STUDIA MATHEMATICA

Online First version

The isomorphism problem for analytic discs with self-crossings on the boundary

by

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Abstract. Suppose V is the unit disc $\mathbb{D}$ embedded in the d-dimensional unit ball $\mathbb{B}_d$ and attached to the unit sphere. Consider the space $\mathcal{H}_V$, the restriction of the Drury–Arveson space to the variety V, and its multiplier algebra $\mathcal{M}_V = \mathrm{Mult}(\mathcal{H}_V)$. The isomorphism problem is the following: Is $V_1 \cong V_2$ equivalent to $\mathcal{M}_{V_1} \cong \mathcal{M}_{V_2}$?

A theorem of Alpay, Putinar and Vinnikov states that for V without self-crossings on the boundary $\mathcal{M}_V$ is the space of bounded analytic functions on V. We consider what happens when there are self-crossings on the boundary and prove that if $\mathcal{M}_{V_1} \cong \mathcal{M}_{V_2}$ algebraically, then V_1 and V_2 must have the same self-crossings up to a unit disc automorphism. We prove that an isomorphism between $\mathcal{M}_{V_1}$ and $\mathcal{M}_{V_2}$ can only be given by a composition with a map from V_1 to V_2. In the case of a single simple self-crossing we show that there are only two possible candidates for this map and find these candidates. Finally, we provide a continuum of V's with the same self-crossing pattern such that their multiplier algebras are all mutually non-isomorphic.

1. Introduction

1.1. The isomorphism problem. The main problem we are interested in is the isomorphism problem for multiplier algebras of varieties in the unit ball. In this section we state this problem in the general setting.

Let $\mathbb{B}_d$, $d \in \mathbb{N} \cup \{\infty\}$, denote the Euclidean open unit ball in $\mathbb{C}^d$, where $\mathbb{C}^d$ means $\ell^2(\mathbb{N})$ for $d = \infty$. The main function space on $\mathbb{B}_d$ we consider is the Drury–Arveson space.

Definition 1. The *Drury–Arveson space on* $\mathbb{B}_d$ is

$$H^2_d = \left\{ f(z) = \sum_{\alpha \in \mathbb{N}_0^d} c_\alpha z^\alpha : \|f\|^2 = \sum_{\alpha \in \mathbb{N}_0^d} |c_\alpha|^2 \frac{\alpha!}{|\alpha|!} < \infty \right\}.$$

2020 *Mathematics Subject Classification*: Primary 46E22; Secondary 30H50, 47B32, 32A35.
Key words and phrases: reproducing kernel Hilbert space, multiplier algebra, isomorphism problem, embedded disc.
Received 10 September 2025; revised 17 February 2026.
Published online 1 September 2026.

DOI: 10.4064/sm250910-27-2

It is a reproducing kernel Hilbert space with kernel

$$\kappa(z,w)=\kappa_w(z)=\frac{1}{1-\langle z,w\rangle_{\mathbb{C}^d}}.$$

This space is one of the multivariate generalizations of the Hardy space $H^2(\mathbb{D})$ on the unit disc, i.e., $H^2(\mathbb{D})=H^2_1$. The Drury–Arveson space naturally arises in operator theory as well as in the theory of complete Pick spaces; see the survey [Sha15] or the more recent [Har23] for an introduction.

We denote by $\mathcal{M}_d=\mathrm{Mult}(H^2_d)$ the multiplier algebra of H^2_d. We say that $V\subset\mathbb{B}_d$ is *a multiplier variety* if it is a joint zero set of functions from $\mathcal{M}_d$, i.e., there exists $E\subset\mathcal{M}_d$ such that

$$V=\{z\in\mathbb{B}_d:\varphi(z)=0 \text{ for all } \varphi\in E\}.$$

Note that for $d<\infty$ we can replace E by a finite set (even of cardinality at most d); see the argument in [Chi89, Chapter 5, Section 7]. In particular, V is an analytic variety in $\mathbb{B}_d$. Throughout this paper we simply refer to multiplier varieties as *varieties*.

For a variety $V\subset\mathbb{B}_d$ the associated Hilbert space of functions on V is $\mathcal{H}_V\subset H^2_d$:

$$\mathcal{H}_V=\mathrm{Cl}\,\mathrm{span}\,\{\kappa_\lambda:\lambda\in V\}=\{f\in H^2_d: f|_V=0\}^\perp.$$

This Hilbert space is a reproducing kernel Hilbert space of functions on V.

Next, we define the algebra associated to V:

$$\mathcal{M}_V=\{\varphi|_V:\varphi\in\mathcal{M}_d\}.$$

By [DRS15, Proposition 2.6], $\mathcal{M}_V$ is exactly the multiplier algebra $\mathrm{Mult}(\mathcal{H}_V)$, and $\mathcal{M}_V$ is completely isometrically isomorphic to the quotient $\mathcal{M}_d/J_V$, where

$$J_V=\{\varphi\in\mathcal{M}_d:\varphi|_V=0\}.$$

Moreover, $\mathcal{M}_V$ is contained in $H^\infty(V)$, the algebra of bounded analytic functions on V, but might not coincide with it.

The isomorphism problem. When is $\mathcal{M}_V\cong\mathcal{M}_W$ equivalent to $V\cong W$?

The answer heavily depends on the notion of isomorphism for both varieties and algebras. We recall a few possible notions. First, V is said to be an *automorphic image* of W in $\mathbb{B}_d$ if there exists an automorphism μ of $\mathbb{B}_d$ such that $V=\mu(W)$. Since μ^{-1} is also an automorphism of $\mathbb{B}_d$, this notion is symmetric with respect to V and W. We say that V and W are *biholomorphic* if there are maps F and G holomorphic in $\mathbb{B}_d$ such that $F|_V$ is a bijection from V to W and $G|_W$ is its inverse. Finally, V and W in $\mathbb{B}_d$ are *multiplier biholomorphic* if they are biholomorphic via maps F,G which are coordinate multipliers, i.e., $F,G\in\mathcal{M}_d\times\cdots\times\mathcal{M}_d$ (d factors).

We refer to the survey by Salomon and Shalit [SS16] for an overview of the available answers to the isomorphism problem:

- $\mathcal{M}_V \cong \mathcal{M}_W$ isometrically if and only if $\mathcal{H}_V \cong \mathcal{H}_W$ as reproducing kernel Hilbert spaces if and only if V is an automorphic image of W, where $d < \infty$ or V and W have the same affine codimension ([SS16, Proposition 4.8, Theorem 4.6], [DRS15]).

  Similarly, Rochberg [Roc19, Theorem 7] has shown that for ordered finite sets V, W, $\mathcal{H}_V \cong \mathcal{H}_W$ if and only if V is an automorphic image of W, adding some concrete quantitative conditions for an isomorphism to exist, such as coinciding "normal forms" or having the same Gram matrix. Rochberg also showed that $V \cong W$ if and only if all triples in V are congruent to all triples in W.
- $\mathcal{M}_V \cong \mathcal{M}_W$ algebraically is equivalent to V and W being multiplier biholomorphic for homogeneous varieties V and W if $d < \infty$ ([SS16, Theorem 5.14], [DRS11], [Har12]).
- $\mathcal{M}_V \cong \mathcal{M}_W$ algebraically implies that V and W are multiplier biholomorphic, for V and W which are irreducible varieties (or finite unions of irreducible varieties and a discrete variety) ([SS16, Theorem 5.5], [DRS15]).
- $V \cong W$ via a biholomorphism that extends to be a 1-to-1 C^2-map on the boundary implies $\mathcal{M}_V \cong \mathcal{M}_W$ algebraically, for V and W – images of finite Riemann surfaces under a holomap that extends to a 1-to-1 C^2-map on the boundary [SS16, Corollary 5.18]. This is the result of an idea from [APV03] being generalized in [ARS08, Section 2.3.6] and culminating in [KMS13].
- For $V \subset \mathbb{B}_\infty$ of the form $V = f(\mathbb{D})$ with
$$f(z) = (b_1 z, b_2 z^2, b_3 z^3, \ldots), \qquad \text{where } b_1 \neq 0 \text{ and } \|b\|_2^2 = 1,$$
  $\mathcal{M}_V = H^\infty(V) \cong H^\infty(\mathbb{D})$ if and only if $\sum n|b_n|^2 < \infty$, while any two such varieties are multiplier biholomorphic, [DHS15, Corollary 7.4].

It is still an open question whether $V \cong W$ via a multiplier biholomorphism implies $\mathcal{M}_V \cong \mathcal{M}_W$ algebraically for sufficiently simple V and W. However, it is known to be false for V, W discrete varieties; see [SS16, Example 5.7].

1.2. Complete Pick spaces. A different way to look at the multiplier algebras $\mathcal{M}_V$ comes from function theory, in particular from the theory of complete Pick spaces.

Let $\mathcal{H}$ be a reproducing kernel Hilbert space on a set X with kernel κ and let $\mathcal{M}$ be its multiplier algebra $\mathrm{Mult}(\mathcal{H})$. Consider the following problem. Given $\lambda_1, \ldots, \lambda_n \in X$ and $a_1, \ldots, a_n \in \mathbb{C}$, does the interpolation problem
$$\varphi(\lambda_j) = a_j, \qquad j = 1, \ldots, n,$$
have a multiplier solution $\varphi \in \mathcal{M}$ with $\|\varphi\|_{\mathcal{M}} \leqslant 1$?

This problem is called the *Pick interpolation problem*. Denote by M_φ the operator on $\mathcal{H}$ given by multiplying by φ. By definition the norm on $\mathcal{M}$ is inherited from the operator norm, hence, $\|\varphi\|_{\mathcal{M}} \leqslant 1$ is equivalent to $I_{\mathcal{H}} - M_\varphi M_\varphi^* \geqslant 0$. Using this fact together with $M_\varphi^* \kappa_x = \overline{\varphi(x)} \kappa_x$ we get a necessary condition for the Pick interpolation problem to have a solution:

$$[(1 - a_j\overline{a_k})\kappa(\lambda_j, \lambda_k)]_{j,k=1}^n \geqslant 0. \tag{1}$$

Pick [Pic15] showed that for the Hardy space $H^2(\mathbb{D})$, which has the Szegő kernel

$$\kappa(z, w) = \frac{1}{1 - z\bar{w}},$$

this positivity condition is also sufficient. Spaces and their kernels for which the positivity (1) is sufficient for the Pick interpolation problem to have a solution are called *Pick spaces* and *Pick kernels* respectively. It is not difficult to see that not all spaces are Pick, e.g., the Bergman space on the unit disc, with the kernel

$$\kappa(z, w) = \frac{1}{(1 - z\bar{w})^2},$$

is not Pick.

Let us now turn our attention to the matrix version of the Pick interpolation problem. We consider $\mathcal{M}^{m\times m}$, the algebra of $m\times m$ multiplier matrices, so that $\mathcal{M}^{m\times m} = \mathrm{Mult}(\mathcal{H}^m)$. Then the $m \times m$ *Pick interpolation problem* is the following: Given $\lambda_1, \ldots, \lambda_n \in X$ and $A_1, \ldots, A_n \in M_m(\mathbb{C})$, does the interpolation problem

$$F(\lambda_j) = A_j, \quad j = 1, \ldots, n,$$

have an $m \times m$ multiplier solution $F \in \mathcal{M}^{m\times m}$ with $\|F\|_{\mathcal{M}^{m\times m}} \leqslant 1$?

Similarly to the $m = 1$ case, it is possible to show the necessity of the positivity condition

$$[(1 - A_jA_k^*)\,\kappa(\lambda_j, \lambda_k)]_{j,k=1}^n \geqslant 0.$$

If this condition is sufficient for all $m \geqslant 1$ we call the space and the respective kernel *complete Pick*. We refer to the monograph of Agler and McCarthy [AM02] for an in-depth introduction to the study of (complete) Pick kernels, and to [Qui94, Chapter 5, Section 3] for an example of a space which is Pick but not complete Pick.

We say that an RKHS $\mathcal{H}$ on X with kernel κ is *irreducible* if $\kappa(x, y)$ does not vanish for $x, y \in X$. The connection between complete Pick spaces and our isomorphism problem is given by the following theorem due to Agler and McCarthy ([AM00, Theorem 4.2], [AM02, Theorem 8.2]).

Theorem A. *Let $\mathcal{H}$ be a separable irreducible reproducing kernel Hilbert space on X. Then $\mathcal{H}$ is complete Pick if and only if for some $d \in \mathbb{N} \cup \{\infty\}$ there is a map $b : X \to \mathbb{B}_d$ and a nowhere vanishing function $\delta : X \to \mathbb{C}$ such that*

$$\kappa(z, w) = \frac{\delta(z)\overline{\delta(w)}}{1 - \langle b(z), b(w)\rangle_{\mathbb{C}^d}}.$$

This means that up to rescaling and composition, $\mathcal{H}$ is essentially $\mathcal{H}_V$ for some variety $V \subset \mathbb{B}_d$. Moreover, $\mathcal{M}$ is isometrically isomorphic to $\mathcal{M}_V$ via the same composition.

We see that the Drury–Arveson space is, in a sense, the universal complete Pick space. Hence, the isomorphism problem is closely related to the study of irreducible complete Pick spaces and their multiplier algebras.

1.3. Analytic discs. In this paper we study the isomorphism problem in the case where V and W are analytic discs attached to the unit sphere.

DEFINITION 2. An *analytic disc attached to the unit sphere* is a variety $V \subset \mathbb{B}_d$, $d < \infty$, for which there exists an injective analytic map $f : \mathbb{D} \to \mathbb{B}_d$ with $f'(z) \neq 0$ for $z \in \mathbb{D}$ such that $V = f(\mathbb{D})$, f extends to C^2 up to $\overline{\mathbb{D}}$ and

$$\|f(x)\| = 1 \iff |x| = 1.$$

We say that f is an *embedding map* of V.

Note that by [DHS15, Corollary 3.2] such discs meet the boundary transversally, i.e.,

$$\langle f(\xi), f'(\xi)\rangle \neq 0, \qquad \xi \in \mathbb{T}.$$

We emphasize that V is defined to be a variety. It is not clear whether for an arbitrary f satisfying properties from Definition 2 the image $V = f(\mathbb{D})$ is a variety.

Instead of working directly with the spaces $\mathcal{H}_V$ and $\mathcal{M}_V$, it is more convenient to pull back these spaces from the variety V to the unit disc $\mathbb{D}$. We denote by $\mathcal{H}_f$ an RKHS on $\mathbb{D}$ that we get from $\mathcal{H}_V$ by composing with f:

$$\mathcal{H}_f = \{h : \mathbb{D} \to \mathbb{C} : h \circ f^{-1} \in \mathcal{H}_V\}, \qquad \|h\|_{\mathcal{H}_f} = \|h \circ f^{-1}\|_{\mathcal{H}_V}.$$

The reproducing kernel of this space is

$$\kappa^f(z, w) = \frac{1}{1 - \langle f(z), f(w)\rangle}. \tag{2}$$

We denote the multiplier algebra of this space by $\mathcal{M}_f$, so that

$$\mathcal{M}_f = \{\varphi : \mathbb{D} \to \mathbb{C} : \varphi \circ f^{-1} \in \mathcal{M}_V\}, \qquad \|\varphi\|_{\mathcal{M}_f} = \|\varphi \circ f^{-1}\|_{\mathcal{M}_V}.$$

The isomorphism problem was partially solved in the case when f extends injectively to $\overline{\mathbb{D}}$. We state the results of [APV03, Proposition 2.2, Theorem 2.3] combined with the above-mentioned fact that the transversality condition is satisfied automatically.

THEOREM B. *Suppose V is an analytic disc attached to the unit sphere, and f is the embedding map of V. If the extension of f to $\overline{\mathbb{D}}$ is injective, then*

- $\mathcal{H}_f = H^2(\mathbb{D})$ *with equivalent norms,*
- $\mathcal{M}_f = H^\infty(\mathbb{D})$ *with equivalent norms.*

We now state our results.

THEOREM 1. *Suppose V, W are analytic discs attached to the unit sphere, and f, g are the respective embedding maps. If $\mathcal{M}_f = \mathcal{M}_g$, then*

$$f(\xi) = f(\zeta) \iff g(\xi) = g(\zeta), \qquad \xi, \zeta \in \mathbb{T}.$$

This means that f and g have the same self-crossings on the boundary.

THEOREM 2. *Suppose V, W are analytic discs attached to the unit sphere, and f, g are the respective embedding maps. If $\mathcal{M}_f \cong \mathcal{M}_g$ algebraically, then there exists an automorphism μ of the unit disc $\mathbb{D}$ such that $\mathcal{M}_f = \mathcal{M}_{g\circ\mu}$ with equivalent norms.*

Combining these two theorems we get the following result.

COROLLARY 1. *Suppose V, W are analytic discs attached to the unit sphere, and f, g are the respective embedding maps. If $\mathcal{M}_f \cong \mathcal{M}_g$ algebraically, then, up to a unit disc automorphism, f and g have the same self-crossings on the boundary.*

If f and g have the same self-crossings on the boundary up to a unit disc automorphism we say that they have the same *self-crossing type*. Corollary 1 means that for f and g with different self-crossing types the multiplier algebras are always non-isomorphic.

Note that now we have a more complete solution in the injective case: Corollary 1 together with Theorem B gives us the following.

COROLLARY 2. *Suppose V, W are analytic discs attached to the unit sphere, f, g are the respective embedding maps and f is injective up to $\overline{\mathbb{D}}$. Then $\mathcal{M}_f \cong \mathcal{M}_g$ algebraically if and only if g is injective up to $\overline{\mathbb{D}}$.*

In general, in light of Corollary 1, we have found a coarse characteristic which separates non-isomorphic cases, so that it remains to solve the isomorphism problem, for embeddings with the same self-crossing type. However, even in the case of embeddings with the same self-crossing type there might be more than one isomorphism class of multiplier algebras, which we will show in the next section.

1.4. First order equivalence and examples. The results from the previous section can be thought of as the zeroth order equivalence, i.e. the alignments of values of the embedding map f. In this section we consider the effect that the derivative of f has on the isomorphism problem.

Define $A_f(\xi) = \langle f(\xi), f'(\xi)\xi \rangle$ for $\xi \in \mathbb{T}$. From [DHS15, Corollary 3.2] we know that $A_f(\xi) > 0$ for embedding maps f. We prove a theorem that shows that this function on the circle is an important semi-invariant for $\mathcal{M}_f$:

THEOREM 3. *Suppose V, W are analytic discs attached to the unit sphere, and f, g are the respective embedding maps. If $\mathcal{M}_f = \mathcal{M}_g$ with equivalent* *norms, then for* $\xi, \zeta \in \mathbb{T}$ *such that* $f(\xi) = f(\zeta)$ *(and so* $g(\xi) = g(\zeta)$ *by Theorem* 1*) we have*

$$\frac{A_f(\xi)}{A_f(\zeta)} = \frac{A_g(\xi)}{A_g(\zeta)}.$$

If there is an *n-point self-crossing*, i.e., $f(\xi_1) = \cdots = f(\xi_n)$, we can apply this theorem pairwise to obtain the following.

COROLLARY 3. *If there is an n-point self-crossing* $f(\xi_1) = \cdots = f(\xi_n)$, *then*

$$[A_f(\xi_1) : \ldots : A_f(\xi_n)] = [A_g(\xi_1) : \ldots : A_g(\xi_n)],$$

by which we mean that there is a scaling factor $c > 0$ *such that* $A_f(\xi_j) = cA_g(\xi_j)$ *for* $j = 1, \ldots, n$.

This theorem allows us to better understand some cases of the isomorphism problem with specific self-crossing types.

First, we consider analytic discs with exactly one simple self-crossing on the boundary. Since we can always apply an automorphism, we assume that $f(-1) = f(1)$ is the self-crossing and otherwise f is injective in $\overline{\mathbb{D}}$. We use Theorem 3 to show that in this case the isomorphism condition is rigid. Set

$$b_r(z) = \frac{z - r}{1 - rz}, \qquad -1 < r < 1,$$

exactly the automorphisms of the unit disc such that $b_r(\pm 1) = \pm 1$.

THEOREM 4. *Suppose* V, W *are analytic discs attached to the unit sphere, and* f, g *are the respective embedding maps with the only self-crossing at* ± 1. *Then* $\mathcal{M}_f \cong \mathcal{M}_g$ *algebraically if and only if* $\mathcal{M}_f = \mathcal{M}_{g \circ \mu}$, *where either*

$$\mu(z) = b_\alpha(z) = \frac{z - \alpha}{1 - \alpha z} \quad \text{or} \quad \mu(z) = -b_\beta(z) = \frac{\beta - z}{1 - \beta z},$$

with constants $\alpha, \beta \in (-1, 1)$ *that can be explicitly computed in terms of* $A_f(\pm 1)$ *and* $A_g(\pm 1)$:

$$\alpha = \frac{\sqrt{A_f(1)A_g(-1)} - \sqrt{A_f(-1)A_g(1)}}{\sqrt{A_f(1)A_g(-1)} + \sqrt{A_f(-1)A_g(1)}},$$

$$\beta = \frac{\sqrt{A_f(1)A_g(1)} - \sqrt{A_f(-1)A_g(-1)}}{\sqrt{A_f(1)A_g(1)} + \sqrt{A_f(-1)A_g(-1)}}.$$

Let us consider an example. Define

$$f_r(z) = \frac{1}{\sqrt{2}}(z^2, b_r(z)^2), \qquad r > 0.$$

Then, according to [DHS15, Theorem 5.2], $V = f(\mathbb{D})$ is a variety with the only self-crossing at ± 1. Using Theorem 4 we show the following.

THEOREM 5. $\mathcal{M}_{f_r} \neq \mathcal{M}_{f_s}$ *for* $r \neq s$.

We see that, in contrast to the injective case, even for embeddings with the same self-crossings the multiplier algebras might not coincide. However, this does not mean that the algebras are not isomorphic, as we still have the freedom to perform unit disc automorphisms. Let us take care of this by considering a more general version of this embedding. Set

$$f_{r,s}(z) = \frac{1}{\sqrt{2}}(b_r(z)^2, b_s(z)^2), \qquad r \neq s,$$

so that $f_r = f_{0,r}$. Note that

$$(b_r \circ b_s)(z) = \frac{z - \frac{r+s}{1+rs}}{1 - \frac{r+s}{1+rs}z} = b_{\frac{r+s}{1+rs}}(z).$$

Hence,

$$(f_{r,s} \circ b_{-r})(z) = \frac{1}{\sqrt{2}}(z^2, b_{\frac{s-r}{1-sr}}(z)^2) = f_{0,\frac{s-r}{1-sr}}(z),$$

meaning that we get the same varieties (take $z \mapsto -z$ in case $s < r$). Now, for $f_{0,r}$ with $r \in (0,1)$ consider

$$t = \frac{-1 + \sqrt{1-r^2}}{r};$$

we have $t \in (-1,0)$ and $f_{0,r} \circ b_t = f_{t,-t}$. In fact, t is unique, so that any $f_{r,s}$ corresponds to a unique $f_{t,-t}$ with $t \in (-1,0)$ via a unit disc automorphism. For such embeddings Theorem 4 takes a simpler form:

Theorem 6. *$\mathcal{M}_{f_{r,-r}} \cong \mathcal{M}_{f_{s,-s}}$ algebraically if and only if $\mathcal{M}_{f_{r,-r}} = \mathcal{M}_{f_{s,-s}}$.*

The author does not know whether $\mathcal{M}_{f_{r,-r}} = \mathcal{M}_{f_{s,-s}}$ for all r and s, but it seems plausible. If there are r and s for which $\mathcal{M}_{f_{r,-r}} \neq \mathcal{M}_{f_{s,-s}}$, then Theorem 6 gives us an example where two embeddings with a single simple self-crossing give rise to non-isomorphic multiplier algebras. This means that whether there is only one isomorphism class of multiplier algebras remains unclear in the single simple self-crossing case.

We finish by proving that already for the single 3-point self-crossing case the converse to Corollary 1 does not hold. There are, in fact, quite a few isomorphism classes of multiplier algebras.

Theorem 7. *There is a continuum of analytic discs attached to the unit sphere with the same 3-point self-crossing and such that their multiplier algebras are all mutually non-isomorphic.*

For the example appearing in this theorem we rely on Theorem 3 to prove that two multiplier algebras are non-isomorphic. It, hence, remains an open question whether the converse to Theorem 3 is true: If the embedding maps have the same self-crossing type and an equivalent derivative behavior, does that imply that the multiplier algebras are isomorphic?

1.5. Organization of the paper. In Section 2 we prove the results stated in Section 1.3: First, we study the spaces $\mathcal{H}_f$ and $\mathcal{M}_f$ in general in Section 2.1. Then, we prove Theorems 1 and 2 in Sections 2.2 and 2.3 respectively.

Section 3 is entirely devoted to the proofs of the results stated in Section 1.4. We prove Theorems 3–7 in Sections 3.1–3.5 respectively.

2. Analytic discs

2.1. Properties of functions in $\mathcal{H}_f$ and $\mathcal{M}_f$. We start by studying some properties which must be satisfied by functions in the Hilbert spaces $\mathcal{H}_f$ and the multiplier algebras $\mathcal{M}_f$.

THEOREM 8. *Suppose V is an analytic disc attached to the unit sphere, and f is its embedding map such that $f(-1) = f(1)$. Let h be a function in $\mathcal{H}_f$ such that the following limits exist:*

$$h(1) = \lim_{r\to 1-} h(r), \qquad h(-1) = \lim_{r\to 1-} h(-r).$$

Then $h(1) = h(-1)$.

In particular, if $h \in \mathcal{H}_f$ extends continuously up to $\overline{\mathbb{D}}$, then $h(1) = h(-1)$.

Proof. By scaling h appropriately, we can assume $\|h\|_{\mathcal{H}_f} \leqslant 1$. Since $\mathcal{H}_f$ is an RKHS with kernel κ^f, we can use [PR16, Theorem 3.11] to describe the functions in $\mathcal{H}_f$ in terms of the kernel:

$$\|h\|_{\mathcal{H}_f} \leqslant 1 \iff \big(\kappa^f(z,w) - h(z)\overline{h(w)}\big)_{z,w\in\mathbb{D}} \geqslant 0.$$

In particular, using (2), for $z, w \in \mathbb{D}$ we obtain

$$\begin{pmatrix} \frac{1}{1-\|f(z)\|^2} - |h(z)|^2 & \frac{1}{1-\langle f(z), f(w)\rangle} - h(z)\overline{h(w)} \\ \frac{1}{1-\langle f(w), f(z)\rangle} - h(w)\overline{h(z)} & \frac{1}{1-\|f(w)\|^2} - |h(w)|^2 \end{pmatrix} \geqslant 0.$$

As this condition implies that the determinant is positive, we have

$$\left(\frac{1}{1-\|f(z)\|^2} - |h(z)|^2\right)\left(\frac{1}{1-\|f(w)\|^2} - |h(w)|^2\right) - \left|\frac{1}{1-\langle f(z), f(w)\rangle} - h(z)\overline{h(w)}\right|^2 \geqslant 0. \tag{3}$$

Now take $z = 1-x, w = -1+y$ for $x, y > 0$. Expanding f around 1 and -1 we get

$$f(1-x) = f(1) - f'(1)x + \frac{f''(1)}{2}x^2 + o(x^2),$$
$$f(-1+y) = f(-1) + f'(-1)y + \frac{f''(-1)}{2}y^2 + o(y^2).$$

Thus,

$$\begin{aligned}(4)\qquad & 1-\|f(1-x)\|^2 \\ &= 2\operatorname{Re}\langle f(1), f'(1)\rangle x - (\|f'(1)\|^2 + \operatorname{Re}\langle f(1), f''(1)\rangle)x^2 + o(x^2).\end{aligned}$$

Similarly,

$$\begin{aligned}(5)\qquad & 1-\|f(-1+y)\|^2 \\ &= -2\operatorname{Re}\langle f(-1), f'(-1)\rangle y - (\|f'(-1)\|^2 + \operatorname{Re}\langle f(-1), f''(-1)\rangle)y^2 + o(y^2).\end{aligned}$$

Note that by [DHS15, Proposition 3.1, Corollary 3.2], $A = \langle f(1), f'(1)\rangle > 0$ and $B = -\langle f(-1), f'(-1)\rangle > 0$. Set

$$\begin{aligned} C &= \|f'(1)\|^2, & 2F &= \langle f(1), f''(1)\rangle, \\ D &= \|f'(-1)\|^2, & 2G &= \langle f(-1), f''(-1)\rangle. \end{aligned}$$

We get

$$1-\|f(1-x)\|^2 = 2Ax - (C + 2\operatorname{Re} F)x^2 + o(x^2), \tag{6}$$

$$1-\|f(-1+y)\|^2 = 2By - (D + 2\operatorname{Re} G)y^2 + o(y^2). \tag{7}$$

Finally, noting that $f(1) = f(-1)$ and setting $E = \langle f'(1), f'(-1)\rangle$, we get

$$1-\langle f(1-x), f(-1+y)\rangle = Ax + By + Exy - \bar{F}x^2 - Gy^2 + o(x^2) + o(y^2). \tag{8}$$

Expanding the brackets in (3) we obtain

$$\frac{1}{1-\|f(z)\|^2}\,\frac{1}{1-\|f(w)\|^2} - \left|\frac{1}{1-\langle f(z), f(w)\rangle}\right|^2 \tag{9}$$

$$\geqslant \frac{|h(w)|^2}{1-\|f(z)\|^2} + \frac{|h(z)|^2}{1-\|f(w)\|^2} - 2\operatorname{Re}\left(\frac{\overline{h(z)}h(w)}{1-\langle f(z), f(w)\rangle}\right). \tag{10}$$

To annihilate the highest order term in (9) we set $Ax = By = t > 0$. We obtain

$$\frac{1}{1-\|f(z)\|^2} = \frac{1}{2t}\left(1 + \frac{C + 2\operatorname{Re} F}{2A^2}t + o(t)\right), \tag{11}$$

$$\frac{1}{1-\|f(w)\|^2} = \frac{1}{2t}\left(1 + \frac{D + 2\operatorname{Re} G}{2B^2}t + o(t)\right), \tag{12}$$

$$\frac{1}{1-\langle f(z), f(w)\rangle} = \frac{1}{2t}\left(1 + \left(\frac{\bar{F}}{2A^2} + \frac{G}{2B^2} - \frac{E}{2AB}\right)t + o(t)\right), \tag{13}$$

and hence

$$\left|\frac{1}{1-\langle f(z), f(w)\rangle}\right|^2 = \frac{1}{4t^2}\left(1 + \left(\frac{\operatorname{Re} F}{A^2} + \frac{\operatorname{Re} G}{B^2} - \frac{\operatorname{Re} E}{AB}\right)t + o(t)\right). \tag{14}$$

Substituting (11), (12) and (14) into (9) we get

$$\tag{15} \frac{1}{1-\|f(z)\|^2}\,\frac{1}{1-\|f(w)\|^2} - \left|\frac{1}{1-\langle f(z), f(w)\rangle}\right|^2 = \frac{1}{8t}\left(\frac{C}{A^2}+\frac{D}{B^2}+\frac{2\operatorname{Re}E}{AB}\right) + o\left(\frac{1}{t}\right).$$

Similarly, expanding only up to $o(1)$ in (11)–(13) and substituting into (10) we obtain

$$\tag{16} \frac{|h(w)|^2}{1-\|f(z)\|^2} + \frac{|h(z)|^2}{1-\|f(w)\|^2} - 2\operatorname{Re}\left(\frac{\overline{h(z)}h(w)}{1-\langle f(z), f(w)\rangle}\right) = \frac{1}{2t}|h(z)-h(w)|^2 + o\left(\frac{1}{t}\right).$$

Thus, replacing (9) with (15) and (10) with (16) we get

$$\frac{1}{8t}\left(\frac{C}{A^2}+\frac{D}{B^2}+\frac{2\operatorname{Re}E}{AB}\right) + o\left(\frac{1}{t}\right) \geqslant \frac{1}{2t}|h(z)-h(w)|^2 + o\left(\frac{1}{t}\right).$$

Explicitly writing $z = 1 - t/A$ and $w = -1 + t/B$, we obtain

$$\tag{17} \left|h\left(1-\frac{t}{A}\right) - h\left(-1+\frac{t}{B}\right)\right|^2 \leqslant \frac{1}{4}\left(\frac{C}{A^2}+\frac{D}{B^2}+\frac{2\operatorname{Re}E}{AB}\right) + o(1), \qquad t > 0.$$

Taking the limit $t \to 0$ and scaling h, we get

$$|h(1)-h(-1)|^2 \leqslant \frac{1}{4}\left(\frac{C}{A^2}+\frac{D}{B^2}+\frac{2\operatorname{Re}E}{AB}\right)\|h\|^2_{\mathcal{H}_f}$$

for all $h \in \mathcal{H}_f$ with radial limits at ± 1. Note that such functions are dense in $\mathcal{H}_f$, since, for example, all κ_z^f for $z \in \mathbb{D}$ (see (2)) are continuous in $\overline{\mathbb{D}}$.

We conclude that there is a bounded functional on $\mathcal{H}_f$, with the corresponding function denoted by $\varphi \in \mathcal{H}_f$, such that $\langle h, \varphi\rangle_{\mathcal{H}_f} = h(1) - h(-1)$ for $h \in \mathcal{H}_f$ that have the limits

$$h(1) = \lim_{x \to 0+} h(1-x), \qquad h(-1) = \lim_{y \to 0+} h(-1+y).$$

To finish the proof it remains to show that $\varphi = 0$. Indeed, for any $z \in \mathbb{D}$,

$$\varphi(z) = \langle \varphi, \kappa_z^f\rangle_{\mathcal{H}_f} = \overline{\langle \kappa_z^f, \varphi\rangle_{\mathcal{H}_f}} = \overline{\kappa_z^f(1) - \kappa_z^f(-1)} = 0. \ \blacksquare$$

COROLLARY 4. *In the setting of Theorem 8 we have*

$$\kappa^f_{1-t/A} - \kappa^f_{-1+t/B} \to 0, \quad t \to 0.$$

in the w^* topology.

Proof. From (17) it follows that

$$\kappa^f_{1-t/A} - \kappa^f_{-1+t/B}, \qquad t \to 0,$$

is bounded in norm. Hence, since the linear combinations of κ^f_z, $z \in \mathbb{D}$, are dense in $\mathcal{H}_f$, the conclusion follows from Theorem 8. ∎

An immediate corollary of Theorem 8 is the following.

THEOREM 9. *Suppose V is an analytic disc attached to the unit sphere, and f is its embedding map. Let $f(\xi) = f(\zeta)$ for two distinct $\xi, \zeta \in \mathbb{T}$. If h is a function in $\mathcal{H}_f$ that is continuous up to $\overline{\mathbb{D}}$, then $h(\xi) = h(\zeta)$.*

Proof. Consider a disc automorphism μ such that $\mu(-1) = \xi$, $\mu(1) = \zeta$. Then $g = f \circ \mu$ satisfies the conditions of Theorem 8. Note that $h \circ \mu$ belongs to $\mathcal{H}_g$ ($\mathcal{H}_f \cong \mathcal{H}_g$ as RKHS via $\circ\mu$). The function $h \circ \mu$ is continuous up to $\overline{\mathbb{D}}$ as well, so it satisfies the conditions of Theorem 8. We conclude that $(h \circ \mu)(-1) = (h \circ \mu)(1)$, which is exactly $h(\xi) = h(\zeta)$. ∎

From this result we can infer that the same must hold for multipliers.

THEOREM 10. *Suppose V is an analytic disc attached to the unit sphere, and f is its embedding map. Let $f(\xi) = f(\zeta)$ for two distinct $\xi, \zeta \in \mathbb{T}$. If φ is a function in $\mathcal{M}_f$ that is continuous up to $\overline{\mathbb{D}}$, then $\varphi(\xi) = \varphi(\zeta)$.*

Proof. Consider $h(z) = \kappa^f_0(z)$. By definition $h(\xi) = h(\zeta) \neq 0$. Note that $\varphi h \in \mathcal{H}_f$ is continuous up to $\overline{\mathbb{D}}$, so that $\varphi(\xi)h(\xi) = \varphi(\zeta)h(\zeta)$. Dividing by $h(\xi) = h(\zeta) \neq 0$ we get $\varphi(\xi) = \varphi(\zeta)$. ∎

2.2. Proof of Theorem 1. The result follows from Theorem 10. Write f as $f = (f_1, \ldots, f_d)$. Note that $z_j \in \mathcal{M}_d$ for $j = 1, \ldots, d$, so that if we compose with f we get $f_j \in \mathcal{M}_f$ for $j = 1, \ldots, d$. Since $\mathcal{M}_f = \mathcal{M}_g$, by Theorem 10 we get $g(\xi) = g(\zeta) \Rightarrow f_j(\xi) = f_j(\zeta)$ for $j = 1, \ldots, d$, which means exactly $f(\xi) = f(\zeta)$. Exchanging f, g and repeating this argument we get the desired equivalence. ∎

2.3. Proof of Theorem 2. First, we show that analytic discs are irreducible.

LEMMA 1. *If V is an analytic disc attached to the unit sphere, then V is irreducible in the sense of* [SS16, Section 5.1]*, i.e., for any regular point $\lambda \in V$ and any $\varepsilon > 0$ the intersection of the zero sets of all multipliers vanishing on $V \cap B_\varepsilon(\lambda)$, a small neighborhood of λ in V, is exactly V.*

Proof. Suppose V is not irreducible. This means that there are $\lambda \in V$ and $\varepsilon > 0$ such that the intersection of the zero sets of all multipliers vanishing on $V \cap B_\varepsilon(\lambda)$ is a proper subset of V. Hence, there are $v \in V$ and $\varphi \in \mathcal{M}_d$ such that $\varphi|_{V \cap B_\varepsilon(\lambda)} = 0$ but $\varphi(v) \neq 0$. Let f be the embedding map of V. Then $v = f(a)$ for some $a \in \mathbb{D}$. The function $\psi = \varphi \circ f$ is analytic in $\mathbb{D}$. Since

$\varphi|_{V\cap B_\varepsilon(\lambda)} = 0$, also $\psi|_{f^{-1}(B_\varepsilon(\lambda))} = 0$. But the set $f^{-1}(B_\varepsilon(\lambda))$ is open in $\mathbb{D}$, which means that $\psi = 0$. This contradicts the fact that $\psi(a) = \varphi(v) \neq 0$. ∎

Now, we can prove Theorem 2. Suppose that $\mathcal{M}_f \cong \mathcal{M}_g$ algebraically. This is equivalent to $\mathcal{M}_V \cong \mathcal{M}_W$. Denote by $\Phi : \mathcal{M}_V \to \mathcal{M}_W$ an isomorphism. By Lemma 1, the varieties V and W are irreducible. Thus, by [SS16, Theorem 5.5], there are maps $F, G : \mathbb{B}_d \to \mathbb{B}_d$ with multiplier coefficients, i.e., $F_j, G_j \in \mathcal{M}_d$, $j = 1, \ldots, d$, such that $F \circ G = \mathrm{id}_V$, $G \circ F = \mathrm{id}_W$ and the composition with these maps gives us Φ, i.e.,

$$\Phi(\varphi) = \varphi \circ F,\ \varphi \in \mathcal{M}_V, \qquad \Phi^{-1}(\psi) = \psi \circ G,\ \psi \in \mathcal{M}_W.$$

To go back to the unit disc we define

$$\mu = g^{-1} \circ G \circ f : \mathbb{D} \to \mathbb{D},$$

so that

$$\mu^{-1} = f^{-1} \circ F \circ g : \mathbb{D} \to \mathbb{D}.$$

Since f^{-1}, g^{-1} are analytic on V, W respectively, we see that μ is an analytic bijection from $\mathbb{D}$ onto itself, and hence a disc automorphism.

By definition,

$$\begin{aligned} \varphi \in \mathcal{M}_{g\circ\mu} &\iff \varphi \circ \mu^{-1} \circ g^{-1} \in \mathcal{M}_W \iff \varphi \circ \mu^{-1} \circ g^{-1} \circ G \in \mathcal{M}_V \\ &\iff \varphi \circ \mu^{-1} \circ g^{-1} \circ G \circ f \in \mathcal{M}_f \iff \varphi \in \mathcal{M}_f. \end{aligned}$$

Since Φ is continuous as a homomorphism of semisimple Banach algebras and the norms on $\mathcal{M}_f, \mathcal{M}_g$ are inherited from $\mathcal{M}_V$, $\mathcal{M}_W$, we conclude that $\mathcal{M}_f = \mathcal{M}_{g\circ\mu}$ with equivalent norms. ∎

3. First order equivalence and examples. We begin this section by describing how the semi-invariant $A_f(\xi) = \langle f(\xi), f'(\xi)\xi\rangle$, $\xi \in \mathbb{T}$, changes under unit disc automorphisms.

Lemma 2. *Suppose f is an embedding map, and μ is a disc automorphism*

$$\mu(z) = \lambda\frac{\alpha - z}{1 - \bar{\alpha}z}, \qquad \lambda \in \mathbb{T},\ \alpha \in \mathbb{D}.$$

Then

$$A_{f\circ\mu}(\xi) = A_f(\mu(\xi))\frac{1 - |\alpha|^2}{|\alpha - \xi|^2}.$$

Proof. Expand

$$\begin{aligned} A_{f\circ\mu}(\xi) &= \langle f(\mu(\xi)), (f\circ\mu)'(\xi)\xi\rangle = \langle f(\mu(\xi)), f'(\mu(\xi))\mu'(\xi)\xi\rangle \\ &= \langle f(\mu(\xi)), f'(\mu(\xi))\mu(\xi)\frac{\mu'(\xi)}{\mu(\xi)}\xi\rangle = \overline{\frac{\mu'(\xi)}{\mu(\xi)}}\xi A_f(\mu(\xi)). \end{aligned}$$

Hence, it is enough to show

$$\frac{\mu'(\xi)}{\mu(\xi)}\xi = \frac{1-|\alpha|^2}{|\alpha-\xi|^2}.$$

Indeed,

$$\frac{\mu'(\xi)}{\mu(\xi)}\xi = \frac{|\alpha|^2-1}{(\alpha-\xi)(1-\bar{\alpha}\xi)}\xi = \frac{|\alpha|^2-1}{(\alpha-\xi)(\bar{\xi}-\bar{\alpha})} = \frac{1-|\alpha|^2}{|\alpha-\xi|^2}. \blacksquare$$

3.1. Proof of Theorem 3. In the proof, for two quantities $A, B > 0$ depending on some parameters, we write $A \asymp B$ whenever there exists a constant $C > 0$ that is independent of the parameters such that $A \leqslant CB$ and $B \leqslant CA$.

In light of Lemma 2, it is enough to prove the theorem for $\xi = 1$, $\zeta = -1$.

Consider a metric on $\mathbb{D}$ induced by $\mathcal{H}_f$ (see [AM02, Lemma 9.9]),

$$d_f(z,w) = \sqrt{1 - \frac{|\kappa_f(z,w)|^2}{\kappa_f(z,z)\kappa_f(w,w)}}.$$

It is easy to see that since $\mathcal{H}_f$ is complete Pick, we have

$$d_f(z,w) = \sup\{|\varphi(z)| : \varphi \in \mathcal{M}_f,\ \|\varphi\|_{\mathcal{M}_f} \leqslant 1,\ \varphi(w) = 0\}.$$

We have a similar metric d_g for g instead of f. Since $\mathcal{M}_f = \mathcal{M}_g$ with equivalent norms, we conclude that the metrics d_f and d_g are equivalent, meaning that the identity map between $(\mathbb{D}, d_f)$ to $(\mathbb{D}, d_g)$ is bi-Lipschitz. Hence, d_f^2 and d_g^2 are equivalent as well, i.e.,

$$1 - \frac{(1-\|f(z)\|^2)(1-\|f(w)\|^2)}{|1-\langle f(z), f(w)\rangle|^2} \asymp 1 - \frac{(1-\|g(z)\|^2)(1-\|g(w)\|^2)}{|1-\langle g(z), g(w)\rangle|^2}.$$

Similarly to Theorem 8 we consider $z = 1-x, w = -1+y$ for $x, y > 0$ with $x, y \to 0$. In a similar fashion to the proof of Theorem 8 we set

$$x = \frac{t}{A_f(1)}, \qquad y = \frac{t}{A_f(-1)},$$

for $t > 0$, $t \to 0$. This way, we have

$$\begin{aligned} 1 - \|f(z)\|^2 &= 2t + o(t), \\ 1 - \|f(w)\|^2 &= 2t + o(t), \\ 1 - \langle f(z), f(w)\rangle &= 2t + o(t). \end{aligned}$$

Thus,

$$\frac{(1-\|f(z)\|^2)(1-\|f(w)\|^2)}{|1-\langle f(z), f(w)\rangle|^2} = \frac{(2t+o(t))(2t+o(t))}{|2t+o(t)|^2} = 1 + o(1), \qquad t \to 0.$$

We conclude that

$$d_f^2\left(1 - \frac{t}{A_f(1)}, -1 + \frac{t}{A_f(-1)}\right) = o(1), \qquad t \to 0.$$

Next, we look at what happens to d_g^2. Set $a = \frac{A_g(1)}{A_f(1)}$, $b = \frac{A_g(-1)}{A_f(-1)}$. We need to prove that $a = b$. We expand

$$1 - \|g(z)\|^2 = 2at + o(t),$$
$$1 - \|g(w)\|^2 = 2bt + o(t),$$
$$1 - \langle g(z), g(w)\rangle = (a+b)t + o(t).$$

Thus,

$$\frac{(1 - \|g(z)\|^2)((1 - \|g(w)\|^2))}{|1 - \langle g(z), g(w)\rangle|^2} = \frac{4ab}{(a+b)^2} + o(1),$$

and so

$$d_g^2\left(1 - \frac{t}{A_f(1)}, -1 + \frac{t}{A_f(-1)}\right) = 1 - \frac{4ab}{(a+b)^2} + o(1), \qquad t \to 0.$$

Since $d_g^2 \asymp d_f^2 = o(1)$ as $t \to 0$, we must have $4ab = (a+b)^2$, which holds if and only if $a = b$. We conclude that $\frac{A_f(1)}{A_g(1)} = \frac{A_f(-1)}{A_g(-1)}$, as we wanted. ∎

3.2. Proof of Theorem 4. The fact that $\mathcal{M}_f = \mathcal{M}_{g\circ\mu}$ implies $\mathcal{M}_f \cong \mathcal{M}_g$ algebraically is evident for any automorphism μ. Thus, we prove the other implication.

Suppose $\mathcal{M}_f \cong \mathcal{M}_g$ algebraically. By Theorem 2, this means that there is a disc automorphism μ such that $\mathcal{M}_f = \mathcal{M}_{g\circ\mu}$ with equivalent norms. We claim that there are only two possible choices for μ as in the statement of the theorem.

First, if $\mathcal{M}_f = \mathcal{M}_{g\circ\mu}$, then, by Theorem 1, f and $g \circ \mu$ must have the same self-crossings on the boundary. But the only self-crossings for f and g are ± 1, which means that $\mu(1) = \pm 1$ and $\mu(-1) = \mp 1$. Hence, either μ or $-\mu$ maps 1 to 1 and -1 to -1. Thus, either

$$\mu(z) = \frac{z - \alpha}{1 - \alpha z}, \qquad \alpha \in (-1, 1),$$

or

$$\mu(z) = \frac{\beta - z}{1 - \beta z}, \quad \beta \in (-1, 1).$$

It remains to prove that α and β are allowed to take only one value each, as in the statement of the theorem. Indeed, if $\mathcal{M}_f = \mathcal{M}_{g\circ\mu}$, then in light of Theorem 3,

$$\frac{A_f(1)}{A_f(-1)} = \frac{A_{g\circ\mu}(1)}{A_{g\circ\mu}(-1)}. \tag{18}$$

If we consider $\mu(z) = (z - \alpha)/(1 - \alpha z)$, then (18) together with Lemma 2 gives us

$$\frac{A_f(1)}{A_f(-1)} = \frac{A_g(1)}{A_g(-1)} \left(\frac{\alpha + 1}{1 - \alpha}\right)^2,$$

so that

$$\alpha = \frac{\sqrt{A_f(1)A_g(-1)} - \sqrt{A_f(-1)A_g(1)}}{\sqrt{A_f(1)A_g(-1)} + \sqrt{A_f(-1)A_g(1)}}. \tag{19}$$

Similarly, if we consider $\mu(z) = (\beta - z)/(1 - \beta z)$, then

$$\frac{A_f(1)}{A_f(-1)} = \frac{A_g(-1)}{A_g(1)} \left(\frac{\beta + 1}{1 - \beta}\right)^2,$$

so that

$$\beta = \frac{\sqrt{A_f(1)A_g(1)} - \sqrt{A_f(-1)A_g(-1)}}{\sqrt{A_f(1)A_g(1)} + \sqrt{A_f(-1)A_g(-1)}}. \ \blacksquare \tag{20}$$

3.3. Proof of Theorem 5. In light of Theorem 3 it is sufficient to prove that for $r > s > 0$,

$$\frac{A_{f_r}(1)}{A_{f_r}(-1)} \neq \frac{A_{f_s}(1)}{A_{f_s}(-1)}.$$

Evaluating at ± 1 we get $f_r(1) = f_r(-1) = \frac{1}{\sqrt{2}}(1, 1)$ and

$$f_r'(1) = \frac{1}{\sqrt{2}}\left(2, 2\frac{1+r}{1-r}\right), \qquad f_r'(-1) = -\frac{1}{\sqrt{2}}\left(2,\ 2\frac{1-r}{1+r}\right).$$

Hence,

$$\frac{A_{f_r}(1)}{A_{f_r}(-1)} = \frac{1 + \frac{1+r}{1-r}}{1 + \frac{1-r}{1+r}} = \frac{1+r}{1-r}$$

is strictly increasing for $r \in (0, 1)$, which finishes the proof. $\blacksquare$

3.4. Proof of Theorem 6. Note that

$$f_{r,-r}(-z) = f_{-r,r}(z) = \frac{1}{\sqrt{2}}(b_{-r}(z)^2, b_r(z)^2), \tag{21}$$

so that if $f_{r,-r}'(1) = (z_1, z_2)$, then $f_{r,-r}'(-1) = -(z_2, z_1)$. Since

$$f_{r,-r}(1) = f_{r,-r}(-1) = \frac{1}{\sqrt{2}}(1, 1),$$

we get

$$A_{f_{r,-r}}(1) = A_{f_{r,-r}}(-1). \tag{22}$$

Let us apply Theorem 4 to $f_{r,-r}$ and $f_{s,-s}$. Calculating α, β from (19), (20) and using (22) we get $\alpha = \beta = 0$. Thus, in light of (21), Theorem 4 implies that $\mathcal{M}_{f_{r,-r}} \cong \mathcal{M}_{f_{s,-s}}$ algebraically if and only if either $\mathcal{M}_{f_{r,-r}} = \mathcal{M}_{f_{s,-s}}$ or $\mathcal{M}_{f_{r,-r}} = \mathcal{M}_{f_{-s,s}}$.

Note that by (21),

$$\langle f_{r,-r}(z), f_{r,-r}(w)\rangle = \langle f_{-r,r}(z), f_{-r,r}(w)\rangle,$$

whence

$$\kappa^{f_{r,-r}} = \kappa^{f_{-r,r}},$$

meaning $\mathcal{H}_{f_{r,-r}} = \mathcal{H}_{f_{-r,r}}$ with the same inner product, which means $\mathcal{M}_{f_{r,-r}} = \mathcal{M}_{f_{-r,r}}$ isometrically.

We conclude that $\mathcal{M}_{f_{r,-r}} \cong \mathcal{M}_{f_{s,-s}}$ algebraically if and only if $\mathcal{M}_{f_{r,-r}} = \mathcal{M}_{f_{s,-s}}$. ∎

3.5. Proof of Theorem 7. Let us denote $1, \omega, \omega^2$ the third roots of unity. We consider embedding maps f with a single 3-point self-crossing $f(\omega) = f(\omega^2) = f(1)$. The basic building block for our embedding maps is a special type of Blaschke product.

LEMMA 3. *The Blaschke product*

$$g_\alpha(z) = z\frac{z-\alpha}{1-\alpha z}\,\frac{z-\beta}{1-\beta z} \tag{23}$$

with $\alpha \in (-1,1)$ *and*

$$\beta = \frac{-\alpha}{1+\alpha}$$

satisfies

$$g_\alpha(1) = g_\alpha(\omega) = g_\alpha(\omega^2) = 1,$$

as well as the symmetry condition $g^* = g$, *where* $g^*(z) = \overline{g(\bar{z})}$.

Proof. The symmetry condition follows from the fact that α and β are real. In particular, $g_\alpha(\omega^2) = g_\alpha(\bar{\omega}) = \overline{g_\alpha(\omega)}$. Note that $g_\alpha(1) = 1$ is evident, hence it is enough to show $g_\alpha(\omega) = 1$; and this follows by direct evaluation. ∎

We consider the embedding maps of the form

$$f(z) = \tfrac{1}{2}(z^3, g_{\alpha_0}(z), g_{\alpha_1}(z), g_{\alpha_2}(z)). \tag{24}$$

First, we need to guarantee that f is indeed an embedding map of an analytic disc attached to the unit sphere. Note that f is analytic in a larger disc and $\|f(x)\| = 1 \Leftrightarrow |x| = 1$. Hence, we need to have $f'(z) \neq 0$ for $z \in \mathbb{D}$, injectivity in $\mathbb{D}$, as well as $V = f(\mathbb{D})$ has to be a variety. For this purpose we fix $\alpha_0 \in (0,1)$; then clearly $f'(z) \neq 0$ for $z \in \mathbb{D}$. If we have injectivity as well, then V is automatically a variety as f is analytic in a larger disc (see the proof of [KMS13, Lemma 6.2]). Thus, the goal is to find α_1 such that $z \mapsto (z^3, g_{\alpha_0}(z), g_{\alpha_1}(z))$ is injective in $\mathbb{D}$; this way we are still free to choose any α_2.

LEMMA 4. *There exists* α_1 *such that* $z \mapsto (z^3, g_{\alpha_0}(z), g_{\alpha_1}(z))$ *is injective in* $\mathbb{D}$.

Proof. Suppose we have $z \neq w \in \mathbb{D}$ such that

$$(z^3, g_{\alpha_0}(z), g_{\alpha_1}(z)) = (w^3, g_{\alpha_0}(w), g_{\alpha_1}(w)).$$

Since $z^3 = w^3$ and $z \neq w$, we have $z, w \neq 0$ and either $w = \omega z$ or $z = \omega w$. By swapping z and w we can always assume that $w = \omega z \neq 0$. Hence,

$$(z^3, g_{\alpha_0}(z), g_{\alpha_1}(z)) = (z^3, g_{\alpha_0}(\omega z), g_{\alpha_1}(\omega z)). \tag{25}$$

The idea is to take α_1 such that (25) does not have any solutions in $\mathbb{D} \setminus \{0\}$, meaning that the assumption we made was wrong and $z \mapsto (z^3, g_{\alpha_0}(z), g_{\alpha_1}(z))$ is injective in $\mathbb{D}$.

Consider the equation $g_{\alpha_0}(z) = g_{\alpha_0}(\omega z)$. Note that since g_{α_0} is analytic in a larger disc there can only be finitely many solutions to this equation in $\mathbb{D} \setminus \{0\}$. To finish the proof choose α_1 such that $g_{\alpha_1}(z) \neq g_{\alpha_1}(\omega z)$ for any such solution z. For a fixed $z \in \mathbb{D} \setminus \{0\}$, consider a function of α_1,

$$h_z(\alpha_1) = g_{\alpha_1}(z) - g_{\alpha_1}(\omega z).$$

By (23) it is a rational function in α_1. By plugging $\alpha_1 = 1$ we see that it is not identically zero, as the solutions of $g_1(z) = g_1(\omega z)$ are $z = 0$ and $z = \omega$, which do not lie in $\mathbb{D} \setminus \{0\}$. This means that there can only be finitely many values of $\alpha_1 \in (-1, 1)$ such that $g_{\alpha_1}(z) = g_{\alpha_1}(\omega z)$; those are the values for α_1 we do not want to take. Repeating this argument for each $z \in \mathbb{D} \setminus \{0\}$ that satisfies $g_{\alpha_0}(z) = g_{\alpha_0}(\omega z)$, we remove at most finitely many points from the interval $(-1, 1)$ and choose α_1 from the resulting set. With this value of α_1 the map $z \mapsto (z^3, g_{\alpha_0}(z), g_{\alpha_1}(z))$ is injective in $\mathbb{D}$. ■

We fix a value of α_1 from Lemma 4, and, since α_2 is a free parameter, we write the embedding functions as

$$f_\alpha(z) = \tfrac{1}{2}(z^3, g_{\alpha_0}(z), g_{\alpha_1}(z), g_\alpha(z)), \qquad \alpha \in (-1, 1). \tag{26}$$

Let us now show that as α approaches 1 from below we get uncountably many analytic discs $f_\alpha(\mathbb{D})$ such that all their multiplier algebras $\mathcal{M}_{f_\alpha}$ are mutually non-isomorphic. By Theorem 2 we know that if they are isomorphic, then the isomorphism is given by composition with a unit disc automorphism μ. By Theorem 1, μ must map the self-crossing $1, \omega, \omega^2$ onto itself.

Lemma 5. *Suppose μ is a unit disc automorphism that maps $\{1, \omega, \omega^2\}$ onto itself. Then either $\mu(z) = z$, $\mu(z) = \omega z$ or $\mu(z) = \omega^2 z$.*

Proof. It is enough to show that $\mu(1) = 1$ implies $\mu(z) = z$, since we can always compose μ with an appropriate rotation. Now, since μ is a disc automorphism it must preserve the order $1, \omega, \omega^2$ on the circle while mapping $\{\omega, \omega^2\}$ onto itself. This is only possible if $\mu(\omega) = \omega$ and $\mu(\omega^2) = \omega^2$, which, together with $\mu(1) = 1$, gives us $\mu(z) = z$. ■

Now we know that the only candidates for isomorphism maps are compositions with trivial rotations. Finally, the main instrument we use is Theorem 3, or, specifically, Corollary 3. For the maps f, g with a single 3-point self-crossing at $1, \omega, \omega^2$ this reduces to the following: $\mathcal{M}_f = \mathcal{M}_g$ implies

$$[A_f(1) : A_f(\omega) : A_f(\omega^2)] = [A_g(1) : A_g(\omega) : A_g(\omega^2)]. \tag{27}$$

Together with Theorem 2, Theorem 1 and Lemma 5 we deduce that $\mathcal{M}_f \cong \mathcal{M}_g$ implies the identity (27) up to composing g with $\mu(z) = z$, $\mu(z) = \omega z$ or $\mu(z) = \omega^2 z$. In light of Lemma 2 this composition reduces to the cyclic permutation of the expression, i.e., at least one of the following three identities holds:

$$\begin{aligned}
[A_f(1) : A_f(\omega) : A_f(\omega^2)] &= [A_g(1) : A_g(\omega) : A_g(\omega^2)],\\
[A_f(1) : A_f(\omega) : A_f(\omega^2)] &= [A_g(\omega^2) : A_g(1) : A_g(\omega)],\\
[A_f(1) : A_f(\omega) : A_f(\omega^2)] &= [A_g(\omega) : A_g(\omega^2) : A_g(1)].
\end{aligned}$$

For our embedding maps f_α this takes a simpler form. Indeed, by definition (24), we have the symmetry $f_\alpha(z) = \overline{f_\alpha(\bar{z})}$, which means that $A_{f_\alpha}(\omega) = A_{f_\alpha}(\omega^2)$, since $\omega^2 = \bar{\omega}$. Thus, if $\mathcal{M}_{f_\alpha} \cong \mathcal{M}_{f_{\tilde{\alpha}}}$, then

$$[A_{f_\alpha}(1) : A_{f_\alpha}(\omega) : A_{f_\alpha}(\omega^2)] = [A_{f_{\tilde{\alpha}}}(1) : A_{f_{\tilde{\alpha}}}(\omega) : A_{f_{\tilde{\alpha}}}(\omega^2)]$$

without any cyclic permutations. It remains to directly compute $A_{f_\alpha}(1) : A_{f_\alpha}(\omega)$ for different α. We will show that $A_{f_\alpha}(1) : A_{f_\alpha}(\omega)$ continuously tends to infinity as $\alpha \to 1^-$. In turn, this will imply that the there are uncountably many α such that the $\mathcal{M}_{f_\alpha}$ are all mutually non-isomorphic.

The continuity directly follows from the fact that $f_\alpha(z)$ and $f'_\alpha(z)$ continuously depend on α for a fixed z, so we only need to show that

$$\frac{A_{f_\alpha}(1)}{A_{f_\alpha}(\omega)} \to \infty, \qquad \alpha \to 1^-.$$

In (26), α only affects the fourth coordinate, hence, it is enough to show that $|g_\alpha(\omega) g'_\alpha(\omega)|$ is bounded while $|g_\alpha(1) g'_\alpha(1)| \to \infty$ as $\alpha \to 1^-$. Note that $g_\alpha(\omega) = g_\alpha(1) = 1$, meaning we need to show that $|g'_\alpha(\omega)/g_\alpha(\omega)|$ is bounded while $|g'_\alpha(1)/g_\alpha(1)| \to \infty$. Directly computing the logarithmic derivative of g_α from (23) we get

$$\frac{g'_\alpha(z)}{g_\alpha(z)} = \frac{1}{z} + \frac{1}{z-\alpha} + \frac{1}{z-\beta} + \frac{\alpha}{1-\alpha z} + \frac{\beta}{1-\beta z}.$$

Note that as $\alpha \to 1$ we get $\beta \to -1/2$, so that

$$\frac{g'_\alpha(\omega)}{g_\alpha(\omega)} \to \frac{1}{\omega} + \frac{1}{\omega - 1} + \frac{1}{\omega + 1/2} + \frac{1}{1-\omega} + \frac{-1/2}{1+\omega/2}, \qquad \alpha \to 1,$$

in particular, it is bounded, while

$$\frac{g'_\alpha(1)}{g_\alpha(1)} = 1 + \frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{\alpha}{1-\alpha} + \frac{\beta}{1-\beta} \to \infty, \qquad \alpha \to 1,$$

as we wanted. We conclude that

$$\frac{A_{f_\alpha}(1)}{A_{f_\alpha}(\omega)} \to \infty, \qquad \alpha \to 1^-,$$

continuously, which means that that there are uncountably many values of α for which the multiplier algebras $\mathcal{M}_{f_\alpha}$ of the analytic discs $f_\alpha(\mathbb{D})$ are mutually non-isomorphic. At the same time all f_α have exactly the same 3-point self-crossing at $\{1, \omega, \omega^2\}$. ∎

Acknowledgements. I would like to express my gratitude to Orr Shalit for his invaluable support and guidance throughout this project, which originated from my master's thesis under his supervision. I am also grateful to Oleg Ivrii, Victor Vinnikov, and Eli Shamovich, whose comments and remarks significantly enhanced the exposition of my thesis, and by extension, this paper. I wish to thank Jeet Sampat for his thoughtful suggestions that helped improve the manuscript. I am also grateful to the anonymous reviewer for their valuable comments and suggestions.

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