \xi_j z_{n_{i_j}}.$$

Clearly $v' \in \langle C\rangle$. We shall show that $v = v'$, which proves the claim.

We first note that $\mathrm{supp}_A(v') = \mathrm{supp}_A(v)$ (which implies that $\mathrm{supp}(v') = \mathrm{supp}(v)$) – observe that $\mathrm{supp}_A(v') = \bigcup_{j<t} \mathrm{supp}_A(z_{n_{i_j}})$. If $m \in \mathrm{supp}_A(v)$, then $[m]_\sim \subseteq \mathrm{supp}_A(v)$ and $[m]_\sim = [n_{i_j}]_\sim = \mathrm{supp}_A(z_{n_{i_j}})$ for some j, so $m \in \mathrm{supp}_A(v')$. Conversely, if $m \in \mathrm{supp}_A(z_{n_{i_j}})$, then since $n_{i_j} \in \mathrm{supp}_A(v)$, we have $[m]_\sim = [n_{i_j}]_\sim \subseteq \mathrm{supp}_A(v)$, so $m \in \mathrm{supp}_A(v)$.

It remains to show that $\lambda_k(v') = \lambda_k(v)$ for all $k \in \mathrm{supp}(v)$. We fix some $j < t$, and observe that

$$\lambda_{k_j}(v') = \frac{\lambda_{k_j}(v)}{\lambda_{k_j}(z_{n_{i_j}})} \cdot \lambda_{k_j}(z_{n_{i_j}}) = \lambda_{k_j}(v).$$

Since $v \in \langle A\rangle$ and $k_j \in \mathrm{supp}(x_{n_{i_j}}) \subseteq \mathrm{supp}(v)$, there exist some vectors $u, w \in \langle A\rangle$ (or $u = 0$ or $w = 0$) such that $u < x_{n_{i_j}} < w$, and

$$v = u + \frac{\lambda_{k_j}(v)}{\lambda_{k_j}(x_{n_{i_j}})} x_{n_{i_j}} + w.$$

Similarly, there exist some vectors $u', w' \in \langle A\rangle$ such that $u' < x_{n_{i_j}} < w'$ (or $u' = 0$ or $w' = 0$), and

$$v' = u' + \frac{\lambda_{k_j}(v)}{\lambda_{k_j}(x_{n_{i_j}})} x_{n_{i_j}} + w'.$$

Therefore, $v - v' \in \langle A\rangle \cap \langle B\rangle$, and $n_{i_j} \notin \mathrm{supp}(v - v')$. But this implies that $[n_{i_j}]_\sim \cap \mathrm{supp}(v - v') = \emptyset$, which is only possible if for all $k \in \mathrm{supp}(z_{n_{i_j}})$, $\lambda_k(v') = \lambda_k(v)$. Therefore, for all $j < t$ and $k \in \mathrm{supp}(z_{n_{i_j}})$, $\lambda_k(v') = \lambda_k(v)$, so $v' = v$, as desired. ∎

This completes the proof of Lemma 2.1. ∎

3. Local Ramsey theory of block subspaces. Smythe [24] explored the local Ramsey theory of block subspaces of a countable vector space, mainly focusing on $\leq^*$-upward closed subsets of $E^{[\infty]}$ (which Smythe calls *families*) and their various properties. We provide an overview of this theory, along with several additional results which will prove useful in later sections.

3.1. Semicoideals. To avoid confusion with other uses of the word "family" such as almost disjoint families, we shall use the term "semicoideals" in place of "families".

DEFINITION 2. A subset $\mathcal{H} \subseteq E^{[\infty]}$ is a *semicoideal* if it is $\leq^*$-upward closed. That is, if $A \in \mathcal{H}$ and $A \leq^* B$, then $B \in \mathcal{H}$.

To cope with the failure of the pigeonhole principle in the setting of countable vector spaces over $\mathbb{F} \neq \mathbb{F}_2$, Smythe introduced a weaker form of the pigeonhole property of semicoideals, which he calls *fullness*.

NOTATION 2. Let $\mathcal{H} \subseteq E^{[\infty]}$ be a semicoideal. Given $A \in \mathcal{H}$, we write

$$\mathcal{H}{\restriction}A := \{B \in \mathcal{H} : B \leq A\}.$$

DEFINITION 3. Let $\mathcal{H} \subseteq E^{[\infty]}$ be a semicoideal.

(1) A subset $\mathcal{D} \subseteq E^{[\infty]}$ is *$\mathcal{H}$-dense* below some $A \in \mathcal{H}$ if for all $B \in \mathcal{H}{\restriction}A$, there exists some $C \leq B$ such that $C \in \mathcal{D}$. A set $Y \subseteq E$ is *$\mathcal{H}$-dense* below A if the set $\{B \in E^{[\infty]} : \langle B\rangle \subseteq Y\}$ is.
(2) We say that $\mathcal{H}$ is *full* if for all $Y \subseteq E$ such that Y is $\mathcal{H}$-dense below some $A \in \mathcal{H}$, there exists some $B \in \mathcal{H}{\restriction}A$ such that $\langle B\rangle \subseteq Y$.

It turns out that if E is a vector space over $\mathbb{F}_2$, then fullness coincides with the local pigeonhole principle property. In this case, we say that $\mathcal{H}$ is a *coideal*.

DEFINITION 4. Let $\mathcal{H} \subseteq E^{[\infty]}$ be a semicoideal. We say that $\mathcal{H}$ is a *coideal* if for all $Y \subseteq E$ and $A \in \mathcal{H}$, there exists some $B \in \mathcal{H}{\restriction}A$ such that $\langle B\rangle \subseteq Y$ or $\langle B\rangle \subseteq Y^c$.

By Hindman's theorem, $E^{[\infty]}$ is a coideal if E is a vector space over $\mathbb{F}_2$.

LEMMA 3.1. *Suppose that E is a vector space over $\mathbb{F}_2$, and let $\mathcal{H} \subseteq E^{[\infty]}$ be a semicoideal. Then $\mathcal{H}$ is full iff $\mathcal{H}$ is a coideal.*

Proof. Suppose that $\mathcal{H}$ is a coideal. Let $Y \subseteq E$, and suppose that Y is $\mathcal{H}$-dense below some $A \in \mathcal{H}$. Since $\mathcal{H}$ is a coideal, there exists some $B \in \mathcal{H}{\restriction}A$ such that $\langle B\rangle \subseteq Y$ or $\langle B\rangle \subseteq Y^c$. Since Y is $\mathcal{H}$-dense below A, the latter option is not possible.

Now suppose that $\mathcal{H}$ is full. Let $Y \subseteq E$, and let $A \in \mathcal{H}$. If Y is $\mathcal{H}$-dense below A, then we are done by the fullness of $\mathcal{H}$, so assume otherwise. This means that there exists some $B \in \mathcal{H}{\restriction}A$ such that for all $C \leq B$, $\langle C\rangle \not\subseteq Y$. By Hindman's theorem, for all $C \in \mathcal{H}{\restriction}B$, there exists some $D \leq C$ such that $\langle D\rangle \subseteq Y^c$ (as $\langle D\rangle \not\subseteq Y$). But this means that Y^c is $\mathcal{H}$-dense below B, so there exists some $C \in \mathcal{H}\restriction B$ such that $\langle C\rangle \subseteq Y^c$, as desired. ∎

3.2. Selectivity properties. In the process of studying semicoideals related to mad families (particularly $\mathcal{H}(\mathcal{A})$, see Definition 11), Smythe extended several local Ramsey-theoretic properties of coideals of $[\mathbb{N}]^\infty$ to semicoideals of $E^{[\infty]}$. Selectivity properties of (ultra)filters (see Definition 21) in $E^{[\infty]}$ have also been studied in [26]. The first such property extends the (p)-property for coideals of $[\mathbb{N}]^\infty$ (see [28, Lemma 7.4]).

Definition 5. Let $\mathcal{H} \subseteq E^{[\infty]}$ be a semicoideal.

(1) Let $(A_n)_{n<\omega}$ be a $\leq$-decreasing sequence in $\mathcal{H}$. We say that $B \in \mathcal{H}$ *weakly diagonalises* $(A_n)_{n<\omega}$ if $B \leq A_0$ and $B \leq^* A_n$ for all n (3).
(2) We say that $\mathcal{H}$ satisfies the *(p)-property*, or that $\mathcal{H}$ is a *(p)-semicoideal*, if every $\leq$-decreasing sequence in $\mathcal{H}$ has a weak diagonalisation in $\mathcal{H}$.

We shall provide a similar extension of the *(q)-property* (see [28, Lemma 7.4]). Smythe [24, Definition 8.11] also made a similar extension, which he termed *spreadness*.

Definition 6. Let $\mathcal{H} \subseteq E^{[\infty]}$ be a semicoideal. We say that $\mathcal{H}$ satisfies the *(q)-property*, or that $\mathcal{H}$ is a *(q)-semicoideal*, if for all $A \in \mathcal{H}$ and for all increasing sequences $(I_m)_{m<\omega}$ of finite intervals, there exists some $B \in \mathcal{H}\restriction A$ such that for all n, there exists some m such that

(1) $\langle r_n(B)\rangle \subseteq \langle r_{\min(I_m)-1}(A)\rangle$,
(2) $B/n \leq A/\max(I_m)$.

In a similar vein, we extend the definition of selectivity of coideals of $[\mathbb{N}]^\infty$ (see [28, Definition 7.3]) to semicoideals of block subspaces.

Definition 7. Let $\mathcal{H} \subseteq E^{[\infty]}$ be a semicoideal.

(1) Let $A \in \mathcal{H}$, and let $(A_n)_{n<\omega}$ be a $\leq$-decreasing sequence in $\mathcal{H}$ below A (i.e. $A_0 \leq A$). We say that $B \in \mathcal{H}\restriction A$ *diagonalises* $(A_n)_{n<\omega}$ *below* A if $B/n \leq A_{d_A(r_n(B))}$ for all $n < \omega$.
(2) We say that $\mathcal{H}$ is *selective* if for all $A \in \mathcal{H}$ and all $\leq$-decreasing chains $(A_n)_{n<\omega}$ of elements in $\mathcal{H}$ below A, there exists a diagonalisation of $(A_n)_{n<\omega}$ below A in $\mathcal{H}$.

Smythe [24, Definition 4.1] also proposed a notion of *strong diagonalisation*, followed by a notion of *strong (p)-property*. The following lemma shows that they are equivalent if E is a vector space over a finite field. It is unclear if they are equivalent in case the underlying field is infinite.

(3) In [24, Definition 2.2], this property was similarly defined using the term *diagonalises*. We shall reserve this term for a stronger form of diagonalisation later (Definition 7)

LEMMA 3.2. *Suppose that E is a vector space over a finite field. Let $\mathcal{H} \subseteq E^{[\infty]}$ be a semicoideal. The following are equivalent:*

(1) *$\mathcal{H}$ is selective.*
(2) *If $(A_a)_{a\in E^{[<\infty]}}$ generates a filter in $\mathcal{H}$, then there exists some $B \in \mathcal{H}$ such that for all n, $B/n \leq A_{r_n(B)}$.*

Proof. (2)⇒(1). Let $(A_n)_{n<\omega}$ be a $\leq$-decreasing sequence in $\mathcal{H}$ below some $A \in \mathcal{H}$. For each $a \in E^{[<\infty]}\restriction A$, we let $A_a := A_{d_A(a)}$, and if $a \notin E^{[<\infty]}\restriction A$ then we let $A_a := A$. By (2), there exists some $B \in \mathcal{H}$ such that $B/n \leq A_{r_n(B)} = A_{d_A(r_n(B))}$ for all n, so B diagonalises $(A_n)_{n<\omega}$ below A.

(1)⇒(2). Suppose that $(A_a)_{a\in E^{[<\infty]}}$ generates a filter in $\mathcal{H}$. Let $A := A_\emptyset$. For each n, let:

$$\mathcal{F}_n := \{a \in E^{[<\infty]}\restriction A : d_A(a) \leq n\}.$$

Since the underlying field is finite, $\mathcal{F}_n$ is finite for all n. Thus, we may inductively define a $\leq$-decreasing sequence $(A_n)_{n<\omega}$ in $\mathcal{H}$ such that $A_n \leq A_a$ for all $a \in \mathcal{F}_n$. Since $\mathcal{H}$ is selective, we may let $B \in \mathcal{H}$ be such that $B/n \leq A_{d_A(r_n(B))} \leq A_{r_n(B)}$ for all n. Then B is the desired element in $\mathcal{H}$. ∎

Note that the assumption that the underlying field is finite is not required in the direction (2)⇒(1).

We conclude the section by examining the relationship among the selectivity property, the (p)-property and the (q)-property.

LEMMA 3.3. *Let $\mathcal{H} \subseteq E^{[\infty]}$ be a (q)-semicoideal. Let $(A_n)_{n<\omega}$ be a $\leq$-decreasing sequence in $\mathcal{H}$. If $B \in \mathcal{H}\restriction A_0$ weakly diagonalises $(A_n)_{n<\omega}$, then there exists some $C \in \mathcal{H}\restriction B$ which diagonalises $(A_n)_{n<\omega}$ below A_0.*

Proof. Let $A := A_0$, and let $B \in \mathcal{H}\restriction A$ weakly diagonalise $(A_n)_{n<\omega}$. Let $(I_m)_{m<\omega}$ be any increasing sequence of intervals such that $B/\max(I_m) \leq A_{d_A(r_{\min(I_m)-1}(B))}$ for all m. By the (q)-property, there exists some $C \in \mathcal{H}\restriction B$ such that for all n, there exists some m such that $\langle r_n(C)\rangle \subseteq \langle r_{\min(I_m)-1}(B)\rangle$ and $C/n \leq B/\max(I_m)$. The first property implies that

$$d_A(r_n(C)) \leq d_A(r_{\min(I_m)}(B)),$$

so by the second property,

$$C/n \leq B/\max(I_m) \leq A_{d_A(r_{\min(I_m)}(B))} \leq A_{d_A(r_n(C))},$$

so C diagonalises $(A_n)_{n<\omega}$ below A_0. ∎

PROPOSITION 3.4. *Let $\mathcal{H} \subseteq E^{[\infty]}$ be a semicoideal. The following are equivalent:*

(1) *$\mathcal{H}$ is selective.*
(2) *$\mathcal{H}$ satisfies both the (p)-property and the (q)-property.*

Proof. (2)⇒(1). Let $(A_n)_{n<\omega}$ be a $\leq$-decreasing sequence in $\mathcal{H}$, and let $A := A_0$. By the (p)-property, there exists some $B \in \mathcal{H}\restriction A$ which weakly diagonalises $(A_n)_{n<\omega}$. Since $\mathcal{H}$ satisfies the (q)-property, by Lemma 3.3 there exists some $C \in \mathcal{H}\restriction B$ which diagonalises $(A_n)_{n<\omega}$. Therefore, $\mathcal{H}$ is selective.

(1)⇒(2). Since every diagonalisation is a weak diagonalisation, $\mathcal{H}$ satisfies the (p)-property. We shall show that $\mathcal{H}$ satisfies the (q)-property. Let $A \in \mathcal{H}$, and let $(I_m)_{m<\omega}$ be any increasing sequence of finite intervals. For each n, let $m(n)$ be the least integer such that $\min(I_{m(n)}) > n$. Let $B \in \mathcal{H}\restriction A$ diagonalise $(A/\max(I_{m(n)}))_{n<\omega}$ below A. Then for any n, if $k := d_A(r_n(B))$, then

$$\langle r_n(B)\rangle \subseteq \langle r_k(A)\rangle \subseteq \langle r_{\min(I_{m(k)-1})}(A)\rangle,$$

and

$$B/n \leq A_{d_A(r_n(B))} = A/\max(I_{m(k)}),$$

as desired. ■

We shall examine the selectivity of semicoideals generated by almost disjoint families in the next section.

4. Almost disjoint families of subspaces.

We recall the following:

DEFINITION 8. Two infinite-dimensional subspaces $V, W \subseteq E$ are *almost disjoint* if $V \cap W$ is a finite-dimensional subspace of E. Otherwise, we say that V and W are *compatible*.

DEFINITION 9. Let $\mathcal{A}$ be a family of infinite-dimensional subspaces of E. We say that $\mathcal{A}$ is *almost disjoint* if all subspaces in $\mathcal{A}$ are pairwise almost disjoint. We say that $\mathcal{A}$ is *maximal almost disjoint* (or just *mad*) if $\mathcal{A}$ is not strictly contained in another almost disjoint family of infinite-dimensional subspaces.

Since every infinite-dimensional subspace of E contains a block subspace, an almost disjoint family of subspaces $\mathcal{A}$ is mad iff there is no block subspace which is almost disjoint from every element of $\mathcal{A}$.

We dedicate this section to some basic definitions/notations and results on almost disjoint families of block subspaces, and their relationship with the local Ramsey theory of semicoideals of block subspaces.

4.1. Uncountability of almost disjoint families.

We begin with the most fundamental question of the theory of mad families – the uncountability of infinite mad families. Consider the following two cardinals:

$\mathfrak{a}^*_{\mathrm{vec},\mathbb{F}} := \min\{|\mathcal{A}| : \mathcal{A}$ is an infinite mad family of subspaces over $\mathbb{F}\}$,

$\mathfrak{a}_{\mathrm{vec},\mathbb{F}} := \min\{|\mathcal{A}| : \mathcal{A}$ is an infinite mad family of block subspaces over $\mathbb{F}\}$.

It is clear that $\mathfrak{a}^*_{\mathrm{vec},\mathbb{F}} \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}} \leq \mathfrak{c}$. By [25, Proposition 2.5], $\aleph_1 \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$ (in fact, $\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$), so $\mathfrak{a}_{\mathrm{vec},\mathbb{F}}$ is a cardinal invariant. One may ask if the same holds for $\mathfrak{a}^*_{\mathrm{vec},\mathbb{F}}$ – that is, if $\aleph_1 \leq \mathfrak{a}^*_{\mathrm{vec},\mathbb{F}}$ is true. Smythe's proof of $\aleph_1 \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$ heavily relies on the following lemma:

Lemma 4.1 ([25, Lemma 2.3]). *Let $A \in E^{[\infty]}$ be an infinite block sequence, and let $a \in E^{[<\infty]}$ be a finite block sequence. Then there exists some $N < \omega$, that depends only on A and $\max(\mathrm{supp}(a))$, such that for all $x > N$ for which $x \notin \langle A\rangle$,*

$$\langle a^\frown x\rangle \cap \langle A\rangle = \langle a\rangle \cap \langle A\rangle.$$

Smythe remarked that removing the "block" requirement from this lemma seems difficult. $\aleph_1 \leq \mathfrak{a}^*_{\mathrm{vec},\mathbb{F}}$ was also stated in [13, Proposition 3.1], but Kolman further assumed that in the countable almost disjoint family of subspaces $\{V_n : n < \omega\}$, $V_n \cap \sum_{i\in I} V_i$ is finite-dimensional for all finite $n \notin I \subseteq \omega$. This is generally not satisfied by families of subspaces that are pairwise almost disjoint. The present author raised this question on MathOverflow, to which user527492 provided an affirmative answer (see [1]).

Lemma 4.2. $\aleph_1 \leq \mathfrak{a}^*_{\mathrm{vec},\mathbb{F}}$.

Proof. Let $\{V_n : n < \omega\}$ be a countable family of almost disjoint subspaces of E. We wish to find some infinite-dimensional subspace W which is almost disjoint from V_n for all n. We shall first prove a useful claim.

Claim 5. *Let $U, V, W \subseteq E$ be subspaces such that V and W are almost disjoint, and U is finite-dimensional. Then $U+V$ and W are almost disjoint.*

Proof. Define a linear map $\phi : V\times W \to E$ by stipulating that $\phi(v,w) := w - v$. Let $\rho : E \to E/U$ be the canonical quotient map, and let $\psi := \rho\circ\phi$. Note that

$$\dim(\ker(\psi)) \leq \dim(\ker(\phi)) + \dim(\ker(\rho)) < \infty.$$

Let $\pi : V \times W \to W$ be the projection map. Observe that for all $w \in W$, we have

$$\begin{aligned} w \in (U+V)\cap W &\iff w = u + v \text{ for some } u \in U,\ v \in V \\ &\iff w - v \in U \text{ for some } v \in V \\ &\iff (v,w) \in \ker(\psi) \text{ for some } v \in V. \end{aligned}$$

This implies that π maps $\ker(\psi)$ surjectively onto $(U+V)\cap W$, and since $\ker(\psi)$ is of finite dimension, so is $(U+V)\cap W$. ■

We shall define the subspace $W = \langle x_n : n < \omega\rangle$ by defining x_n inductively as follows: Let $x_0 \in V_0$ be any non-zero vector. Now assume that we have defined a linearly independent set $\{x_i : i < n\}$ such that for all $k < n$, we have

$$\langle x_i : i < n\rangle \cap V_k \subseteq \langle x_i : i \leq k\rangle.$$

We wish to find some vector $x_n \in V_n$, linearly independent of $\{x_i : i < n\}$, such that for all $k < n$, we have

$$\langle x_i : i \leq n \rangle \cap V_k \subseteq \langle x_i : i \leq k \rangle.$$

If we can complete this induction step, then we are done as we then se that $W \cap V_n \subseteq \langle x_0, \dots, x_n \rangle$ is finite-dimensional for all n.

For each $k < n$, let $U_k := V_k + \langle x_i : i < n \rangle$. By Claim 5, $V_n \cap U_k$ is of finite dimension. Since no infinite-dimensional vector space is a finite union of finite-dimensional subspaces, we have

$$V_n \setminus \bigcup_{k<n} V_n \cap U_k \neq \emptyset,$$

so we may let x_n be any vector in this set. We shall show that this vector x_n works. We first note that $\{x_i : i \leq n\}$ is linearly independent, as $\langle x_i : i < n \rangle \subseteq U_0$ and $x_n \notin U_0$. Now let $k < n$ and $x \in \langle x_i : i \leq n \rangle \cap V_k$, so we have $x = \sum_{i \leq n} \xi_i x_i \in V_k$. If $\xi_n \neq 0$, then

$$x_n = \xi_n^{-1}\Big(x - \sum_{i<n} \xi_i x_i\Big) \in V_n \cap U_k,$$

contradicting the choice of x_n. Therefore, $\xi_n = 0$, which implies that

$$x \in \langle x_i : i < n \rangle \cap V_k = \langle x_i : i \leq k \rangle \cap V_k,$$

as desired. ■

Smythe [25] asked whether $\mathfrak{a}^*_{\mathrm{vec},\mathbb{F}} = \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$ holds in ZFC, and this problem remains open to the author's knowledge. In the same paper, Smythe showed that no finite almost disjoint family of block subspaces of cardinality greater than 1 is maximal. We extend this fact to arbitrary subspaces.

Proposition 4.3. *Let $\mathcal{A}$ be a finite almost disjoint family of subspaces of E such that $|\mathcal{A}| > 1$. Then $\mathcal{A}$ is not maximal.*

Proof. Let $\mathcal{A} = \{V_k : k < n\}$ be a finite almost disjoint family of subspaces of E.

Claim 6. $E \neq \bigcup_{k<n} V_k$.

Proof. We first assume that for all $j \neq k$, $V_j \cap V_k = \{0\}$. Let $\mathbb{F}$ be the field that E is defined over. If $\mathbb{F}$ is infinite, then we are done, as any subspace over an infinite field is not a finite union of proper subspaces. Assume that $|\mathbb{F}| = N < \infty$, and suppose for a contradiction that $E = \bigcup_{k<n} V_k$. Let M be some large number (which we shall fix later), and let $W := \langle e_n : n < M \rangle$. Let $U_k := W \cap V_k$, and by taking M large enough we may assume that U_k is non-trivial for all k. Note that $W = \bigcup_{k<n} U_k$.

Observe that $\dim(W) = M$, so if $U, U' \subseteq W$ are two subspaces such that $\dim(U), \dim(U') > M/2$, then U and U' intersect non-trivially. Since $\{U_k : k < n\}$ is a collection of proper subspaces of W with pairwise trivial intersections, and $n > 2$, there is at most one k such that $\dim(U_k) > M/2$. We may assume that if such a k exists, then $\dim(U_0) > M/2$. Note that since U_k is non-trivial for $k > 0$ and $U_0 \cap U_k \neq \emptyset$, we have $\dim(U_0) \leq M-1$. Consequently, for each $k > 0$, U_k contains at most $N^{M/2}$ vectors, which implies that

$$\Big|\bigcup_{k<n} U_k\Big| \leq (n-1)\cdot N^{M/2} + N^{M-1}.$$

On the other hand, $|W| = N^M$. Since

$$\begin{aligned} N^M - ((n-1)N^{M/2} + N^{M-1}) &= (N-1)N^{M-1} - (n-1)N^{M/2} \\ &\geq N^{M-1} - (n-1)N^{M/2} \\ &= (N^{M/2-1} - n + 1)N^{M/2} \\ &> 0 \quad \text{for } M \text{ large enough,} \end{aligned}$$

we have $W \supsetneq \bigcup_{k<n} U_k$, a contradiction.

For the general case, suppose that $\{V_k : k < n\}$ is almost disjoint and $E = \bigcup_{k<n} V_k$. Let M be large enough that $M > \max(\operatorname{supp}(x))$ for all $x \in V_j \cap V_k$ and $j \neq k$. Let $W' := \langle e_n : n \geq M\rangle$, and notice that $W' = \bigcup_{k<n} W' \cap V_k$. Then $\{W' \cap V_k : k < n\}$ is a collection of infinite-dimensional subspaces of W' with pairwise trivial intersections, contradicting the argument above. ∎

We shall now construct some $V = \langle x_m : m < \omega\rangle \subseteq E$ which intersects V_k trivially for every $k < n$ by defining x_m inductively. Let $x_0 \in E \setminus \bigcup_{k<n} V_k$ be any vector. Now suppose that we have defined x_i for $i < m$ such that for all $k < n$,

$$\langle x_i : i < m\rangle \cap V_k = \{0\}.$$

For each $k < n$, let $U_k := V_k + \langle x_i : i < m\rangle$. By Claim 5 in the proof of Lemma 4.2, $\{U_k : k < n\}$ is an almost disjoint family of subspaces of E, so by applying Claim 6 to $\{U_k : k < n\}$, $E \neq \bigcup_{k<n} U_k$. Let $x_m \in E \setminus \bigcup_{k<n} U_k$. Note that $\{x_i : i \leq m\}$ is linearly independent, as $x_i \in U_0$ for all $i < m$ but $x_m \notin U_0$. We shall show that

$$\langle x_i : i \leq m\rangle \cap V_k = \{0\}.$$

If we can complete this induction step, then we are done as $V \cap V_k = \bigcup_{m<\omega} \langle x_i : i < m\rangle \cap V_k = \{0\}$ for all $k < n$.

Let $y \in \langle x_i : i \leq m\rangle \cap V_k$ for some $k < n$. Then $y = \sum_{i\leq m} \xi_i x_i$ for some $\xi_i \in \mathbb{F}$. If $\xi_m \neq 0$, then

$$x_m = \xi_m^{-1}\Big(y - \sum_{i<m} \xi_i x_i\Big) \in U_k,$$

contradicting the choice of x_m. Therefore, $\xi_m = 0$, which implies that

$$y \in \langle x_i : i < m \rangle \cap V_k = \{0\},$$

as desired. ■

4.2. Ideals and semicoideals generated by almost disjoint families. If $\mathcal{A}$ is an almost disjoint family in $[\omega]^\omega$, then the following families of sets are often studied:

$$\begin{aligned} \mathcal{I}(\mathcal{A}) &:= \{Y \subseteq \omega : Y \subseteq^* A_0 \cup \cdots \cup A_{n-1} \text{ for some } A_0, \ldots, A_{n-1} \in \mathcal{A}\}, \\ \mathcal{I}^+(\mathcal{A}) &:= \mathcal{P}(\omega) \setminus \mathcal{I}(\mathcal{A}), \\ \mathcal{I}^{++}(\mathcal{A}) &:= \{Y \subseteq \omega : \text{the set } \{A \in \mathcal{A} : Y \cap A \text{ is infinite}\} \text{ is infinite}\}, \\ \mathcal{H}(\mathcal{A}) &:= \mathcal{I}^{++}(\mathcal{A}). \end{aligned}$$

It is simple to verify that $\mathcal{I}^+(\mathcal{A}) = \mathcal{I}^{++}(\mathcal{A})$ iff $\mathcal{A}$ is maximal. Motivated by this, Smythe introduced a version of $\mathcal{H}(\mathcal{A})$ for almost disjoint families of subspaces. $\mathcal{H}(\mathcal{A})$ is a semicoideal, and Smythe studied the conditions that $\mathcal{A}$ must satisfy to make $\mathcal{H}(\mathcal{A})$ a full semicoideal.

Definition 10. Let $Y \subseteq E$.

(1) Y is *big* if there exists some $A \in E^{[\infty]}$ such that $\langle A \rangle \subseteq Y$.
(2) Y is *small* if it is not big.
(3) Y is *very small* if for all $Z \subseteq E$ such that Z is small, $Y \cup Z$ is small.

Note that if $V \subseteq E$ is a subspace, then V is big iff V is infinite-dimensional.

If E is a vector space over $\mathbb{F}_2$, then by Hindman's theorem, every small set is very small. On the other hand, if E is a vector space over $\mathbb{F} \neq \mathbb{F}_2$, then this is no longer true – fix any $\mu \in \mathbb{F} \setminus \{0, 1\}$. We may define the set

$$Y := \{x \in E : x = e_n + y \text{ for some } e_n < y \text{ or } y = 0\}.$$

Observe that Y and Y^c are both small – for any $A \in E^{[\infty]}$, let $x \in \langle A \rangle$ be any non-zero vector. Then $x = \lambda_n(x) e_n + y$ for some $e_n < y$ or $y = 0$ and $\lambda_n(x) \neq 0$, so $\frac{1}{\lambda_n(x)} x \in \langle A \rangle \cap Y$ and $\frac{\mu}{\lambda_n(x)} x \in \langle A \rangle \cap Y^c$. However, since $E = Y \cup Y^c$ and E is big, Y is not very small.

Lemma 4.4. *Let $A \in E^{[\infty]}$ and let $V \subseteq E$ be a subspace. Then $\langle A \rangle \cap V$ is small iff $\langle A/N \rangle \cap V = \emptyset$ for some N. Consequently:*

(1) *If $A, B \in E^{[\infty]}$, then A and B are almost disjoint iff $\langle A/N \rangle \cap \langle B \rangle = \emptyset$ for some N.*
(2) *If A/N and B are almost disjoint for some N, then A and B are almost disjoint.*

Proof. Suppose that $\langle A \rangle \cap V$ is big. Let $B \in E^{[\infty]}$ be such that $\langle B \rangle \subseteq \langle A \rangle \cap V$. For any N, there exists some M such that $B/M \leq A/N$. This implies that $\langle B/M \rangle \subseteq \langle A/N \rangle \cap V$, so in particular $\langle A/N \rangle \cap V \neq \emptyset$ for all N.

Conversely, suppose that $\langle A/N\rangle \cap V \neq \emptyset$ for all N. We construct some $B = (y_n)_{n<\omega} \in E^{[\infty]}$ such that $\langle B\rangle \subseteq \langle A\rangle \cap V$ inductively as follows: Assuming that $y_0, \dots, y_{n-1}$ have been defined so far, let N be large enough that $\operatorname{supp}(y_{n-1}) < A/N$. We let $y_n \in \langle A/N\rangle \cap V$ be any element. Since $\langle A\rangle \cap V$ is a subspace, we have $\langle B\rangle \subseteq \langle A\rangle \cap V$, so $\langle A\rangle \cap V$ is big. ∎

LEMMA 4.5. *Let $U, V \subseteq E$ be subspaces.*

(1) *If $U \cap V$ is small, then $U \cap V$ is very small.*
(2) *If $U \cap V$ is small, then $U \setminus V$ is very small.*

Consequently, if U is infinite-dimensional, then it is not possible to have both $U \cap V$ and $U \setminus V$ small.

Proof. Assume that $U \cap V$ is small. Let $Y \subseteq E$, and suppose that $\langle A\rangle \subseteq (U \cap V) \cup Y$ for some $A = (x_n)_{n<\omega} \in E^{[\infty]}$. Since $U \cap V$ is a finite-dimensional space, we may let N be large enough that $\operatorname{supp}(x_N) > y$ for all $y \in U \cap V$. Then $\langle A/N\rangle \cap (U \cap V) = \emptyset$, so $\langle A/N\rangle \subseteq Y$. Therefore, Y is big.

Now assume that $U \setminus V$ is small. Let $Y \subseteq E$, and suppose that $\langle A\rangle \subseteq (U \setminus V) \cup Y$. We consider three cases. For the first case, suppose that $\langle A\rangle \cap U$ is small. By Lemma 4.4, let N be large enough that $\langle A/N\rangle \cap U = \emptyset$. Then this implies that $\langle A/N\rangle \subseteq Y$, so Y is big.

For the second case, suppose that $\langle A\rangle \cap V$ is big. Let $B \in E^{[\infty]}$ be such that $\langle B\rangle \subseteq \langle A\rangle \cap V$. Then $\langle B\rangle \subseteq (U \setminus V) \cup Y$, and since $\langle B\rangle \cap (U \setminus V) = \emptyset$, we have $\langle B\rangle \subseteq Y$. Therefore, Y is big.

For the last case, suppose that $\langle A\rangle \cap U$ is big but $\langle A\rangle \cap V$ is small. Let $B \in E^{[\infty]}$ be such that $\langle B\rangle \subseteq \langle A\rangle \cap U$, and note that $\langle B\rangle \cap V$ is also small. By Lemma 4.4, let N be large enough that $\langle B/N\rangle \cap V = \emptyset$. Since $\langle B/N\rangle \subseteq U$, this implies that $\langle B/N\rangle \subseteq U \setminus V$, contradicting our assumption that $U \setminus V$ is small. ∎

DEFINITION 11. Let $\mathcal{A}$ be an almost disjoint family of subspaces. We then define the following:

$$\begin{aligned}
\mathcal{I}(\mathcal{A}) &:= \left\{ Y \subseteq E : \begin{matrix} Y \setminus (V_0 \cup \dots \cup V_{n-1}) \text{ is small} \\ \text{for some } V_0, \dots, V_{n-1} \in \mathcal{A} \end{matrix} \right\}, \\
\mathcal{I}^+(\mathcal{A}) &:= \mathcal{P}(E) \setminus \mathcal{I}(\mathcal{A}), \\
\mathcal{I}^{++}(\mathcal{A}) &:= \{ Y \subseteq E : \text{the set } \{V \in \mathcal{A} : Y \cap V \text{ is big}\} \text{ is infinite}\}, \\
\mathcal{H}^-(\mathcal{A}) &:= \{ B \in E^{[\infty]} : \langle B\rangle \in \mathcal{I}^+(\mathcal{A})\}, \\
\mathcal{H}(\mathcal{A}) &:= \{ B \in E^{[\infty]} : \langle B\rangle \in \mathcal{I}^{++}(\mathcal{A})\}.
\end{aligned}$$

LEMMA 4.6. *Let $\mathcal{A}$ be an infinite almost disjoint family of subspaces. Then $\mathcal{I}^{++}(\mathcal{A}) \subseteq \mathcal{I}^+(\mathcal{A})$ (so $\mathcal{H}(\mathcal{A}) \subseteq \mathcal{H}^-(\mathcal{A})$).*

Proof. Suppose that $Y \in \mathcal{I}(\mathcal{A})$, and let $V_0, \dots, V_{n-1} \in \mathcal{A}$ be such that $Y \setminus (V_0 \cup \dots \cup V_{n-1})$ is small. Then for any $U \in \mathcal{A}$ such that $U \neq V_i$ for any $i < n$, we have

$$Y \cap U \subseteq \Big(Y \setminus \bigcup_{i<n} V_i\Big) \cup \Big(U \cap \bigcup_{i<n} V_i\Big) = \Big(Y \setminus \bigcup_{i<n} V_i\Big) \cup \bigcup_{i<n} V_i \cap U.$$

By Lemma 4.5, $V_i \cap U$ is very small for all $i < n$, so $\bigcup_{i<n} V_i \cap U$ is very small. Since $Y \setminus \bigcup_{i<n} V_i$ is small, $Y \cap U$ is small. Therefore,

$$\{U \in \mathcal{A} : Y \cap U \text{ is big}\} \subseteq \{V_0, \dots, V_{n-1}\},$$

so $Y \notin \mathcal{I}^{++}(\mathcal{A})$. ∎

LEMMA 4.7. *Let $\mathcal{A}$ be an infinite almost disjoint family of subspaces, and let $U \subseteq E$ be a subspace. The following are equivalent:*

(1) $U \in \mathcal{I}^{+}(\mathcal{A})$.
(2) $U \in \mathcal{I}^{++}(\mathcal{A})$ *or there exists some $B \in E^{[\infty]}$, with $\langle B\rangle \subseteq U$, such that B is almost disjoint from every element of $\mathcal{A}$.*

Consequently, if $\mathcal{A}$ is mad, then $\mathcal{I}^{+}(\mathcal{A}) = \mathcal{I}^{++}(\mathcal{A})$ (hence $\mathcal{H}^{-}(\mathcal{A}) = \mathcal{H}(\mathcal{A})$).

Proof. (1)⇒(2). Let $U \in \mathcal{I}^{+}(\mathcal{A})$. Suppose that for all $B \in E^{[\infty]}$ such that $\langle B\rangle \subseteq U$, B is not almost disjoint from some element of $\mathcal{A}$. We let $V_0 \in \mathcal{A}$ be compatible with U (i.e. $U \cap V_0$ is big), and let $B_0 \in E^{[\infty]}$ be such that $\langle B_0\rangle \subseteq U \cap V_0$. Clearly $\langle B_0\rangle \in \mathcal{I}(\mathcal{A})$, so $U \setminus \langle B_0\rangle \in \mathcal{I}^{+}(\mathcal{A})$ (for otherwise $U \in \mathcal{I}(\mathcal{A})$). Let $B_1' \in E^{[\infty]}$ be such that $\langle B_1'\rangle \subseteq U \setminus \langle B_0\rangle$. By our assumption, B_1' is compatible with some $V_1 \in \mathcal{A}$, so let $B_1 \in E^{[\infty]}$ be such that $\langle B_1\rangle \subseteq \langle B_1'\rangle \cap V_1$. By a similar reasoning to the above, we have $U \setminus (\langle B_0\rangle \cup \langle B_1\rangle) \in \mathcal{I}^{+}(\mathcal{A})$. We may repeat this process indefinitely, and we find that $\{V_n : n < \omega\} \subseteq \mathcal{A}$ is such that U is compatible with V_n for all n. Therefore, $U \in \mathcal{I}^{++}(\mathcal{A})$.

(2)⇒(1). If $U \in \mathcal{I}^{++}(\mathcal{A}) \subseteq \mathcal{I}^{+}(\mathcal{A})$, then we are done, so assume that there exists some $B \in E^{[\infty]}$ such that $\langle B\rangle \subseteq U$ and B is almost disjoint from every element of $\mathcal{A}$. Then for any $V_0, \dots, V_{n-1} \in \mathcal{A}$, by Lemma 4.4 we may let N be large enough that $\langle B/N\rangle \cap V_i = \emptyset$ for all $i < n$. This implies that $\langle B/N\rangle \subseteq U \setminus \bigcup_{i<n} V_i$. Therefore, $B/N \in \mathcal{H}^{-}(\mathcal{A})$, so $U \in \mathcal{I}^{+}(\mathcal{A})$ as $\mathcal{I}^{+}(\mathcal{A})$ is $\le$-upward closed. ∎

We conclude this part with three more lemmas that we shall use repeatedly.

LEMMA 4.8. *Let $A \in E^{[\infty]}$, and suppose that $\{B_n : n < \omega\} \subseteq E^{[\infty]}$ is such that $B_n \le A$ for all n. Then there exists some $B \le A$ such that B is compatible with B_n for all n, and*

$$(*) \qquad \langle B\rangle \subseteq \Big\langle \bigcup_{n<\omega} \langle B_n\rangle \Big\rangle \subseteq \langle A\rangle.$$

Proof. Let $\Gamma : \omega \to \omega$ be any infinite-to-one surjection, i.e. a surjection such that $\Gamma^{-1}(m)$ is infinite for all $m < \omega$. We define $B = (x_n)_{n<\omega}$ inductively as follows: If $(x_k)_{k<n}$ has been defined, let $x_n \in \langle B_{\Gamma(n)}\rangle$ be any element such that $(x_k)_{k<n} < x_n$. Then $(x_n)_{n\in\Gamma^{-1}(m)}$ witnesses that B and B_m are compatible for all $m < \omega$, and since $x_n \in \bigcup_{m<\omega}\langle B_m\rangle$ for all n, the inclusions $(*)$ are satisfied. ∎

LEMMA 4.9. *Let $\mathcal{A}$ be an infinite almost disjoint family of subspaces, and let $U \subseteq E$ be a subspace. Suppose further that $U \in \mathcal{I}^+(\mathcal{A})$ and there does not exist any $B \in E^{[\infty]}$ such that $\langle B\rangle \subseteq U$ and B is almost disjoint from every element of $\mathcal{A}$. Then there exists an infinite set $\{V_n : n < \omega\} \subseteq \mathcal{A}$ and an infinite disjoint family $\{B_n : n < \omega\}$ such that for all n, $\langle B_n\rangle \subseteq V_n \cap U$. .*

Proof. This is similar to showing (1)⇒(2) of Lemma 4.7. As $U \in \mathcal{I}^+(\mathcal{A})$, U is big, so there exists some B_0' such that $\langle B_0'\rangle \subseteq U$. By our assumption, B_0' is compatible with some $V_0 \in \mathcal{A}$. Let B_0 be such that $\langle B_0\rangle \subseteq \langle B_0'\rangle \cap V_0$. Since $U \in \mathcal{I}^+(\mathcal{A})$, $U \setminus V_0$ is big. Given some $n < \omega$, suppose that we have defined an infinite disjoint family $\{B_k : k < n\}$ and $\{V_k : k < n\} \subseteq \mathcal{A}$ such that $\langle B_k\rangle \subseteq V_k \cap U$ for all $k < n$. Note that $U \setminus \bigcup_{k<n}\langle B_k\rangle$ is not small as $U \in \mathcal{I}^+(\mathcal{A})$. Thus, we may let B_n' be such that $\langle B_n'\rangle \subseteq U \setminus \bigcup_{k<n}\langle B_k\rangle$, and let $V_n \in \mathcal{A}$ be compatible with B_n'. We then let B_n be such that $\langle B_n\rangle \subseteq \langle B_n'\rangle \cap V_n$, and $\{B_n : n < \omega\}$ and $\{V_n : n < \omega\}$ are as desired. ∎

LEMMA 4.10. *Let $\mathcal{A}$ be an infinite almost disjoint family of subspaces, and let $U \subseteq E$ be a subspace. If $U \in \mathcal{I}^+(\mathcal{A})$, then there exists some $B \in \mathcal{H}^-(\mathcal{A})$ such that $\langle B\rangle \subseteq U$. If furthermore $U \in \mathcal{I}^{++}(\mathcal{A})$, then B may be chosen such that $B \in \mathcal{H}(\mathcal{A})$.*

Proof. If $U \notin \mathcal{I}^{++}(\mathcal{A})$, then by Lemma 4.7 there exists some $B \in E^{[\infty]}$ such that $\langle B\rangle \subseteq U$ and B is almost disjoint from every element of $\mathcal{A}$. Then $B \in \mathcal{H}^-(\mathcal{A})$ and $\langle B\rangle \subseteq U$, so we are done. Otherwise, let $\{V_n : n < \omega\} \subseteq \mathcal{A}$ be such that for all n, $\langle B_n\rangle \subseteq V_n \cap U$ for some $B_n \in E^{[\infty]}$. By Lemma 4.8, there exists some $B \in E^{[\infty]}$ such that

$$\langle B\rangle \subseteq \Big\langle \bigcup_{n<\omega} \langle B_n\rangle \Big\rangle \subseteq U,$$

and B is compatible with B_n for all n. This implies that B is compatible with V_n for all n, so $B \in \mathcal{H}(\mathcal{A})$ and $\langle B\rangle \subseteq U$, as desired. ∎

4.3. Selectivity properties revisited. We now return to studying the combinatorial properties of the semicoideals $\mathcal{H}^-(\mathcal{A})$ and $\mathcal{H}(\mathcal{A})$. In Mathias' process of proving that there are no analytic mad families of $[\mathbb{N}]^\infty$, it is shown that if $\mathcal{A}$ is an almost disjoint family in $[\mathbb{N}]^\infty$, then $\mathcal{H}^-(\mathcal{A})$ is a selective coideal. Smythe [25] proved that if $\mathcal{A}$ is a mad family of subspaces, then $\mathcal{H}(\mathcal{A})$ is a (p)-semicoideal. We prove a few extensions of this result.

Proposition 4.11. *Let $\mathcal{A}$ be an infinite almost disjoint family of subspaces. Then $\mathcal{H}^-(\mathcal{A})$ is a (p)-semicoideal.*

Proof. Let $(B_n)_{n<\omega}$ be a $\leq$-decreasing sequence in $\mathcal{H}^-(\mathcal{A})$. If there exists some C which weakly diagonalises $(B_n)_{n<\omega}$ and C is almost disjoint from every element of $\mathcal{A}$, then $C \in \mathcal{H}^-(\mathcal{A})$ and we are done, so assume otherwise. Let C_0 be a weak diagonalisation of $(B_n)_{n<\omega}$. By our assumption, C_0 is compatible with some $V_0 \in \mathcal{A}$, so by taking a further block subspace of C_0 if necessary, we may assume that $\langle C_0 \rangle \subseteq V_0$. Now assume, for the induction hypothesis, that we have defined C_k, V_k for $k < m$ such that

- $C_k \leq B_k$ is a weak diagonalisation of $(B_n)_{n<\omega}$,
- $\langle C_k \rangle \subseteq V_k$ and $V_k \in \mathcal{A}$,
- if $i, j < m$ then $V_i \neq V_j$.

Claim 7. *There exists a $\leq$-decreasing sequence $(B_n^m)_{n<\omega}$ such that for all n,*

(1) $B_n^m \leq B_n$,
(2) B_n^m *is almost disjoint from V_k for all $k < m$.*

Proof. For any n, since $B_n \in \mathcal{H}^-(\mathcal{A})$, there exists some $D_n \leq B_n$ that is disjoint from V_k for all $k < m$. Let $\Gamma : \omega \to \omega$ be any infinite-to-one surjection. We define $D = (x_n)_{n<\omega}$ as follows: Since $D_{\Gamma(0)}$ is almost disjoint from V_k for all $k < m$, we may let $x_0 \in \langle D_{\Gamma(0)} \rangle \setminus \bigcup_{k<m} V_k$ be any vector (which exists by Lemma 4.5). This implies that $\langle x_0 \rangle \cap V_k = \emptyset$ for all $k < m$. Now assume that $(x_i)_{i<n}$ has been defined, and for all $k < m$,

$$\langle x_0, \ldots, x_{n-1} \rangle \cap V_k = \emptyset.$$

By Claim 5 in the proof of Lemma 4.2, for each $k < m$, since $D_{\Gamma(n)}$ and V_k are almost disjoint, we see that $D_{\Gamma(n)}$ and $V_k + \langle x_0, \ldots, x_{n-1} \rangle$ are also almost disjoint. Let $x_n \in \langle D_{\Gamma(n)} \rangle \setminus \bigcup_{k<m} (V_k + \langle x_0, \ldots, x_{n-1} \rangle)$ be any vector (again, it exists by Lemma 4.5). We claim that for all $k < m$,

$$\langle x_0, \ldots, x_n \rangle \cap V_k = \emptyset.$$

Otherwise, for any $k < m$ let $y \in \langle x_0, \ldots, x_n \rangle \cap V_k$ be any vector, so $y = \sum_{i \leq n} \xi_i x_i$. Since $\langle x_0, \ldots, x_{n-1} \rangle \cap V_k = \emptyset$, we have $\xi_n \neq 0$. But this implies that

$$x_n = \frac{1}{\xi_n}\Big(y - \sum_{i<n} \xi_i x_i\Big) \in V_k + \langle x_0, \ldots, x_{n-1} \rangle,$$

contradicting our choice of x_n. Then $D = (x_n)_{n<\omega}$ is disjoint from V_k for all $k < m$, as

$$\langle D \rangle \cap V_k = \bigcup_{i<\omega} \langle x_0, \ldots, x_i \rangle \cap V_k = \emptyset.$$

Let $B_n^m := (x_i)_{\Gamma(i)\geq n}$. Since Γ is infinite-to-one, B_n^m is an infinite block sequence for all n, m. Then $B_n^m \leq B_n$, $(B_n^m)_{n<\omega}$ is a $\leq$-decreasing sequence, and $B_n^m \leq D$, so B_n^m is almost disjoint from V_k for all $k < m$. ∎

Let $C_m \leq B_m^m$ be a weak diagonalisation of $(B_n^m)_{n<\omega}$. By our assumption at the beginning, there exists some $V_m \in \mathcal{A}$ which is compatible with C_m, and we may assume that $\langle C_m \rangle \subseteq V_m$. Since $C_m \leq B_m^m$, and B_m^m is almost disjoint from V_k for all $k < m$, we have $V_k \neq V_m$ for all $k < m$. This completes the inductive definition of C_m, V_m for $m < \omega$.

Let $\Gamma : \omega \to \omega$ be an infinite-to-one surjection, and define $C := (y_i)_{i<\omega}$ as follows: Let $y_0 \in \langle C_{\Gamma(0)} \rangle$, and let $y_i \in \langle C_{\Gamma(i)}/i \rangle$ be any vector such that $y_{i-1} < y_i$. Observe that for all m, we have

$$\langle y_i : \Gamma(i) = m \rangle \subseteq \langle C \rangle \cap \langle C_m \rangle \subseteq \langle C \rangle \cap V_m,$$

so C is compatible with V_m for all m. Thus, $C \in \mathcal{H}^-(\mathcal{A})$. It remains to show that C weakly diagonalises $(B_n)_{n<\omega}$. Fix any n, and let N be large enough that $C_k/N \leq B_n$ for all $k < n$. For all $i \geq N$, we have $y_i \in \langle C_{\Gamma(i)}/i \rangle \subseteq \langle C_{\Gamma(i)}/N \rangle$. If $\Gamma(i) < n$, then $y_i \in \langle C_{\Gamma(i)}/N \rangle \subseteq \langle B_n \rangle$ by our choice of N. If $\Gamma(i) \geq n$, then $C_{\Gamma(i)} \leq B_{\Gamma(i)}^{\Gamma(i)} \leq B_{\Gamma(i)} \leq B_n$, so $y_i \in \langle B_n \rangle$. Therefore, $C/N \leq B_n$. ∎

Proposition 4.12. *Let $\mathcal{A}$ be an infinite almost disjoint family of subspaces. Then $\mathcal{H}^-(\mathcal{A})$ is a selective semicoideal.*

Proof. By Proposition 4.11, it remains to show that $\mathcal{H}^-(\mathcal{A})$ is a (q)-semicoideal. Let $A \in \mathcal{H}^-(\mathcal{A})$, and let $(I_m)_{m<\omega}$ be an increasing sequence of finite intervals. We split our proof into two cases.

If there is some $B \leq A$ which is almost disjoint from every element of $\mathcal{A}$, then $C \in \mathcal{H}^-(\mathcal{A})$ for all $C \leq B$, so any "sufficiently sparse" subsequence of B would do the trick. More precisely, if $B = (y_n)_{n<\omega}$, then we define a subsequence of integers $(n_k)_{k<\omega}$ as follows: Let $n_0 := 0$, and if n_k is defined, let m be the least integer such that $n_k < \min(I_m)$. We then define $n_{k+1} := \max(I_m) + 1$, and $(y_{n_k})_{k<\omega} \in \mathcal{H}^-(\mathcal{A})$ satisfies the conclusion of the (q)-property for A and $(I_m)_{m<\omega}$.

Suppose there is no $B \leq A$ which is almost disjoint from every element of $\mathcal{A}$. By Lemma 4.9, there exists an infinite $\{V_n : n < \omega\} \subseteq \mathcal{A}$ and an infinite disjoint family $\{B_n : n < \omega\}$ such that $\langle B_n \rangle \subseteq \langle A \rangle \cap V_n$ for all n. Let $\Gamma : \omega \to \omega$ be any infinite-to-one surjection. We write $A = (x_n)_{n<\omega}$, and define $B = (y_n)_{n<\omega}$ below A as follows: If $(y_k)_{k<n}$ has been defined, we let m be the least integer such that $\langle y_k : k < n \rangle \subseteq \langle x_0, \dots, x_{\min(I_m)-1} \rangle$. We let $y_n \in \langle B_{\Gamma(n)}/\max(I_m) \rangle$ be any element. Then $(y_n)_{n\in\Gamma^{-1}(m)}$ witnesses that B and B_m are compatible for all $m < \omega$, and since $y_n \in \langle A \rangle$ for all n, we have $B \leq A$ and $B \in \mathcal{H}^-(\mathcal{A})$. ∎

5. Constructions of mad families of subspaces

Definition 12. A mad family $\mathcal{A}$ of subspaces is *full* if $\mathcal{H}(\mathcal{A})$ is full (see Definitions 3 and 11).

We dedicate Sections 5.1–5.2 to the proof of Theorem 1.5.

5.1. Block-splitting families. We introduce/recall the cardinal invariants which are relevant to our proof of Theorem 1.5. If $f, g \in \omega^\omega$, we write $f \leq^* g$ if $f(n) \leq g(n)$ for almost all $n < \omega$. We say that $\mathcal{F} \subseteq \omega^\omega$ is *bounded* if there exists some $g \in \omega^\omega$ such that $f \leq^* g$ for all $f \in \mathcal{F}$, and *unbounded* otherwise. The *bounding number* $\mathfrak{b}$ is defined as

$$\mathfrak{b} := \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ is unbounded}\}.$$

If $S \in [\omega]^\omega$ and $A \in [\omega]^\omega$, we say that S *splits* A if $A \cap S$ and $A \setminus S$ are both infinite. A *splitting family* is a family $\mathcal{S} \subseteq [\omega]^\omega$ such that for all $A \in [\omega]^\omega$, there exists some $S \in \mathcal{S}$ which splits A. The *splitting number* is defined as

$$\mathfrak{s} := \min\{|\mathcal{S}| : \mathcal{S} \subseteq [\omega]^\omega \text{ is a splitting family}\}.$$

It turns out that both of these cardinals are closely related to *block-splitting families.* If $\mathcal{P} = \{P_n : n < \omega\}$ is an interval partition, we say that a set $S \in [\omega]^\omega$ *splits* $\mathcal{P}$ if $\{n < \omega : P_n \subseteq S\}$ and $\{n < \omega : P_n \cap S = \emptyset\}$ are both infinite. A *block-splitting family* is a family $\mathcal{S} \subseteq [\omega]^\omega$ such that every interval partition is split by some $S \in \mathcal{S}$. The following lemma is due to Kamburelis–Węglorz.

Lemma 5.1 ([12]). $\max\{\mathfrak{b}, \mathfrak{s}\} = \min\{|\mathcal{S}| : \mathcal{S}$ *is a block-splitting family*$\}$.

We abbreviate $\mathfrak{bs} = \max\{\mathfrak{b}, \mathfrak{s}\}$. By [25, Corollary 2.11], the identity $\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$ holds for any countable field $\mathbb{F}$. Since $\mathfrak{b} \leq \mathrm{non}(\mathcal{M})$ and $\mathfrak{s} \leq \mathrm{non}(\mathcal{M})$ (see, for instance, [6]), we may conclude the following.

Lemma 5.2. $\mathfrak{bs} \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$.

5.2. Proof of Theorem 1.5

Lemma 5.3. *There exists a collection $\{Z_\alpha : \alpha < \mathfrak{bs}\}$ of block subspaces of E such that for all almost disjoint families $\mathcal{A}$ of subspaces and $B \in \mathcal{H}^-(\mathcal{A})$, the following properties hold:*

(i) *For all $\alpha < \mathfrak{bs}$, $\langle B\rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$ or $\langle B\rangle \cap Z_\alpha^c \in \mathcal{I}^+(\mathcal{A})$.*
(ii) *If $\alpha < \mathfrak{bs}$ and $\langle B\rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$, then there exists some $C_\alpha^0 \in \mathcal{H}^-(\mathcal{A})$ such that $\langle C_\alpha^0\rangle \subseteq \langle B\rangle \cap Z_\alpha$.*
(iii) *If $\alpha < \mathfrak{bs}$ and $\langle B\rangle \cap Z_\alpha^c \in \mathcal{I}^+(\mathcal{A})$, then there exists some $C_\alpha^1 \in \mathcal{H}^-(\mathcal{A})$ such that $\langle C_\alpha^1\rangle \subseteq \langle B\rangle \cap Z_\alpha^c$.*
(iv) *There exists some $\alpha < \mathfrak{bs}$ such that $\langle B\rangle \cap Z_\alpha \in \mathcal{I}^+(\mathcal{A})$ and $\langle B\rangle \cap Z_\alpha^c \in \mathcal{I}^+(\mathcal{A})$.*

Proof. Let $\{S_\alpha : \alpha < \mathfrak{bs}\} \subseteq [\omega]^\omega$ be a block-splitting family, and let $Z_\alpha := \langle (e_n)_{n \in S_\alpha} \rangle$. We remark that $\langle (e_n)_{n \notin S_\alpha} \rangle \subseteq Z_\alpha^c$. We shall show that $\{Z_\alpha : \alpha < \mathfrak{bs}\}$ satisfies the four properties above.

Property (i). Suppose that $\langle B \rangle \cap Z_\alpha \in \mathcal{I}(\mathcal{A})$ and $\langle B \rangle \cap Z_\alpha^c \in \mathcal{I}(\mathcal{A})$. Let $\{V_i : i < n\} \subseteq \mathcal{A}$ be such that $(\langle B \rangle \cap Z_\alpha) \setminus \bigcup_{i<n} V_i$ and $(\langle B \rangle \cap Z_\alpha^c) \setminus \bigcup_{i<n} V_i$ are small. Since $B \in \mathcal{H}^-(\mathcal{A})$, there exists some $C \leq B$ such that

$$\langle C \rangle \subseteq \langle B \rangle \setminus \bigcup_{i<n} V_i.$$

However, since

$$\langle C \rangle \cap Z_\alpha \subseteq (\langle B \rangle \cap Z_\alpha) \setminus \bigcup_{i<n} V_i, \qquad \langle C \rangle \cap Z_\alpha^c \subseteq (\langle B \rangle \cap Z_\alpha^c) \setminus \bigcup_{i<n} V_i,$$

we see that $\langle C \rangle \cap Z_\alpha$ and $\langle C \rangle \cap Z_\alpha^c$ are small. This contradicts Lemma 4.5.

Property (ii). This follows from Lemma 2.1, where we may let $C_\alpha^0 \in E^{[\infty]}$ be such that $\langle C_\alpha^0 \rangle = \langle B \rangle \cap \langle (e_n)_{n \in S_\alpha} \rangle$.

Property (iii). Suppose that $\langle B \rangle \cap Z_\alpha^c \in \mathcal{I}^+(\mathcal{A})$. If there exists some $C \in E^{[\infty]}$ such that $\langle C \rangle \subseteq \langle B \rangle \cap Z_\alpha^c$ and C is almost disjoint from every element of $\mathcal{A}$, then $C \in \mathcal{H}^-(\mathcal{A})$ and we are done, so assume otherwise. In a way similar to the one presented in the proof of Lemma 4.9, we define V_n, B_n for $n < \omega$ such that $\bigcup_{n<\omega} \langle B_n \rangle \subseteq \langle B \rangle \cap Z_\alpha^c$, $\{B_n : n < \omega\}$ is a disjoint family of block subspaces, and $V_n \in \mathcal{A}$ is compatible with B_n for all n. Let C_n be such that $\langle C_n \rangle \subseteq V_n \cap \langle B_n \rangle$.

For each n, we write $C_n := (x_{n,m})_{m<\omega}$. Since $\langle C_n \rangle \subseteq Z_\alpha^c$, this means that for all m, $\lambda_{l_{n,m}}(x_{n,m}) \neq 0$ for some $l_{m,n} \notin S_\alpha$. Now let $\Gamma : \omega \to \omega \times \omega$ be any function such that

- for all n, there are infinitely many k such that $\Gamma(k) = (n, m)$ for some m,
- if $k < k'$, $\Gamma(k) = (n, m)$ and $\Gamma(k') = (n', m')$, then $x_{n,m} < x_{n',m'}$.

This implies that $C := (x_{\Gamma(k)})_{k<\omega}$ is a well-defined infinite block sequence, where if $\Gamma(k) = (n, m)$ then $x_{\Gamma(k)} := x_{n,m}$. Then C is compatible with every C_n, so $C \in \mathcal{H}^-(\mathcal{A})$. Furthermore, we have

$$\langle C \rangle \subseteq \Big\langle \bigcup_{n<\omega} \langle C_n \rangle \Big\rangle \subseteq \langle B \rangle,$$

so $C \leq B$. Finally, if $x \in \langle C \rangle$, then $\lambda_l(x) \neq 0$ for some $l \notin S_\alpha$, so $x \notin Z_\alpha$. In other words, $\langle C \rangle \subseteq Z_\alpha^c$. Therefore, $C \in \mathcal{H}^-(\mathcal{A})$ and $\langle C \rangle \subseteq \langle B \rangle \cap Z_\alpha^c$, as desired.

Property (iv). We first suppose that B is almost disjoint from every element of $\mathcal{A}$. If $B = (x_n)_{n<\omega}$, we define, for $n < \omega$,

$$P_0 := \{0, \ldots, \max(\operatorname{supp}(x_0))\},$$
$$P_{n+1} := \{\max(\operatorname{supp}(x_n)) + 1, \ldots, \max(\operatorname{supp}(x_{n+1}))\}.$$

We observe that $\mathcal{P} := \{P_n : n < \omega\}$ is an interval partition such that $\operatorname{supp}(x_n) \subseteq P_n$ for all n. Since $\{S_\alpha : \alpha < \mathfrak{bs}\}$ is a block-splitting family, there exists some $\alpha < \mathfrak{bs}$ that splits $\mathcal{P}$. Let $T_0 := \{n < \omega : P_n \subseteq S_\alpha\}$ and $T_1 := \{n < \omega : P_n \cap S_\alpha = \emptyset\}$. Define $B_0 := (x_n)_{n \in T_0}$ and $B_1 := (x_n)_{n \in T_1}$. Since $B_0, B_1 \leq B$, and B is almost disjoint from every element of $\mathcal{A}$, so are B_0, B_1 and so $B_0, B_1 \in \mathcal{H}^-(\mathcal{A})$. If $n \in T_0$, then $\operatorname{supp}(x_n) \subseteq P_n \subseteq S_\alpha$, $x_n \in (e_n)_{n \in S_\alpha}$. Therefore, $\langle B_0 \rangle \subseteq \langle B \rangle \cap Z_\alpha$. By a similar reasoning, we may also conclude that $\langle B_1 \rangle \subseteq \langle B \rangle \cap Z_\alpha^c$.

Now suppose that there is no $C \leq B$ which is almost disjoint from every element of $\mathcal{A}$ (if such a C exists, we may apply the argument in the previous paragraph to C, and use the fact that $\mathcal{I}^+(\mathcal{A})$ is $\subseteq$-upward closed). By Lemma 4.9, let $\{V_n : n < \omega\} \subseteq \mathcal{A}$ be an infinite set and let $\{B_n : n < \omega\}$ be an infinite disjoint family such that $\langle B_n \rangle \subseteq \langle B \rangle \cap V_n$ for all n. Let $\Gamma : \omega \to \omega$ be an infinite-to-one surjection. We define an infinite block sequence $(x_n)_{n<\omega}$ inductively as follows: If $(x_k)_{k<n}$ has been defined, let $x_n \in \langle B_{\Gamma(n)} \rangle$ be any vector $x_{n-1} < x_n$. Having defined $(x_n)_{n<\omega}$, we now define an interval partition $\mathcal{P} := \{P_m : m < \omega\}$ as follows: Let $P_0 := \{0, \ldots, \max(\operatorname{supp}(x_{n_0}))\}$, where n_0 is the least integer such that $\Gamma(n_0) = 0$. If P_m has been defined with $\max(P_m) = n_m$, let $\min(P_{m+1}) = n_m + 1$, and let $\max(P_{m+1})$ be large enough that for all $i \leq m+1$, there exists some n such that $\Gamma(n) = i$ and $\operatorname{supp}(x_n) \subseteq P_{m+1}$. This is possible as Γ is infinite-to-one.

Now let $\alpha < \mathfrak{bs}$ be such that S_α splits $\mathcal{P}$. For each i, let $T_i^0 := \{n < \omega : \Gamma(n) = i \wedge \operatorname{supp}(x_n) \subseteq S_\alpha\}$ and $T_i^1 := \{n < \omega : \Gamma(n) = i \wedge \operatorname{supp}(x_n) \cap S_\alpha = \emptyset\}$. By our construction of $\mathcal{P}$, we see that T_i^0 and T_i^1 are infinite for all i. By a similar reasoning to the first case, we may conclude that $\langle B \rangle \cap Z_\alpha$ and $\langle B \rangle \cap Z_\alpha^c$ are compatible with V_i for all i, so they are both in $\mathcal{I}^+(\mathcal{A})$. ■

Proof of Theorem 1.5. Let $\{Z_\alpha : \alpha < \mathfrak{bs}\}$ be a family of subsets of vectors satisfying the conclusions of Lemma 5.3. We write $Z_\alpha^0 := Z_\alpha$ and $Z_\alpha^1 := Z_\alpha^c$ for all α. With this, for a fixed almost disjoint family $\mathcal{A} \subseteq E^{[\infty]}$ and for $B \in \mathcal{H}^-(\mathcal{A})$, there exist $\alpha < \mathfrak{bs}$ and $\tau_B^{\mathcal{A}} \in 2^\alpha$ such that

- if $\beta < \alpha$, then $\langle B \rangle \cap Z_\beta^{1-\tau_B^{\mathcal{A}}(\beta)} \in \mathcal{I}(\mathcal{A})$ (so $\langle B \rangle \cap Z_\beta^{\tau_B^{\mathcal{A}}(\beta)} \in \mathcal{I}^+(\mathcal{A})$ by property (i) of Lemma 5.3),
- $\langle B \rangle \cap Z_\alpha^0 \in \mathcal{I}^+(\mathcal{A})$ and $\langle B \rangle \cap Z_\alpha^1 \in \mathcal{I}^+(\mathcal{A})$ (such an α exists by property (iv) of Lemma 5.3).

Observe that for a fixed almost disjoint family $\mathcal{A}$, $\tau_B^{\mathcal{A}} \in 2^{<\mathfrak{bs}}$ is unique, and if $B' \leq B$ and $B' \in \mathcal{H}^-(\mathcal{A})$, then $\tau_B^{\mathcal{A}} \subseteq \tau_{B'}^{\mathcal{A}}$. Let $\operatorname{Lim}(\mathfrak{c})$ be the set of limit ordinals below $\mathfrak{c}$. Let $\{Y_\alpha : \alpha \in \operatorname{Lim}(\mathfrak{c})\}$ be an enumeration of $\mathcal{P}(E)$, and let $\{B_\alpha : \alpha \in \operatorname{Lim}(\mathfrak{c})\}$ be an enumeration of $E^{[\infty]}$ in a way that for any $B \in E^{[\infty]}$ and $Y \subseteq E$, there are cofinally many $\alpha \in \operatorname{Lim}(\mathfrak{c})$ such that $(Y_\alpha, B_\alpha) = (Y, B)$. We shall define our mad family $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ recursively, where $\mathcal{A}_\alpha :=$

$\{A_\beta : \beta < \alpha\}$, along with a collection $\{A'_\alpha : \alpha \in \mathrm{Lim}(\mathfrak{c})\}$ of block sequences, and an injective sequence $\{\sigma_\alpha : \alpha \in \mathrm{Lim}(\mathfrak{c})\}$ of elements of $2^{<\mathfrak{bs}}$ such that the following hold:

- If $\beta < \alpha$ are limit ordinals, then $\sigma_\alpha \not\sqsubseteq \sigma_\beta$.
- A'_α is almost disjoint from A_β for all $\beta < \alpha$.
- If $\xi < \mathrm{dom}(\sigma_\alpha)$, then $\langle A'_\alpha\rangle \setminus Z_\xi^{\sigma_\alpha(\xi)}$ is small (hence, by Lemma 4.5, is very small).
- $\{A_{\alpha+n} : n < \omega\}$ is a disjoint family below A'_α.

For our base case, we let $\mathcal{A}_\omega$ be any countable almost disjoint family. Assume that $\alpha \geq \omega$ is a limit ordinal, and that $A_{\beta+n}$, A'_β and σ_β have been defined for limit $\beta < \alpha$ and $n < \omega$, and satisfy the requirements stated above so far. We define the element B depending on the following scenarios:

(a) If $B_\alpha \notin \mathcal{H}(\mathcal{A}_\alpha)$, then let $B := B_\alpha$ if $B_\alpha \in \mathcal{H}^-(\mathcal{A}_\alpha)$. Otherwise, let B be any element of $\mathcal{H}^-(\mathcal{A}_\alpha)$.
(b) Suppose that $B_\alpha \in \mathcal{H}(\mathcal{A}_\alpha)$ (so $B_\alpha \in \mathcal{H}^-(\mathcal{A}_\alpha)$ by Lemma 4.6), and let

$$\mathcal{C}_\alpha := \{A \in \mathcal{A}_\alpha : B_\alpha \text{ and } A \text{ are compatible}\}.$$

Suppose further that there exists some $B'_\alpha \leq B_\alpha$ that is almost disjoint from every element of $\mathcal{C}_\alpha$, and $\langle B'_\alpha\rangle \subseteq Y_\alpha$. Then we let $B := B'_\alpha$.
(c) Suppose that $B_\alpha \in \mathcal{H}(\mathcal{A}_\alpha)$ (so $B_\alpha \in \mathcal{H}^-(\mathcal{A}_\alpha)$), and for all $B \leq B_\alpha$ which are almost disjoint from every element of $\mathcal{C}_\alpha$, there is no $B' \leq B$ such that $\langle B'\rangle \not\subseteq Y_\alpha$. Then we let $B := B_\alpha$.

We inductively define $\{C_s : s \in 2^{<\omega}\} \subseteq \mathcal{H}^-(\mathcal{A}_\alpha)$, $\{\eta_s : s \in 2^{<\omega}\} \subseteq 2^{<\mathfrak{bs}}$ and ordinals $\{\alpha_s : s \in 2^{<\omega}\}$ as follows:

- $C_\emptyset := B$,
- $\eta_s := \tau_{C_s}^{\mathcal{A}_\alpha}$ and $\alpha_s := \mathrm{dom}(\eta_s)$,
- $C_{s^\frown 0}, C_{s^\frown 1} \in \mathcal{H}^-(\mathcal{A}_\alpha)$ are such that $\langle C_{s^\frown 0}\rangle \subseteq \langle C_s\rangle \cap Z^0_{\alpha_s}$ and $\langle C_{s^\frown 1}\rangle \subseteq \langle C_s\rangle \cap Z^1_{\alpha_s}$ (which both exist by properties (ii) and (iii) of Lemma 5.3).

We then define $\eta_f := \bigcup_{n<\omega} \eta_{f\restriction n}$ for any $f \in 2^\omega$, and note that $\eta_f \in 2^{<\mathfrak{bs}}$ as $\mathfrak{bs}$ has uncountable cofinality (as $\mathfrak{b}$ is regular and $\mathfrak{s}$ has uncountable cofinality, see [9, Corollary 21]). We record the following observations:

- If $s \sqsubseteq t$, then $C_s \leq C_t$ and $\eta_s \sqsubseteq \eta_t$.
- If s and t are incompatible, then so are η_s and η_t.
- Consequently, if $f \neq g$, then η_f and η_g are incompatible.

Since $\alpha < \mathfrak{c}$, we may let $f \in 2^\omega$ be such that $\eta_f \not\sqsubseteq \sigma_\beta$ for all limit ordinals $\beta < \alpha$. We now have a $\leq$-decreasing sequence $(C_{f\restriction n})_{n<\omega}$ in $\mathcal{H}^-(\mathcal{A}_\alpha)$, so by Proposition 4.11 there exists some $D \in \mathcal{H}^-(\mathcal{A}_\alpha)$ such that $D \leq^* C_{f\restriction n}$ for all n.

Let $\sigma_\alpha := \eta_f$. Observe that for all $\xi < \mathrm{dom}(\eta_f)$, if $n < \omega$ is such that $\xi \in \mathrm{dom}(\eta_{f\restriction n})$, and if N is such that $D/N \leq C_{f\restriction n}$, then

$$\langle D/N \rangle \cap Z_\xi^{1-\eta_f(\xi)} = \langle D/N \rangle \cap Z_\xi^{1-\eta_{f\restriction n}(\xi)} \subseteq \langle C_{f\restriction n} \rangle \cap Z_\xi^{1-\eta_{f\restriction n}(\xi)} \in \mathcal{I}(\mathcal{A}_\alpha),$$

so that for all $\xi < \mathrm{dom}(\sigma_\alpha)$, $\langle D \rangle \cap Z_\xi^{1-\sigma_\alpha(\xi)} \in \mathcal{I}(\mathcal{A})$ (as $\langle D \rangle \setminus \langle D/N \rangle$ is very small, by Lemma 4.5). Therefore, for each $\xi < \mathrm{dom}(\sigma_\alpha)$ we may find a finite subset $\mathcal{F}_\xi \subseteq \mathcal{A}_\alpha$ such that $(\langle D \rangle \cap Z_\xi^{1-\sigma_\alpha(\xi)}) \setminus \bigcup_{A \in \mathcal{F}_\xi} \langle A \rangle$ is small. Now let

$$\mathcal{B}_0 := \{A_{\beta+n} : \sigma_\beta \sqsubseteq \sigma_\alpha \text{ and } n < \omega\}, \quad \mathcal{B}_1 := \bigcup_{\xi < \mathrm{dom}(\sigma_\alpha)} \mathcal{F}_\xi, \quad \mathcal{B} := \mathcal{B}_0 \cup \mathcal{B}_1.$$

Note that $\mathcal{B} \subseteq \mathcal{A}_\alpha$, so it is an almost disjoint family.

Claim 8. $|\mathcal{B}| < \mathfrak{bs}$.

Proof. It suffices to show that $|\mathcal{B}_0| < \mathfrak{bs}$ and $|\mathcal{B}_1| < \mathfrak{bs}$. Since $\mathrm{dom}(\sigma_\alpha) < \mathfrak{bs}$, there are $< \mathfrak{bs}$ many initial segments of σ_α. This implies that there are $< \mathfrak{bs}$ many $\beta < \alpha$ such that $\sigma_\beta \sqsubseteq \sigma_\alpha$, because if $\beta, \gamma < \alpha$ and $\beta \neq \gamma$, then $\sigma_\beta \neq \sigma_\gamma$ by the induction hypothesis. Therefore, $|\mathcal{B}_0| < \mathfrak{bs}$. Since $\mathcal{B}_1$ is a union of $\mathrm{dom}(\sigma_\alpha) < \mathfrak{bs}$ many finite sets, we have $|\mathcal{B}_1| < \mathfrak{bs}$ as well. ∎

By Lemma 2.1, the set

$$\{\langle B' \rangle \cap \langle D \rangle : B' \in \mathcal{B} \text{ and } \langle B' \rangle \cap \langle D \rangle \text{ is infinite-dimensional}\}$$

is an almost disjoint family of block subspaces of $\langle D \rangle$. Note that $\langle D \rangle$ is not in the set, as otherwise this implies that $\langle D \rangle \subseteq A_\beta$ for some $\beta < \alpha$, contradicting that $\mathcal{D} \in \mathcal{H}^-(\mathcal{A}_\alpha)$. By Lemma 5.2, $|\mathcal{B}| < \mathfrak{bs} \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$, so there exists some $A'_\alpha \leq D$ such that A'_α is almost disjoint from every element of $\mathcal{B}$.

Claim 9.

(1) *For all $\xi \in \mathrm{dom}(\sigma_\alpha)$, $\langle A'_\alpha \rangle \setminus Z_\xi^{\sigma_\alpha(\xi)}$ is small.*
(2) *For all $\beta < \alpha$, A'_α and A_β are almost disjoint.*

Proof. (1) Note that $\langle A'_\alpha \rangle \setminus Z_\xi^{\sigma_\alpha(\xi)} = \langle A'_\alpha \rangle \cap Z_\xi^{1-\sigma_\alpha(\xi)}$. We observe that $\langle A'_\alpha \rangle \cap Z_\xi^{1-\sigma_\alpha(\xi)} \setminus \bigcup_{A \in \mathcal{F}_\xi} \langle A \rangle$ is small by the above, and since A'_α is almost disjoint from every $A \in \mathcal{F}_\xi$, we have

$$\langle A'_\alpha \rangle \cap Z_\xi^{1-\sigma_\alpha(\xi)} \cap \bigcup_{A \in \mathcal{F}_\xi} \langle A \rangle \subseteq \bigcup_{A \in \mathcal{F}_\xi} \langle A'_\alpha \rangle \cap \langle A \rangle.$$

By Lemma 4.5, $\langle A'_\alpha \rangle \cap \langle A \rangle$ is very small for each $A \in \mathcal{F}_\xi$, so $\langle A'_\alpha \rangle \cap Z_\xi^{1-\sigma_\alpha(\xi)}$ is small.

(2) Let $\beta + n < \alpha$, where β is a limit ordinal. If $A_{\beta+n} \in \mathcal{B}_0$, then A_α is almost disjoint from $A_{\beta+n}$ by construction, so we are done. Now assume that $A_{\beta+n} \notin \mathcal{B}_0$. Let ξ be the least ordinal such that $\sigma_\beta(\xi) \neq \sigma_\alpha(\xi)$. By induction

hypothesis, $W := \langle A'_\beta\rangle \setminus Z_\xi^{\sigma_\beta(\xi)} = \langle A'_\beta\rangle \setminus Z_\xi^{1-\sigma_\alpha(\xi)}$ is small (and hence very small). Since $A_{\beta+n} \leq A'_\beta$, we have for some small set W',

$$\langle A'_\alpha\rangle \cap \langle A_{\beta+n}\rangle \subseteq W \cup (\langle A'_\alpha\rangle \cap Z_\xi^{1-\sigma_\alpha(\xi)}) \subseteq W \cup W' \cup \bigcup_{A\in F_\xi} \langle A'_\alpha\rangle \cap \langle A\rangle,$$

which is also small as all the components in the union are very small, as desired. ∎

We complete the induction by letting $\{A_{\alpha+n} : n < \omega\}$ be any infinite disjoint family below A'_α. We note that $\mathcal{A}$ is a mad family – otherwise, if B is almost disjoint from every element of $\mathcal{A}$, then $B \in \mathcal{H}^-(\mathcal{A}_\alpha)$ for all $\alpha < \mathfrak{c}$. Let α be any limit ordinal such that $B_\alpha = B$; then $A_\alpha \leq B$, a contradiction. It remains to show that $\mathcal{A}$ is full.

To do so, let $Y \subseteq E$, and suppose that Y is $\mathcal{H}(\mathcal{A})$-dense below some $B \in \mathcal{H}(\mathcal{A})$. Since $\mathcal{H}(\mathcal{A}) = \bigcup_{\alpha<\mathfrak{c}} \mathcal{H}(\mathcal{A}_\alpha)$, we may let α be large enough that $(Y_\alpha, B_\alpha) = (Y, B)$ and $B \in \mathcal{H}(\mathcal{A}_\alpha)$. This means that in the induction step, we have either case (b) or (c). Observe that (c) is not possible – since $A'_\alpha \leq B_\alpha$ is almost disjoint from every element of $\mathcal{C}_\alpha$ as $\mathcal{C}_\alpha \subseteq \mathcal{A}_\alpha$, and $A'_\alpha \in \mathcal{H}(\mathcal{A})$ as A'_α is compatible with $A_{\alpha+n}$ for all n. However, there is no $B' \leq A'_\alpha$ for which $\langle B'\rangle \subseteq Y_\alpha$, contradicting that Y is $\mathcal{H}(\mathcal{A})$-dense below B. This means that case (b) must have occurred in this induction step. Since $B_{\alpha'}$ is compatible with $A_{\alpha+n}$ for all n, we have $B_{\alpha'} \in \mathcal{H}(\mathcal{A})$ and $\langle B'_\alpha\rangle \subseteq Y_\alpha$, as desired. ∎

5.3. Completely separable mad family of subspaces. We dedicate this section to defining a notion of completely separable almost disjoint families of subspaces and proving Theorem 1.6.

Definition 13. An almost disjoint family $\mathcal{A}$ of subspaces is *completely separable* if for any subspace $U \subseteq E$, if $U \in \mathcal{I}^+(\mathcal{A})$ then there exists some $V \in \mathcal{A}$ such that $V \subseteq U$.

Lemma 5.4. *A completely separable almost disjoint family of subspaces is maximal.*

Proof. Suppose that $\mathcal{A}$ is a completely separable almost disjoint family of subspaces that is not maximal. Let $U \subseteq E$ be a subspace almost disjoint from every $V \in \mathcal{A}$. For any $V_0, \dots, V_{n-1} \in \mathcal{A}$, we have

$$U \setminus \bigcup_{i<n} V_i = U \setminus \bigcup_{i<n} U \cap V_i.$$

By Lemma 4.5, $U \cap V_i$ is very small for all $i < n$, so $\bigcup_{i<n} U \cap V_i$ is very small. Since U is big, we se that $U \setminus \bigcup_{i<n} V_i$ is big. Since $V_0, \dots, V_{n-1}$ are arbitrary, $U \in \mathcal{I}^+(\mathcal{A})$. By the complete separability of $\mathcal{A}$, this implies that $V \subseteq U$ for some $V \in \mathcal{A}$, contradicting that U and V are almost disjoint. ∎

One may consider the alternative definition where we remove the subspace requirement of U in Definition 13. However, it turns out that no such almost disjoint families of subspaces exist.

PROPOSITION 5.5. *Let $\mathcal{A}$ be an infinite almost disjoint family of subspaces. There exists some $Y \in \mathcal{I}^{++}(\mathcal{A})$ such that for all $A \in \mathcal{A}$, $\langle A \rangle \not\subseteq Y$.*

Proof. Let $\{V_n : n < \omega\} \subseteq \mathcal{A}$. For each n, let B_n, C_n be block subspaces of V_n such that $\langle B_n \rangle \cap \langle C_n \rangle = \emptyset$. Let $\Gamma : \omega \to \omega$ be an infinite-to-one surjection. We define a block sequence $(x_m)_{m<\omega}$ as follows: Let $x_0 \in \langle C_{\Gamma(0)} \rangle$ be any vector. If x_m has been defined, let $x_{m+1} \in \langle C_{\Gamma(m+1)} \rangle$ be any vector such that $x_m < x_{m+1}$. We let $D_n := (x_m)_{m \in \Gamma^{-1}(n)}$, which is an infinite block sequence as Γ is infinite-to-one. We write $D_n = (y^n_k)_{k<\omega}$ Note that the block subspaces D_n have the following properties:

- For all $z \in \langle D_n \rangle$, $\mathrm{supp}(z) \subseteq \bigcup_{k<\omega} \mathrm{supp}(y^n_k)$.
- If $m \neq n$, then $\bigcup_{k<\omega} \mathrm{supp}(y^n_k) \cap \bigcup_{j<\omega} \mathrm{supp}(y^m_j) = \emptyset$.

We define
$$Y := \bigcup_{n<\omega} \langle D_n \rangle.$$
Clearly $Y \in \mathcal{I}^{++}(\mathcal{A})$, as $\langle D_n \rangle \subseteq V_n \cap Y$ for all n.

CLAIM 10. *If $B \in E^{[\infty]}$ and $\langle B \rangle \subseteq Y$, then $B \leq D_n$ for some n.*

Proof. We write $B = (z_k)_{k<\omega}$, and suppose that $z_0 \in \langle D_n \rangle$. We shall show that $z_k \in \langle D_n \rangle$ for all k, which implies that $\langle B \rangle \subseteq \langle D_n \rangle$. Suppose that $z_k \in \langle D_m \rangle$ for some $m \neq n$ and $k > 0$. Then $z_0 + z_k \notin \langle D_n \rangle$, as
$$\mathrm{supp}(z_0 + z_k) = \mathrm{supp}(z_0) \cup \mathrm{supp}(z_k) \not\subseteq \bigcup_{j<\omega} \mathrm{supp}(y^n_j).$$
Similarly, $z_0 + z_k \notin \langle D_m \rangle$. For each $i \neq m, n$, since
$$\mathrm{supp}(z_0) \cap \bigcup_{j<\omega} \mathrm{supp}(y^i_j) = \emptyset,$$
we may conclude that $z_0 + z_k \notin \langle D_i \rangle$ either. Therefore, $z_0 + z_k \in \langle B \rangle \setminus Y$, a contradiction. ■

We may now finish the proof of Proposition 5.5. Suppose that there exists some $V \in \mathcal{A}$ such that $V \subseteq Y$. By Claim 10, we have $V \subseteq \langle D_n \rangle \subseteq V_n$ for some n. Since $\mathcal{A}$ is an almost disjoint family, we see that $V = V_n$. But this implies that $\langle B_n \rangle \subseteq V_n \subseteq \langle D_n \rangle \subseteq \langle C_n \rangle$, which contradicts that $\langle B_n \rangle \cap \langle C_n \rangle = \emptyset$. ■

The proof of Theorem 1.5 may be adjusted to obtain a completely separable mad family of block subspaces, with the main change being at the induction step. We describe the details here.

Proof of Theorem 1.6. Let $\{Z_\alpha : \alpha < \mathfrak{hs}\}$ be a family of subsets of vectors satisfying the conclusions of Lemma 5.3. We write $Z_\alpha^0 := Z_\alpha$ and $Z_\alpha^1 := Z_\alpha^c$ for all α. For a fixed almost disjoint family $\mathcal{A} \subseteq E^{[\infty]}$ and $B \in \mathcal{H}^-(\mathcal{A})$, there exist $\alpha < \mathfrak{hs}$ and $\tau_B^{\mathcal{A}} \in 2^\alpha$ such that

- if $\beta < \alpha$, then $\langle B\rangle \cap Z_\beta^{1-\tau_B^{\mathcal{A}}(\beta)} \in \mathcal{I}(\mathcal{A})$ (so $\langle B\rangle \cap Z_\beta^{\tau_B^{\mathcal{A}}(\beta)} \in \mathcal{I}^+(\mathcal{A})$ by property (i) of Lemma 5.3),
- $\langle B\rangle \cap Z_\alpha^0 \in \mathcal{I}^+(\mathcal{A})$ and $\langle B\rangle \cap Z_\alpha^1 \in \mathcal{I}^+(\mathcal{A})$ (such an α exists by property (iv) of Lemma 5.3).

Observe that for a fixed almost disjoint family $\mathcal{A}$, $\tau_B^{\mathcal{A}} \in 2^{<\mathfrak{hs}}$ is unique, and if $B' \leq B$ and $B' \in \mathcal{H}^-(\mathcal{A})$, then $\tau_B^{\mathcal{A}} \subseteq \tau_{B'}^{\mathcal{A}}$. Let $\{U_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all subspaces of E. We may need that for any subspace $U \subseteq E$, there are cofinally many $\alpha < \mathfrak{c}$ such that $U_\alpha = U$. We shall define our mad family $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ recursively, where $\mathcal{A}_\alpha := \{A_\beta : \beta < \alpha\}$, along with an injective sequence $\{\sigma_\alpha : \alpha < \mathfrak{c}\}$ of elements of $2^{<\mathfrak{hs}}$ such that

- if $\beta < \alpha$, then $\sigma_\alpha \not\sqsubseteq \sigma_\beta$,
- if $\xi < \mathrm{dom}(\sigma_\alpha)$, then $\langle A_\alpha\rangle \setminus Z_\xi^{\sigma_\alpha(\xi)}$ is small (hence, by Lemma 4.5, is very small).

For our base case, we let $\mathcal{A}_\omega$ be any countable almost disjoint family. Assume that A_β, σ_β have been defined for $\beta < \alpha$ such that they satisfy the requirements stated above so far. We inductively define $\{C_s : s \in 2^{<\omega}\} \subseteq \mathcal{H}^-(\mathcal{A}_\alpha)$, $\{\eta_s : s \in 2^{<\omega}\} \subseteq 2^{<\mathfrak{hs}}$ and ordinals $\{\alpha_s : s \in 2^{<\omega}\}$ as follows:

- If $U_\alpha \in \mathcal{I}^+(\mathcal{A}_\alpha)$, let $C_\emptyset \in \mathcal{H}^-(\mathcal{A}_\alpha)$ such that $\langle C_\emptyset\rangle \subseteq U_\alpha$. This is possible by Lemma 4.10. If $U_\alpha \notin \mathcal{I}^+(\mathcal{A}_\alpha)$, let $C_\emptyset \in \mathcal{H}^-(\mathcal{A}_\alpha)$ be any block subspace.
- $\eta_s := \tau_{C_s}^{\mathcal{A}_\alpha}$ and $\alpha_s := \mathrm{dom}(\eta_s)$.
- $C_{s\frown 0}, C_{s\frown 1} \in \mathcal{H}^-(\mathcal{A}_\alpha)$ are such that $\langle C_{s\frown 0}\rangle \subseteq \langle C_s\rangle \cap Z_{\alpha_s}^0$ and $\langle C_{s\frown 1}\rangle \subseteq \langle C_s\rangle \cap Z_{\alpha_s}^1$ (which both exist by properties (ii) and (iii) of Lemma 5.3).

We then define $\eta_f := \bigcup_{n<\omega} \eta_{f\restriction n}$ for any $f \in 2^\omega$, and note that $\eta_f \in 2^{<\mathfrak{hs}}$ as $\mathfrak{hs}$ has uncountable cofinality. We record the following observations:

- If $s \sqsubseteq t$, then $C_s \leq C_t$ and $\eta_s \sqsubseteq \eta_t$.
- If s and t are incompatible, then so are η_s and η_t.
- Consequently, if $f \neq g$, then η_f and η_g are incompatible.

Since $\alpha < \mathfrak{c}$, we may let $f \in 2^\omega$ be such that $\eta_f \not\sqsubseteq \sigma_\beta$ for all limit ordinals $\beta < \alpha$. We now have a $\leq$-decreasing sequence $(C_{f\restriction n})_{n<\omega}$ in $\mathcal{H}^-(\mathcal{A}_\alpha)$, so by Proposition 4.11 there exists some $D \in \mathcal{H}^-(\mathcal{A}_\alpha)$ such that $D \leq^* C_{f\restriction n}$ for all n.

Let $\sigma_\alpha := \eta_f$. Observe that for all $\xi < \mathrm{dom}(\eta_f)$, if $m < \omega$ is such that $\xi \in \mathrm{dom}(\eta_{f\restriction n})$, and $D/N \leq C_{f\restriction n}$, then

$$\langle D/N\rangle \cap Z_\xi^{1-\eta_f(\xi)} = \langle D/N\rangle \cap Z_\xi^{1-\eta_{f\restriction n}(\xi)} \subseteq \langle C_{f\restriction n}\rangle \cap Z_\xi^{1-\eta_{f\restriction n}(\xi)} \in \mathcal{I}(\mathcal{A}_\alpha),$$

so that for all $\xi < \mathrm{dom}(\sigma_\alpha)$, $\langle D\rangle \cap Z_\xi^{1-\sigma_\alpha(\xi)} \in \mathcal{I}(\mathcal{A})$ (as $\langle D\rangle \setminus \langle D/N\rangle$ is very small, by Lemma 4.5). Therefore, for each $\xi < \mathrm{dom}(\sigma_\alpha)$ we may find a finite subset $\mathcal{F}_\xi \subseteq \mathcal{A}_\alpha$ such that $(\langle D\rangle \cap Z_\xi^{1-\sigma_\alpha(\xi)}) \setminus \bigcup_{A\in\mathcal{F}_\xi}\langle A\rangle$ is small. Now let

$$\mathcal{B}_0 := \{A_{\beta+n} : \sigma_\beta \sqsubseteq \sigma_\alpha\}, \quad \mathcal{B}_1 := \bigcup_{\xi<\mathrm{dom}(\sigma_\alpha)} \mathcal{F}_\xi, \quad \mathcal{B} := \mathcal{B}_0 \cup \mathcal{B}_1.$$

One may show, in the same way of proving Claim 8, that $|\mathcal{B}| < \mathfrak{bs}$. By Lemma 2.1, the set

$$\{\langle B'\rangle \cap \langle D\rangle : B' \in \mathcal{B} \text{ and } \langle B'\rangle \cap \langle D\rangle \text{ is infinite-dimensional}\}$$

is an almost disjoint family of block subspaces of $\langle D\rangle$. Since $D \in \mathcal{H}^-(\mathcal{A}_\alpha)$, we see that $\langle D\rangle$ is not in this set. By Lemma 5.2, $|\mathcal{B}| < \mathfrak{bs} \le \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$, so there exists some $A_\alpha \le D$ such that A_α is almost disjoint from every element of $\mathcal{B}$.

Claim 11.

(1) *For all $\xi \in \mathrm{dom}(\sigma_\alpha)$, $\langle A_\alpha\rangle \setminus Z_\xi^{\sigma_\alpha(\xi)}$ is small.*
(2) *For all $\beta < \alpha$, A_α and A_β are almost disjoint.*

Proof. (1) Note that $\langle A_\alpha\rangle \setminus Z_\xi^{\sigma_\alpha(\xi)} = \langle A_\alpha\rangle \cap Z_\xi^{1-\sigma_\alpha(\xi)}$. We observe that $\langle A_\alpha\rangle \cap Z_\xi^{1-\sigma_\alpha(\xi)} \setminus \bigcup_{A\in\mathcal{F}_\xi}\langle A\rangle$ is small by the above, and since A_α is almost disjoint from every $A \in \mathcal{F}_\xi$, we have

$$\langle A_\alpha\rangle \cap Z_\xi^{1-\sigma_\alpha(\xi)} \cap \bigcup_{A\in\mathcal{F}_\xi}\langle A\rangle \subseteq \bigcup_{A\in\mathcal{F}_\xi}\langle A_\alpha\rangle \cap \langle A\rangle.$$

By Lemma 4.5, $\langle A_\alpha\rangle \cap \langle A\rangle$ is very small for each $A \in \mathcal{F}_\xi$, so $\langle A_\alpha\rangle \cap Z_\xi^{1-\sigma_\alpha(\xi)}$ is small.

(2) Let $\beta < \alpha$. If $A_\beta \in \mathcal{B}_0$, then A_α is almost disjoint from A_β by construction, so we are done. Now assume that $A_\beta \notin \mathcal{B}_0$. Let ξ be the least ordinal such that $\sigma_\beta(\xi) \ne \sigma_\alpha(\xi)$. By induction hypothesis, we see that $W := \langle A_\beta\rangle \setminus Z_\xi^{\sigma_\beta(\xi)} = \langle A_\beta\rangle \setminus Z_\xi^{1-\sigma_\alpha(\xi)}$ is small (and hence very small). Thus, for some small set W',

$$\langle A_\alpha\rangle \cap \langle A_\beta\rangle \subseteq W \cup (\langle A_\alpha\rangle \cap Z_\xi^{1-\sigma_\alpha(\xi)}) \subseteq W \cup W' \cup \bigcup_{A\in F_\xi}\langle A_\alpha\rangle \cap \langle A\rangle,$$

which is also small as all the components in the union are very small, as desired. ■

This completes the induction. If $U \in \mathcal{I}^+(\mathcal{A})$, then $\mathcal{I}^+(\mathcal{A}_\alpha)$ for all α, and there exists some α with $U_\alpha = U$. This implies that $\langle A_\alpha\rangle \subseteq U$, as desired. ■

6. Topological Ramsey spaces and the abstract Mathias forcing. We dedicate this section to an overview of the axioms of topological Ramsey spaces, and the abstract Mathias forcing developed in [7].

6.1. Topological Ramsey spaces. We recap the four axioms **A1**–**A4** presented by Todorčević [28], which are sufficient conditions for a triple $(\mathcal{R}, \leq, r)$ to be a topological Ramsey space. Here, $\mathcal{R}$ is a non-empty set, $\leq$ a quasi-order on $\mathcal{R}$, and $r : \mathcal{R} \times \omega \to \mathcal{AR}$ a surjective function. We also define a sequence of maps $r_n : \mathcal{R} \to \mathcal{AR}$ by $r_n(A) := r(A, n)$ for all $A \in \mathcal{R}$. Let $\mathcal{AR}_n \subseteq \mathcal{AR}$ be the image of r_n (i.e. $a \in \mathcal{AR}_n$ iff $a = r_n(A)$ for some $A \in \mathcal{R}$).

The four axioms are as follows:

A1. (1) $r_0(A) = \emptyset$ for all $A \in \mathcal{AR}$.
(2) $A \neq B$ implies $r_n(A) \neq r_n(B)$ for some n.
(3) $r_n(A) = r_m(B)$ implies $n = m$ and $r_k(A) = r_k(B)$ for all $k < n$.

For each $a \in \mathcal{AR}$, let $\mathrm{lh}(a)$ denote the unique n for which $a \in \mathcal{AR}_n$. By axiom **A1**(3), this n is well-defined. Given $a, b \in \mathcal{AR}$, we write $a \sqsubseteq b$ if there exists some $A \in \mathcal{R}$ such that $a = r_n(A)$ and $b = r_m(A)$ for some $n \leq m$.

A2. There is a quasi-ordering $\leq_{\mathrm{fin}}$ on $\mathcal{AR}$ such that

(1) $\{a \in \mathcal{AR} : a \leq_{\mathrm{fin}} b\}$ is finite for all $b \in \mathcal{AR}$,
(2) $A \leq B$ iff $\forall n\ \exists m\ [r_n(A) \leq_{\mathrm{fin}} r_m(B)]$,
(3) $\forall a, b \in \mathcal{AR}\ [a \sqsubseteq b \wedge b \leq_{\mathrm{fin}} c \to \exists d \sqsubseteq c\, [a \leq_{\mathrm{fin}} d]]$.

A3. We may define the *Ellentuck neighbourhoods* as follows: For any $A \in \mathcal{R}$, $a \in \mathcal{AR}$ and $n \in \mathbb{N}$, we let

$$[a, A] := \{B \in \mathcal{R} : B \leq A \wedge \exists n\, [r_n(B) = a]\},$$
$$[n, A] := [r_n(A), A].$$

Then the depth function defined by, for $B \in \mathcal{R}$ and $a \in \mathcal{AR}$,

$$\mathrm{depth}_B(a) := \begin{cases} \min\{n < \omega : a \leq_{\mathrm{fin}} r_n(B)\} & \text{if such } n \text{ exist,} \\ \infty, & \text{otherwise,} \end{cases}$$

satisfies the following:

(1) If $\mathrm{depth}_B(a) < \infty$, then for all $A \in [\mathrm{depth}_B(a), B]$, $[a, A] \neq \emptyset$.
(2) If $A \leq B$ and $[a, A] \neq \emptyset$, then there exists $A' \in [\mathrm{depth}_B(a), B]$ such that $\emptyset \neq [a, A'] \subseteq [a, A]$.

For each $A \in \mathcal{R}$, we let

$$\mathcal{AR}{\restriction}A := \{a \in \mathcal{AR} : \exists n\, [a \leq_{\mathrm{fin}} r_n(A)]\}.$$

If $a \in \mathcal{AR}{\restriction}A$, we also define

$$\mathcal{AR}{\restriction}[a, A] := \{b \in \mathcal{AR}{\restriction}A : a \sqsubseteq b\},$$
$$r_n[a, A] := \{b \in \mathcal{AR}{\restriction}[a, A] : \mathrm{lh}(b) = n\}.$$

A4. If $\mathrm{depth}_B(a) < \infty$ and if $\mathcal{O} \subseteq \mathcal{AR}_{\mathrm{lh}(a)+1}$, then there exists $A \in [\mathrm{depth}_B(a), B]$ such that $r_{\mathrm{lh}(a)+1}[a, A] \subseteq \mathcal{O}$ or $r_{\mathrm{lh}(a)+1}[a, A] \subseteq \mathcal{O}^c$.

Note that by axiom **A1**, we may identify each element $A \in \mathcal{R}$ as a sequence of elements of $\mathcal{AR}$, via the map $A \mapsto (r_n(A))_{n<\omega}$. Therefore, we may identify $\mathcal{R}$ as a subset of $\mathcal{AR}^{\mathbb{N}}$. In turn, $\mathcal{AR}^{\mathbb{N}}$ may be equipped with a natural metric topology by considering the first difference metric – that is, if $(a_n)_{n<\omega}, (b_n)_{n<\omega} \in \mathcal{AR}^{\mathbb{N}}$, then

$$d((a_n)_{n<\omega}, (b_n)_{n<\omega}) = 2^{-\min\{n:\, a_n \neq b_n\}}.$$

Definition 14. A triple $(\mathcal{R}, \leq, r)$ is said to be a *closed triple* if $\mathcal{R}$ is a metrically closed subset of $\mathcal{AR}^{\mathbb{N}}$. A closed triple $(\mathcal{R}, \leq, r)$ is an **A2**-*space* if $(\mathcal{R}, \leq, r)$ satisfies axioms **A1**–**A3**.

An **A2**-space $(\mathcal{R}, \leq, r)$ satisfying **A4** is a topological Ramsey space, as it satisfies the abstract Ellentuck theorem – that is, a subset $\mathcal{X} \subseteq \mathcal{R}$ is Ramsey iff it has the property of Baire relative to the Ellentuck topology. This is the abstract Ellentuck theorem by Todorčević – see [28, Theorem 5.4].

Mathias' original proof of his celebrated theorem in [14] (i.e. Theorem 1.7 here for $[\omega]^{\omega}$) uses the (original) Mathias forcing localised to a coideal $\mathcal{H}$. As we shall use a similar approach to prove Theorem 1.7, it is necessary to extend the notion of a coideal to topological Ramsey spaces.

Definition 15 ([7, Definition 3.2]). Let $(\mathcal{R}, \leq, r)$ be a closed triple satisfying **A1**–**A4**. A set $\mathcal{H} \subseteq \mathcal{R}$ is a *coideal* if it satisfies the following:

(1) (Upward closure) If $A \in \mathcal{H}$ and $A \leq B$, then $B \in \mathcal{H}$.
(2) (**A3** mod $\mathcal{H}$) For all $A \in \mathcal{H}$ and $a \in \mathcal{AR}{\restriction}A$, we have:
  (a) $[a, B] \cap \mathcal{H} \neq \emptyset$ for all $B \in [\mathrm{depth}_A(a), A] \cap \mathcal{H}$.
  (b) If $B \in \mathcal{H}{\restriction}A$ and $[a, B] \neq \emptyset$, then there exists $A' \in [\mathrm{depth}_A(a), A] \cap \mathcal{H}$ such that $\emptyset \neq [a, A'] \subseteq [a, B]$.
(3) (**A4** mod $\mathcal{H}$) Fix $A \in \mathcal{H}$ and $a \in \mathcal{AR}{\restriction}A$. For any $\mathcal{O} \subseteq \mathcal{AR}_{\mathrm{lh}(a)+1}$, there exists $B \in [\mathrm{depth}_A(a), A] \cap \mathcal{H}$ such that $r_{\mathrm{lh}(a)+1}[a, B] \subseteq \mathcal{O}$ or $r_{\mathrm{lh}(a)+1}[a, B] \subseteq \mathcal{O}^c$.

A notion of semiselectivity for a coideal $\mathcal{H}$ was also given in [7, Definitions 3.4–3.7], which says that for all $A \in \mathcal{H}$ and a family of dense open sets, there exists some $B \in \mathcal{H}{\restriction}A$ that diagonalises the family. We shall propose a slightly stronger version of semiselectivity, which instead asks that for all $A \in \mathcal{H}$, $a \in \mathcal{AR}{\restriction}A$ and a family of dense open sets, there exists some $B \in \mathcal{H}{\restriction}[a, A]$ such that B diagonalises the family.

Definition 16. Let $(\mathcal{R}, \leq, r)$ be a closed triple satisfying **A1**–**A4**, and let $\mathcal{H} \subseteq \mathcal{R}$ be a coideal. Let $A \in \mathcal{H}$ and $a \in \mathcal{AR}{\restriction}A$.

(1) A family of subsets $\vec{\mathcal{D}} = \{\mathcal{D}_b\}_{b \in \mathcal{AR}{\restriction}[a,A]}$ is *dense open below* $[a, A]$ *in* $\mathcal{H}$ if for all $b \in \mathcal{AR}{\restriction}[a, A]$, $\mathcal{D}_b$ is a $\leq$-downward closed subset of $\mathcal{H}{\restriction}[b, A]$, and for all $B \in \mathcal{H}{\restriction}[b, A]$, there exists some $C \in \mathcal{H}{\restriction}[b, B]$ such that $C \in \mathcal{D}_b$.

(2) Let $\vec{\mathcal{D}} = \{\mathcal{D}_b\}_{b\in\mathcal{AR}\restriction[a,A]}$ be dense open below $[a, A]$ in $\mathcal{H}$. We say that $B \in \mathcal{H}\restriction[a, A]$ *diagonalises* $\vec{\mathcal{D}}$ if for all $b \in \mathcal{AR}\restriction[a, A]$, there exists some $A_b \in \mathcal{D}_b$ such that $[b, B] \subseteq [b, A_b]$.
(3) We say that $\mathcal{H}$ is *semiselective* if for all $A \in \mathcal{H}$ and $a \in \mathcal{AR}\restriction A$, every dense open family below $[a, A]$ in $\mathcal{H}$ has a diagonalisation in $\mathcal{H}$.

The following is stated in [31, Lemma 3.11]. We replicate the proof here.

LEMMA 6.1. *Let $(\mathcal{R}, \leq, r)$ be a closed triple satisfying* **A1**–**A4**. *Then $\mathcal{R}$ is a semiselective coideal.*

Proof. Note that $\mathcal{R}$ is a coideal by definition. Fix some $A \in \mathcal{R}$ and $a \in \mathcal{AR}\restriction A$. Suppose that $\vec{\mathcal{D}} = \{\mathcal{D}_b\}_{b\in\mathcal{AR}\restriction[a,A]}$ is dense open below $[a, A]$. We shall define a fusion sequence $(A_n)_{n<\omega}$ in $[a, A]$, with $a_{n+1} := r_{\mathrm{lh}(a)+n+1}(A_n)$, such that $A_{n+1} \in [a_{n+1}, A_n]$: Let $A_0 := A$, and suppose that A_n has been defined. Let $\{b_i : i < N\}$ enumerate the set of all $b \in \mathcal{AR}\restriction A_n$ such that $a \sqsubseteq b$ and $b \leq_{\mathrm{fin}} a_{n+1}$. Let $A^0_{n+1} := A_n$. If $A^i_{n+1} \in [a_{n+1}, A_n]$ has been defined, let $B_{n+1} \in \mathcal{D}_{b_i}$ be such that $B_{n+1} \in [b_i, A^i_{n+1}]$, which exists as $\mathcal{D}_{b_i}$ is dense open in $[b_i, A]$. By **A3**, we then let $A^{i+1}_{n+1} \in [a_{n+1}, A^i_n]$ be such that $[b_i, A^{i+1}_{n+1}] \subseteq [b_i, B_{n+1}]$. We complete the induction by letting $A_{n+1} := A^N_{n+1}$. Let B be the limit of the fusion sequence $(A_n)_{n<\omega}$. Since $\mathcal{R}$ is metrically closed in $\mathcal{AR}^{\mathbb{N}}$, $B \in \mathcal{R}$ is well-defined, and so B diagonalises $\vec{\mathcal{D}}$. ■

We will see that the semiselectivity property plays an important role in both combinatorial forcing and the abstract Mathias forcing.

6.2. Combinatorial forcing. Let $(\mathcal{R}, \leq, r)$ be a closed triple satisfying **A1**–**A4**, and let $\mathcal{H} \subseteq \mathcal{R}$ be a coideal. We also fix a subset $\mathcal{X} \subseteq \mathcal{R}$. Unless stated otherwise, we assume that $\mathcal{H}$ is semiselective.

DEFINITION 17. Let $A \in \mathcal{H}$ and let $a \in \mathcal{AR}$. We say that

(1) A *accepts* a if $[a, A] \subseteq \mathcal{X}$,
(2) A *rejects* a if $a \in \mathcal{AR}\restriction A$, and for all $B \in \mathcal{H}\restriction[a, A]$, B does not accept a,
(3) A *decides* a if A accepts or rejects a.

This localised variant of combinatorial forcing for topological Ramsey spaces was introduced in [7], along with various basic properties of this combinatorial forcing. The details of the proofs of these properties with our definition of semiselectivity (Definition 16) remain mostly unchanged, but we shall provide an overview for the sake of completeness.

LEMMA 6.2. *Let $A \in \mathcal{H}$ and let $a \in \mathcal{AR}$.*

(1) *If $a \notin \mathcal{AR}\restriction A$, then A accepts a (as $[a, A] = \emptyset$).*
(2) *If A accepts a and $B \in \mathcal{H}\restriction A$, then B accepts a.*
(3) *If A rejects a, $B \in \mathcal{H}\restriction A$ and $a \in \mathcal{AR}\restriction B$, then B rejects a.*
(4) *If A decides a and $B \in \mathcal{H}\restriction A$, then B decides a.*

(5) *If* $a \in \mathcal{AR}\restriction A$, *then there exists* $B \in \mathcal{H}\restriction[\mathrm{depth}_A(a), A]$ *which decides* B.
(6) *If* $a \in \mathcal{AR}\restriction A$ *and* A *decides* a, *then for all* $B \in \mathcal{H}\restriction[\mathrm{depth}_A(a), A]$, B *decides* a *in the same way* A *does.*
(7) A *accepts* a *iff* A *accepts every* $b \in r_{\mathrm{lh}(a)+1}[a, A]$.

Proof. The proofs are straightforward and mostly the same as for the non-localised variant of combinatorial forcing – we refer the readers to [28, Lemma 4.31]. ∎

LEMMA 6.3. *For all* $A \in \mathcal{H}$ *and* $a \in \mathcal{AR}\restriction A$, *then there exists some* $B \in \mathcal{H}\restriction[a, A]$ *which decides every* $b \in \mathcal{AR}\restriction[a, A]$.

Proof. For each $b \in \mathcal{AR}\restriction[a, A]$, let

$$\mathcal{D}_b := \{B \in \mathcal{H}\restriction[b, A] : B \text{ decides } b\}.$$

By Lemma 6.2(5), $\mathcal{D}_b$ is dense open in $\mathcal{H}\restriction[b, A]$. Since $\mathcal{H}$ is semiselective, there exists some $B \in \mathcal{H}\restriction[a, A]$ which diagonalises $\{\mathcal{D}_b\}_{b\in\mathcal{AR}\restriction[a,A]}$. If $b \in \mathcal{AR}\restriction B$, then $[b, B] \subseteq [b, A_b]$ for some $A_b \in \mathcal{D}_b$. Since A_b decides b, $B \in \mathcal{H}\restriction A_b$ and $b \in \mathcal{AR}\restriction B$, we see by Lemma 6.2(2, 3) that B decides b as well. Note that if $b \notin \mathcal{AR}\restriction B$, then $[b, B] = \emptyset$, so B trivially decides b. ∎

LEMMA 6.4. *If* A *rejects* a, *then there exists some* $B \in \mathcal{H}\restriction[a, A]$ *which rejects every* $b \in r_{\mathrm{lh}(a)+1}[a, A]$.

Proof. By Lemma 6.3, we may let $A' \in \mathcal{H}[a, A]$ decide every $b \in \mathcal{AR}\restriction[a, A]$. Let

$$\mathcal{O} := \{b \in r_{\mathrm{lh}(a)+1}[a, A] : A' \text{ rejects } b\}.$$

By **A4** mod $\mathcal{H}$, there exists some $B \in \mathcal{H}\restriction[a, A]$ such that $r_{\mathrm{lh}(a)+1}[a, B] \subseteq \mathcal{O}$ or $r_{\mathrm{lh}(a)+1}[a, B] \subseteq \mathcal{O}^c$. It remains to show that the latter case is impossible. Since A' decides every $b \in \mathcal{AR}\restriction A$, so does B. Therefore, if $r_{\mathrm{lh}(a)+1}[a, B] \subseteq \mathcal{O}^c$ then B accepts every $b \in r_{\mathrm{lh}(a)+1}[a, B]$. By Lemma 6.2(7), this implies that B accepts a, contradicting that A rejects a. ∎

LEMMA 6.5. *If* A *rejects* a, *then there exists some* $B \in \mathcal{H}\restriction[a, A]$ *which rejects every* $b \in \mathcal{AR}\restriction[a, A]$.

Proof. Again, by Lemma 6.3 we may let $A' \in \mathcal{H}\restriction[a, A]$ decide every $b \in \mathcal{AR}\restriction[a, A]$. For each $b \in \mathcal{AR}\restriction[b, A]$, we let

$$\mathcal{D}_b := \{B \in \mathcal{H}\restriction[b, A] : B \text{ accepts } b \text{ or } B \text{ rejects every } b' \in r_{\mathrm{lh}(b)+1}[b, A]\}.$$

By Lemma 6.4, $\mathcal{D}_b$ is dense open in $\mathcal{H}\restriction[a, A]$, so by the semiselectivity property of $\mathcal{H}$ there exists some $B \in \mathcal{H}\restriction[a, A]$ which diagonalises $\{\mathcal{D}_b\}_{b\in\mathcal{AR}\restriction[a,A]}$. We shall induct on $\mathrm{lh}(b)$ and show that B rejects every $b \in \mathcal{AR}\restriction[a, B]$ (if $b \in \mathcal{AR}\restriction[a, A]$ but $b \notin \mathcal{AR}\restriction[a, B]$, then B trivially rejects b).

For the base case $\mathrm{lh}(b) = \mathrm{lh}(a)$ (i.e. $b = a$), it follows that A rejects a, and that $B \in \mathcal{H}\restriction[a, A]$ (cf. Lemma 6.2(3)). If B rejects some $b \in \mathcal{AR}\restriction[a, A]$, where $\mathrm{lh}(b) = n \geq \mathrm{lh}(a)$, then since $[b, B] \subseteq [b, A_b]$ for some $A_b \in \mathcal{D}_b$, we see that B rejects every $b' \in r_{\mathrm{lh}(a)+1}[b, B]$, completing the induction step. Therefore, B rejects every $b \in \mathcal{AR}{\restriction}[a, B]$. ∎

The abstract combinatorial forcing allows us to prove the abstract local Ellentuck theorem – see [7, Theorem 3.12]. We do not require the full strength of this result in the present paper. Instead, we give an outline of the proof that if $\mathcal{H}$ is semiselective, then every metrically open set $\mathcal{X} \subseteq \mathcal{R}$ is $\mathcal{H}$-Ramsey.

DEFINITION 18. A set $\mathcal{X} \subseteq \mathcal{R}$ is *$\mathcal{H}$-Ramsey* if for all $A \in \mathcal{R}$ and $a \in \mathcal{AR}{\restriction}A$, there exists some $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \subseteq \mathcal{X}^c$.

PROPOSITION 6.6. *Let $\mathcal{H} \subseteq \mathcal{R}$ be a semiselective coideal. If $\mathcal{X} \subseteq \mathcal{R}$ is metrically open, then $\mathcal{X}$ is $\mathcal{H}$-Ramsey.*

Proof. Let $A \in \mathcal{H}$ and let $a \in \mathcal{AR}{\restriction}A$. We consider combinatorial forcing with respect to $\mathcal{X}$. If A does not reject a, then $[a, B] \subseteq \mathcal{X}$ for some $B \in \mathcal{H}{\restriction}[a, A]$, so we are done. Otherwise, A rejects a, so by Lemma 6.5 there exists some $B \in \mathcal{H}{\restriction}[a, A]$ such that B rejects every $b \in \mathcal{AR}{\restriction}[a, A]$. We claim that $[a, B] \subseteq \mathcal{X}^c$. Otherwise, we may let $C \in [a, B] \cap \mathcal{X}$. Since $\mathcal{X}$ is metrically open, there exists some $b \in \mathcal{AR}{\restriction}[a, B]$ such that $C \in [b] \subseteq \mathcal{X}$, where $[b] := \{D \in \mathcal{R} : b \sqsubseteq D\}$. In particular, we find that $[b, B] \subseteq \mathcal{X}$, contradicting that B rejects b. ∎

6.3. Abstract Mathias forcing. The *abstract Mathias forcing localised to a coideal $\mathcal{H}$* is defined as the forcing poset $(\mathbb{M}_{\mathcal{H}}, \leq)$, where

$$\mathbb{M}_{\mathcal{H}} = \{(a, A) : A \in \mathcal{R} \text{ and } a \in \mathcal{AR}{\restriction}A\},$$

and where $(b, B) \leq (a, A)$ iff $[b, B] \subseteq [a, A]$. We also write $(b, B) \leq_0 (a, A)$ if $(b, B) \leq (a, A)$ and $b = a$.

Recall that given a closed triple $(\mathcal{R}, \leq, r)$ satisfying **A1**–**A4**, $\mathcal{R}$ is a closed subset of the product space $\mathcal{AR}^{\mathbb{N}}$, where $\mathcal{AR}$ is equipped with the discrete topology. Thus, there exists some absolute formula φ such that

$$\mathcal{R} = \{A \in \mathcal{AR}^{\mathbb{N}} : \varphi(A)\}.$$

Therefore, for any model M, we may write

$$\mathcal{R}^M := \{A \in \mathcal{AR}^{\mathbb{N}} : \varphi^M(A)\}.$$

DEFINITION 19. An element $A \in \mathcal{R}$ is said to be *$\mathbb{M}_{\mathcal{H}}$-generic* (over a ground model $\mathbf{V}$) if

$$G_A := \{(a, B) \in \mathbb{M}_{\mathcal{H}} : A \in [a, B]\}$$

is a generic filter of $\mathbb{M}_{\mathcal{H}}$ (over $\mathbf{V}$).

LEMMA 6.7. *Let $\mathbb{M}_{\mathcal{H}}$ be the Mathias forcing localised to a coideal $\mathcal{H}$.*

(1) *If G is a generic filter of $\mathbb{M}_{\mathcal{H}}$, then there exists a unique $\mathbb{M}_{\mathcal{H}}$-generic $A_G \in \mathcal{R}$ such that $G = \{(a, A) \in \mathbb{M}_{\mathcal{H}} : A_G \in [a, A]\}$.*

(2) *If* $A \in \mathcal{R}$ *is* $M_{\mathcal{H}}$*-generic, then* $A_{G_A} = A$.
(3) *If* G *is a generic filter of* $\mathbb{M}_{\mathcal{H}}$*, then* $G_{A_G} = G$.

Proof. We only prove (1)–(2) follow from the uniqueness of A_G in (1), and (3) follows easily from the definitions. Let $\mathcal{D}_n := \{(a, A) \in \mathbb{M}_{\mathcal{H}} : \mathrm{lh}(a) \geq n\}$. We see that $\mathcal{D}_n$ is dense open in $\mathbb{M}_{\mathcal{H}}$ for all n – if $(a, A) \notin \mathcal{D}_n$ (i.e. $\mathrm{lh}(a) < n$) and $b \in r_n[a, A]$, then by **A3** mod $\mathcal{H}$ we have $[b, A] \cap \mathcal{H} \neq \emptyset$. Let $B \in [b, A] \cap \mathcal{H}$; then we have $[b, B] \subseteq [a, A]$, so $(b, B) \leq (a, A)$ and $(b, B) \in \mathcal{D}_n$. Therefore, $G \cap \mathcal{D}_n \neq \emptyset$ for all n, and since G is a filter, for all $(a, A), (b, B) \in G$, $a \sqsubseteq b$ or $b \sqsubseteq a$. Thus, there exists a unique $A_G \in \mathcal{R}$ such that for all n, $(r_n(A_G), A) \in G$ for some A. It is easy to verify $G = G_{A_G}$. ∎

Definition 20. Let $(\mathcal{R}, \leq, r)$ be a closed triple satisfying **A1**–**A4**, and let $\mathcal{H} \subseteq \mathcal{R}$ be a coideal.

(1) $\mathbb{M}_{\mathcal{H}}$ satisfies the *Mathias property* if whenever $A \in \mathcal{R}$ is $\mathbb{M}_{\mathcal{H}}$-generic over a ground model $\mathbf{V}$ and $B \leq A$, also B is $\mathbb{M}_{\mathcal{H}}$-generic over $\mathbf{V}$.
(2) $\mathbb{M}_{\mathcal{H}}$ satisfies the *Prikry property* if whenever φ is a sentence in the forcing language, then for all $p \in \mathbb{M}_{\mathcal{H}}$, there exists some $q \leq_0 p$ such that $q \Vdash \varphi$ or $q \Vdash \neg\varphi$.

Given a subset $\mathcal{D} \subseteq \mathbb{M}_{\mathcal{H}}$, we let

$$\overline{\mathcal{D}} := \bigcup_{(a,A) \in \mathcal{D}} [a, A].$$

Note that $\overline{\mathcal{D}}$ is an Ellentuck-open subset of $\mathcal{R}$ in $\mathbf{V}$. Thus, if $\mathcal{D}$ is dense open in $\mathbb{M}_{\mathcal{H}}$, then for all $A \in \mathcal{H}$ and $a \in \mathcal{AR}{\restriction}A$, A does not reject a with respect to $\overline{\mathcal{D}}$ (otherwise, by Lemma 6.5 there exists some $B \in [a, A]$ which rejects every $b \in \mathcal{AR}{\restriction}[a, A]$, contradicting that $\mathcal{D}$ is dense in $\mathbb{M}_{\mathcal{H}}$). In other words, for all $A \in \mathcal{H}$ and $a \in \mathcal{AR}{\restriction}A$, there exists some $B \in \mathcal{H}{\restriction}[a, A]$ such that $[a, B] \subseteq \overline{\mathcal{D}}$.

Notation 3. Given $a, b \in \mathcal{AR}$, we write

$$b \preceq_{\mathrm{fin}} a \iff b \leq_{\mathrm{fin}} a \text{ and } \forall a' \sqsubseteq a\, [a' \neq a \to b \not\leq_{\mathrm{fin}} a'].$$

Proposition 6.8. *If* $\mathcal{H}$ *is semiselective, then* $\mathbb{M}_{\mathcal{H}}$ *has the Mathias property.*

Proof. Let A_G be an $\mathbb{M}_{\mathcal{H}}$-generic element of $\mathcal{R}$ (over some ground model $\mathbf{V}$), and let $D \leq A_G$. We need to show that the filter

$$H := \{(a, A) \in \mathbb{M}_{\mathcal{H}} : D \in [a, A]\}$$

is generic. Let $\mathcal{D} \subseteq \mathbb{M}_{\mathcal{H}}$ be a dense open subset (in the ground model $\mathbf{V}$). For each $a \in \mathcal{AR}$, we let

$$\mathcal{E}_a := \{B \in \mathcal{H} : a \in \mathcal{AR}{\restriction}B \text{ and } \forall b \preceq_{\mathrm{fin}} a\, [[b, B] \subseteq \overline{\mathcal{D}}]\}.$$

CLAIM 12. *For all $A \in \mathcal{H}$ such that $a \in \mathcal{AR}\restriction A$, $\mathcal{E}_a$ is dense open in $\mathcal{H}\restriction[a, A]$.*

Proof. It is clear that $\mathcal{E}_a$ is open, so we shall show that it is dense in $\mathcal{H}\restriction[a, A]$. By the comment above, let $A' \in \mathcal{H}\restriction[a, A]$ be such that $[a, A'] \subseteq \overline{\mathcal{D}}$. Let $\{b_i : 1 \leq i \leq N\}$ be an enumeration of all $b \preceq_{\text{fin}} a$. Let $A_0 := A'$, and suppose that we have defined $A_i \in [a, A']$. We find that $b_{i+1} \in \mathcal{AR}\restriction A_i$ (as $b_{i+1} \leq_{\text{fin}} a \sqsubseteq A_i$), so by the observation stated prior to the claim there exists some $A'_{i+1} \in \mathcal{H}\restriction[b_{i+1}, A_i]$ which accepts b_{i+1}, A_i with respect to $\overline{\mathcal{D}}$, i.e. $[b_{i+1}, A'_{i+1}] \subseteq \overline{\mathcal{D}}$. By **A3** mod $\mathcal{H}$, there exists some $A_{i+1} \in \mathcal{H}\restriction[a, A]$ such that $[a_{i+1}, A_{i+1}] \subseteq [a_{i+1}, A'_{i+1}]$. This completes the induction, and so $(a, A_N) \in \mathcal{D}$, as desired. ∎

Fix any $(a, A) \in G$. Since $\mathcal{H}$ is semiselective, for all $(b, B) \leq (a, A)$ in $\mathbb{M}_{\mathcal{H}}$, there exists some $C \in \mathcal{H}\restriction[b, B]$ which diagonalises $\vec{\mathcal{E}_b} := \{\mathcal{E}_c\}_{c \in \mathcal{AR}\restriction[b,A]}$ below A. That is, the set

$$\overline{\mathcal{D}} := \{(b, B) \in \mathbb{M}_{\mathcal{H}} : B \text{ diagonalises } \vec{\mathcal{E}_b} \text{ below } A\}$$

is dense open in $\mathbb{M}_{\mathcal{H}}$. Since G is generic, we may let $(b, B) \in G \cap \mathcal{D}'$. Then $A_G \in [b, B]$, and for all $c \in \mathcal{AR}\restriction[b, A_G]$ and $d \preceq_{\text{fin}} c$, $[d, A_G] \subseteq [d, B] \subseteq \overline{\mathcal{D}}$. Let n be the least integer such that $\operatorname{depth}_{A_G}(r_n(D)) =: m \geq \operatorname{lh}(b)$. Then $r_n(D) \preceq_{\text{fin}} r_m(A_G) \in \mathcal{AR}\restriction[b, B]$, so $D \in [r_n(D), A_G] \subseteq [r_n(D), B] \subseteq \overline{\mathcal{D}}$. This implies that $(r_n(D), B) \in H \cap \mathcal{D}$, and since $\mathcal{D}$ is arbitrary, H is a generic filter, as required. ∎

PROPOSITION 6.9. *If $\mathcal{H}$ is semiselective, then $\mathbb{M}_{\mathcal{H}}$ has the Prikry property.*

Proof. Let φ be a sentence in the forcing language. Let

$$\mathcal{D} := \{q \in \mathbb{M}_{\mathcal{H}} : q \Vdash \varphi\}.$$

We consider doing combinatorial forcing with respect to the set $\overline{\mathcal{D}}$. Let $(a, A) \in \mathbb{M}_{\mathcal{H}}$, and by Lemma 6.2(2, 3) there exists some $B \in [a, A]$ which decides a. If B accepts a, then $[a, B] \subseteq \overline{\mathcal{D}}$, so $(a, B) \in \mathcal{D}$ as $\mathcal{D}$ is open. In this case, $(a, B) \leq_0 (a, A)$ and $(a, B) \Vdash \varphi$. Otherwise, by Lemma 6.5 there exists some $B \in [a, A]$ which rejects every $b \in \mathcal{AR}\restriction[a, A]$. In other words, for all $(b, C) \leq (a, B)$, $(b, C) \notin \mathcal{D}_0$, i.e. (b, C) does not force φ. But this precisely means that $(a, B) \Vdash \neg\varphi$, as needed. ∎

7. Topological Ramsey space of block subspaces. By Hindman's theorem, the pigeonhole principle holds for $E^{[\infty]}$ if E is a vector space over $\mathbb{F}_2$. That is, for all $Y \subseteq E$ and $A \in E^{[\infty]}$, there exists some $B \leq A$ such that $\langle B \rangle \subseteq Y$ or $\langle B \rangle \subseteq Y^c$. Consequently, if for each $A := (x_n)_{n<\omega} \in E^{[\infty]}$ we let

$$r_n(A) := (x_0, \ldots, x_{n-1}),$$

then $(E^{[\infty]}, \leq, r)$ is a closed triple satisfying **A1**–**A4**. We shall discuss the intersection between the local Ramsey theory of $(E^{[\infty]}, \leq, r)$ as a topological Ramsey space and the local Ramsey theory of block subspaces developed by Smythe, followed by an application of the abstract Mathias forcing to prove Theorem 1.7.

The symbols $<$ and $\leq$ will appear repeatedly throughout this section. To avoid confusion, the symbol $<$ shall be reserved for Notation 1(5, 6), while $\leq$ shall be reserved for the quasi-order $\leq$ in $(E^{[\infty]}, \leq, r)$.

7.1. Semicoideals of block subspaces. Readers may notice that there are two separate definitions for a coideal $\mathcal{H} \subseteq E^{[\infty]}$, i.e. Definition 4 and Definition 15. It turns out that these two definitions are equivalent. We remark that in this setting, if $A \in E^{[\infty]}$ and $a \in E^{[<\infty]}\restriction A$ (i.e. $a \in E^{[<\infty]}$ and $\langle a\rangle \subseteq \langle A\rangle$), we have that $\mathrm{depth}_A(a) = d_A(a)$ (see Notation 1).

Lemma 7.1. *Let $\mathcal{H} \subseteq E^{[\infty]}$. The following are equivalent:*

(1) *$\mathcal{H}$ is a semicoideal.*
(2) *$\mathcal{H}$ is $\leq$-upward closed and* **A3** *mod $\mathcal{H}$ holds.*

Proof. (2)⇒(1). Let $A = (x_n)_{n<\omega} \in \mathcal{H}$ and $B \in E^{[\infty]}$ be such that $A \leq^* B$. Then there exists some N such that $A/N \leq B$. Therefore, if we let $a := (x_N)$, then $a < C$ for all $C \in [a, C]$, so $[a, A] \subseteq [a, B]$. By **A3** mod $\mathcal{H}$ there exists some $C \in [a, A] \cap \mathcal{H}$. Then $C \in [a, B]$, and in particular $C \leq B$, so $B \in \mathcal{H}$ by $\leq$-upward closure.

(1)⇒(2). Suppose that $\mathcal{H}$ is $\leq^*$-upward closed. Clearly, this implies that $\mathcal{H}$ is $\leq$-upward closed. We shall verify that **A3** mod $\mathcal{H}$ holds. Let $A \in \mathcal{H}$ and $a \in E^{[<\infty]}\restriction A$.

(a) For any $B \in [d_A(a), A] \cap \mathcal{H}$, we have $B \leq^* B/a$ (as $B/a \leq B/a$), so $B/a \in \mathcal{H}$. Since $B/a \leq a^\frown(B/a)$, we see that $a^\frown(B/a) \in \mathcal{H}$ by $\leq$-upward closure. But $a^\frown(B/a) \leq^* B$, so $a^\frown(B/a) \in [a, B] \cap \mathcal{H}$, as required.

(b) Suppose that $B \in \mathcal{H}\restriction A$ and $[a, B] \neq \emptyset$. Let $N := d_A(a)$, and let M be large enough that $B/M \leq A/N$. Then $C := r_N(A)^\frown(B/m) \in [d_A(a), A] \cap \mathcal{H}$ is well-defined, and

$$\begin{aligned} D \in [a, C] &\implies a \sqsubseteq D \wedge D \leq C \implies a \sqsubseteq D \wedge D/a \leq C/a \\ &\implies a \sqsubseteq D \wedge D/a \leq B/M \implies a \sqsubseteq D \wedge D/a \leq A/N \\ &\implies D \in [a, A], \end{aligned}$$

as desired. ∎

Lemma 7.2. *Let $\mathcal{H} \subseteq E^{[\infty]}$ be a semicoideal. Then $\mathcal{H}$ is a coideal (as in Definition 4) iff* **A4** *mod $\mathcal{H}$ holds.*

Proof. Definition 4 is exactly the statement **A4** mod $\mathcal{H}$ for $a = \emptyset$, so it suffices to show that the converse holds. Let $A \in E^{[\infty]}$, $a \in E^{[<\infty]}\restriction A$, and let $\mathcal{O} \subseteq E^{[\mathrm{lh}(a)+1]}$. Define

$$Y := \{x \in E : a < x \wedge a^\frown x \in \mathcal{O}\}.$$

Since $\mathcal{H}$ is a coideal (as in Definition 4), there exists some $B' \leq A$ such that $\langle B' \rangle \subseteq Y$ or $\langle B' \rangle \subseteq Y^c$. Let M be large enough that $B'/M \leq A/d_A(a)$, and note that $a < B'/M$. Let $B := B'/M$, and let $C := a^\frown B \in [a, A]$. Observe that

$$b \in r_{\mathrm{lh}(a)+1}[a, C] \iff b = a^\frown x \text{ for some } x \in \langle B \rangle.$$

Therefore, if $\langle B \rangle \subseteq Y$, then $r_{\mathrm{lh}(a)+1}[a, C] \subseteq \mathcal{O}$, and if $\langle B \rangle \subseteq Y^c$, then $r_{\mathrm{lh}(a)+1}[a, C] \subseteq \mathcal{O}^c$. Thus, **A4** mod $\mathcal{H}$ holds. ∎

We now address the overlap in the selectivity properties. There are two natural questions:

- Is Definition 16 equivalent to the natural extension of semiselectivity of coideals of $[\omega]^\omega$ to semicoideals of block subspaces (i.e. [28, Lemma 7.10])?
- Is a selective (semi)coideal (as in Definition 7) semiselective?

We provide a positive answer to both of these problems.

LEMMA 7.3. *Let $\mathcal{H} \subseteq E^{[\infty]}$ be a semicoideal. The following are equivalent:*

(1) *$\mathcal{H}$ is semiselective (as in Definition 16).*
(2) *If $\{\mathcal{D}_n\}_{n<\omega}$ is a collection of dense open subsets of $\mathcal{H}$ (under $\leq$), then for all $A \in \mathcal{H}$ there exists some $B \in \mathcal{H}{\upharpoonright}A$ such that $B/n \in \mathcal{D}_{d_A(r_n(B))}$ for all n.*

In statement (2) of the lemma, we also say that B *diagonalises* $\{\mathcal{D}_n\}_{n<\omega}$ *below* A.

Proof of Lemma 7.3. (1)⇒(2). Let $\{\mathcal{D}_n\}_{n<\omega}$ be a collection of dense open subsets of $\mathcal{H}$, and let $A \in \mathcal{H}$. For each $b \in \mathcal{AR}{\upharpoonright}A$, we let

$$\mathcal{D}_b := \{b^\frown(B/b) : B \in \mathcal{D}_{d_A(b)} \text{ and } B \leq A\}.$$

It is clear that $\mathcal{D}_b \subseteq \mathcal{H}{\upharpoonright}[b, A]$ for all $b \in \mathcal{AR}{\upharpoonright}A$, and that $\mathcal{D}_b$ is $\leq$-downward closed in $\mathcal{H}{\upharpoonright}[b, A]$. We see that $\mathcal{D}_b$ is dense in $\mathcal{H}{\upharpoonright}[b, A]$ – If $B \in \mathcal{H}{\upharpoonright}[b, A]$, then since $\mathcal{D}_{d_A(b)}$ is dense open in $\mathcal{H}{\upharpoonright}A$, there exists some $C \in \mathcal{D}_{d_A(b)}$ such that $C \leq B$. Then $b^\frown(C/b) \in \mathcal{D}_b$ and $b^\frown(C/b) \leq B$. Since $\mathcal{H}$ is semiselective, there exists some $B \in \mathcal{H}{\upharpoonright}A$ such that for all $b \in \mathcal{AR}{\upharpoonright}B$, $[b, B] \subseteq [b, A_b]$ for some $A_b \in \mathcal{D}_b$. In particular, for all n we deduce that $B \in [r_n(B), B] \subseteq [r_n(B), A_{r_n(B)}]$ for some $A_{r_n(B)} \in \mathcal{D}_{r_n(B)}$, so $B/n \leq A_{r_n(B)}/r_n(B) \in \mathcal{D}_{d_A(r_n(B))}$.

(2)⇒(1). Let $A \in \mathcal{H}$ and $a \in \mathcal{AR}\restriction A$. Let $\vec{\mathcal{D}} = \{\mathcal{D}_b\}_{b\in\mathcal{AR}\restriction A}$ be a dense open family below $[a, A]$. For each $b \in \mathcal{AR}\restriction[a, A]$, let

$$\mathcal{D}'_b := \{B/b : B \in \mathcal{D}_b\}, \qquad \mathcal{D}_n := \bigcap_{\substack{b\in\mathcal{AR}\restriction[a,A]\\ d_A(b)=d_A(a)+n}} \mathcal{D}'_b.$$

We claim that $\mathcal{D}'_b$ is dense open in $\mathcal{H}\restriction A$ for all $b \in \mathcal{AR}\restriction[a, A]$. This implies that $\mathcal{D}_n$ is dense open in $\mathcal{H}\restriction A$, as it is a finite intersection of dense open subsets of $\mathcal{H}\restriction A$. Let $B \in \mathcal{H}\restriction A$, and let $b \in \mathcal{AR}\restriction[a, A]$. Then $b^\frown(B/b) \in \mathcal{H}\restriction[b, A]$, and since $\mathcal{D}_b$ is dense open in $\mathcal{H}\restriction[b, A]$, there exists some $C \in \mathcal{D}_b$ such that $C \le b^\frown(B/b)$. This implies that $C/b \le B/b$, and $C/b \in \mathcal{D}'_b$, as desired.

By our assumption, there exists some $B\in\mathcal{H}\restriction A$ such that $B/n\in\mathcal{D}_{d_A(r_n(B))}$ for all n. Observe that $a < B$, as $B \in \mathcal{D}_0 \subseteq \mathcal{D}_a$, so $B \le C/a$ for some $C \in \mathcal{D}_a$. Therefore, $a^\frown B \in \mathcal{R}$ is well-defined. We claim that $a^\frown B$ diagonalises $\vec{\mathcal{D}}$. Let $b \in \mathcal{AR}\restriction[a, a^\frown B]$, and suppose that $d_A(b) = d_A(a) + n$. Let b' be such that $b = a^\frown b'$, and note that $d_A(b') = d_A(b)$. Since $\langle b'\rangle \subseteq \langle r_{d_B(b')}(B)\rangle$, we have $d_A(b') \le d_A(r_{d_B(b')}(B))$. Therefore

$$(a^\frown B)/b = B/b' = B/d_B(b') \in \mathcal{D}_{d_A(r_{d_B(b')}(B))} \subseteq \mathcal{D}_{d_A(b')} = \mathcal{D}_{d_A(b)} \subseteq \mathcal{D}'_b,$$

so $A_b := b^\frown((a^\frown B)/b) \in \mathcal{D}_b$. Observe that $[b, B] = [b, A_b]$, and since b is arbitrary, B diagonalises $\vec{\mathcal{D}}$. ∎

Lemma 7.4. *Let $\mathcal{H} \subseteq E^{[\infty]}$ be a semicoideal. If $\mathcal{H}$ is selective, then it is semiselective.*

Proof. Let $\{\mathcal{D}_n\}_{n<\omega}$ be a family of dense open subsets of $\mathcal{H}$, and let $A \in \mathcal{H}$. Define a $\le$-decreasing sequence in $\mathcal{H}$ as follows: Let $A_0 \in \mathcal{D}_0$ be such that $A_0 \le A$. If $A_n \in \mathcal{D}_n$ has been defined, then let $A_{n+1} \in \mathcal{D}_{n+1}$ be such that $A_{n+1} \le A_n$. By the selectivity of $\mathcal{H}$, there exists some $B \in \mathcal{H}\restriction A$ such that $B/n \le A_{d_A(r_n(B))} \in \mathcal{D}_{d_A(r_n(B))}$ for all n. Since $\mathcal{D}_n$'s are all $\le$-downward closed, $B/n \in \mathcal{D}_{d_A(r_n(B))}$ for all n. ∎

An abstract notion of selectivity was also proposed in [7], but the definition appears to be problematic. For instance, [7, Proposition 5.4] claims that every selective coideal is semiselective, but there appears to be a gap in the proof–it is unclear how the element $A^{2,1} \in \mathcal{D}_{a_1^2}\restriction A^{1,n_1}$ may be obtained. It seems unlikely that this version of selectivity coincides with our definition of selectivity.

Definition 21. Let $\mathcal{H} \subseteq E^{[\infty]}$ be a semicoideal.

(1) $\mathcal{H}$ is a *filter* if $(\mathcal{H}, \le)$ is a filter (as a partial order), i.e. for all $A, B \in \mathcal{H}$ there exists some $C \in \mathcal{H}$ such that $C \le A$ and $C \le B$.
(2) $\mathcal{H}$ is an *ultrafilter* if it is a coideal and a filter.

We shall dedicate the remaining portion of this subsection to showing that if $\mathcal{U} \subseteq E^{[\infty]}$ is a selective ultrafilter, then a block sequence $D \in (E^{[\infty]})^{\mathbf{V}[G]}$ is $\mathbb{M}_{\mathcal{U}}$-generic iff $D \leq^* A$ for all $A \in \mathcal{U}$. The proof is very similar to the case for the Ellentuck space – a proof may be found, for instance, in [10, Chapter 26].

The following proposition, in the setting of topological Ramsey spaces, was stated in [7, Theorem 6.7], extending the original theorem in the setting $[\omega]^\omega$ in [8]. We shall re-prove this result for block subspaces.

Proposition 7.5. *Suppose that E is a vector space over $\mathbb{F}_2$, and that $\mathcal{H} \subseteq E^{[\infty]}$ is a coideal. The following are equivalent:*

(1) *$\mathcal{H}$ is semiselective.*
(2) *$(\mathcal{H}, \leq^*)$ is σ-distributive, and every $(\mathcal{H}, \leq^*)$-generic filter $\mathcal{U}$ is a selective ultrafilter.*

Proof. (1)$\Rightarrow$(2). Let $\{\mathcal{D}_n\}_{n<\omega}$ be a family of dense open subsets of $\mathcal{H}$ under the partial order $\leq^*$. Note that each $\mathcal{D}_n$ is also dense open in $\mathcal{H}$ under $\leq$, so by semiselectivity there exists some $B \in \mathcal{H}$ which diagonalises $\{\mathcal{D}_n\}_{n<\omega}$ (below some $A \in \mathcal{H}$). Since $\mathcal{D}_n$ is $\leq^*$-downward closed, this implies that $B \in \mathcal{D}_n$ for all n, so $(\mathcal{H}, \leq^*)$ is σ-distributive.

Let $\mathcal{U}$ be an $(\mathcal{H}, \leq^*)$-generic filter. It is clear that $\mathcal{U}$ is a filter (as in Definition 21). We need to show that $\mathcal{U}$ is a coideal (and hence an ultrafilter). Let $Y \subseteq E$, and let $A \in \mathcal{U} \subseteq \mathcal{H}$. Since Y is countable and $(\mathcal{H}, \leq^*)$ adds no reals, Y is in the ground model. By Hindman's theorem, the set

$$\mathcal{D} := \{B \in \mathcal{H}{\upharpoonright}A : \langle B\rangle \subseteq Y \text{ or } \langle B\rangle \subseteq Y^c\}$$

is dense open below A in $(\mathcal{H}, \leq^*)$. Therefore, the genericity of $\mathcal{U}$ implies that $\mathcal{U} \cap \mathcal{D} \neq \emptyset$, i.e. there exists some $B \leq A$ in $\mathcal{U}$ such that $\langle B\rangle \subseteq Y$ or $\langle B\rangle \subseteq Y^c$. Hence, $\mathcal{U}$ is a coideal.

We now show that $\mathcal{U}$ is selective. Let $(A_n)_{n<\omega}$ be a $\leq$-decreasing sequence in $\mathcal{U}$, and let $A := A_0$. Since $(\mathcal{H}, \leq^*)$ adds no reals, the sequence $(A_n)_{n<\omega}$ is in the ground model. For each n, let

$$\mathcal{D}_n := \{B \in \mathcal{H}{\upharpoonright}A : B \leq A_n \text{ or } \nexists C \in \mathcal{H}{\upharpoonright}A\,[C \leq B \text{ and } C \leq A_n]\}.$$

It is clear that $\mathcal{D}_n$ is dense open below A in $\mathcal{H}$ for all n. Since $\mathcal{H}$ is semiselective, we see that the set

$$\mathcal{E} := \{B \in \mathcal{H}{\upharpoonright}A : B \text{ diagonalises } \{\mathcal{D}_n\}_{n<\omega} \text{ below } A\}$$

is dense open below A in $\mathcal{H}$. By the genericity of $\mathcal{U}$, let $B \in \mathcal{U} \cap \mathcal{E}$. Then $B/n \in \mathcal{D}_{d_A(r_n(B))}$ for all n, and since $\mathcal{U}$ is a filter, B is compatible with A_n for all n in $\mathcal{U}$, so $B/n \leq A_{d_A(r_n(B))}$ for all n. Thus, B diagonalises $(A_n)_{n<\omega}$ below A.

(2)$\Rightarrow$(1). Let $\{\mathcal{D}_n\}_{n<\omega}$ be a family of dense open subsets of $\mathcal{H}$, and let $A \in \mathcal{H}$. Let $\mathcal{U}$ be an $(\mathcal{H}, \leq^*)$-generic filter containing A. We define a

$\leq$-decreasing sequence $(A_n)_{n<\omega}$ in $\mathcal{U}$ as follows: By the genericity of $\mathcal{U}$, let $A_0 \in \mathcal{U} \cap \mathcal{D}_0$. Assume that $A_n \in \mathcal{U} \cap \mathcal{D}_n$ has been defined. Since $\mathcal{D}_{n+1} \cap \mathcal{H} \upharpoonright A_n$ is dense open below A_n in $\mathcal{H}$ and $A_n \in \mathcal{U}$, there exists some $A_{n+1} \in \mathcal{U} \upharpoonright A_n$ such that $A_{n+1} \in \mathcal{U} \cap \mathcal{D}_{n+1}$. By the selectivity of $\mathcal{U}$, there exists some $B \in \mathcal{U} \upharpoonright A$ such that $B/n \leq A_{d_A(r_n(B))} \in \mathcal{D}_{d_A(r_n(B))}$ for all n. Therefore, $B \in \mathcal{H} \upharpoonright A$ diagonalises $\{\mathcal{D}_n\}_{n<\omega}$ below A. ∎

Definition 22. Let $(\mathcal{R}, \leq, r)$ be a closed triple satisfying **A1**–**A4**, and let $\mathcal{H} \subseteq \mathcal{R}$. Let $\mathcal{D} \subseteq \mathbb{M}_{\mathcal{H}}$. We say that $(a, A) \in \mathbb{M}_{\mathcal{H}}$ *captures* $\mathcal{D}$ if for all $B \in \mathcal{H} \upharpoonright [a, A]$, there exists some initial segment $b \in \mathcal{AR} \upharpoonright [a, B]$ of B such that $(b, B) \in \mathcal{D}$.

We remark that if $\mathcal{AR}$ is countable (e.g. $E^{[<\infty]}$ is countable), then the statement "(a, A) captures $\mathcal{D}$" is Π_1 with respect to the Polish topology of $\mathcal{R}$. Since $\mathcal{R}$ is a subspace of $\mathcal{AR}^{\mathbb{N}}$, and $\mathcal{AR}$ is equipped with the discrete topology, we may apply Mostowski's absoluteness theorem to conclude that "(a, A) captures $\mathcal{D}$" is absolute.

Lemma 7.6. *Suppose that E is a vector space over $\mathbb{F}_2$. Let $\mathcal{U} \subseteq E^{[\infty]}$ be a selective ultrafilter. Let $\mathcal{D} \subseteq \mathbb{M}_{\mathcal{U}}$ be dense open. For all $a \in E^{[<\infty]}$ and $A \in E^{[\infty]}$ such that $a \in E^{[<\infty]} \upharpoonright A$, there exists some $B \in \mathcal{U} \upharpoonright [a, A]$ such that (a, B) captures $\mathcal{D}$.*

Proof. For each $b \in E^{[<\infty]} \upharpoonright [a, A]$, let $A_b \in \mathcal{U} \upharpoonright [b, A]$ be any block sequence such that $(b, A_b) \in \mathcal{D}$ if it exists – otherwise, we let $A_b := A$. Since $\mathcal{U}$ is an ultrafilter, we may inductively define a $\leq$-decreasing sequence in $\mathcal{U} \upharpoonright [a, A]$ such that $A_n \leq A_b/a$ for all $b \in E^{[<\infty]} \upharpoonright [a, A]$ in which $d_A(b) \leq n$. Since $\mathcal{U}$ is selective, there exists some $B' \in \mathcal{H} \upharpoonright (A/a)$ which diagonalises $(A_n)_{n<\omega}$ below A. Let $B := a^\frown B' \in \mathcal{U} \upharpoonright [a, A]$, and consider the set

$$\mathcal{X} := \{C \in E^{[\infty]} : C \in [a, B] \to \exists b \in E^{[<\infty]} \upharpoonright [a, C]\, [(b, B) \in \mathcal{D}]\}.$$

Since $\mathcal{X}$ is a metrically open subset of $E^{[\infty]}$, by Proposition 6.6 there exists some $C \in \mathcal{U} \upharpoonright [a, B]$ such that $[a, C] \subseteq \mathcal{X}$ or $[a, C] \subseteq \mathcal{X}^c$. Observe that if $[a, C] \subseteq \mathcal{X}$, then (a, C) captures $\mathcal{D}$, so we are done. It remains to show that $[a, C] \subseteq \mathcal{X}^c$ is not possible. Since $\mathcal{D}$ is dense in $\mathbb{M}_{\mathcal{H}}$, there exists some condition $(b, D) \leq (a, C)$ in $\mathcal{D}$. This means that $(b, A_b) \in \mathcal{D}$, and since $B/b \leq A_{d_A(b)} \leq A_b$, we have $(b, B) \in \mathcal{D}$. In other words, there exists some $b \in E^{[<\infty]} \upharpoonright C$ such that $(b, B) \in \mathcal{D}$, so $C \in \mathcal{X}$. ∎

Proposition 7.7. *Suppose that E is a vector space over $\mathbb{F}_2$. Let $\mathcal{U} \subseteq E^{[\infty]}$ be a selective ultrafilter. Then a block sequence D is $\mathbb{M}_{\mathcal{U}}$-generic iff $D \leq^* A$ for all $A \in \mathcal{U}$.*

Proof. Given a block sequence D, we let

$$G := \{(a, A) \in \mathbb{M}_{\mathcal{U}} : D \in [a, A]\}.$$

($\Rightarrow$) Suppose that D is $\mathbb{M}_{\mathcal{U}}$-generic, i.e. G is a generic filter. For any $A \in \mathcal{U}$ and $a \in E^{[<\infty]}\restriction A$, the set

$$\mathcal{D}_A := \{(b, B) \in \mathbb{M}_{\mathcal{U}} : B \leq^* A \text{ and } b \in E^{[<\infty]}\restriction B\}$$

is dense open in $\mathbb{M}_{\mathcal{U}}$. Indeed, for any $(b, B) \in \mathbb{M}_{\mathcal{U}}$, let $C \in \mathcal{U}$ be such that $C \leq B$ and $C \leq A$. Then $(a, a^\frown(C/a)) \in \mathcal{D}_A$, and $C \leq^* B$ and $C \leq^* B$. Since G is generic, $G \cap \mathcal{D}_A \neq \emptyset$, so we may let $(b, B) \in \mathcal{D}_A$ be such that $D \in [b, B]$. This implies that $D \leq^* B \leq^* A$, as desired.

($\Leftarrow$) Suppose that $D \leq^* A$ for all $A \in \mathcal{U}$. Let $\mathcal{D} \subseteq \mathbb{M}_{\mathcal{U}}$ be dense open. To show that D is generic, it suffices to show that $D \in [r_n(D), B]$ for some $(r_n(G), B) \in \mathcal{D}$. By Lemma 7.6, let $A := A_\emptyset \in \mathcal{U}$ be such that $(\emptyset, A)$ captures $\mathcal{D}$; then for each $a \in E^{[<\infty]}$ we let $A_a \in \mathcal{U}$ be such that (a, A_a) captures $\mathcal{D}$. We then inductively define a $\leq$-decreasing sequence $(A_n)_{n<\omega}$ in $\mathcal{U}$ with $A_n \leq A_a$ for all $d_A(a) \leq n$. Since $\mathcal{U}$ is selective, we may let $B \in \mathcal{U}\restriction A$ be such that $B/a \leq A_{d_A(a)} \leq A_a$ for all $a \in E^{[<\infty]}\restriction B$.

We know that $D \leq^* B$, so let $a \in E^{[<\infty]}\restriction D \cap E^{[<\infty]} \cap B$ be such that $D/a \leq B/a \leq A_a$. Therefore, (a, B) captures $\mathcal{D}$ in the ground model, so (a, B) captures $\mathcal{D}$ in the generic extension (see the remark after Definition 22). Since $D \in [a, B]$, we have $(r_n(D), B) \in \mathcal{D}$ for some n, as desired. ∎

7.2. Proof of Theorem 1.7 and Corollary 1.8. Recall that for an uncountable cardinal κ, the forcing poset $\mathrm{Coll}(\omega, <\kappa)$ is the Lévy collapse, which collapses κ to ω_1. We shall follow the proof layout of [16, Theorem 4.1]. The following lemma is due to Solovay, and a proof may be found in [27]. For the rest of this section, we fix a ground model $\mathbf{V}$ of ZFC, and all vector spaces are over $\mathbb{F}_2$.

LEMMA 7.8. *Let κ be an inaccessible cardinal, and let $\mathbb{P} \in \mathbf{V}$ be a forcing poset of size $< \kappa$. Let G be a $\mathrm{Coll}(\omega, <\kappa)$-generic filter. If $H \in \mathbf{V}[G]$ is $\mathbb{P}$-generic over $\mathbf{V}$, then there exists a filter G^* which is $\mathrm{Coll}(\omega, <\kappa)$-generic over $\mathbf{V}[H]$, and $\mathbf{V}[H][G^*] = \mathbf{V}[G]$.*

Proof of Theorem 1.7. Let $\mathcal{H} \in \mathbf{V}[G]$ be a selective coideal in $E^{[\infty]}$, and let $\mathcal{X} \in \mathbf{L}(\mathbb{R})^{\mathbf{V}[G]}$ be a subset of $E^{[\infty]}$. Since every set in $\mathbf{L}(\mathbb{R})^{\mathbf{V}[G]}$ is ordinal-definable from some real in $\mathbf{V}[G]$, there exists a ternary formula ϕ, some real $r \in \mathbf{V}[G]$ and a sequence of ordinals α such that for all $B \in (E^{[\infty]})^{\mathbf{V}[G]}$,

$$B \in \mathcal{X} \iff \mathbf{V}[G] \models \phi[B, r, \alpha].$$

We need to show that for all $A \in \mathcal{H}$ and $a \in E^{[<\infty]}\restriction A$, there exists some $B \in \mathcal{H}\restriction[a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \subseteq \mathcal{X}^c$. We fix some $A \in \mathcal{H}$ and $a \in E^{[<\infty]}\restriction A$.

Since κ is Mahlo, we may fix some inaccessible $\lambda < \kappa$ such that $\mathcal{H} \cap \mathbf{V}[G\restriction\lambda] \in \mathbf{V}[G\restriction\lambda]$, and $\mathbf{V}[G\restriction\lambda] \models \mathcal{H} \cap \mathbf{V}[G\restriction\lambda]$ is a selective coideal, and denote $\overline{\mathcal{H}} := \mathcal{H} \cap \mathbf{V}[G\restriction\lambda]$. By the κ-chain condition of $\mathrm{Coll}(\omega, <\kappa)$, we may also assume that $A \in \overline{\mathcal{H}}$ and $r \in \mathbf{V}[G\restriction\lambda]$. Let ϕ^* be the formula such that $\phi^*(x, y, w)$ holds iff $\Vdash_{\mathrm{Coll}(\omega,<\kappa)} \phi(\dot{x}, \check{y}, \check{w})$. Let

$$\overline{\mathcal{X}} := \{B \in E^{[\infty]} \cap \mathbf{V}[G\restriction\lambda] : V[G\restriction\lambda] \models \phi^*[B, r, \alpha]\}.$$

Note that $\overline{\mathcal{X}} \in \mathbf{V}[G\restriction\lambda]$. Furthermore, since the Lévy collapse is homogeneous, we have, for all $B \in E^{[\infty]} \cap \mathbf{V}[G\restriction\lambda]$,

$$B \in \mathcal{X} \iff \mathbf{V}[G] \models \phi[B, r, \alpha] \iff \Vdash_{\mathrm{Coll}(\omega,<\kappa)} \phi[B, r, \alpha] \iff B \in \overline{\mathcal{X}}.$$

Thus, $\overline{\mathcal{X}} = \mathcal{X} \cap \mathbf{V}[G\restriction\lambda]$.

Let $\mathcal{U}$ be a $(\overline{\mathcal{H}}, \leq^*)$-generic filter such that $A \in \mathcal{U}$. Note that by Proposition 7.5, $\mathcal{U}$ is a selective ultrafilter. Since $(\overline{\mathcal{H}}, \leq^*)$ adds no reals, we have $E^{[\infty]} \cap \mathbf{V}[G\restriction\lambda][\mathcal{U}] = E^{[\infty]} \cap \mathbf{V}[G\restriction\lambda]$. This implies that $\overline{\mathcal{X}} = \mathcal{X} \cap \mathbf{V}[G\restriction\lambda][\mathcal{U}]$. Furthermore,

$$\mathbf{V}[G\restriction\lambda][\mathcal{U}][G\restriction[\lambda, \kappa)] = \mathbf{V}[G\restriction\lambda][G\restriction[\lambda, \kappa)][\mathcal{U}] = \mathbf{V}[G][\mathcal{U}].$$

Claim 13. *Let $B \in E^{[\infty]} \cap \mathbf{V}[G][\mathcal{U}]$ be $\mathbb{M}_{\mathcal{U}}$-generic over $\mathbf{V}[G\restriction\lambda][\mathcal{U}]$. Then*

$$B \in \mathcal{X} \iff V[G\restriction\lambda][\mathcal{U}][B] \models \phi^*[B, r, \alpha].$$

Proof. We work in $\mathbf{V}[G][\mathcal{U}]$. Since κ is inaccessible, $\mathrm{Coll}(\omega, <\lambda) * \dot{\mathbb{M}}_{\mathcal{U}}$ is a poset in $\mathbf{V}$ of size $<\kappa$, so by Lemma 7.8 applied to the model $\mathbf{V}[G\restriction\lambda][\mathcal{U}]$ there exists a filter G^*, $\mathrm{Coll}(\omega, <\kappa)$-generic over $\mathbf{V}[G\restriction\lambda][\mathcal{U}][B]$, such that

$$\mathbf{V}[G][\mathcal{U}] = \mathbf{V}[G\restriction\lambda][\mathcal{U}][G\restriction[\lambda, \kappa)] = \mathbf{V}[G\restriction\lambda][\mathcal{U}][B][G^*].$$

Therefore,

$$\begin{aligned} B \in \mathcal{X} &\iff \mathbf{V}[G][\mathcal{U}] \models \phi[B, r, \alpha] \iff \Vdash \phi(\dot{B}, \check{r}, \check{\alpha}) \text{ in } \mathbf{V}[G\restriction\lambda][\mathcal{U}][B] \\ &\iff \mathbf{V}[G\restriction\lambda][\mathcal{U}][B] \models \phi^*[B, r, \alpha], \end{aligned}$$

where the second equivalence follows from the homogeneity of the Lévy collapse. ∎

We consider the forcing notion $\mathbb{M}_{\mathcal{U}}$ in $\mathbf{V}[G\restriction\lambda][\mathcal{U}]$, and note that $(a, A) \in \mathbb{M}_{\mathcal{U}}$. By the Prikry property of $\mathbb{M}_{\mathcal{U}}$ (Proposition 6.9), there exists some condition $(a, A') \leq_0 (a, A)$ in $\mathbb{M}_{\mathcal{U}}$ which decides the formula $\phi^*(\dot{A}_H, \check{r}, \check{\alpha})$, where $\dot{A}_H$ is the canonical name for an $\mathbb{M}_{\mathcal{U}}$-generic element of $E^{[\infty]}$. Since $\mathbf{V}[G][\mathcal{U}] \models \mathcal{U}$ is countable and $\mathcal{U}$ is an ultrafilter, there exists some $A_H \in [a, A'] \cap \mathbf{V}[G]$ such that $A_H \leq^* B$ for all $B \in \mathcal{U}$. By Proposition 7.7, A_H is $\mathbb{M}_{\mathcal{U}}$-generic over $\mathbf{V}[G\restriction\lambda][\mathcal{U}]$. We shall finish the proof by showing that $[a, A_H] \subseteq \mathcal{X}$ or $[a, A_H] \subseteq \mathcal{X}^c$.

Suppose that $(a, A') \Vdash \phi^*(\dot{A}_H, \check{r}, \check{\alpha})$, and let $B \in [a, A_H]$. By the Mathias property of $\mathbb{M}_{\mathcal{U}}$, B is also $\mathbb{M}_{\mathcal{U}}$-generic. Since $B \in [a, A']$, the condition (a, A') is in the $\mathbb{M}_{\mathcal{U}}$-generic filter generated by B, so by Claim 13, $B \in \mathcal{X}$. Therefore, $[a, A_H] \subseteq \mathcal{X}$. Alternatively, if $(a, A') \Vdash \neg\phi^*(\dot{A}_H, \check{r}, \check{\alpha})$, then $[a, A_H] \subseteq \mathcal{X}^c$. ∎

We may now prove Corollary 1.8. Given an almost disjoint family $\mathcal{A}$ of subspaces, we let

$$\overline{\mathcal{A}} := \{B \in E^{[\infty]} : \langle B \rangle \subseteq V \text{ for some } V \in \mathcal{A}\}.$$

Observe that $\mathcal{H}^-(\mathcal{A}) \cap \overline{\mathcal{A}} = \emptyset$.

Proof of Corollary 1.8. Suppose for a contradiction that $\mathcal{A} \in \mathbf{L}(\mathbb{R})^{\mathbf{V}[G]}$ is a full mad family of subspaces. Note that $\overline{\mathcal{A}} \in \mathbf{L}(\mathbb{R})^{\mathbf{V}[G]}$. By Proposition 4.12, $\mathcal{H}(\mathcal{A})$ is a selective coideal, so by Theorem 1.7 we see that $\mathcal{A}$ is $\mathcal{H}$-Ramsey. Let $B \in \mathcal{H}(\mathcal{A})$, and let $C \in \mathcal{H}(\mathcal{A}) \restriction A$ be such that $[\emptyset, C] \subseteq \overline{\mathcal{A}}$ or $[\emptyset, C] \subseteq \overline{\mathcal{A}}^c$. Since $C \in \mathcal{H}(\mathcal{A})$ and $\mathcal{H}(\mathcal{A}) \cap \overline{\mathcal{A}} = \emptyset$, the first case is not possible. On the other hand, since $\mathcal{A}$ is mad, there exists some $V \in \mathcal{A}$ which is compatible with C, i.e. there exists some $D \in E^{[\infty]}$ such that $D \leq C$ and $D \leq A$. Then $D \in [\emptyset, C] \cap \overline{\mathcal{A}}$, another contradiction. ∎

We conclude this section by addressing the difficulty in removing the additional large cardinal hypothesis on κ. Neeman–Norwood [16] successfully reduced the required large cardinal strength of κ in the original Mathias' theorem [14] from being Mahlo to just inaccessible, with the additional requirement that the selective coideal $\mathcal{H}$ lies in $\mathbf{L}(\mathbb{R})^{\mathbf{V}[G]}$. Their approach is as follows:

1. For an ultrafilter $\mathcal{U}$ on ω, they define the *diagonalisation forcing* $\mathbb{P}_{\mathcal{U}}$, and show that $\mathbb{P}_{\mathcal{U}}$ satisfies the Prikry and Mathias properties (where $\mathcal{U}$ need not be selective).
2. Let $\mathrm{Coll}(\omega, {<}\kappa)$ be the Lévy collapse of an inaccessible cardinal κ, and let G be a $\mathrm{Coll}(\omega, {<}\kappa)$-generic filter. There exists some $\beta < \kappa$ such that $\overline{\mathcal{H}} = \mathcal{H} \cap \mathbf{V}[G \restriction \beta] \in \mathbf{V}[G \restriction \beta]$. Then $\overline{\mathcal{H}}$ is a coideal on ω, but it may not be selective. Let $\mathcal{U} \subseteq \overline{\mathcal{H}}$ be any ultrafilter on ω in $\mathbf{V}[G \restriction \beta]$.
3. Repeat the argument of Mathias with Mathias forcing $\mathbb{M}_{\mathcal{U}}$ replaced by diagonalisation forcing $\mathbb{P}_{\mathcal{U}}$.

Using the definition of ultrafilter for infinite block sequences (see Definition 21), it is possible to define a version of diagonalisation forcing for infinite block sequences, and it does satisfy the Prikry and Mathias properties. The issue lies in the second step – if $\mathcal{H}$ is a selective coideal in $E^{[\infty]}$ in $\mathbf{V}[G]$ (or $\mathbf{L}[G]^{\mathbf{V}[G]}$), it is unclear if $\overline{\mathcal{H}} = \mathcal{H} \cap \mathbf{V}[G \restriction \beta]$ is a coideal in $E^{[\infty]}$. It is even less clear if we may obtain an ultrafilter $\mathcal{U} \subseteq \overline{\mathcal{H}}$ in $\mathbf{V}[G \restriction \beta]$ – the only current known construction of ultrafilters in $E^{[\infty]}$ assumes set-theoretic hypotheses such as CH or MA, and defines an ultrafilter as the $\leq$-upward closure of a $\leq^*$-decreasing chain of block sequences of size continuum (see [24, Theorem 5.3] and [4]). This requires one to repeatedly obtain weak diagonalisations (as in Definition 5) at countable limit ordinal stages of the induction. However, as $\overline{\mathcal{H}}$ need not satisfy the (p)-property, this construction may not be performed inside $\overline{\mathcal{H}}$ to obtain an ultrafilter $\mathcal{U} \subseteq \overline{\mathcal{H}}$.

8. Further remarks and open problems. Although Theorem 1.5 asserts the existence of a full mad family of subspaces in ZFC, it remains unknown if every mad family of subspaces in ZFC is full.

PROBLEM 8.1. *Is there a non-full mad family of subspaces?*

While there are various results about the cardinal $\mathfrak{a}_{\mathrm{vec},\mathbb{F}}$, the cardinal $\mathfrak{a}^*_{\mathrm{vec},\mathbb{F}}$ is not well-understood. In particular, while it is known that $\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_{\mathrm{vec},\mathbb{F}}$ holds in ZFC, the $\mathfrak{a}^*_{\mathrm{vec},\mathbb{F}}$ variant of the problem remains unsolved.

PROBLEM 8.2. *Is* $\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}^*_{\mathrm{vec},\mathbb{F}}$ *true in* ZFC*?*

Most of the current approaches to the non-existence of mad families in $[\omega]^\omega$ employ some Ramsey-theoretic techniques (such as [14, 16, 21]). Consequently, it is difficult to use these approaches to prove the non-existence of mad families of block subspaces over $\mathbb{F} \neq \mathbb{F}_2$. Perhaps the approach highlighted in [29] may be used to answer the following question:

PROBLEM 8.3. *Let κ be an inaccessible cardinal, and let G be* $\mathrm{Coll}(\omega, {<}\kappa)$*-generic. Are there mad families of subspaces* (*over any countable field*) *in* $\mathbf{L}(\mathbb{R})^{\mathbf{V}[G]}$*? What if κ is Mahlo?*

Acknowledgements. The author would like to express his deepest gratitude to Spencer Unger and Iian Smythe for their helpful comments, suggestions and corrections.

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