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Operator models and analytic subordination for operator-valued free convolution powers

by

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Abstract. We revisit the theory of operator-valued free convolution powers given by a completely positive map η. We first give a general result, with a new analytic proof, that the η-free convolution power of the law of X is realized by V^*XV for any operator V satisfying certain conditions, which unifies Nica and Speicher's construction in the scalar-valued setting and Shlyakhtenko's construction in the operator-valued setting. Second, we provide an analog, for the setting of η-free convolution powers, of the analytic subordination for conditional expectations that holds for additive free convolution. Finally, we describe a Hilbert-space manipulation that explains the equivalence between the n-fold additive free convolution and the convolution power with respect to $\eta = n\,\mathrm{id}$.

1. Introduction. The classical convolution of probability measures describes the distribution of the sum of independent variables realizing the measures. For a probability measure μ, one can therefore make sense of μ^{*n} for $n \in \mathbb{N}$ as the sum of n independent random variables with identical distributions μ. Since convolution of probability measures corresponds to multiplication of their Fourier transforms, there is an immediate candidate to define as μ^{*t} for $t > 0$ by taking an appropriate power on the Fourier transform side; however, even if one begins with a probability measure there is not in general a guarantee that the output of this operation is a positive measure of total mass 1. This does happen, however, for the so-called *infinitely divisible* probability measures: for such a measure μ one has a Lévy process consisting of measures $(\mu^{*t})_{t>0}$ which in particular is a semigroup with respect to convolution.

In the non-commutative setting of free probability, one likewise has a notion of free convolution: the law $\mu \boxplus \nu$ is the law of the sum of freely inde-

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pendent random variables having laws μ and ν individually. Corresponding to the Fourier transform of measures is Voiculescu's R-transform, which associates to a probability measure μ an analytic function R_μ and linearizes free convolution: $R_{\mu \boxplus \nu} = R_\mu + R_\nu$. (Strictly speaking, R is therefore analogous to the logarithm of the Fourier transform.) One arrives at the notion of $\boxplus$-infinitely divisible law μ, namely a law for which tR_μ is the R-transform of a probability measure for every $t > 0$, and a corresponding free Lévy process. The central limit distributions in these settings are prime examples of infinite divisibility: if γ_t is the centered Gaussian distribution of variance t, one has $\gamma_t = \gamma_1^{*t}$; similarly, if σ_t is the centered semicircular distribution of variance t, one has $\sigma_t = \sigma_1^{\boxplus t}$. These form semigroups with respect to $*$ and $\boxplus$, respectively. A result of Bercovici and Pata shows that there is a deep connection between the $\boxplus$-infinitely divisible laws and the $*$-infinitely divisible laws: there is a bijection between these two sets of laws which preserves domains of partial attraction [8].

Surprisingly, for every probability measure μ and every $t \geq 1$, the function tR_μ is the R-transform of some probability measure $\mu^{\boxplus t}$, which is called the tth *free convolution power of* μ; this is in stark contrast to the classical setting where real-valued convolution powers often do not exist. This fact was first shown for sufficiently large t by Bercovici and Voiculescu [9], and then proved for all $t \geq 1$ by Nica and Speicher [21]. Moreover, Nica and Speicher were able to produce an explicit model of these distributions: if $x = x^*$ is an element of a tracial non-commutative probability space $(\mathcal{A}, \phi)$ with law μ and $p \in \mathcal{A}$ is a projection freely independent from x with $\phi(p) = 1/t$, then pxp has law $\mu^{\boxplus t}$ in the non-commutative probability space $(p\mathcal{A}p, t\phi|_{p\mathcal{A}p})$.

This paper studies free convolution powers in the setting of operator-valued free probability, where the algebra of scalars $\mathbb{C}$ is replaced by a C*-algebra $\mathcal{B}$, and the state by a conditional expectation onto $\mathcal{B}$. In this setting, the central limit distribution – the semicircular law – is still characterized by its first and second moment, but the second moment is now a completely positive map $\eta : \mathcal{B} \to \mathcal{B}$ [28]. Motivated by scalar-valued free convolution powers, Anshelevich, Belinschi, Fevrier, and Nica [3] showed that there is a natural operation $\mu \mapsto \mu^{\boxplus \eta}$ with the desirable property that if σ is a $\mathcal{B}$-valued semicircular distribution with variance $\mathrm{id}_{\mathcal{B}}$ then $\sigma^{\boxplus \eta}$ is the $\mathcal{B}$-valued semicircular distribution with variance η. Their convolution is defined for arbitrary $\mathcal{B}$-valued distributions μ when $\eta - \mathrm{id}_{\mathcal{B}}$ is completely positive (mimicking the scalar-valued condition that $t \geq 1$), and for certain distributions for arbitrary η completely positive. It was later shown by Shlyakhtenko [24] that $\eta - \mathrm{id}_{\mathcal{B}}$ being completely positive is necessary to ensure that the η-free convolution power $\cdot^{\boxplus \eta}$ preserves positivity for all (multivariate) distributions. The precise definition of η-free convolution power is given at the start of Section 3, but it essentially amounts to a condition on $\mathcal{B}$-valued R-transforms:

$$R_{\mu^{\boxplus\eta}} = \eta \circ R_\mu,$$

and a similar condition for matrix amplifications, on an appropriate domain. (In the scalar-valued case, the completely positive maps $\mathbb{C} \to \mathbb{C}$ are just multiplication by elements of $[0, \infty)$, and so this extends the usual notion of free convolution powers.)

Shlyakhtenko [24] exhibited a construction of η-convolution powers: given $\mathcal{B}$ and $\eta : \mathcal{B} \to \mathcal{B}$ such that $\eta - \mathrm{id}_{\mathcal{B}}$ is completely positive, he produces an operator V living on a free Fock space so that whenever μ is a $\mathcal{B}$-valued distribution realized by a variable X free from V (in a potentially larger space), V^*XV has distribution $\mu^{\boxplus\eta}$.

Our first main result, Theorem 3.1, gives conditions for an operator V which guarantee that, when V is free from X which has distribution μ, the distribution of V^*XV is $\mu^{\boxplus\eta}$. Shlyakhtenko's construction satisfies the conditions of our theorem, but the properties we demand are weaker than those needed for the proof used in [24]. In particular, this allows us to construct a suitable operator V living on $\mathcal{B} \oplus \mathcal{B} \otimes_{\eta-\mathrm{id}} \mathcal{B}$ (see Lemma 3.2 and the preceding discussion), and to make the connection to the scalar case more explicit. The main stroke of our proof is as follows: writing ν for the distribution of V^*XV, we show that $\eta \circ R_\mu$ satisfies the functional equation defining R_ν in terms of its Cauchy transform G_ν, implying that $\nu = \mu^{\boxplus\eta}$. This approach is inspired by the proof of the additivity of the (scalar) R-transform over $\boxplus$ given by Haagerup [13, §3] and Lehner [18, Theorem 3.1], which was also adapted to the operator-valued case by Dykema [12, Lemma 4.5].

Our second main result, Theorem 4.1, gives a formula relating analytic subordination and conditional expectations in the setting of operator-valued convolution powers. Analytic subordination of Cauchy transforms has been an important tool for studying additive free convolution, beginning with the works of Voiculescu [27] and Biane [10]. Given probability measures μ, ν on $\mathbb{R}$ there are analytic functions ω_μ, ω_ν such that

$$G_\mu \circ \omega_\mu(z) = G_\nu \circ \omega_\nu(z) = G_{\mu\boxplus\nu}(z).$$

The theory was further developed and codified by Voiculescu [29]. A key point in this theory is that for free variables X and Y, the conditional expectation of a resolvent in $X+Y$ onto the von Neumann algebra generated by X is a resolvent in X, namely, $E_{\mathrm{W}^*(X)}[(z-X+Y)^{-1}] = (\omega_\mu(z)-X)^{-1}$. The theory of analytic subordination for additive free convolution was adapted to the $\mathcal{B}$-valued setting in [29, 6, 19, 5]. Our Theorem 4.1 shows that an analogous conditional expectation formula holds in the setting of operator-valued convolution powers with respect to a completely positive map η, even though this cannot always be realized as a free sum. Namely, if X and V are as in Theorem 3.1 and $\mathcal{E}_1$ is the conditional expectation onto the algebra generated by X, then

$$\mathcal{E}_1[V(z - V^*XV)^{-1}V^*] = (F(z) - X)^{-1},$$

where $F = G_\mu^{-1} \circ G_{\mu^{\boxplus\eta}}$ is the $\mathcal{B}$-valued subordination function.

As a final point of interest, we examine the particular case of n-fold convolution with $n \in \mathbb{N}$ in §5. The distribution of the n-fold free convolution can be modeled in two ways, either by the sum of n copies of the original variable that are freely independent over $\mathcal{B}$, or as a convolution power corresponding to the completely positive map $\eta = n\,\mathrm{id}$. We give an explicit transformation of the $\mathcal{B}$-$\mathcal{B}$-correspondences that relates these two models, hence obtaining a new heuristic for why these operations should be the same.

The structure of this paper is as follows. In Section 2 we recall much of the background material needed for the rest of the paper and establish notation. Section 3 contains our realization of operator-valued convolution powers. Section 4 shows that our model can be used to produce subordination functions in a manner analogous to the scalar-valued setting, satisfying the expected conditional expectation relation. Finally, in Section 5 we investigate the particular case where $\eta = n\,\mathrm{id}$ and show that an isomorphism of the underlying $\mathcal{B}$-$\mathcal{B}$-correspondence produces the familiar model of n freely independent variables being summed.

2. Preliminaries. In this section, we review the necessary background on $\mathcal{B}$-valued non-commutative probability theory. For general background on free probability, see [1, §5], [20], and for the operator-valued setting, see [25], [6], [14, §§2–5].

2.1. $\mathcal{B}$-valued non-commutative probability spaces. In non-commutative probability theory, one replaces the (commutative) algebra L^∞ of a probability space with a general unital C*- or von Neumann algebra $\mathcal{A}$. The expectation is then replaced by a state $\phi : \mathcal{A} \to \mathbb{C}$. Often $\mathcal{A}$ is represented concretely as bounded operators on a Hilbert space H, and the state ϕ is given by a vector ξ, i.e. $\phi(a) = \langle \xi, a\xi \rangle$. Even if $(\mathcal{A}, \phi)$ is given abstractly without a particular representation on a Hilbert space, a Hilbert space H_ϕ can be obtained by the GNS construction equipping $\mathcal{A}$ with the sesquilinear form $\langle a, b \rangle_\phi = \phi(a^*b)$, and there is a representation $\pi_\phi : A \to B(H_\phi)$ given by left multiplication such that $\phi(a) = \langle 1, \pi_\phi(a)1 \rangle_{H_\phi}$.

We work in the setting of $\mathcal{B}$-valued non-commutative probability, that is, we are given a C*-algebra $\mathcal{B}$, and all the scalar-valued quantities such as the expectation and the inner product on a Hilbert space are replaced by $\mathcal{B}$-valued versions. We assume familiarity with the basic theory of unital C*-algebras, matrices over a C*-algebra, and completely positive maps. We refer to [11, Chapter II] for a summary of results and for references.

Definition 2.1 ($\mathcal{B}$-valued probability space). Let $\mathcal{B}$ be a unital C*-algebra. A *$\mathcal{B}$-valued probability space* is a pair $(\mathcal{A} \supseteq \mathcal{B}, E : \mathcal{A} \to \mathcal{B})$ where

$\mathcal{A}$ is a unital C*-algebra and $\mathcal{B} \subseteq \mathcal{A}$ is a unital inclusion (injective $*$-homomorphism), and $E : \mathcal{A} \to \mathcal{B}$ is a *conditional expectation*, that is:

- $E|_{\mathcal{B}} = \mathrm{id}_{\mathcal{B}}$;
- E is a $\mathcal{B}$-$\mathcal{B}$-bimodule map, so $E[b_1 a b_2] = b_1 E[a] b_2$ for $a \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$;
- E is completely positive, that is, if $n \geq 1$ and $A \in M_n(\mathcal{A})$, then $E^{(n)}[A^* A] \geq 0$ in $M_n(\mathcal{B})$, where $E^{(n)} : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ is the entrywise application of E.

Next, we recall C*-correspondences, the $\mathcal{B}$-valued analog of representations on Hilbert spaces; this will be important in §5 where we perform "$\mathcal{B}$-valued Hilbert space" manipulations. For background on correspondences, see [22], [17], [11, §§II.7.1–II.7.2], and the references therein. We start with right Hilbert $\mathcal{B}$-modules, which are an analog of Hilbert spaces with a $\mathcal{B}$-valued right $\mathcal{B}$-linear inner product.

Definition 2.2. Let $\mathcal{B}$ be a unital C*-algebra. If $\mathcal{H}$ is a right $\mathcal{B}$-module, then a *$\mathcal{B}$-valued semi-inner product* is a map $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{B}$ such that for $h_1, h_2 \in \mathcal{H}$:

(1) $h_2 \mapsto \langle h_1, h_2\rangle$ is a right $\mathcal{B}$-module map;
(2) $\langle h_2, h_1\rangle = \langle h_1, h_2\rangle^*$;
(3) $\langle h_1, h_1\rangle \geq 0$.

One can show that the semi-inner product must satisfy an analog of the Cauchy–Schwarz inequality and hence $\|h\| := \|\langle h, h\rangle\|_{\mathcal{B}}^{1/2}$ defines a seminorm on $\mathcal{H}$. We also have $\|hb\| \leq \|h\|\,\|b\|_{\mathcal{B}}$ for $h \in \mathcal{H}$ and $b \in \mathcal{B}$.

Definition 2.3. If this seminorm is in fact a norm which makes $\mathcal{H}$ a Banach space, then we say that $\mathcal{H}$ is a *right Hilbert $\mathcal{B}$-module*. In general, if $\mathcal{H}$ has a $\mathcal{B}$-valued semi-inner product, then the completion of $\mathcal{H}/\{h : \|h\| = 0\}$ is a right Hilbert $\mathcal{B}$-module with the right $\mathcal{B}$-action and the $\mathcal{B}$-valued inner product induced in the natural way from those of $\mathcal{H}$. We refer to this module as the *separation-completion* of $\mathcal{H}$ with respect to $\langle\cdot,\cdot\rangle$.

Definition 2.4. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert $\mathcal{B}$-modules. We say that a linear map $T : \mathcal{H}_1 \to \mathcal{H}_2$ is *right $\mathcal{B}$-modular* if $T(hb) = (Th)b$ for $h \in \mathcal{H}_1$ and $b \in \mathcal{B}$. We say that T is *adjointable* if there exists a map $T^* : \mathcal{H}_2 \to \mathcal{H}_1$ such that

$$\langle Th_1, h_2\rangle = \langle h_1, T^* h_2\rangle \quad \text{for all } h_1 \in \mathcal{H}_1 \text{ and } h_2 \in \mathcal{H}_2.$$

We denote by $\mathcal{L}(\mathcal{H})$ the space of bounded, right $\mathcal{B}$-modular, adjointable operators on a right Hilbert $\mathcal{B}$-module $\mathcal{H}$. When we want to emphasize the right $\mathcal{B}$-module structure of $\mathcal{H}$, we may write $\mathcal{H}_{\mathcal{B}}$ and $\mathcal{L}(\mathcal{H}_{\mathcal{B}})$.

Unlike the case of scalar-valued Hilbert spaces, adjointability is *not* automatic. One can check that $\mathcal{L}(\mathcal{H})$ is a C*-algebra (see for instance [17, p. 8]). Now we can define the $\mathcal{B}$-valued analog of representations on a Hilbert space.

Definition 2.5. Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. An *$\mathcal{A}$-$\mathcal{B}$-correspondence* is a right Hilbert $\mathcal{B}$-module $\mathcal{H}$ together with a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$. Such a representation endows $\mathcal{H}$ with the structure of an $\mathcal{A}$-$\mathcal{B}$-bimodule, and so we will write $a\xi b$ for $\pi(a)[\xi b]$ where $\xi \in \mathcal{H}$, $a \in \mathcal{A}$, and $b \in \mathcal{B}$.

Every completely positive map $\psi : \mathcal{A} \to \mathcal{B}$ naturally gives rise to an $\mathcal{A}$-$\mathcal{B}$-correspondence $\mathcal{A} \otimes_\psi \mathcal{B}$ as follows.

Lemma 2.6 ([22, Theorem 5.2], see also [17, Theorem 5.6]). *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras and $\psi : \mathcal{A} \to \mathcal{B}$ be completely positive. On the algebraic tensor product $\mathcal{A} \otimes_{\text{alg}} \mathcal{B}$, define a $\mathcal{B}$-valued sesquilinear map by*

$$\langle a \otimes b, a' \otimes b' \rangle_\psi = b^* \psi(a^* a') b'.$$

Then $\langle \cdot, \cdot \rangle_\psi$ is a $\mathcal{B}$-valued semi-inner product. The associated separation-completion $\mathcal{A} \otimes_\psi \mathcal{B}$ has an $\mathcal{A}$-$\mathcal{B}$-correspondence structure where the left action of $\mathcal{A}$ is given on simple tensors by $a(a' \otimes b') = aa' \otimes b'$.

Next, we discuss the relationship between $\mathcal{B}$-valued correspondences and $\mathcal{B}$-valued probability spaces. Suppose $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{H}$ is an $\mathcal{A}$-$\mathcal{B}$-correspondence. If $\xi \in \mathcal{H}$, then under what conditions is the map $a \mapsto \langle \xi, a\xi \rangle$ a conditional expectation? The following is proved in [15, Lemma 2.10], and part of it was given also in [19, Proposition 2.10].

Lemma 2.7. *Let $\mathcal{B} \subseteq \mathcal{A}$ be a unital inclusion of unital C*-algebras. Let $\mathcal{H}$ be an $\mathcal{A}$-$\mathcal{B}$-correspondence and $\xi \in \mathcal{H}$, and define $E : \mathcal{A} \to \mathcal{B}$ by $E(a) := \langle \xi, a\xi \rangle$. Then the following are equivalent:*

(1) *E is a $\mathcal{B}$-valued conditional expectation.*
(2) *$\langle \xi, b\xi \rangle = b$ for every $b \in \mathcal{B}$.*
(3) *$\langle \xi, \xi \rangle = 1$ and $b\xi = \xi b$ for every $b \in \mathcal{B}$.*

Remark 2.8. This lemma is closely related to the well-known fact that if $\mathcal{B} \subseteq \mathcal{A}$ and $E : \mathcal{A} \to \mathcal{B}$ is completely positive with $E|_\mathcal{B} = \text{id}$, then E is automatically a conditional expectation. Indeed, letting $\xi = 1 \otimes 1$ in $\mathcal{A} \otimes_E \mathcal{B}$, we see that Lemma 2.7(2) is equivalent to $E|_\mathcal{B} = \text{id}$, and so E is a conditional expectation.

Definition 2.9. In the setting of Lemma 2.7, if ξ satisfies conditions (1)–(3), then we call ξ a *$\mathcal{B}$-central unit vector* and we call $(\mathcal{H}, \xi)$ a *pointed $\mathcal{A}$-$\mathcal{B}$-correspondence.*

Hence, for $\mathcal{B} \subseteq \mathcal{A}$, every pointed $\mathcal{A}$-$\mathcal{B}$-correspondence makes $\mathcal{A}$ into a $\mathcal{B}$-valued probability space. Conversely, given a $\mathcal{B}$-valued probability space $(\mathcal{A}, E)$, one can construct a canonical $\mathcal{A}$-$\mathcal{B}$-correspondence $L^2(\mathcal{A}, E)$ by the operator-valued GNS construction as follows. Note that $\mathcal{A}$ is a right $\mathcal{B}$-module and we can define a $\mathcal{B}$-valued semi-inner product on $\mathcal{A}$ by $\langle a_1, a_2 \rangle = E[a_1^* a_2]$. We denote the separation-completion with respect to this inner product by $L^2(\mathcal{A}, E)$. One can check that $L^2(\mathcal{A}, E)$ is an $\mathcal{A}$-$\mathcal{B}$-correspondence. If we denote by ξ the equivalence class of the vector $1 \in \mathcal{A}$, then we have

$$E[a] = \langle \xi, a\xi \rangle.$$

Thus, every $\mathcal{B}$-valued non-commutative probability space can be realized using some pointed $\mathcal{A}$-$\mathcal{B}$-correspondence.

In some contexts of $\mathcal{B}$-valued probability spaces, one may require an additional condition that the $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H}_{\mathcal{B}})$ is injective, i.e., the representation of $\mathcal{A}$ is faithful. This is equivalent to saying that $E[a_1 a a_2] = 0$ for all $a_1, a_2 \in \mathcal{A}$ only if $a = 0$ (since elements from $\mathcal{A}$ are dense in $L^2(\mathcal{A}, E)$). Generally, if $(\mathcal{H}, \xi)$ is a pointed $\mathcal{A}$-$\mathcal{B}$-correspondence, then the representation of $\mathcal{A}$ restricts to a faithful representation of $\mathcal{B}$ since $\langle \xi, b\xi \rangle = b$ for $b \in \mathcal{B}$. Hence, $\pi(\mathcal{A}) \supseteq \mathcal{B}$ can be viewed as a $\mathcal{B}$-valued probability space and $\mathcal{H}$ as a $\pi(\mathcal{A})$-$\mathcal{B}$-correspondence. In §5, we will simply assume a pointed $\mathcal{B}$-$\mathcal{B}$-correspondence $(\mathcal{H}, \xi)$ is given and that $\mathcal{A}$ is a C*-subalgebra of $\mathcal{L}(\mathcal{H}_{\mathcal{B}})$ containing $\mathcal{B}$.

2.2. $\mathcal{B}$-valued laws and analytic transforms. In order to discuss free convolution powers, we must first describe the analog of probability distributions in the $\mathcal{B}$-valued setting. These are an analog of *compactly supported* measures on $\mathbb{R}$ (the $\mathcal{B}$-valued analog of measures with unbounded support is beset with technical issues). A measure with compact support on $\mathbb{R}$ is uniquely determined by its moments (i.e. the expectation of polynomials), and thus can be viewed as a positive linear functional μ on the polynomial algebra $\mathbb{C}[x]$ satisfying $|\mu[x^k]| \leq R^k$ for some R. The description in the $\mathcal{B}$-valued setting is analogous, with $\mathbb{C}[x]$ replaced by $\mathcal{B}$-valued polynomials.

Definition 2.10. Let $\mathcal{B}$ be a unital C*-algebra. We define the *non-commutative polynomial algebra* $\mathcal{B}\langle x \rangle$ to be the universal unital $*$-algebra generated by $\mathcal{B}$ and a self-adjoint indeterminate x. As a vector space, $\mathcal{B}\langle x \rangle$ is spanned by the non-commutative monomials $b_0 x b_1 \dots x b_k$ for $k \geq 0$ and $b_j \in \mathcal{B}$. Note that $\mathcal{B} \subseteq \mathcal{B}\langle x \rangle$ as unital $*$-algebras and in particular $\mathcal{B}\langle x \rangle$ is a $\mathcal{B}$-$\mathcal{B}$-bimodule.

Definition 2.11 ($\mathcal{B}$-valued laws, [28, 23, 2]). Let $(\mathcal{A}, E)$ be a $\mathcal{B}$-valued probability space and let $X \in \mathcal{A}$ be self-adjoint. The *$\mathcal{B}$-valued law of X* is the map $\mu_X : \mathcal{B}\langle x \rangle \to \mathcal{B}$ given by $f \mapsto E[f(X)]$. In particular, if $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H}_{\mathcal{B}})$ for some pointed $\mathcal{B}$-$\mathcal{B}$-correspondence $(\mathcal{H}, \xi)$, then $\mu[f] = \langle \xi, f(X)\xi \rangle$.

Just as in the case of probability measures on $\mathbb{R}$, there is an abstract characterization of the linear functionals $\mu : \mathcal{B}\langle x \rangle \to \mathcal{B}$ that arise as the $\mathcal{B}$-valued law of some self-adjoint element with $\|X\| \leq R$, namely, μ is a completely positive $\mathcal{B}$-bimodule map satisfying the conditions $\mu|_{\mathcal{B}} = \mathrm{id}_{\mathcal{B}}$ and $\|\mu(b_0 x b_1 \dots x b_k)\| \leq R^k \|b_0\| \dots \|b_k\|$ for all $k \geq 0$ and $b_0, \dots, b_k \in \mathcal{B}$. For details, see [23, Proposition 1.2], [31, Proposition 2.9], and [15, Theorem 2.17].

We also have $\mathcal{B}$-valued versions of the Cauchy–Stieltjes transform and other analytic functions associated to a probability measure. For a probability measure μ on $\mathbb{R}$ and $\operatorname{im}(z) > 0$, the Cauchy–Stieltjes transform is given by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z - x}\, d\mu(x).$$

The definition in the $\mathcal{B}$-valued setting is similar. Let μ be the $\mathcal{B}$-valued law of some self-adjoint X in a $\mathcal{B}$-valued probability space $(\mathcal{A}, E)$. Then we want to define, for certain $z \in \mathcal{B}$,

$$G_\mu(z) = E[(z - X)^{-1}].$$

For this to make sense, $z - X$ must be invertible in $\mathcal{A}$. But in fact, if we denote $\operatorname{im}(z) = (z - z^*)/(2i)$, then $\operatorname{im}(z) \geq \epsilon 1$ implies that z is invertible with $\|z^{-1}\| \leq 1/\epsilon$. And so since X is self-adjoint, i.e. $\operatorname{im}(X) = 0$, we also obtain $\|(z - X)^{-1}\| \leq 1/\epsilon$.

There is one more wrinkle in the definition of G_μ in the $\mathcal{B}$-valued setting. Namely, knowing the values of $G_\mu(z)$ is not sufficient to determine the $\mathcal{B}$-valued law μ uniquely. Indeed, looking at the power series of $G_\mu(z)$ "at infinity" we would only obtain moments of the form $\mu[z^{-1}xz^{-1} \dots xz^{-1}]$, but for the $\mathcal{B}$-valued law, we need all moments of the form $\mu[b_0 x b_1 \dots x b_n]$. This problem can be solved by studying matrix amplifications.

Definition 2.12. Let μ be the $\mathcal{B}$-valued law of some self-adjoint X in a $\mathcal{B}$-valued probability space $(\mathcal{A}, E)$. Let

$$\mathbb{H}^{(n)}(\mathcal{B}) = \{z \in M_n(\mathcal{B}) : \operatorname{im}(z) \geq \epsilon 1 \text{ for some } \epsilon > 0\}.$$

Let $G_\mu^{(n)} : \mathbb{H}^{(n)}(\mathcal{B}) \to M_n(\mathcal{B})$ be given by

$$G_\mu^{(n)}(z) = E^{(n)}[(z - X^{(n)})^{-1}],$$

where $X^{(n)}$ denotes the image of X in $M_n(\mathcal{A})$ under the canonical mapping $\mathcal{A} \to M_n(\mathcal{A})$, that is, $X^{(n)} = X \otimes I_n$.

Note that the function

$$\tilde{G}_\mu^{(n)}(z) := G_\mu^{(n)}(z^{-1}) = E^{(n)}[(z^{-1} - X^{(n)})^{-1}] = E^{(n)}[z(1 - X^{(n)}z)^{-1}]$$

extends to the neighborhood $B_{1/R}(0)$ in $M_n(\mathcal{B})$ where $R = \|X\|$. The moments of μ can then be recovered from $\tilde{G}_\mu^{(n)}$ as follows: letting

$$z = \begin{bmatrix} 0 & b_0 & 0 & \dots & 0 & 0 \\ 0 & 0 & b_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & b_n \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \in M_{n+2}(\mathcal{B}),$$

we have

$$[\tilde{G}_\mu^{(n+2)}(z)]_{1,n+2} = \mu[b_0 x b_1 \dots x b_n].$$

The study of the $\mathcal{B}$-valued Cauchy transform motivates the definition of $\mathcal{B}$-valued non-commutative functions or $\mathcal{B}$-valued fully matricial functions, which are a $\mathcal{B}$-valued analog of complex-analytic functions. For background on these fully matricial (a.k.a. non-commutative) functions, see [30], [16], [14, §3].

The appropriate domain for these functions are so-called fully matricial domains:

Definition 2.13. Let $\mathcal{B}$ be a C*-algebra. A *fully matricial domain* is a sequence of sets $\Omega^{(n)} \subseteq M_n(\mathcal{B})$ such that the following conditions hold:

(1) If $z \in \Omega^{(n)}$ and $w \in \Omega^{(m)}$, then $z \oplus w \in \Omega^{(n+m)}$.
(2) Let $z \in \Omega^{(n)}$. For each m, let $z^{(m)}$ be the direct sum of m copies of z. Then there is some $\delta > 0$ such that $B_\delta(z^{(m)}) \subseteq \Omega^{(nm)}$ for all $m \geq 1$.

For example, it is immediate to check that $\mathbb{H}^{(n)}(\mathcal{B})$ is a fully matricial domain, and so is the sequence $\Omega^{(n)}$ given by the ball $B_\delta(0)$ in $M_n(\mathcal{B})$.

Definition 2.14. Let $\Omega = (\Omega^{(n)})_{n\in\mathbb{N}}$ be a fully matricial domain over $\mathcal{B}$. A *fully matricial function* on Ω is a sequence of functions $F^{(n)} : \Omega^{(n)} \to M_n(\mathcal{B})$ satisfying the following conditions:

(1) If $z \in \Omega^{(n)}$ and $w \in \Omega^{(m)}$, then $F^{(m+n)}(z \oplus w) = F^{(n)}(z) \oplus F^{(m)}(w)$.
(2) If $z \in \Omega^{(n)}$ and $S \in M_n(\mathbb{C})$ is invertible and $SzS^{-1} \in \Omega^{(n)}$, then we have $F^{(n)}(SzS^{-1}) = SF^{(n)}(z)S^{-1}$.
(3) Given $z \in \Omega^{(n)}$, there exists some $\delta > 0$ and $M > 0$ such that for all $m \in \mathbb{N}$, for all $w \in B_\delta(z^{(m)})$, we have $\|F^{(nm)}(w)\| \leq M$.

One can check easily that the Cauchy transform $G_\mu(z)$ is a fully matricial function. Interestingly, the conditions (1) and (2) in Definition 2.14 are purely algebraic and (3) is merely a local boundedness condition, but these conditions automatically imply the existence of local power series expansions. The inverse function theorem for fully matricial functions can be used to show that for a $\mathcal{B}$-valued law μ, the transform $\tilde{G}_\mu(z)$ is invertible in a neighborhood of zero, and thus to define the $\mathcal{B}$-valued analog of the R-transform, which plays a central role in the study of free convolution. Here we single out the fact that we need for later use.

Proposition 2.15 ([12, Proposition 4.1], [23, proof of Corollary 5.4], [14, Lemma 4.5.9]). *Let μ be the $\mathcal{B}$-valued law of some self-adjoint operator X in a $\mathcal{B}$-valued probability space $(\mathcal{A}, E)$ with $\|X\| \leq r$. Then $\tilde{G}_\mu^{(n)}$ has a unique inverse function defined on $B_{c/r}(0)$ in $M_n(\mathcal{B})$ where c is a universal constant* (*we can take $c = 3 - 2\sqrt{2}$*). *The inverse function is fully matricial. Moreover, given $\delta > 0$, μ is uniquely determined by the values of $(\tilde{G}_\mu^{(n)})^{-1}(z)$ for $z \in B_\delta(0)$ of $M_n(\mathcal{B})$ and $n \in \mathbb{N}$.*

Definition 2.16 ([28, §4.8]). Let μ be a $\mathcal{B}$-valued law. The *R-transform* of μ is the function $R_\mu^{(n)}(z) = (G_\mu^{(n)})^{-1}(z) - z^{-1}$ defined in a ball around zero in $M_n(\mathcal{B})$.

2.3. $\mathcal{B}$-valued free independence and free products

Definition 2.17 (Free independence). Let $(\mathcal{A}, E)$ be a $\mathcal{B}$-valued probability space. Then unital subalgebras $\mathcal{A}_1, \dots, \mathcal{A}_N$ containing $\mathcal{B}$ are said to be *freely independent* (*over $\mathcal{B}$*) if we have

$$E[a_1 \dots a_k] = 0$$

whenever $a_j \in \mathcal{A}_{i(j)}$ with $E[a_j] = 0$, provided that all consecutive indices $i(j)$ and $i(j+1)$ are distinct. Moreover, if $X_1, \dots, X_N$ are self-adjoint elements of $\mathcal{A}$, we say that $X_1, \dots, X_N$ are freely independent (over $\mathcal{B}$) if the subalgebras $\mathcal{A}_j$ generated by $\mathcal{B}$ and X_j are freely independent (over $\mathcal{B}$).

The following fundamental result in free probability relates the R-transform and free independence.

Proposition 2.18 ([28, §4.11], [12, Lemma 4.5]). *Suppose that X and Y are freely independent. For ease of notation, let R_X denote the R-transform of the law of X. Then there is some $r > 0$ such that $R_{X+Y}^{(n)}(z) = R_X^{(n)}(z) + R_Y^{(n)}(z)$ for z in $B_r(0)$. In particular, the law μ_{X+Y} is uniquely determined by μ_X and μ_Y.*

In this case, the law of $X + Y$ is said to be the *$\mathcal{B}$-valued free convolution* of μ_X and μ_Y, or $\mu_{X+Y} = \mu_X \boxplus \mu_Y$. Since the law of $X + Y$ is uniquely determined, this construction can be used to define the free convolution $\mu \boxplus \nu$ for any two given $\mathcal{B}$-valued laws μ and ν, provided we can show that there exists some $\mathcal{B}$-valued non-commutative probability space $(\mathcal{A}, E)$ and some X, $Y \in \mathcal{A}$ that are freely independent and realize the laws μ and ν respectively. This follows from the construction of $\mathcal{B}$-valued free products that we will now describe.

Suppose we are given $\mathcal{B}$-valued probability spaces $(\mathcal{A}_j, E_j)$ for $j = 1, \dots, N$. Assume, as at the end of §2.1, that $\mathcal{A}_j \subseteq \mathcal{L}((\mathcal{H}_j)_\mathcal{B})$ for some pointed $\mathcal{A}_j$-$\mathcal{B}$-correspondence $(\mathcal{H}_j, \xi_j)$. Then we want to construct some pointed $\mathcal{B}$-$\mathcal{B}$-correspondence $(\mathcal{H}, \xi)$ and $*$-homomorphisms $\rho_j : \mathcal{L}((\mathcal{H}_j)_\mathcal{B}) \to \mathcal{L}(\mathcal{H}_\mathcal{B})$

such that $\rho_j(\mathcal{L}((\mathcal{H}_j)_{\mathcal{B}})$ are freely independent over $\mathcal{B}$. Two ingredients are needed for this construction: first, decomposing $\mathcal{H}_j$ into the span of ξ_j and its orthogonal complement, and second, tensor products of correspondences.

Let $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{H}$ be an $\mathcal{A}$-$\mathcal{B}$-correspondence; in general, if $\mathcal{K} \subseteq \mathcal{H}$ is an $\mathcal{A}$-$\mathcal{B}$-submodule, then the orthogonal complement $\mathcal{K}^\perp = \{h : \langle h, k\rangle = 0$ for all $k \in \mathcal{K}\}$ is also an $\mathcal{A}$-$\mathcal{B}$-correspondence, but $\mathcal{K} + \mathcal{K}^\perp$ might not span all of $\mathcal{H}$ (this is related to the fact that adjointability is not automatic for bounded right $\mathcal{B}$-module maps). However, we do have such a decomposition in the special case where $\mathcal{K}$ is the span of a $\mathcal{B}$-central unit vector in a $\mathcal{B}$-$\mathcal{B}$-correspondence. The following lemma is proved as in [23, proof of Remark 3.3] or [14, Lemma 2.5.7].

Lemma 2.19. *Let $\mathcal{H}$ be a $\mathcal{B}$-$\mathcal{B}$-correspondence and ξ a $\mathcal{B}$-central unit vector. Let $\mathcal{H}^\circ = \{h \in \mathcal{H} : \langle h, \xi\rangle = 0\}$. Then we have $\mathcal{H} = \mathcal{B}\xi \oplus \mathcal{H}^\circ$ as $\mathcal{B}$-$\mathcal{B}$-correspondences.*

We will also need the following notion of *tensor products of* C*-*correspondences.*

Construction 2.20. Let $\mathcal{B}_0, \dots, \mathcal{B}_n$ be unital C*-algebras, and suppose that $\mathcal{H}_j$ is a Hilbert $\mathcal{B}_{j-1}$-$\mathcal{B}_j$-correspondence for each j. We can form the algebraic tensor product

$$\mathcal{H}_1 \otimes_{\mathrm{alg},\mathcal{B}_1} \cdots \otimes_{\mathrm{alg},\mathcal{B}_{n-1}} \mathcal{H}_n$$

in the sense of algebraic bimodules. We define a $\mathcal{B}_n$-valued semi-inner product by

$$\langle h_1 \otimes \cdots \otimes h_n, h_1' \otimes \cdots \otimes h_n'\rangle = \langle h_n, \langle h_{n-1}, \langle \dots, \langle h_1, h_1'\rangle \dots\rangle h_{n-1}'\rangle h_n'\rangle.$$

In other words, we first evaluate $\langle h_1, h_1'\rangle \in \mathcal{B}_1$, then evaluate $\langle h_1, h_1'\rangle h_2'$ using the left $\mathcal{B}_1$-module structure on $\mathcal{H}_2$, then compute $\langle h_2, \langle h_1, h_1'\rangle h_2'\rangle \in \mathcal{B}_2$ and so forth. Positivity of the inner product is checked by using complete positivity in the standard way (see e.g. [17, Proposition 4.5], [14, §2.3.2]).

We denote the separation-completion with respect to this inner product by

$$\mathcal{H}_1 \otimes_{\mathcal{B}_1} \cdots \otimes_{\mathcal{B}_{n-1}} \mathcal{H}_n$$

and one can show that this is a $\mathcal{B}_0$-$\mathcal{B}_n$-correspondence in the obvious way. As one would expect, these tensor products satisfy the associativity up to a canonical isomorphism, and they distribute over direct sums of correspondences in each argument.

We now come to the construction of free products of pointed $\mathcal{B}$-$\mathcal{B}$-correspondences as defined in [26, §5]. Variants of the construction were also explained in [28, §§2–3] and [25, §3.3, §3.5]. We use similar terminology to [14, §5]. Suppose that $(\mathcal{H}_1, \xi_1), \ldots, (\mathcal{H}_N, \xi_N)$ are pointed $\mathcal{B}$-$\mathcal{B}$-correspondences. Define

$$\mathcal{H} = \mathop{*}_{j=1}^{N} (\mathcal{H}_j, \xi_j) := \mathcal{B}\xi \oplus \bigoplus_{k \geq 1} \bigoplus_{\substack{j_1, \ldots, j_k \in [N] \\ j_r \neq j_{r+1}}} \mathcal{H}_{j_1}^\circ \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} \mathcal{H}_{j_k}^\circ.$$

Here $\mathcal{B}\xi$ represents a copy of $\mathcal{B}$ as a $\mathcal{B}$-$\mathcal{B}$-correspondence with ξ corresponding to the vector 1.

Next, we define $*$-homomorphisms $\rho_j : B(\mathcal{H}_j) \to B(\mathcal{H})$ as follows. For each j, there is a natural decomposition of $\mathcal{H}$ as

$$\mathcal{H} \cong \mathcal{H}_j \otimes_{\mathcal{B}} \mathcal{M}_j,$$

where

$$\mathcal{M}_j = \mathcal{B} \oplus \bigoplus_{k \geq 1} \bigoplus_{\substack{j_1, \ldots, j_k \in [N] \\ j_r \neq j_{r+1} \\ j_1 \neq j}} \mathcal{H}_{j_1}^\circ \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} \mathcal{H}_{j_k}^\circ.$$

In each case, if we denote by U_j the (unitary) isomorphism $\mathcal{H} \to \mathcal{H}_j \otimes_{\mathcal{B}} \mathcal{M}_j$, we define the $*$-homomorphism ρ_j by

$$\rho_j(a) = U_j^*(a \otimes \mathrm{id}_{\mathcal{M}_j})U_j.$$

Note that ρ_j is injective because $\rho_j(a)$ restricted to the direct summands $\mathcal{B} \oplus \mathcal{H}_j^\circ$ in $\mathcal{H}$ is a itself conjugated by the obvious isomorphism $\mathcal{B} \oplus \mathcal{H}_j^\circ \to \mathcal{H}_j$. Moreover, by the same token,

$$\langle \xi, \rho_j(a)\xi \rangle = \langle \xi_j, a\xi_j \rangle,$$

which means that ρ_j is expectation-preserving. It is also easy to check that ρ_j is a $\mathcal{B}$-$\mathcal{B}$-bimodule map, where $B(\mathcal{H}_j)$ and $B(\mathcal{H})$ are given a $\mathcal{B}$-$\mathcal{B}$-bimodule structure through the embeddings $\mathcal{B} \to B(\mathcal{H}_j)$ and $\mathcal{B} \to B(\mathcal{H})$ given by the left $\mathcal{B}$-module structure of $\mathcal{H}_j$ and $\mathcal{H}$.

THEOREM 2.21 (see e.g. [14, §5.3.2]). *Let $(\mathcal{H}_j, \xi_j)$ for $j = 1, \ldots, N$ be $\mathcal{B}$-$\mathcal{B}$-correspondences with $\mathcal{B}$-central unit vectors. Let $(\mathcal{H}, \xi) = *_{j=1}^N (\mathcal{H}_j, \xi_j)$. Let $E : B(\mathcal{H}) \to \mathcal{B}$ be the expectation given by the vector ξ. Then the algebras $\rho_1(B(\mathcal{H}_1)), \ldots, \rho_N(B(\mathcal{H}_N))$ are freely independent in the $\mathcal{B}$-valued probability space $(B(\mathcal{H}), E)$.*

The construction of the free product of correspondences also naturally produces conditional expectations from $\mathcal{L}(\mathcal{H}_{\mathcal{B}})$ to $\mathcal{L}((\mathcal{H}_j)_{\mathcal{B}})$ as follows.

PROPOSITION 2.22. *Let $(\mathcal{H}, \xi)$ be the $\mathcal{B}$-valued free product of pointed $\mathcal{B}$-$\mathcal{B}$-correspondences $(\mathcal{H}_j, \xi_j)$. Let $V : \mathcal{H}_j = \mathcal{B}\xi_j \oplus \mathcal{H}_j^\circ \to \mathcal{H}$ be the natural inclusion mapping $\mathcal{H}_j$ onto the summands $\mathcal{B}\xi$ and $\mathcal{H}_j^\circ$ in $\mathcal{H}$.*

(1) *Let $\mathcal{E}_j : \mathcal{L}(\mathcal{H}_{\mathcal{B}}) \to \mathcal{L}((\mathcal{H}_j)_{\mathcal{B}})$ be given by $T \mapsto V^*TV$. Then $\mathcal{E}_j$ is a conditional expectation (with the identification of $\mathcal{A}_j$ with $\rho_j(\mathcal{A}_j)$).*

(2) *Suppose that* $T = \rho_{i(1)}(a_1) \dots \rho_{i(k)}(a_k)$ *where* $k \geq 1$, $a_j \in \mathcal{L}((\mathcal{H}_{i(j)})_{\mathcal{B}})$, $i(1) \neq i(2) \neq \cdots \neq i(k)$, *and* $\langle \xi_\ell, a_\ell \xi_\ell \rangle = 0$ *for each* ℓ. *Then* $\mathcal{E}_j[T] = 0$ *unless* $k = 1$ *and* $i(1) = j$.

(3) *Given unital* C^**-algebras* $\mathcal{A}_j$ *with* $\mathcal{B} \subseteq \mathcal{A}_j \subseteq \mathcal{L}((\mathcal{H}_j)_{\mathcal{B}})$, *the expectation* $\mathcal{E}_j$ *maps* $\mathrm{C}^*(\rho_1(\mathcal{A}_1), \dots, \rho_N(\mathcal{A}_N))$ *into* $\mathcal{A}_j$.

Proof. (1) This follows by direct computation.

(2) First, suppose that $i(k) \neq j$. Then by construction of the free product space, we have

$$\rho_{i(1)}(a_1) \dots \rho_{i(k)}(a_k)[\mathcal{B}\xi \oplus \mathcal{H}_j^\circ] \subseteq \mathcal{H}_{i(1)}^\circ \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} \mathcal{H}_{i(j)}^\circ \otimes (\mathcal{B}\xi \oplus \mathcal{H}_j^\circ),$$

and hence the image of $\rho_{i(1)}(a_1) \dots \rho_{i(k)}(a_k)V$ is orthogonal to $\mathcal{B}\xi \oplus \mathcal{H}_j^\circ$, and so $V^* \rho_{i(1)}(a_1) \dots \rho_{i(k)}(a_k)V = 0$. Thus, the result holds in this case.

Next, suppose that $i(k) = j$. If $k = 1$, then there is nothing to prove. So suppose $k > 1$. Let $T' = \rho_{i(1)}(a_1) \dots \rho_{i(k-1)}(a_{k-1})$. Then $\mathcal{E}_j[T] = \mathcal{E}_j[T']\rho_j(a_k)$ by the bimodule property for $\mathcal{E}_j$. Then we note that $\mathcal{E}_j[T'] = 0$ by the same reasoning as above.

(3) We recall that the $*$-algebra generated by $\rho_j(\mathcal{A}_j)$ for $j = 1, \dots N$ is *spanned* by elements of the form $\rho_{i(1)}(a_1) \dots \rho_{i(k)}(a_k)$ where $k \geq 1$ and $a_j \in \mathcal{L}((\mathcal{H}_{i(j)})_{\mathcal{B}})$ and $i(1) \neq i(2) \neq \cdots \neq i(k)$ and $\langle \xi_j, a_j \xi_j \rangle = 0$. (One proves that arbitrary products of elements from $\mathcal{A}_j$ are in the span of these reduced products, by induction on the length of the product; see the proof of [28, Proposition 1.3] or [14, Lemma 5.2.8].) By the previous step, $\mathcal{E}_j[\rho_{i(1)}(a_1) \dots \rho_{i(k)}(a_k)]$ is equal to zero unless $k = 1$ and $i(1) = j$, in which case it is $\mathcal{E}[\rho_j(a_1)] = a_1$. In either case, the image is in $\mathcal{A}_j$. Hence, $\mathcal{E}_j$ maps the $*$-algebra generated by $\rho_1(\mathcal{A}_1), \dots, \rho_N(\mathcal{A}_N)$ into $\mathcal{A}_j$. By density/continuity, it maps the C^*-algebra generated by $\rho_1(\mathcal{A}_1), \dots, \rho_N(\mathcal{A}_N)$ into $\mathcal{A}_j$. ∎

3. Realization of operator-valued convolution powers. For $\mathcal{B}$-valued laws μ and ν, and $\eta : \mathcal{B} \to \mathcal{B}$ completely positive, we say that ν is the *η-free convolution power* of μ, or $\nu = \mu^{\boxplus \eta}$, if we have $R_\nu^{(n)} = \eta^{(n)} \circ R_\mu^{(n)}$ on $B_\delta(0)$ in $M_n(\mathcal{B})$ for all n, for some $\delta > 0$. In this case, by Proposition 2.15, ν is uniquely determined by μ and η. Given μ and η, the η free convolution power of μ does not always exist in general, but it does always exist if $\eta - \mathrm{id}$ is completely positive [3, 24]. In fact, when $\eta - \mathrm{id}$ is completely positive, the following theorem describes a general construction to realize the η-free convolution power of μ.

THEOREM 3.1. *Let* $(\mathcal{A}, E)$ *be a* $\mathcal{B}$*-valued probability space. Let* $\eta : \mathcal{B} \to \mathcal{B}$ *be a linear map such that* $\eta - \mathrm{id}$ *is completely positive. Assume that* X *and* V *are freely independent over* $\mathcal{B}$*,* X *is a self-adjoint operator with* $\mathcal{B}$*-valued distribution* μ*, and* V *satisfies*

$$V b_1 V^* b_2 V = V b_1 \eta(b_2) \quad \textit{for } b_1, b_2 \in \mathcal{B}, \tag{3.1}$$

$$E[VbV^*] = b \quad \textit{for } b \in \mathcal{B}. \tag{3.2}$$

Consider the map $\tilde{E}[T] = E[VTV^*]$ *for* $T \in \mathcal{A}$ *(which is a conditional expectation by* (3.2) *and Remark* 2.8*). Let* ν *be the* $\mathcal{B}$*-valued law of* V^*XV *with respect to the expectation* $\tilde{E}$*. Then* $\nu = \mu^{\boxplus\eta}$*.*

In the case $\mathcal{B} = \mathbb{C}$, condition (3.1) is equivalent to $\eta(1)^{-1/2}V$ being a partial isometry (and in general, if $\eta(1) \in \mathbb{C}$, then (3.1) implies that $\eta(1)^{-1/2}V$ is a partial isometry). In particular, when $\mathcal{B} = \mathbb{C}$, we can take $V = P/E[P]^{1/2}$ for a projection P and correspondingly $\eta(1) = E[P]^{-1}$. This leads to $\tilde{E}[VTV^*] = E[PTP]/E[P]$, or $\tilde{E}$ is the natural state on the corner $P\mathcal{A}P$. We thus recover the construction of free convolution powers in the scalar case from [21].

For general $\mathcal{B}$ and $\eta \geq \mathrm{id}$, Shlyakhtenko [24] gave a free Fock space construction of an operator V that satisfies the stronger condition $V^*bV = \eta(b)$ for $b \in \mathcal{B}$ in place of (3.1), and also satisfies (3.2). This stronger condition unfortunately does not permit V to be a scalar multiple of a projection in the case $\mathcal{B} = \mathbb{C}$, since if $V = \lambda P$, then $\lambda^2 P = V^*1V = \eta(1) \in \mathbb{C}$, which forces $P = 0$ or 1. Besides, an operator V satisfying the weaker (3.1) and (3.2) can be constructed in a simpler way as follows.

Lemma 3.2. *Let* $\eta : \mathcal{B} \to \mathcal{B}$ *such that* $\eta - \mathrm{id}$ *is completely positive. Then there exists a* $\mathcal{B}$*-valued probability space* $(\mathcal{A}, E)$ *and an operator* V *satisfying* (3.1) *and* (3.2)*.*

Proof. Let $\psi = \eta - \mathrm{id}$. Let $\mathcal{H}^\circ = \mathcal{B} \otimes_\psi \mathcal{B}$ be the $\mathcal{B}$-$\mathcal{B}$-correspondence associated to ψ from Lemma 2.6, and let ζ be the vector $1 \otimes 1$ in $\mathcal{H}^\circ$. Let $\mathcal{H} = \mathcal{B} \oplus \mathcal{H}^\circ$, where $\mathcal{B}$ is viewed as the trivial $\mathcal{B}$-$\mathcal{B}$-correspondence. Let $\mathcal{A}$ be the space of right $\mathcal{B}$-linear and adjointable operators on $\mathcal{H}$, and let $E : \mathcal{A} \to \mathcal{B}$ be given by $E[T] = \langle 1 \oplus 0, T(1 \oplus 0)\rangle$. Let $V \in \mathcal{A}$ be given by

$$V(b \oplus h) = b \oplus \zeta b.$$

It is straightforward to check that V is adjointable with

$$V^*(b \oplus h) = (b + \langle \zeta, h \rangle) \oplus 0.$$

To check (3.1), note that for $b_1, b_2, b \in \mathcal{B}$ and $h \in \mathcal{H}^\circ$,

$$\begin{aligned}
V b_1 V^* b_2 V(b \oplus h) &= V b_1 V^* b_2 (b \oplus \zeta b) \\
&= V b_1 V^* (b_2 b \oplus b_2 \zeta b) \\
&= V b_1 ((b_2 b + \langle \zeta, b_2 \zeta b \rangle) \oplus 0) \\
&= V b_1 ((b_2 + \langle \zeta, b_2 \zeta \rangle) b \oplus 0) \\
&= V b_1 (\eta(b_2) b \oplus 0) \\
&= V b_1 \eta(b_2)(b \oplus h),
\end{aligned}$$

where the last line follows because

$$Vb_1\eta(b_2)(0 \oplus h) = V(0 \oplus b_1\eta(b_2)h) = 0.$$

Thus, (3.1) holds. Moreover, since $V^*(1 \oplus 0) = (1 \oplus 0)$, we deduce that $E[VTV^*] = E[T]$ for $T \in \mathcal{A}$, and in particular $E[VbV^*] = b$ for $b \in \mathcal{B}$, so (3.2) holds. ∎

REMARK 3.3. Conversely, the condition that $\eta \geq \mathrm{id}$ is necessary in order for there to exist an operator V satisfying (3.1) and (3.2). Indeed, if such a V exists, then for $b \in \mathcal{B}$,

$$\begin{aligned}\eta(b^*b) &= E[V\eta(b^*b)V^*] \\ &= E[VV^*b^*bVV^*] \\ &= E[(bVV^*)^*(bVV^*)] \\ &\geq E[bVV^*]^*E[bVV^*] \\ &= b^*b.\end{aligned}$$

The same equation also holds for $b \in M_n(\mathcal{B})$ and $V^{(n)} = V \otimes I_n \in M_n(\mathcal{B})$, and so $\eta - \mathrm{id}$ is completely positive. Hence, although the proof of Theorem 3.1 below does not directly use that $\eta \geq \mathrm{id}$, the statement is vacuous outside this case.

REMARK 3.4. It is also natural to ask when the operator V can be realized in a *tracial* von Neumann algebra. Unfortunately, the situations where this can happen are rather restrictive, even with the weakened conditions (3.1) and (3.2). Indeed, assume that $\mathcal{A}$ is equipped with a faithful trace τ and that E is a trace-preserving conditional expectation. Then for $b_1, b_2 \in \mathcal{B}$, applying in order traciality, (3.1), traciality, and (3.2), we have

$$\begin{aligned}\tau(V^*VV^*b_1Vb_2) &= \tau(V^*b_1Vb_2V^*V), \\ \tau(V^*V\eta(b_1)b_2) &= \tau(V^*b_1Vb_2\eta(1)), \\ \tau(V\eta(b_1)b_2V^*) &= \tau(b_1Vb_2\eta(1)V^*), \\ \tau(\eta(b_1)b_2) &= \tau(b_1b_2\eta(1)) = \tau(\eta(1)b_1b_2),\end{aligned}$$

and since b_2 is arbitrary, this yields $\eta(b_1) = \eta(1)b_1$. By taking adjoints, $\eta(b) = b\eta(1)$ as well, and so $\eta(1)$ is in the center of $\mathcal{B}$. Hence, the only situation where these relations can be satisfied using a trace-preserving conditional expectation is if η is given by multiplication by an element of the center of $\mathcal{B}$.

Now we turn to the proof of Theorem 3.1. Using a similar method to Haagerup [13, §3] and Lehner's [18, Theorem 3.1] analytic proof that the R-transform is additive under free convolution, we are able to prove the result from scratch using only the definition of free independence.

Proof of Theorem 3.1. Let ν be the $\mathcal{B}$-valued law of V^*XV. We must show that $R_\nu^{(n)}(z) = \eta^{(n)} \circ R_\mu^{(n)}(z)$ for z in $B_\delta(0)$ in $M_n(\mathcal{B})$ for all n, for some $\delta > 0$. Note that the R-transform being $(\tilde{G}_\mu^{(n)})^{-1}(z) - z$ can be expressed as $G_\mu^{(n)}(z^{-1} + R_\mu^{(n)}(z)) = z$ for invertible $z \in B_\delta(0)$. In light of analytic continuation (see e.g. [14, Corollary 3.9.7]), since the set of invertible z contains an open ball in $B_\delta(0)$, it suffices to prove that $G_\nu^{(n)}(z^{-1} + \eta^{(n)} \circ R_\mu^{(n)}(z)) = z$ for invertible z in a neighborhood of 0. We can also write this as

$$E^{(n)}[V^{(n)}(z^{-1} + \eta^{(n)} \circ R_\mu^{(n)}(z) - (V^{(n)})^* X^{(n)} V^{(n)})^{-1} (V^{(n)})^*] = z,$$

where $X^{(n)}$ and $V^{(n)}$ are the images of X and V under the inclusion $\mathcal{A} \to M_n(\mathcal{A})$. In the following argument, we will omit the superscript (n) for ease of notation. This should not cause confusion since the argument is a manipulation of functional and power series identities and n remains fixed throughout.

Now for z in a neighborhood of zero,

$$\begin{aligned} V(z^{-1} + \eta \circ R_\mu(z) - V^*XV)^{-1}V^* &= V(1 - z(V^*XV - \eta \circ R_\mu(z)))^{-1} zV^* \\ &= \sum_{k=0}^\infty V[z(V^*XV - \eta \circ R_\mu(z))]^k zV^*. \end{aligned}$$

When we expand $V[z(V^*XV - \eta \circ R_\mu(z))]^k$ by the distributive property, we get terms of the form V times a product of zV^*XV's and $z\eta \circ R_\mu(z)$'s. Applying the identity (3.1) repeatedly as $Vz\eta \circ R_\mu(z) = VzV^*R_\mu(z)V$ beginning with the leftmost occurrence of $z\eta \circ R_\mu(z)$ (which necessarily follows a V), we may replace all occurrences of $\eta \circ R_\mu(z)$ with $V^*R_\mu(z)V$ in the formula above. This yields

$$\begin{aligned} V(z^{-1} + \eta \circ R_\mu(z) - V^*XV)^{-1}V^* &= \sum_{k=0}^\infty V[z(V^*XV - \eta \circ R_\mu(z))]^k zV^* \\ &= \sum_{k=0}^\infty V[z(V^*XV - V^*R_\mu(z)V]^k zV^* \\ &= \sum_{k=0}^\infty [VzV^*(X - R_\mu(z))]^k VzV^* \\ &= [1 - VzV^*(X - R_\mu(z))]^{-1} VzV^*. \end{aligned}$$

This is the operator whose expectation we want to equal z. Now our goal is to expand this operator in terms of alternating strings of centered elements that are freely independent, which we know have expectation zero. Thus, we introduce a certain element from $C^*(\mathcal{B}, X)$ with expectation zero. Let

$$\begin{aligned}\Phi_X(z) &= [1 - z(X - R_\mu(z))]^{-1} - 1\\ &= [1 - z(X - R_\mu(z))]^{-1} - [1 - z(X - R_\mu(z))][1 - z(X - R_\mu(z))]^{-1}\\ &= z(X - R_\mu(z))[1 - z(X - R_\mu(z))]^{-1}.\end{aligned}$$

Note that $\Phi_X(z)$ and $z^{-1}\Phi_X(z) = (X - R_\mu(z))[1 - z(X - R_\mu(z))]^{-1}$ are well-defined in a neighborhood of zero. Moreover,

$$E[\Phi_X(z)] = E[(z^{-1} + R_\mu(z) - X)^{-1}z^{-1}] - 1 = G_\mu(z^{-1} + R_\mu(z))z^{-1} - 1 = 0.$$

We also note that $V^*zV - z$ has expectation zero by (3.2).

To express $[1 - VzV^*(X - R_\mu(z))]^{-1}VzV^*$ in terms of $\Phi_X(z)$, we first write

$$\begin{aligned}&1 - VzV^*(X - R_\mu(z))\\ &\quad= 1 - z(X - R_\mu(z)) - (VzV^* - z)(X - R_\mu(z))\\ &\quad= \left[1 - (VzV^* - z)(X - R_\mu(z))[1 - z(X - R_\mu(z))]^{-1}\right][1 - z(X - R_\mu(z))]\\ &\quad= \left[1 - (VzV^* - z)z^{-1}\Phi_X(z)\right][1 - z(X - R_\mu(z))].\end{aligned}$$

Hence,

$$\begin{aligned}&[1 - VzV^*(X - R_\mu(z))]^{-1}VzV^*\\ &\qquad= [1 - z(X - R_\mu(z))]^{-1}\left[1 - (VzV^* - z)z^{-1}\Phi_X(z)\right]^{-1}VzV^*\\ &\qquad= (1 + \Phi_X(z))\left[1 - (VzV^* - z)z^{-1}\Phi_X(z)\right]^{-1}[z + (VzV^* - z)].\end{aligned}$$

Then expanding the geometric series, we get

$$(1 + \Phi_X(z))\sum_{k=0}^{\infty}[(VzV^* - z)z^{-1}\Phi_X(z)]^k[z + (VzV^* - z)],$$

or equivalently

$$\sum_{k=0}^{\infty}[(VzV^* - z)z^{-1}\Phi_X(z)]^k z \tag{3.3}$$

$$+\sum_{k=0}^{\infty}[(VzV^* - z)z^{-1}\Phi_X(z)]^k(VzV^* - z) \tag{3.4}$$

$$+\sum_{k=0}^{\infty}\Phi_X(z)[(VzV^* - z)z^{-1}\Phi_X(z)]^k z \tag{3.5}$$

$$+\sum_{k=0}^{\infty}\Phi_X(z)[(VzV^* - z)z^{-1}\Phi_X(z)]^k(VzV^* - z). \tag{3.6}$$

In the first sum, the $k = 0$ term is equal to z. All the other terms in the four sums are alternating products of freely independent elements with expectation zero, so that freeness implies that they have expectation zero.

Hence,

$$G_\nu(z^{-1} + \eta \circ R_\mu(z)) = E[[1 - VzV^*(X - R_\mu(z))]^{-1} VzV^*] = z,$$

as desired. ■

4. Subordination and conditional expectations. Our next goal is to prove an analog of Biane's theorem on analytic subordination and conditional expectations for free additive convolution [10, Theorem 3.1]. For motivation, we first sketch the result in the scalar-valued case; here we assume that the non-commutative probability spaces in question are tracial von Neumann algebras and we denote by $\mathrm{W}^*(X_1, \dots, X_m)$ the von Neumann subalgebra generated by $X_1, \dots, X_n$. Biane showed that if X and Y are freely independent self-adjoint elements with distributions μ and ν respectively, then there is an analytic function $F : \mathbb{H} \to \mathbb{H}$ such that

$$G_{\mu \boxplus \nu}(z) = G_\mu \circ F(z),$$

but even better than that,

$$E_{\mathrm{W}^*(X)}[(z - X + Y)^{-1}] = (F(z) - X)^{-1},$$

where $E_{\mathrm{W}^*(X)}$ is the conditional expectation from $\mathrm{W}^*(X, Y)$ to $\mathrm{W}^*(X)$. Furthermore, Biane stated this equation in terms of a certain Markov kernel giving "transition probabilities from X to $X + Y$." In the von Neumann algebraic framework, a Markov kernel is equivalent to a unital completely positive map Φ from one von Neumann algebra to another. In this case, the mapping goes

$$\begin{aligned} \Phi : L^\infty(\mathrm{Spec}(X + Y), \mu \boxplus \nu) &\xrightarrow{\cong} \mathrm{W}^*(X + Y) \xrightarrow{\mathrm{incl}} \mathrm{W}^*(X, Y) \\ &\xrightarrow{E_{\mathrm{W}^*(X)}} \mathrm{W}^*(X) \xrightarrow{\cong} L^\infty(\mathrm{Spec}(X), \mu), \end{aligned}$$

where the second map is the inclusion and the third map is the conditional expectation. Biane's result says that the function $x \mapsto (z - x)^{-1}$ in $L^\infty(\mathrm{Spec}(X + Y), \mu \boxplus \nu)$ is sent by Φ to the function $x \mapsto (F(z) - x)^{-1}$ in $L^\infty(\mathrm{Spec}(X), \mu)$.

We pursue an analogous result in the setting of free convolution powers. First, consider the scalar-valued setting. Let $t \geq 1$, and let P be a projection of trace $1/t$ freely independent of X. The role of $X + Y$ in the story for additive convolution will now be played by $tPXP$ which is an element of the von Neumann algebra $P\mathrm{W}^*(X, P)P$, where the trace is given by $\tau_P(T) = \tau(PTP)/\tau(P) = t\tau(PTP)$. We then consider the completely positive map $\Phi : L^\infty(\mathrm{Spec}(tPXP), \mu^{\boxplus t}) \to L^\infty(\mathrm{Spec}(X), \mu))$ given as follows:

$$\begin{aligned} \Phi : L^\infty(\mathrm{Spec}(tPXP), \mu^{\boxplus t}) &\xrightarrow{\cong} \mathrm{W}^*(tPXP) \xrightarrow{\mathrm{incl}} P\mathrm{W}^*(X, P)P \\ &\xrightarrow{t\,\mathrm{incl}} \mathrm{W}^*(X, P) \xrightarrow{E_{\mathrm{W}^*(X)}} \mathrm{W}^*(X) \xrightarrow{\cong} L^\infty(\mathrm{Spec}(X), \mu). \end{aligned}$$

The multiplication by t in the middle map is necessary to make Φ unital; as further motivation, consider that t times the inclusion $P\mathrm{W}^*(X,P)P \to \mathrm{W}^*(X,P)$ is the adjoint of the compression $\mathrm{W}^*(X,P) \to P\mathrm{W}^*(X,P)P$ with respect to the GNS inner products associated to the trace. In this framework, the analog of Biane's result on subordination and conditional expectation would be

$$E_{\mathrm{W}^*(X)}[t(zP - tPXP)^{-1}] = (F(z) - X)^{-1},$$

where F is a function satisfying $G_{\mu^{\boxplus t}} = G_\mu \circ F$. Here we view P as the unit in $P\mathrm{W}^*(X,P)P$. We can write this equivalently as

$$E_{\mathrm{W}^*(X)}[tP(z - tPXP)^{-1}P] = (F(z) - X)^{-1}.$$

In the more general $\mathcal{B}$-valued setting of Theorem 3.1, we need to replace P by the "partial-isometry-like" operator V. Moreover, conditional expectations onto C*-subalgebras do not exist in general, and the partial isometry V often cannot be realized with respect to a conditional expectation E that preserves some trace by Remark 3.4. Therefore, we will restrict our attention to a $\mathcal{B}$-valued probability space $(\mathcal{A}, E)$ which is the free product of $(\mathcal{A}_1, E_1)$ and $(\mathcal{A}_2, E_2)$ since the free product has a canonical conditional expectation $\mathcal{E}_1 : \mathcal{A} \to \mathcal{A}_1$ described in Proposition 2.22 (this proposition holds without assuming that either of E_1 and E_2 is faithful or preserves any trace). The $\mathcal{B}$-valued subordination/conditional expectation relation that we want is thus

$$\mathcal{E}_1[V(z - V^*XV)^{-1}V^*] = (F(z) - X)^{-1}.$$

We will prove the existence of such an F using the same techniques as in the proof of Theorem 3.1, which again were inspired by Haagerup and Lehner's proof in the case of additive convolution (although it is also possible to give a more combinatorial proof along similar lines to Biane's argument for additive convolution [10, Proposition 3.2]). We remark that for operator-valued convolution powers, unlike the case of additive free convolution, the existence of a subordination function $F : \mathbb{H}(\mathcal{B}) \to \mathbb{H}(\mathcal{B})$ such that $G_\nu = G_\mu \circ F$ is immediate, since manipulation of functional identities shows that $F(z) = \eta^{-1}(z) + (\mathrm{id} - \eta^{-1})(F_\nu(z))$. However, the conditional expectation formula is not obvious from this fact. In any case, our proof below gives an independent argument for the existence of F satisfying $G_\nu = G_\mu \circ F$.

Finally, while we have presented the result for additive convolution powers in parallel with additive convolution, one could also view Theorem 4.1 as a subordination result for a special case of multiplicative convolution since the operator model is multiplicative in nature. For general results on subordination for multiplicative convolution, see [10, Theorems 3.5 and 3.6], [4, §3], and [7].

THEOREM 4.1. *Let $(\mathcal{A}_1, E_1)$ and $(\mathcal{A}_2, E_2)$ be $\mathcal{B}$-valued probability spaces and $(\mathcal{A}, E)$ their free product. Let $X \in \mathcal{A}_1$ be self-adjoint and let $V \in \mathcal{A}_2$*

satisfy (3.1) *and* (3.2). *Let* μ *be the* $\mathcal{B}$*-valued law of* X *and let* ν *be the* $\mathcal{B}$*-valued law of* V^*XV *with respect to the conditional expectation* $\tilde{E}[T] = E[VTV^*]$.

Then there exists a unique fully matricial function $F : \mathbb{H}(\mathcal{B}) \to \mathbb{H}(\mathcal{B})$ *such that* $G_\nu = G_\mu \circ F$, *and moreover* F *satisfies*

$$\mathcal{E}_1[V(z - V^*XV)^{-1}V^*] = (F(z) - X)^{-1} \quad \text{for } z \in \mathbb{H}(\mathcal{B})$$

where $\mathcal{E}_1 : \mathcal{A} \to \mathcal{A}_1$ *is the canonical conditional expectation from the free product construction* (*Proposition* 2.22).

Proof. As before, we omit the superscripts (n) for ease of notation. First, we show that for z in a neighborhood of zero, we have

$$\mathcal{E}_1[V(G_\nu^{-1}(z) - V^*XV)^{-1}V^*] = (G_\mu^{-1}(z) - X)^{-1}. \tag{4.1}$$

Recall that $G_\nu^{-1}(z) = z^{-1} + R_\nu(z) = z^{-1} + \eta \circ R_\mu(z)$. From the proof of Theorem 3.1,

$$V(G_\nu^{-1}(z) - V^*XV)^{-1}V^* = V(z^{-1} + \eta \circ R_\mu(z) - V^*XV)^{-1}V^*$$

can be expressed as the sum (3.3)–(3.6). Now the conditional expectation onto $\mathcal{A}_1$ of an alternating product of centered terms from $\mathcal{A}_1$ and $\mathcal{A}_2$ is zero unless the product has length zero, or the product has a single term from $\mathcal{A}_1$. Therefore, the only terms in (3.3)–(3.6) that survive under the conditional expectation are the $k = 0$ terms in (3.3) and (3.5). Thus,

$$\mathcal{E}_1[V(G_\nu^{-1}(z) - V^*XV)^{-1}V^*] = z + \Phi_X(z)z.$$

Recalling the definition of $\Phi_X(z)$, we obtain

$$\begin{aligned} z + \Phi_X(z)z &= (1 + \Phi_X(z))z = [1 - z(X - R_\mu(z))]^{-1}z \\ &= [z^{-1} + R_\mu(z) - X]^{-1} = (G_\mu^{-1}(z) - X)^{-1}. \end{aligned}$$

This shows (4.1). By substituting $G_\nu(z)$ instead of z into (4.1), we deduce that if z^{-1} is sufficiently small, then

$$\mathcal{E}_1[V(z - V^*XV)^{-1}V^*] = (G_\mu^{-1} \circ G_\nu(z) - X)^{-1}.$$

This is the identity that we want with $F = G_\mu^{-1} \circ G_\nu$.

Next, we must prove that F extends to all of $\mathbb{H}(\mathcal{B})$. Let $z \in \mathbb{H}(\mathcal{B})$. Then $\operatorname{im}(z) \geq \epsilon 1$ for some $\epsilon > 0$. Now $\operatorname{im}(z - V^*XV) = \operatorname{im}(z) \geq \epsilon 1$. Thus,

$$\begin{aligned} \operatorname{im}(z - V^*XV)^{-1} &= \operatorname{im}[(z - V^*XV)^{-1}(z^* - V^*XV)(z^* - V^*XV)^{-1}] \\ &= (z - V^*XV)^{-1} \operatorname{im}(z^* - V^*XV)(z^* - V^*XV)^{-1} \\ &= -(z - V^*XV)^{-1} \operatorname{im}(z)(z^* - V^*XV)^{-1} \\ &\leq -\epsilon(z - V^*XV)^{-1}(z^* - V^*XV)^{-1} \\ &= -\epsilon|(z - V^*XV)|^{-2} \leq -\epsilon\|z - V^*XV\|^{-2}1. \end{aligned}$$

Therefore,

$$\operatorname{im} \mathcal{E}_1[V(z - V^*XV)^{-1}V^*] \le -\epsilon \|z - V^*XV\|^{-2} \mathcal{E}_1[VV^*].$$

Since VV^* is freely independent of $\mathcal{A}_1$ in the free product, we get $\mathcal{E}_1[VV^*] = E[VV^*] = 1$. Thus,

$$\operatorname{im} \mathcal{E}_1[V(z - V^*XV)^{-1}V^*] \le -\epsilon \|z - V^*XV\|^{-2} \mathcal{E}_1[VV^*].$$

Hence, $\mathcal{E}_1[V(z - V^*XV)^{-1}V^*]$ has imaginary part less than a negative multiple of 1, and so by similar reasoning to one before, $\mathcal{E}_1[V(z - V^*XV)^{-1}V^*]$ is invertible in $\mathcal{L}(H_1)$ and its inverse has imaginary part bounded below by a positive multiple of the identity. Let $\widehat{F} : \mathbb{H}(\mathcal{B}) \to \mathbb{H}(\mathcal{A}_1)$ be given by

$$\widehat{F}(z) = (\mathcal{E}_1[V(z - V^*XV)^{-1}V^*])^{-1} + X.$$

We claim that $\widehat{F}$ defines a fully matricial function $\mathbb{H}(\mathcal{B}) \to \mathbb{H}(\mathcal{A}_1)$. Preservation of direct sums and similarity by scalar matrices in Definition 2.14(1)–(2) follows by straightforward algebraic computations since multiplication by $X^{(n)}$ and $V^{(n)}$ respects direct sums and similarities. The local boundedness condition, Definition 2.14(3), follows from the fact that $\widehat{F}(z)$ is uniformly bounded for all z with $\operatorname{im}(z) \ge \epsilon$ and $\|z\| \le R$ as a consequence of the estimates we made in the process of showing invertibility of $z - V^*XV$ and $\mathcal{E}_1[V(z - V^*XV)^{-1}V^*]$.

Now if z is invertible with $\|z^{-1}\|$ sufficiently small, we have $\widehat{F}(z) = F(z)$, so that $\widehat{F}(z)$ actually takes values in $\mathbb{H}(\mathcal{B})$. Hence, when z^{-1} is sufficiently small,

$$\widehat{F}(z) = E[\widehat{F}(z)].$$

By the identity theorem (see e.g. [14, Corollary 3.9.7]), this equality extends to all of $\mathbb{H}(\mathcal{B})$. Therefore, $\widehat{F}(z)$ always maps $\mathbb{H}(\mathcal{B})$ into $\mathbb{H}(\mathcal{B})$, and hence provides the desired extension of $F(z)$ to all of $\mathbb{H}(\mathcal{B})$. Moreover, since $G_\nu = G_\mu \circ F$ holds when $z \in \mathbb{H}(\mathcal{B})$ with $\|z^{-1}\|$ sufficiently small, again by the identity theorem this relation extends to all of $\mathbb{H}(\mathcal{B})$. Uniqueness of F follows again by the identity theorem because $F(z) = G_\mu^{-1} \circ G_\nu(z)$ when z^{-1} is sufficiently small. ∎

5. On convolution powers and sums. In the case where $\eta = n\,\mathrm{id}$, the free convolution power $\mu^{\boxplus\eta}$ is simply the n-fold free additive convolution $\mu \boxplus \cdots \boxplus \mu$. Thus, it is natural to ask how the construction of $\mu^{\boxplus n}$ in Theorem 3.1 relates to the construction of free additive convolution by adding copies of the original operator in a free product space. We will now explain the relationship through an explicit transformation of the underlying $\mathcal{B}$-$\mathcal{B}$-correspondences (Hilbert spaces in the scalar-valued case).

THEOREM 5.1. *Let X be an operator on the $\mathcal{B}$-$\mathcal{B}$-correspondence $\mathcal{H}$ with the expectation given by a unit vector ξ. Write $\mathcal{H} = \mathcal{B}\xi \oplus \mathcal{H}^\circ$. Consider the* *free product correspondence* $*_{j=1}^n(\mathcal{H}_j, \xi_j)$ *where* $(\mathcal{H}_j, \xi_j)$ *is a copy of* $(\mathcal{H}, \xi)$. *Let* X_j *be the* j*th copy of* X *acting on the free product.*

Let $\mathcal{K}$ *be the* $\mathcal{B}$-$\mathcal{B}$*-correspondence* $\mathcal{B}^{\oplus n}$, *and let* e_j *be the* j*th canonical basis vector. Let* $\mathcal{K}^\circ = 0 \oplus \mathcal{B}^{\oplus(n-1)}$ *which is the orthogonal complement of* $\mathcal{B}e_1$. *Let* $V \in \mathcal{L}(\mathcal{K})$ *be the operator*

$$V(b_1 \oplus \cdots \oplus b_n) = \left(\frac{b_1 + \cdots + b_n}{\sqrt{n}}\right)^{\oplus n}.$$

Form the free product correspondence $(\mathcal{K}, e_1) * (\mathcal{H}, \xi)$, *and view* V *and* X *as operators on* $(\mathcal{K}, e_1) * (\mathcal{H}, \xi)$.

There exists an isomorphism of $\mathcal{B}$-$\mathcal{B}$*-correspondences*

$$\Phi : \mathop{*}_{j=1}^{n}(\mathcal{H}_j, \xi_j) \to (V(\mathcal{H} * \mathcal{K}), V(\xi))$$

(*explicitly constructed in the proof below*) *such that*

$$\Phi \circ (X_1 + \cdots + X_n) = VXV \circ \Phi. \tag{5.1}$$

Remark 5.2. Note that according to Theorem 3.1, if $\mu^{\boxplus n\,\mathrm{id}}$ is the $\mathcal{B}$-valued law of X, then $\mu^{\oplus n}$ is the law of the operator VXV on the pointed $\mathcal{B}$-$\mathcal{B}$-correspondence $V[(\mathcal{H}, \xi) * (\mathcal{K}, e_1)]$, where the unit vector is given by V applied to the original unit vector. Meanwhile, $\mu^{\boxplus n}$ is also the free convolution of n copies of μ, and so it is the $\mathcal{B}$-valued law of the operator $X_1 + \cdots + X_n$ on $*_{j=1}^n(\mathcal{H}_j, \xi_j)$. Thus, VXV and $X_1 + \cdots + X_n$ must have the same $\mathcal{B}$-valued law, and Theorem 5.1 is asserting that this fact can be witnessed by an explicit spatial isomorphism.

Proof of Theorem 5.1. In other words, V is $\sqrt{n}$ times the projection onto the $\mathcal{B}\varepsilon$, where $\varepsilon = (1 \oplus \cdots \oplus 1)/\sqrt{n}$. We denote this projection by $P = V/\sqrt{n}$. Note that $V(e_j) = \varepsilon$ for $j = 1, \dots, n$.

Note that $(\mathcal{H}, \xi) * (\mathcal{K}, e_1)$ can be written as

$$(\mathcal{H}, \xi) * (\mathcal{K}, e_1) = \Big(\bigoplus_{k=0}^{\infty} \mathcal{K} \otimes_{\mathcal{B}} (\mathcal{H}^\circ \otimes_{\mathcal{B}} \mathcal{K}^\circ)^{\otimes_{\mathcal{B}} k} \otimes_{\mathcal{B}} \mathcal{H}, e_1 \otimes \xi\Big), \tag{5.2}$$

where $e_1 \otimes \xi$ is the vector in the $k = 0$ summand $\mathcal{K} \otimes \mathcal{H}$. Since $\mathcal{K}^\circ = \bigoplus_{j=2}^n \mathcal{B}e_j$, we can express the underlying correspondence as

$$\bigoplus_{k=0}^{\infty} \bigoplus_{j_1, \dots, j_k = 2, \dots, n} \mathcal{K} \otimes_{\mathcal{B}} (\mathcal{H}^\circ \otimes_{\mathcal{B}} \mathcal{B}e_{j_1}) \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} (\mathcal{H}^\circ \otimes_{\mathcal{B}} \mathcal{B}e_{j_k}) \otimes_{\mathcal{B}} \mathcal{H}.$$

Thus,

$$V[(\mathcal{H}, \xi) * (\mathcal{K}, e_1)] =$$

$$\Big(\bigoplus_{k=0}^{\infty} \bigoplus_{j_1, \dots, j_k = 2, \dots, n} \mathcal{B}\varepsilon \otimes_{\mathcal{B}} (\mathcal{H}^\circ \otimes_{\mathcal{B}} \mathcal{B}e_{j_1}) \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} (\mathcal{H}^\circ \otimes_{\mathcal{B}} \mathcal{B}e_{j_k}) \otimes_{\mathcal{B}} \mathcal{H},\ e_1 \otimes \xi\Big).$$

Meanwhile, we write

$$\mathop{*}_{j=1}^{n}(\mathcal{H}_j,\xi_j)=\Big(\bigoplus_{k=0}^{\infty}\bigoplus_{j_1\neq j_2\neq\cdots\neq j_k\neq 1}\mathcal{H}_{j_1}^{\circ}\otimes_{\mathcal{B}}\cdots\otimes_{\mathcal{B}}\mathcal{H}_{j_k}^{\circ}\otimes_{\mathcal{B}}\mathcal{H}_1,\xi_1\Big),$$

where ξ_1 is the vector in the $k=0$ summand $\mathcal{H}_1$. Let Φ be the map given on each of the summands in $*_{j=1}^{n}(\mathcal{H}_j,\xi_j)$ as follows:

$$\begin{aligned}
&\mathcal{H}_{j_1}^{\circ}\otimes_{\mathcal{B}}\cdots\otimes_{\mathcal{B}}\mathcal{H}_{j_k}^{\circ}\otimes_{\mathcal{B}}\mathcal{H}_1\\
&\qquad\cong\mathcal{H}^{\circ}\otimes_{\mathcal{B}}\cdots\otimes_{\mathcal{B}}\mathcal{H}^{\circ}\otimes_{\mathcal{B}}\mathcal{H}\\
&\qquad\cong\mathcal{B}\otimes_{\mathcal{B}}(\mathcal{H}^{\circ}\otimes_{\mathcal{B}}\mathcal{B})\otimes_{\mathcal{B}}\cdots\otimes_{\mathcal{B}}(\mathcal{H}^{\circ}\otimes_{\mathcal{B}}\mathcal{B})\otimes_{\mathcal{B}}\mathcal{H}\\
&\qquad\cong\mathcal{B}\varepsilon\otimes_{\mathcal{B}}(\mathcal{H}^{\circ}\otimes_{\mathcal{B}}\mathcal{B}e_{j_1-j_2+1})\otimes_{\mathcal{B}}\dots\\
&\qquad\quad\cdots\otimes_{\mathcal{B}}(\mathcal{H}^{\circ}\otimes_{\mathcal{B}}\mathcal{B}e_{j_{k-1}-j_k+1})\otimes_{\mathcal{B}}(\mathcal{H}^{\circ}\otimes_{\mathcal{B}}\mathcal{B}e_{j_k})\otimes_{\mathcal{B}}\mathcal{H},
\end{aligned}$$

where the indices are considered modulo n. Since the string is alternating, $j_{i-1}-j_i+1$ will never equal 1 modulo n, and this is why the term on the right-hand side is one of the summands occurring in $V[(\mathcal{H},\xi)*(\mathcal{K},e_1)]$. Note that $\Phi(\xi_1)=\varepsilon\otimes\xi$ as desired.

Now we prove the intertwining relation (5.1). Fix a vector

$$\theta=h_1\otimes\cdots\otimes h_k\otimes h_0\in\mathcal{H}_{j_1}^{\circ}\otimes_{\mathcal{B}}\cdots\otimes_{\mathcal{B}}\mathcal{H}_{j_k}^{\circ}\otimes_{\mathcal{B}}\mathcal{H}_1\subseteq\mathop{*}_{j=1}^{n}(\mathcal{H}_j,\xi_j).$$

Let us first consider the generic case where $k>0$. Viewing h_1 as an element of $\mathcal{H}$, write

$$X\xi=b\xi\oplus\zeta,\quad Xh_1=b'\oplus h_1'$$

where $b,b'\in\mathcal{B}$ and $\zeta,h_1'\in\mathcal{H}^{\circ}$. Then we have

$$X_{j_1}[h_1\otimes\cdots\otimes h_k\otimes h_0]=b'h_2\otimes\cdots\otimes h_k\otimes h_0+h_1'\otimes h_2\otimes\cdots\otimes h_k\otimes h_0.$$

Moreover, if $j\neq j_1$, then

$$\begin{aligned}
&X_j[h_1\otimes\cdots\otimes h_k\otimes h_0]\\
&\qquad=bh_1\oplus h_2\otimes\cdots\otimes h_k\otimes h_0+\zeta_j\otimes h_1\otimes h_2\otimes\cdots\otimes h_k\otimes h_0,
\end{aligned}$$

where ζ_j is the copy of ζ in $\mathcal{H}_j^{\circ}$. Thus,

$$\begin{aligned}
(X_1+\cdots+X_n)[\theta]=&\;b'h_2\otimes\cdots\otimes h_k\otimes h_0\\
&+h_1'\otimes h_2\otimes\cdots\otimes h_k\otimes h_0\\
&+(n-1)bh_1\otimes h_2\otimes\cdots\otimes h_k\otimes h_0\\
&+\sum_{j\neq j_1}\zeta_j\otimes h_1\otimes h_2\otimes\cdots\otimes h_k\otimes h_0.
\end{aligned}$$

Then applying the map Φ, we get

$$\begin{aligned}
&\Phi \circ (X_1 + \dots + X_n)[\theta] \\
&\quad = \varepsilon \otimes (b' h_2 \otimes e_{j_2 - j_3 + 1}) \otimes \dots \otimes (h_k \otimes e_{j_k}) \otimes h_0 \\
&\qquad + \varepsilon \otimes (h_1' \otimes e_{j_1 - j_2 + 1}) \otimes h_2 \otimes \dots \otimes (h_k \otimes e_{j_k}) \otimes h_0 \\
&\qquad + \varepsilon \otimes ((n-1) b h_1 \otimes e_{j_1 - j_2 + 1})) \otimes (h_2 \otimes e_{j_2 - j_3 + 1}) \otimes \dots \otimes (h_k \otimes e_{j_k}) \otimes h_0 \\
&\qquad + \sum_{j \neq j_1} \varepsilon \otimes (\zeta \otimes e_{j - j_1 + 1}) \otimes (h_1 \otimes e_{j_1 - j_2 + 1}) \otimes (h_2 \otimes e_{j_2 - j_3 + 1}) \otimes \dots \\
&\qquad\qquad \dots \otimes (h_k \otimes e_{j_k}) \otimes h_0.
\end{aligned}$$

Furthermore, we can reindex the sum over $j \neq j_1$ of $e_{j - j_1 + 1}$ as the sum over $j \neq 1$ of e_j.

Now on the other hand, we need to compute $VXV \circ \Phi$ on our original vector. First,

$$\Phi[h_1 \otimes \dots \otimes h_k \otimes h_0] = \varepsilon \otimes (h_1 \otimes e_{j_1 - j_2 + 1}) \otimes \dots \otimes (h_k \otimes e_{j_k}) \otimes h_0.$$

The vector is already in the image of V, so $V\Phi[\theta] = \sqrt{n}\Phi[\theta]$. Now we compute the application of the operator X, which is defined by decomposing the $\mathcal{B}$-$\mathcal{B}$-correspondence $(\mathcal{H}, \xi) * (\mathcal{K}, e_1)$ into a tensor product of $\mathcal{H}$ on the left with other terms. To this end, we write

$$\begin{aligned}
\sqrt{n}\Phi[\theta] &= \sum_{j=1}^n e_j \otimes (h_1 \otimes e_{j_1 - j_2 + 1}) \otimes \dots \otimes (h_k \otimes e_{j_k}) \otimes h_0 \\
&= (h_1 \otimes e_{j_1 - j_2 + 1}) \otimes \dots \otimes (h_k \otimes e_{j_k}) \otimes h_0 \\
&\quad + \sum_{j=2}^n e_j \otimes (h_1 \otimes e_{j_1 - j_2 + 1}) \otimes \dots \otimes (h_k \otimes e_{j_k}) \otimes h_0,
\end{aligned}$$

where we have separated out the e_1 term and written it without the e_1 because e_1 is the state vector. We apply the operator X and obtain

$$\begin{aligned}
XV\Phi[\theta] &= b' e_{j_1 - j_2 + 1} \otimes (h_2 \otimes e_{j_3 - j_2 + 1}) \otimes \dots \otimes (h_k \otimes e_{j_k}) \otimes h_0 \\
&\quad + (h_1' \otimes e_{j_1 - j_2 + 1}) \otimes \dots \otimes (h_k \otimes e_{j_k}) \otimes h_0 \\
&\quad + \sum_{j=2}^n b e_j \otimes (h_1 \otimes e_{j_1 - j_2 + 1}) \otimes \dots \otimes (h_k \otimes e_{j_k}) \otimes h_0 \\
&\quad + \sum_{j=2}^n (\zeta \otimes e_j) \otimes (h_1 \otimes e_{j_1 - j_2 + 1}) \otimes \dots \otimes (h_k \otimes e_{j_k}) \otimes h_0.
\end{aligned}$$

When we write this again in terms of the expansion (5.2), we tack on $e_1 \otimes$ to the front of the second and fourth lines. Then applying V again gives

$$\begin{aligned}
VXV\Phi[\theta] &= b' V(e_{j_1 - j_2 + 1}) \otimes (h_2 \otimes e_{j_3 - j_2 + 1}) \otimes \dots \otimes (h_k \otimes e_{j_k}) \otimes h_0 \\
&\quad + V(e_1) \otimes (h_1' \otimes e_{j_1 - j_2 + 1}) \otimes \dots \otimes (h_k \otimes e_{j_k}) \otimes h_0
\end{aligned}$$

$$+\sum_{j=2}^{n} bV(e_j)\otimes(h_1\otimes e_{j_1-j_2+1})\otimes\cdots\otimes(h_k\otimes e_{j_k})\otimes h_0$$
$$+\sum_{j=2}^{n} V(e_1)\otimes(\zeta\otimes e_j)\otimes(h_1\otimes e_{j_1-j_2+1})\otimes\cdots\otimes(h_k\otimes e_{j_k})\otimes h_0.$$

Since $V(e_j)=\varepsilon$ and $b\varepsilon\otimes\phi=\varepsilon b\otimes\phi=\varepsilon\otimes b\phi$ for vectors ϕ, our computation of $VXV\Phi[\theta]$ agrees with our previous expression for $\Phi\circ(X_1+\cdots+X_n)[\theta]$.

In the cases where $k=0$, we can further split into the case of a vector h_0 from $\mathcal{H}^\circ$ and a multiple of ξ. These arguments proceed in a similar way, and are left as an exercise for the reader. ■

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