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On modules of the Hardy space of the Hartogs triangle

by

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Abstract. We investigate the structure of doubly commuting submodules and quotient modules of the Hardy space $H^2(\triangle_H)$ over the Hartogs triangle. We establish a complete classification of doubly commuting submodules. In addition, we characterize all doubly commuting quotient modules of the form $(\theta_1(z/w)\theta_2(w)H^2(\triangle_H))^\perp$, where θ_1 and θ_2 are inner functions on the unit disc. This is achieved by introducing the concept of φ-doubly commuting quotient modules on the Hardy space $H^2(\mathbb{D}^2)$. We further explore the essential normality and double commutativity of quotient modules of the form $(pH^2(\triangle_H))^\perp$ under some mild assumptions on p, where p is a polynomial in two variables.

1. Introduction. The *Hartogs triangle*, denoted by $\triangle_H$, is defined as

$$\triangle_H := \{(z,w) \in \mathbb{C}^2 : |z| < |w| < 1\}.$$

It is a bounded and pseudoconvex domain. However, unlike the unit polydisc or the unit ball, it is neither monomially convex nor polynomially convex (see [26]). Consider the biholomorphism φ from $\triangle_H$ onto $\mathbb{D} \times \mathbb{D}^*$ given by the map

$$\varphi(z,w) = (\varphi_1(z,w), \varphi_2(z,w)) = (z/w, w), \qquad (z,w) \in \triangle_H. \tag{1.1}$$

Here $\mathbb{D}^* = \mathbb{D}\setminus\{0\}$. Note that the Jacobian J_φ of φ is given by $J_\varphi(z,w) = 1/w$ for all $(z,w) \in \triangle_H$. Furthermore, the inverse map φ^{-1} and the Jacobian $J_{\varphi^{-1}}$ of φ^{-1} are given by

$$\varphi^{-1}(z,w) = (zw, w), \quad J_{\varphi^{-1}}(z,w) = w, \quad (z,w) \in \mathbb{D} \times \mathbb{D}^*. \tag{1.2}$$

Recall from [21, Section 3] (see also [3, Section 7]) that the Hardy space $H^2(\triangle_H)$ on $\triangle_H$ is defined as

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$$H^2(\triangle_H) = \Big\{ f \in \mathrm{Hol}(\triangle_H) : \sup_{s,t\in(0,1)} \int_0^{2\pi}\int_0^{2\pi} |f(ste^{i\theta}, te^{i\gamma})|^2 \frac{st^2}{4\pi^2}\, d\theta\, d\gamma < \infty \Big\}.$$

Also, recall that the Hardy space $H^2(\mathbb{D}^2)$ on the unit bidisc $\mathbb{D}^2$ is defined as

$$H^2(\mathbb{D}^2) := \Big\{ f \in \mathrm{Hol}(\mathbb{D}^2) : \sup_{t\in[0,1)} \int_{\mathbb{T}^2} |f(tz_1, tz_2)|^2\, dm(z) < \infty \Big\},$$

where $\mathrm{d}m(z)$ is the normalized Lebesgue measure on $\mathbb{T}^2$. The space $H^2(\mathbb{D}^2)$ may be identified with the closed subspace $H^2(\mathbb{T}^2)$ of $L^2(\mathbb{T}^2)$ [23, Theorem 3.4.3].

The distinguished boundary $\partial_d\triangle_H$ of $\triangle_H$ is $\mathbb{T}^2$ [18, Remark 2.2]. It is shown in [21, Section 3] (see also [18, Proposition 2.4]) that the Hardy space $H^2(\triangle_H)$ is isometrically isomorphic to the closed subspace $H^2(\partial_d\triangle_H)$ of $L^2(\mathbb{T}^2)$ given by

$$H^2(\partial_d\triangle_H) := \Big\{ f(\zeta) = \sum_{\alpha\in\mathfrak{I}} a_\alpha \zeta^\alpha : \sum_{\alpha\in\mathfrak{I}} |a_\alpha|^2 < \infty \Big\},$$

where $\zeta = (\zeta_1, \zeta_2) \in \mathbb{T}^2$ and $\mathfrak{I} = \{\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2 : \alpha_1 \geq 0,\ \alpha_1+\alpha_2+1 \geq 0\}$.

We now recall the following relation from [21, Proposition 3.4] between the inner products of $H^2(\triangle_H)$ and $H^2(\mathbb{D}^2)$: for $f, g \in H^2(\triangle_H)$, we have

$$\langle f, g\rangle_{H^2(\triangle_H)} = \langle \Psi(f), \Psi(g)\rangle_{H^2(\mathbb{D}^2)}, \tag{1.3}$$

where the map $\Psi : H^2(\partial_d\triangle_H) \to H^2(\mathbb{T}^2)$ is unitary and given by

$$\Psi(f) = J_{\varphi^{-1}} \cdot f \circ \varphi^{-1}, \qquad f \in H^2(\partial_d\triangle_H). \tag{1.4}$$

Note that $\Psi^{-1} : H^2(\mathbb{T}^2) \to H^2(\partial_d\triangle_H)$ is given by

$$\Psi^{-1}(f) = J_\varphi \cdot f \circ \varphi, \qquad f \in H^2(\mathbb{T}^2).$$

Consider the closed subspace $H^2_+(\triangle_H)$ of $H^2(\triangle_H)$ defined by

$$H^2_+(\triangle_H) := \Big\{ f \in H^2(\triangle_H) : f(z,w) = \sum_{i,j\geq 0} a_{i,j} z^i w^j \Big\}. \tag{1.5}$$

By Abel's lemma [25, Lemma 1.5.8], the map $R : H^2(\mathbb{D}^2) \to H^2_+(\triangle_H)$ given by $R(f) = f|_{\triangle_H}$ is unitary. Hence, we may identify $H^2(\mathbb{D}^2)$ as a subspace of $H^2(\triangle_H)$ via the map R.

We denote by $H^\infty(\Omega)$ the Banach algebra of all bounded holomorphic functions on Ω, endowed with the sup norm, where $\Omega = \mathbb{D}^2$ or $\triangle_H$.

Remark 1.1. Note that $H^\infty(\mathbb{D}^2) \subsetneq H^\infty(\triangle_H)$, and moreover $\phi \in H^\infty(\triangle_H)$ if and only if $\phi \circ \varphi^{-1} \in H^\infty(\mathbb{D}^2)$ (see [18, Remark 3.1]). Since $\varphi^{-1}(\mathbb{T}^2) = \mathbb{T}^2$, $\phi \in H^\infty(\triangle_H)$ is inner if and only if $\phi \circ \varphi^{-1} \in H^\infty(\mathbb{D}^2)$ is inner (see [18]).

1.1. Notations and conventions. Let $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{N}$ denote the set of integers, nonnegative integers, and natural numbers, respectively. Let $\mathbb{D}$ and $\mathbb{T}$ denote the open unit disc and the unit circle in $\mathbb{C}$, respectively. For a bounded linear operator T on a Hilbert space $\mathcal{H}$, we denote its range and kernel by $\operatorname{ran}(T)$ and $\ker(T)$, respectively. For a closed subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$, the notation $\mathcal{H} \ominus \mathcal{M}$ denotes the orthogonal complement of $\mathcal{M}$ in $\mathcal{H}$. We denote by $\mathbb{C}[z,w]$ the ring of polynomials in the variables z and w with complex coefficients.

Let $\mathcal{H}$ be a Hilbert module [9] over $\mathbb{C}[z,w]$, and let M_z and M_w denote the module multiplication operators on $\mathcal{H}$. A closed subspace $\mathcal{S}$ (resp., $\mathcal{Q}$) of $\mathcal{H}$ is said to be a submodule (resp., quotient module) of $\mathcal{H}$ if $M_\theta(\mathcal{S}) \subseteq \mathcal{S}$ (resp., $M_\theta^*(\mathcal{Q}) \subseteq \mathcal{Q}$) for all $\theta \in \{z,w\}$. A submodule $\mathcal{S}$ (or a quotient module $\mathcal{Q}$) is said to be *nontrivial* if $\mathcal{S} \neq \{0\}$ (resp., $\mathcal{Q} \neq \{0\}$).

The module multiplication operators on a submodule $\mathcal{S}$ and a quotient module $\mathcal{Q}$ of a Hilbert module $\mathcal{H}$ are given by the restrictions (S_z, S_w) and compressions (Q_z, Q_w) of the module multiplication operators on $\mathcal{H}$, respectively:

$$S_z = M_z|_{\mathcal{S}}, \quad S_w = M_w|_{\mathcal{S}}, \quad Q_z = P_{\mathcal{Q}} M_z|_{\mathcal{Q}}, \quad Q_w = P_{\mathcal{Q}} M_w|_{\mathcal{Q}}.$$

Here and throughout, for any closed subspace $\mathcal{M}$ of a Hilbert space $\mathcal{H}$, we denote by $P_{\mathcal{M}}$ the orthogonal projection from $\mathcal{H}$ onto $\mathcal{M}$.

We further say that

- a submodule $\mathcal{S}$ (or a quotient module $\mathcal{Q}$) is *doubly commuting* if $[S_z^*, S_w] = 0$ (resp., $[Q_z^*, Q_w] = 0$);
- a quotient module $\mathcal{Q}$ is *essentially normal* if $[Q_\phi^*, Q_\psi]$ is compact for all $\phi, \psi \in \{z,w\}$.

Here $[\cdot,\cdot]$ denotes the commutator $[A,B] := AB - BA$ for bounded linear operators A, B on $\mathcal{H}$.

In this article, we primarily study two Hilbert modules, $H^2(\mathbb{D}^2)$ and $H^2(\triangle_H)$. The operators M_z, M_w (resp., M_1, M_2) denote multiplication by z and w on $H^2(\triangle_H)$ (resp., $H^2(\mathbb{D}^2)$), respectively. It is shown in [3, Proposition 8.3] that

$$M_1 M_2 = \Psi M_z \Psi^{-1} \quad \text{and} \quad M_2 = \Psi M_w \Psi^{-1}. \tag{1.6}$$

We use $\mathcal{S}$ and $\mathfrak{S}$ to denote submodules of $H^2(\triangle_H)$ and $H^2(\mathbb{D}^2)$, respectively, with $\mathcal{Q}$ and $\mathfrak{Q}$ representing the corresponding quotient modules. The module multiplication operators on the submodules of $H^2(\triangle_H)$ and $H^2(\mathbb{D}^2)$ are denoted by (S_z, S_w) and (S_1, S_2), respectively, while those on the corresponding quotient modules are denoted by (Q_z, Q_w) and (Q_1, Q_2).

1.2. Main results. The theory of invariant subspaces has remained a central area of investigation in operator theory for several decades. Classical results, most notably Beurling's theorem on the invariant subspaces of the Hardy space $H^2(\mathbb{D})$ [2], have laid the foundation for extensive developments in both the one-variable and multivariable settings. In several complex variables, the invariant subspace problem becomes significantly more intricate due to the absence of a full analog of Beurling's theorem. Nonetheless, substantial progress has been made in specific cases, such as doubly commuting shifts (see, for instance, [23, 20, 13] and references therein), and in the study of backward shift-invariant subspaces of $H^2(\mathbb{D}^n)$ (see, for instance, [10, 11, 17, 16, 24] and references therein). The essential normality of polynomially generated quotient modules of $H^2(\mathbb{D}^n)$ has been thoroughly studied; see, for example, [8, 15, 14, 6]. These directions continue to inspire the structural analysis of submodules and quotient modules of Hilbert spaces of analytic functions. While the literature on invariant subspaces and essential normality is vast, we restrict our attention to the works most relevant to our present considerations.

This work is devoted to the classification of doubly commuting submodules and a partial classification of doubly commuting quotient modules, with an analysis of the essentially normal quotient modules of the form $(pH^2(\triangle_H))^\perp$, where p is a polynomial. We now state the main results of the paper in the order they are proved:

Section 2: In [20, Theorem 2], Mandrekar showed that a nontrivial submodule $\mathfrak{S}$ of $H^2(\mathbb{D}^2)$ is doubly commuting if and only if $\mathfrak{S} = qH^2(\mathbb{D}^2)$ for some inner function q on $\mathbb{D}^2$. Motivated by this result, we prove the following for $H^2(\triangle_H)$.

Theorem 1.2. *A nontrivial submodule $\mathcal{S}$ of $H^2(\triangle_H)$ is doubly commuting if and only if $\mathcal{S} = qH^2_+(\triangle_H)$ for some unimodular function q on $\triangle_H$.*

As a consequence, the rank of a doubly commuting submodule of $H^2(\triangle_H)$ is 1 (Corollary 2.5). Furthermore, a submodule of the form $qH^2(\triangle_H)$ is never doubly commuting, where q is an inner function on $\triangle_H$ (Proposition 2.1).

Section 3: Izuchi, Nakazi and Seto [17, Theorem 2.1] provided a characterization of doubly commuting quotient modules of $H^2(\mathbb{D}^2)$ (see also [24, Corollary 3.3]). We prove the following for $H^2(\triangle_H)$:

Theorem 1.3. *For inner functions θ_1 and θ_2 on $\mathbb{D}$, the quotient module $H^2(\triangle_H) \ominus \theta_1(z/w)\theta_2(w)H^2(\triangle_H)$ is doubly commuting if and only if one of the following holds:*

(i) *θ_1 and θ_2 are constant functions.*
(ii) *θ_1 is constant and $\theta_2(w) = c\frac{w-b}{1-\bar{b}w}$ for some $b \in \mathbb{D}$ and $c \in \mathbb{T}$.*
(iii) *θ_2 is constant and $\theta_1(z) = c_1 z$ for some $c_1 \in \mathbb{T}$.*
(iv) *$\theta_1(z) = c_1 z$ and $\theta_2(w) = c_2 w$ for some $c_1, c_2 \in \mathbb{T}$.*

To achieve this, we introduce the notion of φ-doubly commuting quotient modules of $H^2(\mathbb{D}^2)$ (see Definition 3.4). We begin by providing a complete characterization of φ-doubly commuting quotient modules within the class of doubly commuting quotient modules (Theorem 3.8). This together with Corollary 3.9, Lemmas 3.12–3.14 and Theorem 3.15 yields a complete characterization of φ-doubly commuting quotient modules of the form $H^2(\mathbb{D}^2) \ominus \theta_1(z)\theta_2(w)H^2(\mathbb{D}^2)$, where θ_1 and θ_2 are inner functions on $\mathbb{D}$ (Theorem 3.11). We end this section by providing a class φ-doubly commuting quotient modules where the inner function is not separating (Proposition 3.16).

Section 4: Douglas and Misra [8] established by a direct calculation that $H^2(\mathbb{D}^2) \ominus (z-w)^2 H^2(\mathbb{D}^2)$ is essentially normal, while $H^2(\mathbb{D}^2) \ominus z^2 H^2(\mathbb{D}^2)$ is not. Later, Guo and Wang [15] provided a characterization of the homogeneous polynomials $p \in \mathbb{C}[z,w]$ for which $H^2(\mathbb{D}^2) \ominus pH^2(\mathbb{D}^2)$ is essentially normal (see also [5, 6, 14], and the references therein). In the following result, we completely characterize polynomials $p \in \mathbb{C}[z,w]$, under some assumptions on p, for which the associated quotient module is not essentially normal:

THEOREM 1.4. *Let p be a polynomial in the variables z and w represented as*

$$p(z,w) = az + bw + cw^2 + w^3 q(w) + z^2 r(z) + \sum_{i,j \geq 1} a_{i,j} z^i w^j,$$

where a, b, c are scalars, and $q(w)$ and $r(z)$ are single variable polynomials such that $\mathcal{Q}_p = H^2(\triangle_H) \ominus pH^2(\triangle_H)$ satisfies (4.11)*. Then $\mathcal{Q}_p$ is not essentially normal if and only if $a = b = 0$.*

To this end, we carry out a detailed analysis of the dimension of $\mathcal{Q}_p \cap \mathfrak{F}_m$ for $m \in \mathbb{Z}_+$ and $p \in \mathbb{C}[z,w]$, where $\mathfrak{F}_m$ denotes the span of the functions $\left\{\frac{1}{w}\left(\frac{z}{w}\right)^{m-j} w^j : 0 \leq j \leq m\right\}$ (see Section 4.1). We conclude the section by giving a class of doubly commuting quotient modules in this setting (Proposition 4.12).

The paper includes many examples and counterexamples, along with several supporting results that may be of independent interest. This paper is nearly self-contained.

2. Submodules: doubly commuting and properties. This section focuses on the study of doubly commuting submodules of $H^2(\triangle_H)$ and explores some of their applications. It is easy to see that $H^2(\mathbb{D}^2)$ is a doubly commuting submodule of itself (see, e.g., [20, Theorem 2]). In contrast, $H^2(\triangle_H)$ is not a doubly commuting submodule of itself. The following result shows that Beurling-type submodules of $H^2(\triangle_H)$ are never doubly commuting.

PROPOSITION 2.1. *For an inner function q on $\triangle_H$, $qH^2(\triangle_H)$ is never a doubly commuting submodule.*

Proof. Let q be an inner function on $\triangle_H$. Then $\mathcal{S} = qH^2(\triangle_H)$ is a submodule of $H^2(\triangle_H)$. Since q is inner, M_q is an isometry on $H^2(\triangle_H)$ (see [18, Corollary 4.4]) and hence $\{q\frac{1}{w}(\frac{z}{w})^i w^j : i,j \geq 0\}$ is an orthonormal basis of $\mathcal{S}$. Now, for any $g \in H^2(\triangle_H)$,

$$\left\langle S_w^* q\frac{1}{w}\left(\frac{z}{w}\right)^i w^j, qg\right\rangle = \left\langle P_{\mathcal{S}} M_w^* q\frac{1}{w}\left(\frac{z}{w}\right)^i w^j, qg\right\rangle$$
$$= \left\langle q\frac{1}{w}\left(\frac{z}{w}\right)^i w^j, qwg\right\rangle = \left\langle \frac{1}{w}\left(\frac{z}{w}\right)^i w^j, wg\right\rangle.$$

This implies that for all $i \geq 0$ and $j \geq 1$, $S_w^* q\frac{1}{w}(\frac{z}{w})^i = 0$, and

$$S_w^* q\frac{1}{w}\left(\frac{z}{w}\right)^i w^j = q\frac{1}{w}\left(\frac{z}{w}\right)^i w^{j-1}.$$

Thus

$$S_w^* S_z q\frac{1}{w}\left(\frac{z}{w}\right)^i = S_w^* q\frac{1}{w}\left(\frac{z}{w}\right)^{i+1} w = q\frac{1}{w}\left(\frac{z}{w}\right)^{i+1},$$

whereas $S_z S_w^* q\frac{1}{w}(\frac{z}{w})^i = 0$. Therefore, $\mathcal{S}$ is not doubly commuting. ■

We say that a function $\phi : \triangle_H \to \mathbb{C}$ is *unimodular* if $|\phi| = 1$ a.e. on $\partial_d(\triangle_H) = \mathbb{T}^2$. Note that not every unimodular Hardy function on $\triangle_H$ is inner: take for example $q(z,w) = 1/w$.

REMARK 2.2. For a unimodular $\phi \in H^2(\triangle_H)$, it is easy to see that the function $\Psi(\phi)$ is inner on $\mathbb{D}^2$ (see (1.4) for Ψ).

It is easy to see that $H_+^2(\triangle_H)$ is a doubly commuting submodule of $H^2(\triangle_H)$ (see (1.5)). Theorem 1.2 characterizes all doubly commuting submodules in terms of $H_+^2(\triangle_H)$. Its proof follows the approach outlined in [20]. Before proceeding, we note the following elementary lemma.

LEMMA 2.3. *For a unimodular function $q : \triangle_H \to \mathbb{C}$, the operator $M_q : H_+^2(\triangle_H) \to H^2(\triangle_H)$ is an isometry.*

Proof. Let $f, g \in H_+^2(\triangle_H)$. Then

$$\begin{aligned}
\langle M_q f, M_q g\rangle_{H^2(\triangle_H)} &\overset{(1.3)}{=} \langle \Psi(qf), \Psi(qg)\rangle_{H^2(\mathbb{D}^2)} \\
&= \langle \Psi(q)\cdot f\circ\varphi^{-1}, \Psi(q)\cdot g\circ\varphi^{-1}\rangle_{H^2(\mathbb{D}^2)} \\
&= \langle f\circ\varphi^{-1}, g\circ\varphi^{-1}\rangle_{H^2(\mathbb{D}^2)} \quad \text{(see Remark 2.2)} \\
&= \langle M_2 f\circ\varphi^{-1}, M_2 g\circ\varphi^{-1}\rangle_{H^2(\mathbb{D}^2)} \\
&= \langle \Psi(f), \Psi(g)\rangle_{H^2(\mathbb{D}^2)} = \langle f, g\rangle_{H^2(\triangle_H)}. \ ■
\end{aligned}$$

Proof of Theorem 1.2. Let $\mathcal{S} = qH_+^2(\triangle_H)$ for some unimodular function q on $\triangle_H$. We claim that S_z and S_w are doubly commuting on $\mathcal{S}$. Since q is unimodular, by Lemma 2.3, M_q is an isometry from $H^2_+(\triangle_H)$ to $H^2(\triangle_H)$. Hence $\{qz^iw^j : i, j \geq 0\}$ is an orthonormal basis of $\mathcal{S}$. Now for any $g \in H^2_+(\triangle_H)$,

$$\langle S_w^* qz^iw^j, qg\rangle = \langle M_w^* qz^iw^j, qg\rangle = \langle qz^iw^j, qwg\rangle = \langle z^iw^j, wg\rangle.$$

This implies that for all $i \geq 0$,

$$S_w^*(qz^iw^j) = \begin{cases} 0 & \text{if } j = 0, \\ qz^iw^{j-1} & \text{if } j \geq 1. \end{cases}$$

Now, for $i \geq 0$ and $j \geq 1$, we have

$$S_w^* S_z qz^i = 0 = S_z S_w^* qz^i \quad \text{and} \quad S_w^* S_z qz^iw^j = qz^{i+1}w^{j-1} = S_z S_w^* qz^iw^j.$$

Therefore, S_z and S_w are doubly commuting on $\mathcal{S}$.

Conversely, let S_z and S_w be doubly commuting shifts on $\mathcal{S}$. Then, in view of [27, Theorem 1], we get

$$\mathcal{S} = \bigoplus_{m,n=0}^{\infty} z^m w^n (\mathcal{W}_1 \cap \mathcal{W}_2), \tag{2.1}$$

where $\mathcal{W}_1 = \mathcal{S} \ominus z\mathcal{S}$ and $\mathcal{W}_2 = \mathcal{S} \ominus w\mathcal{S}$. Since S_z and S_w are doubly commuting shifts, from [27, Theorem 1, (i)⇔(ii)] and [27, Corollary 1], we further obtain $\mathcal{W}_1 \cap \mathcal{W}_2 \neq \{0\}$. We now claim that $\mathcal{W}_1 \cap \mathcal{W}_2$ is one-dimensional. It suffices to show that $\dim(\mathcal{W}_1 \cap \mathcal{W}_2) \leq 1$. Let $g_1, g_2 \in \mathcal{W}_1 \cap \mathcal{W}_2$, and define $\psi_1 = g_1 \circ \varphi^{-1}$ and $\psi_2 = g_2 \circ \varphi^{-1}$ (see (1.2) for φ^{-1}).

Since S_z and S_w are doubly commuting, by [19, Lemma 6.2] we have $S_z\mathcal{W}_2 \subseteq \mathcal{W}_2$ and $S_w\mathcal{W}_1 \subseteq \mathcal{W}_1$. This, together with (1.3), implies

$$\langle \Psi(z^m g_1), \Psi(w^n g_2)\rangle_{H^2(\mathbb{D}^2)} = 0 \quad \forall m \geq 0,\ n > 0,$$

which further reduces to

$$\int_{\mathbb{T}^2} z^m w^{m-n} \psi_1(z,w)\overline{\psi_2(z,w)}\, d\sigma(z,w) = 0 \quad \forall m \geq 0,\ n > 0, \tag{2.2}$$

where $d\sigma(z,w)$ denotes the normalized Lebesgue measure on $\mathbb{T}^2$. We also have

$$\langle \Psi(w^n g_1), \Psi(z^m g_2)\rangle_{H^2(\mathbb{D}^2)} = 0 \quad \forall m \geq 0,\ n > 0,$$

which is the same as saying

$$\int_{\mathbb{T}^2} z^m w^{m+n} \psi_1(z,w)\overline{\psi_2(z,w)}\, d\sigma(z,w) = 0 \quad \forall m \leq 0,\ n > 0. \tag{2.3}$$

We also note that (2.2) and (2.3) hold for $m > 0$, $n \geq 0$ and $m < 0$, $n \geq 0$, respectively. Moreover, since $g_2 \in \mathcal{W}_2$ and $z^m w^n g_1 \in w\mathcal{S}$ for all $m \geq 0$

and $n > 0$, it follows that

$$\int_{\mathbb{T}^2} z^m w^{m+n} \psi_1(z,w)\overline{\psi_2(z,w)}\, d\sigma(z,w) = 0 \quad \forall m \geq 0,\, n > 0. \tag{2.4}$$

Similarly, since $g_1 \in \mathcal{W}_2$ and $z^m w^n g_2 \in w\mathcal{S}$ for all $m \geq 0$ and $n > 0$, we obtain

$$\int_{\mathbb{T}^2} z^m w^{m+n} \psi_1(z,w)\overline{\psi_2(z,w)} d\sigma(z,w) = 0 \quad \forall m \leq 0,\, n < 0. \tag{2.5}$$

Consequently, from (2.2)–(2.5), we deduce that

$$\int_{\mathbb{T}^2} z^m w^n \psi_1(z,w)\overline{\psi_2(z,w)}\, d\sigma(z,w) = 0 \quad \forall m, n \neq (0,0). \tag{2.6}$$

This implies that $\psi_1\overline{\psi_2} = c$ a.e. on $\mathbb{T}^2$. Now by arguing similarly to the proof of [20, Theorem 2], we find that $\mathcal{W}_1 \cap \mathcal{W}_2$ is one-dimensional. Let q generate $\mathcal{W}_1 \cap \mathcal{W}_2$; one can choose $q \circ \varphi^{-1}$ to be unimodular on $\mathbb{T}^2$. Hence, $q \in H^2(\triangle_H)$ is also unimodular. Finally, (2.1) gives the result. ∎

We now present an alternative proof of Theorem 1.2.

Alternative proof of Theorem 1.2. Note that S_z and S_w are shifts on any submodule $\mathcal{S}$ of $H^2(\triangle_H)$. Now the result follows from [13, Corollary 4]. ∎

The following example shows that the function q is not required to be inner in Theorem 1.2, in contrast to the classification of doubly commuting submodules of $H^2(\mathbb{D}^2)$, where q must be inner (see [20, Theorem 2]).

Example 2.4. Consider the submodule $\mathcal{S} = \frac{1}{w} H^2_+(\triangle_H)$, whose orthonormal basis is given by $\{z^i w^j : i \geq 0, j \geq -1\}$. A routine calculation shows that $[S_z, S_w^*] = 0$. Moreover, $\mathcal{S}$ is an example of a doubly commuting submodule of $H^2(\triangle_H)$ that properly contains $H^2_+(\triangle_H)$.

In view of Theorem 1.2, we have the following.

Corollary 2.5. *The rank of a doubly commuting submodule of $H^2(\triangle_H)$ is* 1.

Let $T = (T_1, T_2)$ be a pair of commuting bounded linear operators on a Hilbert space $\mathcal{H}$. For a subset $E \subseteq \mathcal{H}$, we define $[E]_T$ to be the closed subspace

$$\overline{\operatorname{span}}\,\{T_1^{k_1} T_2^{k_2} E : k_1, k_2 \in \mathbb{Z}_+\}$$

of $\mathcal{H}$. The *rank* of T is defined to be

$$\operatorname{rank}(T) = \min\{\#E : [E]_T = \mathcal{H},\ E \subseteq \mathcal{H}\}.$$

Here, $\#E$ denotes the number of elements in E. Let $\mathcal{S}$ be a submodule of $H^2(\triangle_H)$. Then the *rank* of $\mathcal{S}$ [9, 16] is

$$\operatorname{rank}(\mathcal{S}) = \operatorname{rank}(S_z, S_w).$$

Note that Corollary 2.5 concerns the cyclic submodules of $H^2(\triangle_H)$. In contrast, [3, Proposition 5.1] shows that the pair (M_z, M_w) is not finitely cyclic on $H^2(\triangle_H)$.

Proof of Corollary 2.5. Let $\mathcal{S}$ be a doubly commuting submodule of $H^2(\triangle_H)$. Then by Theorem 1.2, $\mathcal{S} = qH^2_+(\triangle_H)$ for some unimodular function q on $\triangle_H$. Since

$$\mathcal{S} = \overline{\operatorname{span}}\,\{S_z^m S_w^n q : m, n \geq 0\},$$

the rank of $\mathcal{S}$ is 1. ∎

Note that $H^2(\mathbb{D}^2) \subset H^2(\triangle_H)$ (see (1.5)), and every submodule of $H^2(\mathbb{D}^2)$ is also a submodule of $H^2(\triangle_H)$. This leads to a natural question: Let $\mathcal{S}$ be a submodule of $H^2(\triangle_H)$. Is $\Psi(\mathcal{S})$ (see (1.4)) necessarily a submodule of $H^2(\mathbb{D}^2)$?

In general, the answer is negative, as shown by the following example.

Example 2.6. Consider the submodule

$$\mathcal{S} = \overline{\operatorname{span}}\left\{\frac{1}{w}\left(\frac{z}{w}\right)^i w^j : 0 \leq i \leq j\right\} \subset H^2(\triangle_H).$$

It is easy to see that $\Psi(\mathcal{S})$ is not invariant under M_1.

The following result characterizes when the image of a submodule of $H^2(\triangle_H)$ under Ψ is itself a submodule of $H^2(\mathbb{D}^2)$.

Proposition 2.7. *Let $\mathcal{S}$ be a submodule of $H^2(\triangle_H)$. Then $\Psi(\mathcal{S})$ is a submodule of $H^2(\mathbb{D}^2)$ if and only if $\mathcal{S}$ is invariant under M_{φ_1}, where φ_1 is given by* (1.1).

Proof. Clearly $\mathcal{S}$ is invariant under M_w if and only if $\Psi(\mathcal{S})$ is invariant under M_2. Now, $\Psi(\mathcal{S})$ is a submodule of $H^2(\mathbb{D}^2)$ if and only if $\Psi(\mathcal{S})$ is invariant under M_1, which holds if and only if $\mathcal{S}$ is invariant under M_{φ_1}. ∎

Proposition 2.8. *For a submodule $\mathfrak{S}$ of $H^2(\mathbb{D}^2)$, $\Psi^{-1}(\mathfrak{S})$ is always a submodule of $H^2(\triangle_H)$.*

Proof. Take $f \in \mathfrak{S}$. Then

$$z\Psi^{-1}(f) = z \cdot \frac{1}{w} \cdot f \circ \varphi = \Psi^{-1}(zwf) \in \Psi^{-1}(\mathfrak{S}),$$

and similarly

$$w\Psi^{-1}(f) = \Psi^{-1}(wf) \in \Psi^{-1}(\mathfrak{S}).$$

This shows that $\Psi^{-1}(\mathfrak{S})$ is a submodule of $H^2(\triangle_H)$. ∎

Corollary 2.9. *Let $\mathcal{S}$ be a doubly commuting submodule of $H^2(\triangle_H)$ that is invariant under M_{φ_1}. Then $\Psi(\mathcal{S})$ is a submodule of $H^2(\mathbb{D}^2)$ that is never doubly commuting.*

Proof. By Theorem 1.2, $\mathcal{S} = qH^2_+(\triangle_H)$ for some unimodular function q on $\triangle_H$. Note that

$$\Psi(\mathcal{S}) = \{w(q \circ \varphi^{-1}) \cdot f \circ \varphi^{-1} : f \in H^2_+(\triangle_H)\}.$$

Since $\mathcal{S}$ is invariant under M_{φ_1}, in view of Proposition 2.7, $\Psi(\mathcal{S})$ is a submodule of $H^2(\mathbb{D}^2)$. Now, assume, if possible, that $\Psi(\mathcal{S})$ is a doubly commuting submodule of $H^2(\mathbb{D}^2)$. Then, by [20, Theorem 2], we have

$$\Psi(\mathcal{S}) = \widetilde{q}H^2(\mathbb{D}^2)$$

for some inner function $\widetilde{q}$ on $\mathbb{D}^2$. Therefore, $\mathcal{S} = \widetilde{q} \circ \varphi H^2(\triangle_H)$, where $\widetilde{q} \circ \varphi$ is inner on $\triangle_H$. This leads to a contradiction (see Proposition 2.1). ∎

The rest of the section is devoted to exploring various properties of submodules of $H^2(\triangle_H)$.

In what follows, we show that the multiplication operators M_z and M_w on $H^2(\triangle_H)$ cannot have a common reducing subspace.

Proposition 2.10. *$H^2(\triangle_H)$ cannot be decomposed as a direct sum of two proper submodules.*

Proof. Suppose $H^2(\triangle_H) = \mathcal{S} \oplus \mathcal{S}^\perp$ for some submodule $\mathcal{S}$ such that $\mathcal{S}^\perp$ is also a submodule. It follows that $\mathcal{S}$ is a reducing subspace of $H^2(\triangle_H)$. Therefore, by [3, Corollary 6.3], the desired conclusion follows. ∎

Recall that two submodules $\mathcal{M}$ and $\mathcal{N}$ are said to have a *positive angle* (see [10, p. 219]) if

$$\sup\{|\langle f, g\rangle| : f \in \mathcal{M},\, g \in \mathcal{N},\, \|f\| = \|g\| = 1\} < 1.$$

We claim that no two submodules can have a positive angle. This follows from the following lemma, together with an argument analogous to that used in [10, Corollary 4.6].

Lemma 2.11. *Let $\mathcal{S}$ be a nontrivial submodule of $H^2(\triangle_H)$. Then the joint minimal unitary dilation of (S_z, S_w) is (N_z, N_w), where N_z and N_w are the operators of multiplication by z and w respectively on $L^2(\mathbb{T}^2)$.*

Proof. Define

$$\hat{\mathcal{S}} = \overline{\operatorname{span}}\{z^i w^j f : f \in \mathcal{S},\, i, j \in \mathbb{Z}\},$$

where the closure is taken in $L^2(\mathbb{T}^2)$. Then (N_z, N_w) on $\hat{\mathcal{S}}$ is the joint minimal unitary dilation of (S_z, S_w) acting on $\mathcal{S}$. Since $\hat{\mathcal{S}} \subseteq L^2(\mathbb{T}^2)$ and is invariant under multiplication by z, w, $\bar{z}$, and $\bar{w}$, it follows from [13, Lemma 3] that $\hat{\mathcal{S}} = 1_B L^2(\mathbb{T}^2)$ for some measurable subset B of $\mathbb{T}^2$. However, $\mathcal{S} \subseteq \hat{\mathcal{S}}$, and a nonzero function in $H^2(\partial_d(\triangle_H))$ cannot vanish on a subset of $\mathbb{T}^2$ of positive measure. Therefore, $\hat{\mathcal{S}} = L^2(\mathbb{T}^2)$. ∎

We conclude this section with a result of independent interest. Specifically, the result shows that for any nonzero submodule $\mathcal{S}$ of $H^2(\triangle_H)$, we have $\mathcal{S}\ominus z\mathcal{S} \not\perp \mathcal{S}\ominus w\mathcal{S}$ (cf. [7] for an analogous result in the polydisc setting).

PROPOSITION 2.12. *Let $\mathcal{S}$ be a submodule of $H^2(\triangle_H)$. Then $\mathcal{S} = \{0\}$ if and only if*

$$(I_{\mathcal{S}} - S_zS_z^*)(I_{\mathcal{S}} - S_wS_w^*) = 0.$$

Proof. Suppose $(I_{\mathcal{S}} - S_zS_z^*)(I_{\mathcal{S}} - S_wS_w^*) = 0$. Then

$$(I_{\mathcal{S}} - S_wS_w^*)\mathcal{S} \subset \ker(I_{\mathcal{S}} - S_zS_z^*),$$

which is the same as saying $\mathcal{S} \ominus w\mathcal{S} \subset z\mathcal{S}$. Consequently, $w^n(\mathcal{S} \ominus w\mathcal{S}) \subset z\mathcal{S}$ for all $n \geq 0$. From the Wold decomposition [28, 22], we have

$$\mathcal{S} = \bigoplus_{n=0}^{\infty} w^n(\mathcal{S} \ominus w\mathcal{S}),$$

and therefore $\mathcal{S} \subset z\mathcal{S}$, that is, $M_z^*\mathcal{S} \subset \mathcal{S}$. On the other hand, by taking adjoints in $(I_{\mathcal{S}} - S_zS_z^*)(I_{\mathcal{S}} - S_wS_w^*) = 0$ we further obtain $M_w^*\mathcal{S} \subset \mathcal{S}$. This shows that $\mathcal{S}$ is a joint reducing subspace. Thus by Proposition 2.10, we get $\mathcal{S} = \{0\}$ or $\mathcal{S} = H^2(\triangle_H)$. But the latter is not possible as

$$(I_{H^2(\triangle_H)} - M_zM_z^*)(I_{H^2(\triangle_H)} - M_wM_w^*)\left(\frac{1}{w}\left(\frac{z}{w}\right)\right) \neq 0. \blacksquare$$

3. Doubly commuting quotient modules of the form $(\theta H^2(\triangle_H))^{\perp}$. In this section, we initiate the study of doubly commuting quotient modules of $H^2(\triangle_H)$ of the form $\mathcal{Q}_\theta = H^2(\triangle_H) \ominus \theta H^2(\triangle_H)$, with θ inner on $\triangle_H$ (see [18, Definition 2.5]). We completely characterize when $\theta \circ \varphi^{-1}$ is a product of two one-variable inner functions. We also show that several existing results concerning quotient modules of $H^2(\mathbb{D}^2)$ do not hold in this context. The study of doubly commuting quotient modules of $H^2(\mathbb{D}^2)$ was initiated by Douglas and Yang [10, 11] (see also [1]). Izuchi, Nakazi, and Seto [17, Theorem 2.1] later classified the doubly commuting quotient modules of $H^2(\mathbb{D}^2)$. This result was subsequently generalized by Sarkar [24, Corollary 3.3], who provided a complete characterization of the doubly commuting quotient modules of $H^2(\mathbb{D}^n)$ for $n \geq 2$.

[17, Theorem 2.1] provides a complete characterization of doubly commuting quotient modules in $H^2(\mathbb{D}^2)$. However, this result does not extend to $H^2(\triangle_H)$. For instance, $H^2(\triangle_H) \ominus z^2H^2(\triangle_H)$ is not doubly commuting.

Let $\mathfrak{S}$ be a submodule of $H^2(\mathbb{D}^2)$. Then, by Proposition 2.8, $\Psi^{-1}(\mathfrak{S})$ is a submodule of $H^2(\triangle_H)$. Moreover,

$$\begin{aligned} f \in H^2(\triangle_H) \ominus \Psi^{-1}(\mathfrak{S}) &\iff \langle f, \Psi^{-1}(h)\rangle_{H^2(\triangle_H)} = 0,\ h \in \mathfrak{S} \\ &\iff \langle \Psi(f), h\rangle_{H^2(\mathbb{D}^2)} = 0,\ h \in \mathfrak{S}. \end{aligned}$$

Hence,

$$H^2(\triangle_H) \ominus \Psi^{-1}(\mathfrak{S}) = \Psi^{-1}(H^2(\mathbb{D}^2) \ominus \mathfrak{S}). \tag{3.1}$$

The following result is an analog of [10, Corollary 4.2] for $H^2(\triangle_H)$. To motivate it, we make a couple of remarks.

Remark 3.1. Consider the quotient modules $\mathfrak{Q} = H^2(\mathbb{D}^2) \ominus \mathfrak{S}$ and $\mathcal{Q} = H^2(\triangle_H) \ominus \mathcal{S}$, where $\mathfrak{S}$ and $\mathcal{S}$ are submodules of $H^2(\mathbb{D}^2)$ and $H^2(\triangle_H)$, respectively. Douglas and Yang [10, Corollary 4.2] proved that $\mathfrak{Q}$ is invariant under M_1 if and only if $\mathfrak{S} = \theta H^2(\mathbb{D}^2)$ for some inner function θ depending on w only. In this situation, $\mathfrak{Q}$ is doubly commuting. However, this fails for $\mathcal{Q}$, as explained below.

(1) Consider $\mathcal{S} = \theta H^2(\triangle_H)$, where $\theta(z, w) = w$. Then $\mathcal{Q}$ is doubly commuting, but fails to be invariant under M_z.
(2) Consider the submodule

$$\mathcal{S} = \overline{\operatorname{span}} \left\{ \frac{1}{w} \left(\frac{z}{w} \right)^m w^n : m, n \in \mathbb{Z}_+,\ m < n \right\} \subset H^2(\triangle_H).$$

Note that $\mathcal{Q} = \overline{\operatorname{span}} \left\{ \frac{1}{w} \left(\frac{z}{w} \right)^m w^n : m \geq n \geq 0 \right\}$. Clearly, $\mathcal{Q}$ is invariant under M_z. However, $\mathcal{Q}$ is not doubly commuting. Note also that $[S_z^*, S_w] = 0$ on $\mathcal{S}$, which implies that although $\mathcal{Q}$ is z-invariant, $\mathcal{S}$ is not of the form $\theta H^2(\triangle_H)$ for any inner function θ on $\triangle_H$ (see Proposition 2.1). ■

Proposition 3.2. *Let $\mathcal{Q}_\theta$ be a quotient module for some inner function θ on $\triangle_H$. Then $\mathcal{Q}_\theta$ is invariant under M_{φ_1} if and only if θ depends only on w (see (1.1) for φ_1).*

Proof. Let θ be an inner function that depends on w only. Let $f \in \mathcal{Q}_\theta$. First note that

$$\left\langle \frac{z}{w} f, \theta \frac{1}{w} w^j \right\rangle = 0 \quad \text{for all } j \geq 0.$$

Now for any $i \geq 1$ and $j \geq 0$,

$$\left\langle \frac{z}{w} f, \theta \frac{1}{w} \left(\frac{z}{w} \right)^i w^j \right\rangle = \left\langle f, \theta \frac{1}{w} \left(\frac{z}{w} \right)^{i-1} w^j \right\rangle = 0.$$

Hence, $\frac{z}{w} f \in \mathcal{Q}_\theta$, showing that $\mathcal{Q}_\theta$ is invariant under M_{φ_1}.

Conversely, let $\mathcal{Q}_\theta$ be invariant under M_{φ_1}. Arguing similarly to (3.1), we obtain

$$\Psi(\mathcal{Q}_\theta) = H^2(\mathbb{D}^2) \ominus \theta \circ \varphi^{-1} H^2(\mathbb{D}^2),$$

which is invariant under M_1. An application of Remark 1.1 and [10, Corollary 4.2] yields the conclusion. ■

We refer the reader again to Remark 3.1(2), which shows that even if $H^2(\triangle_H) \ominus \mathcal{S}$ is invariant under M_{φ_1}, the submodule $\mathcal{S}$ need not be of the form $\theta H^2(\triangle_H)$ for any inner function θ on $\triangle_H$. We now establish a result that plays a key role in our analysis.

LEMMA 3.3. *Let $\mathfrak{S}$ be a submodule of $H^2(\mathbb{D}^2)$. Then $\mathcal{Q} = H^2(\triangle_H) \ominus \Psi^{-1}(\mathfrak{S})$ is a quotient module of $H^2(\triangle_H)$ and*

$$Q_1Q_2 = \Psi Q_z \Psi^{-1} \quad \text{and} \quad Q_2 = \Psi Q_w \Psi^{-1}.$$

Proof. Let $\mathfrak{Q} = H^2(\mathbb{D}^2)\ominus\mathfrak{S}$. By Proposition 2.8, $\Psi^{-1}(\mathfrak{S})$ is a submodule of $H^2(\triangle_H)$. It follows from (3.1) that $\mathcal{Q} = \Psi^{-1}(\mathfrak{Q})$. For any $f \in H^2(\mathbb{D}^2)$, we have

$$\begin{aligned}\Psi P_{\mathcal{Q}}\Psi^{-1}(f) &= \Psi P_{\mathcal{Q}}\Psi^{-1}((I - P_{\mathfrak{Q}})f + P_{\mathfrak{Q}}f)\\ &= \Psi P_{\mathcal{Q}}\Psi^{-1}(P_{\mathfrak{Q}}f) = \Psi\Psi^{-1}(P_{\mathfrak{Q}}f) = P_{\mathfrak{Q}}f,\end{aligned}$$

where $P_{\mathcal{Q}} : H^2(\triangle_H) \to \mathcal{Q}$ and $P_{\mathfrak{Q}} : H^2(\mathbb{D}^2) \to \mathfrak{Q}$ are the orthogonal projections on $\mathcal{Q}$ and $\mathfrak{Q}$, respectively. Thus we have

$$\Psi P_{\mathcal{Q}}\Psi^{-1} = P_{\mathfrak{Q}}. \tag{3.2}$$

Now for any $f \in \mathfrak{Q}$, we have

$$\begin{aligned}\Psi Q_w\Psi^{-1}(f) &= \Psi P_{\mathcal{Q}}M_w\Psi^{-1}(f)\\ &\overset{(1.6)}{=} \Psi P_{\mathcal{Q}}\Psi^{-1}M_2\Psi\Psi^{-1}(f) \overset{(3.2)}{=} P_{\mathfrak{Q}}M_2f = Q_2f\end{aligned}$$

and

$$\begin{aligned}\Psi Q_z\Psi^{-1}(f) = \Psi P_{\mathcal{Q}}M_z\Psi^{-1}(f) &\overset{(1.6)}{=} \Psi P_{\mathcal{Q}}\Psi^{-1}M_1M_2\Psi\Psi^{-1}(f) \overset{(3.2)}{=} P_{\mathfrak{Q}}M_1M_2f\\ &= P_{\mathfrak{Q}}M_1M_2f - P_{\mathfrak{Q}}M_1P_{\mathfrak{S}}M_2f\\ &= P_{\mathfrak{Q}}M_1P_{\mathfrak{Q}}M_2f = Q_1Q_2f. \ \blacksquare\end{aligned}$$

Under the assumptions of Lemma 3.3, we have the following identity:

$$[Q_1Q_2, Q_2^*] = \Psi[Q_z, Q_w^*]\Psi^{-1}.$$

As a consequence, we obtain the equivalence

$$[Q_1Q_2, Q_2^*] = 0 \text{ on } \mathfrak{Q} \quad \text{if and only if} \quad [Q_z, Q_w^*] = 0 \text{ on } \Psi^{-1}(\mathfrak{Q}). \tag{3.3}$$

In view of (3.3), we investigate those quotient modules $\mathfrak{Q}$ of $H^2(\mathbb{D}^2)$ that satisfy

$$[Q_1Q_2, Q_2^*] = 0,$$

which further helps in characterizing certain doubly commuting quotient modules of $H^2(\triangle_H)$ (see Theorem 1.3). To this end, we introduce the notion of φ-doubly commuting quotient modules of $H^2(\mathbb{D}^2)$.

DEFINITION 3.4. A quotient module $\mathfrak{Q}$ of $H^2(\mathbb{D}^2)$ is said to be *φ-doubly commuting* if (Q_1Q_2, Q_2) is doubly commuting, that is,

$$Q_1Q_2Q_2^* = Q_2^*Q_1Q_2.$$

3.1. Characterization of φ-doubly commuting quotient modules of $H^2(\mathbb{D}^2)$. The Hardy space on the bidisc admits the tensor product decomposition

$$H^2(\mathbb{D}^2) \cong H^2(\mathbb{D}) \otimes H^2(\mathbb{D}),$$

with $z^m w^n$ identified as $z^m \otimes w^n$, and $H^2(\mathbb{D})$ denotes the Hardy space of the unit disc. Let T_z and T_w denote the multiplication operators on $H^2(\mathbb{D})$ corresponding to the variables z and w, respectively. Then the coordinate multiplication operators M_1 and M_2 on $H^2(\mathbb{D}^2)$ act as

$$M_1 = T_z \otimes I \quad \text{and} \quad M_2 = I \otimes T_w.$$

Next, consider a quotient module $\mathfrak{Q} = \mathfrak{Q}_1 \otimes \mathfrak{Q}_2$ of $H^2(\mathbb{D}^2)$, where $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are quotient modules of $H^2(\mathbb{D})$. Following [24, p. 3], the module multiplication operators on $\mathfrak{Q}$ are

$$Q_1 = P_{\mathfrak{Q}_1} T_z|_{\mathfrak{Q}_1} \otimes I_{\mathfrak{Q}_2} \quad \text{and} \quad Q_2 = I_{\mathfrak{Q}_1} \otimes P_{\mathfrak{Q}_2} T_w|_{\mathfrak{Q}_2}. \tag{3.4}$$

Then

$$\begin{aligned} [Q_2, Q_2^*] &= I_{\mathfrak{Q}_1} \otimes [P_{\mathfrak{Q}_2} T_w T_w^*|_{\mathfrak{Q}_2} - P_{\mathfrak{Q}_2} T_w^* P_{\mathfrak{Q}_2} T_w|_{\mathfrak{Q}_2}] \\ &= I_{\mathfrak{Q}_1} \otimes [P_{\mathfrak{Q}_2} T_w|_{\mathfrak{Q}_2}, T_w^*|_{\mathfrak{Q}_2}], \end{aligned} \tag{3.5}$$

which along with (3.4) gives

$$Q_1[Q_2, Q_2^*] = P_{\mathfrak{Q}_1} T_z|_{\mathfrak{Q}_1} \otimes [P_{\mathfrak{Q}_2} T_w|_{\mathfrak{Q}_2}, T_w^*|_{\mathfrak{Q}_2}]. \tag{3.6}$$

From now on, we will use (3.4)–(3.6), along with the usual definition of $H^2(\mathbb{D}^2)$, interchangeably wherever necessary.

LEMMA 3.5. *Let $\mathfrak{Q} = \mathfrak{Q}_1 \otimes \mathfrak{Q}_2$ be a nontrivial quotient module of $H^2(\mathbb{D}^2)$, where $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are quotient modules of $H^2(\mathbb{D})$. Then Q_2 is normal if and only if $\mathfrak{Q}_2$ is one-dimensional.*

Proof. Let Q_2 be normal. From (3.5), it follows that

$$[P_{\mathfrak{Q}_2} T_w|_{\mathfrak{Q}_2}, T_w^*|_{\mathfrak{Q}_2}] = 0,$$

which due to [6, Lemma 2.4] implies that $\mathfrak{Q}_2$ is one-dimensional.

Conversely, let $\mathfrak{Q}_2$ be one-dimensional. Then, by [6, Lemma 2.4] and (3.5), Q_2 is normal. ∎

For $a \in \mathbb{D}$, let f_a denote the automorphism of $\mathbb{D}$ given by

$$f_a(z) = c\frac{z-a}{1-\bar{a}z}, \qquad z \in \mathbb{D},\ c \in \mathbb{T}. \tag{3.7}$$

For future use, define

$$k_a(z,w) = \frac{1}{1-\bar{a}z} \quad \text{and} \quad m_a(z,w) = \frac{1}{1-\bar{a}w}, \qquad (z,w) \in \mathbb{D}^2, \tag{3.8}$$

where $a \in \mathbb{D}$.

If $\mathfrak{Q}$ is a doubly commuting quotient module of $H^2(\mathbb{D}^2)$, then

$$[Q_1Q_2, Q_2^*] = Q_1Q_2Q_2^* - Q_2^*Q_1Q_2 = Q_1[Q_2, Q_2^*]. \tag{3.9}$$

Note that if $[Q_2, Q_2^*] = 0$ or $\operatorname{ran}[Q_2, Q_2^*] \subseteq \ker(Q_1)$, then by (3.9), $\mathfrak{Q}$ is φ-doubly commuting. The following examples illustrate that both conditions can occur.

EXAMPLE 3.6. Consider the submodule $\mathfrak{S}_1$ of $H^2(\mathbb{D}^2)$ given by

$$\overline{\operatorname{span}}\,\{z^iw^j : i, j \geq 0,\ i + j \geq 1\}.$$

Then $\mathfrak{Q}_1 = H^2(\mathbb{D}^2) \ominus \mathfrak{S}_1$ is spanned by 1 and is doubly commuting. Moreover, $[Q_2, Q_2^*] = 0$. Further, consider the submodule $\mathfrak{S}_2 = zH^2(\mathbb{D}^2)$. Then $\mathfrak{Q}_2 = \overline{\operatorname{span}}\,\{w^n : n \geq 0\}$. Note that $\mathfrak{Q}_2$ is doubly commuting and $[Q_2, Q_2^*]1 \neq 0$, but $\ker(Q_1) = \mathfrak{Q}_2$.

In what follows, we characterize the φ-doubly commuting quotient modules of $H^2(\mathbb{D}^2)$ among all doubly commuting quotient modules of $H^2(\mathbb{D}^2)$. We begin with an example of a doubly commuting quotient module which is not φ-doubly commuting.

EXAMPLE 3.7. Let $\mathfrak{Q} = H^2(\mathbb{D}^2) \ominus f_a(z)H^2(\mathbb{D}^2)$ for some $a \in \mathbb{D}$ and $a \neq 0$. By [17, Theorem 2.1], $\mathfrak{Q}$ is doubly commuting. Note that $k_a \in \mathfrak{Q}$ (see (3.8)). By [4, Lemma 12], it follows that $wk_a \in \mathfrak{Q}$. Now,

$$[Q_1Q_2, Q_2^*]k_a \overset{(3.9)}{=} (Q_1Q_2Q_2^* - Q_1Q_2^*Q_2)k_a = -Q_1Q_2^*Q_2k_a = -Q_1k_a,$$

which is nonzero since $a \neq 0$. Hence, $\mathfrak{Q}$ is not φ-doubly commuting.

THEOREM 3.8. *Let $\mathfrak{Q} = \mathfrak{Q}_1 \otimes \mathfrak{Q}_2$ be a nontrivial doubly commuting quotient module of $H^2(\mathbb{D}^2)$. Then $\mathfrak{Q}$ is φ-doubly commuting if and only if one of the following holds:*

(i) $\mathfrak{Q}_1 = H^2(\mathbb{D})$ *and* $\mathfrak{Q}_2 = H^2(\mathbb{D}) \ominus f_a(w)H^2(\mathbb{D})$,
(ii) $\mathfrak{Q}_1 = H^2(\mathbb{D}) \ominus f_0(z)H^2(\mathbb{D})$ *and* $\mathfrak{Q}_2 = H^2(\mathbb{D})$,
(iii) $\mathfrak{Q}_1 = (H^2(\mathbb{D}) \ominus \theta_1(z)H^2(\mathbb{D}))$ *and* $\mathfrak{Q}_2 = (H^2(\mathbb{D}) \ominus \theta_2(w)H^2(\mathbb{D}))$, *where either $\theta_1(z) = f_0(z)$ and θ_2 is an arbitrary inner function, or θ_1 is arbitrary and $\theta_2(w) = f_a(w)$ for some $a \in \mathbb{D}$.*

Proof. We prove only the necessity part, as the sufficiency part is immediate. Let $\mathfrak{Q}$ be a nontrivial doubly commuting quotient module of $H^2(\mathbb{D}^2)$. From [17, Theorem 2.1] (see also, [24, Corollary 3.3]), one of the following four cases occurs:

(a) $\mathfrak{Q} = H^2(\mathbb{D}) \otimes H^2(\mathbb{D})$,
(b) $\mathfrak{Q} = H^2(\mathbb{D}) \otimes (H^2(\mathbb{D}) \ominus \theta_2(w)H^2(\mathbb{D}))$,
(c) $\mathfrak{Q} = (H^2(\mathbb{D}) \ominus \theta_1(z)H^2(\mathbb{D})) \otimes H^2(\mathbb{D})$,
(d) $\mathfrak{Q} = (H^2(\mathbb{D}) \ominus \theta_1(z)H^2(\mathbb{D})) \otimes (H^2(\mathbb{D}) \ominus \theta_2(w)H^2(\mathbb{D}))$,

where θ_1 and θ_2 are nonconstant inner functions on $\mathbb{D}$. Note that for (a), we have

$$[Q_1Q_2, Q_2^*](1 \otimes 1) \neq 0.$$

Hence $\mathfrak{Q} = H^2(\mathbb{D}) \otimes H^2(\mathbb{D})$ is not φ-doubly commuting.

We now consider the remaining cases. Let $\mathfrak{Q}$ be φ-doubly commuting (see Definition 3.4).

Case (b): $\mathfrak{Q} = H^2(\mathbb{D}) \otimes (H^2(\mathbb{D}) \ominus \theta_2(w)H^2(\mathbb{D}))$. Then $\mathfrak{Q}$ is invariant under M_1. This yields $\ker(Q_1) = \{0\}$. Since $\mathfrak{Q}$ is φ-doubly commuting, by (3.9) and (3.6),

$$Q_1[Q_2, Q_2^*](1 \otimes f) = 0 \quad \text{yields} \quad [P_{\mathfrak{Q}_2}T_w|_{\mathfrak{Q}_2}, T_w^*|_{\mathfrak{Q}_2}]f = 0 \quad \text{for all } f \in \mathfrak{Q}_2.$$

Consequently, $P_{\mathfrak{Q}_2}T_w|_{\mathfrak{Q}_2}$ is normal on $\mathfrak{Q}_2$. This together with (3.5) shows that Q_2 is normal. Therefore, by Lemma 3.5, $H^2(\mathbb{D}) \ominus \theta_2(w)H^2(\mathbb{D})$ is one-dimensional. Furthermore, by [12, Propositions 5.19 and 5.16], it follows that $\theta_2 = f_a$ for some $a \in \mathbb{D}$ (see (3.7)).

Case (c): $\mathfrak{Q} = (H^2(\mathbb{D}) \ominus \theta_1(z)H^2(\mathbb{D})) \otimes H^2(\mathbb{D})$. Then $\mathfrak{Q}$ is invariant under M_2. Thus, using (3.5), we get

$$[Q_2, Q_2^*] = I_{\mathfrak{Q}_1} \otimes [T_w, T_w^*].$$

It follows that for any $f \otimes 1 \in \mathfrak{Q}$, we have

$$Q_1[Q_2, Q_2^*](f \otimes 1) = Q_1(I_{\mathfrak{Q}_1} \otimes [T_w, T_w^*])(f \otimes 1) = Q_1(f \otimes -1). \tag{3.10}$$

Since $\mathfrak{Q}$ is φ-doubly commuting, it follows from (3.10) that $P_{\mathfrak{Q}_1}T_z|_{\mathfrak{Q}_1}$ is a zero operator on $H^2(\mathbb{D}) \ominus \theta_1(z)H^2(\mathbb{D})$. This with [12, Proposition 9.13] implies that $H^2(\mathbb{D}) \ominus \theta_1(z)H^2(\mathbb{D})$ is at most one-dimensional. Since θ_1 is nonconstant, it follows from [12, Proposition 5.19] that $\theta_1 = f_a$ for some $a \in \mathbb{D}$. Note that, if $a \neq 0$, then Example 3.7 shows that $\mathfrak{Q}$ is not φ-doubly commuting. Hence $\theta_1 = f_0$.

Case (d): $\mathfrak{Q} = (H^2(\mathbb{D})\ominus\theta_1(z)H^2(\mathbb{D}))\otimes(H^2(\mathbb{D})\ominus\theta_2(w)H^2(\mathbb{D}))$. Now, the φ-double commutativity of $\mathfrak{Q}$ and (3.6) implies that for each $f\otimes g \in \mathfrak{Q}_1\otimes\mathfrak{Q}_2$,

$$P_{\mathfrak{Q}_1}T_z f = 0 \quad \text{or} \quad [P_{\mathfrak{Q}_2}T_w|_{\mathfrak{Q}_2}, T_w^*|_{\mathfrak{Q}_2}]g = 0.$$

If $[P_{\mathfrak{Q}_2}T_w|_{\mathfrak{Q}_2}, T_w^*|_{\mathfrak{Q}_2}] = 0$ on $\mathfrak{Q}_2$, then it follows from Lemma 3.5 that $\theta_2 = f_a$ for some $a \in \mathbb{D}$. In this case, θ_1 can be any inner function. We now assume that there exists $\tilde{g} \in \mathfrak{Q}_2$ such that $[P_{\mathfrak{Q}_2}T_w|_{\mathfrak{Q}_2}, T_w^*|_{\mathfrak{Q}_2}]\tilde{g} \neq 0$. Then

$$Q_1[Q_2, Q_2^*](f \otimes \tilde{g}) = P_{\mathfrak{Q}_1}T_z f \otimes [P_{\mathfrak{Q}_2}T_w|_{\mathfrak{Q}_2}, T_w^*|_{\mathfrak{Q}_2}]\tilde{g} = 0$$

implies that $P_{\mathfrak{Q}_1}T_z|_{\mathfrak{Q}_1}$ is a zero operator on $\mathfrak{Q}_1 = H^2(\mathbb{D})\ominus\theta_1(z)H^2(\mathbb{D})$. Now, by following the arguments used in Case (c), we obtain $\theta_1 = f_0$. In this case, the inner function θ_2 is arbitrary. ∎

Recall that an inner function on $\mathbb{D}$ is called *singular* if it has no zeros in $\mathbb{D}$. The following is an application of Theorem 3.8.

COROLLARY 3.9. *Let S_1 and S_2 be singular inner functions on $\mathbb{D}$. Then $\mathfrak{Q} = H^2(\mathbb{D}^2) \ominus S_1(z)S_2(w)H^2(\mathbb{D}^2)$ is φ-doubly commuting if and only if S_1 and S_2 are both constant.*

Proof. Suppose $\mathfrak{Q}$ is φ-doubly commuting. If possible, suppose S_1 is nonconstant. Consider $f(z,w) = Q_1^*(S_1(z) - S_1(0))$. Since S_1 is nonconstant, $f \neq 0$. Moreover, $f \in \mathfrak{Q}$. Note that $Q_1Q_2Q_2^*f(z,w) = 0$. On the other hand,

$$Q_2^*Q_2Q_1f(z,w) = Q_2^*Q_2P_{\mathfrak{Q}}[S_1(z) - S_1(0)] = S_1(z) - S_1(0) \neq 0,$$

since S_1 is nonconstant. We arrive at a contradiction (see Definition 3.4).

Now, assume if possible that S_1 is constant and S_2 is nonconstant. Then $\mathfrak{Q} = H^2(\mathbb{D}^2) \ominus S_2(w)H^2(\mathbb{D}^2)$ is doubly commuting, which is not φ-doubly commuting by Theorem 3.8. ∎

The following example illustrates that not all φ-doubly commuting quotient modules of $H^2(\mathbb{D}^2)$ are doubly commuting.

EXAMPLE 3.10. Let $\mathfrak{S} = \overline{\operatorname{span}}\,\{z^iw^j : i,j \geq 0,\ i+j \geq 2\}$. Then $\mathfrak{S}$ is a submodule of $H^2(\mathbb{D}^2)$. Moreover, the corresponding quotient module $\mathfrak{Q}$ is $\overline{\operatorname{span}}\,\{1,z,w\}$. It is easy to see that $\mathfrak{Q}$ is φ-doubly commuting. However, $\mathfrak{Q}$ is not doubly commuting.

The following result characterizes φ-doubly commuting quotient modules of the form $H^2(\mathbb{D}^2) \ominus \theta_1(z)\theta_2(w)H^2(\mathbb{D}^2)$, where θ_1 and θ_2 are inner functions on $\mathbb{D}$. These quotient modules are not doubly commuting (see [17, Theorem 2.1]) when θ_1 and θ_2 are nonconstant inner functions.

THEOREM 3.11. *For inner functions θ_1 and θ_2 on $\mathbb{D}$, the quotient module $\mathfrak{Q} = H^2(\mathbb{D}^2) \ominus \theta_1(z)\theta_2(w)H^2(\mathbb{D}^2)$ is φ-doubly commuting if and only if one of the following holds:*

(i) *θ_1 and θ_2 are constant functions.*
(ii) *θ_1 is constant and $\theta_2 = f_b$ for some $b \in \mathbb{D}$.*
(iii) *θ_2 is constant and $\theta_1(z) = c_1z$ for some $c_1 \in \mathbb{T}$.*
(iv) *$\theta_1(z) = c_1z$ and $\theta_2(w) = c_2w$ for some $c_1, c_2 \in \mathbb{T}$.*

We begin by proving a couple of technical lemmas that will play a key role in the characterization.

LEMMA 3.12. *Let $\mathfrak{Q} = H^2(\mathbb{D}^2) \ominus \theta_1(z)\theta_2(w)H^2(\mathbb{D}^2)$, where θ_1 and θ_2 are inner functions on $\mathbb{D}$ such that $\theta_1(a) = 0$ for some $a \in \mathbb{D}$, with $a \neq 0$. Then $\mathfrak{Q}$ is not φ-doubly commuting.*

Proof. Since $\theta_1(a) = 0$, there exists an inner function θ_3 on $\mathbb{D}$ such that $\theta_1 = f_a\theta_3$ (see (3.7)). We now consider several cases for θ_2.

CASE (i): *θ_2 is a constant function.* Consider $f(z,w) = k_a(z,w)\theta_3(z)$ (see (3.8)). It is easy to see that $f \in \mathfrak{Q}$. Note that $Q_1Q_2Q_2^*f(z,w) = 0$, whereas

$$\begin{aligned} Q_2^*Q_1Q_2f(z,w) &= Q_2^*Q_1Q_2(k_a(z,w)\theta_3(z)) = Q_2^*P_{\mathfrak{Q}}(zwk_a(z,w)\theta_3(z)) \\ &= ak_a(z,w)\theta_3(z)) \neq 0, \end{aligned}$$

since $a \neq 0$. Hence $\mathfrak{Q}$ is not φ-doubly commuting.

Case (ii): *θ_2 is nonconstant and $\theta_2(0) \neq 0$.* Consider the function $f(z,w) = k_a(z,w)\theta_3(z)\theta_2(w)$. It is evident that $f \in \mathfrak{Q}$. Now

$$\begin{aligned} Q_2^*Q_1Q_2f(z,w) &= Q_2^*Q_1Q_2(k_a(z,w)\theta_3(z)\theta_2(w)) \\ &= Q_2^*P_{\mathfrak{Q}}(zwk_a(z,w)\theta_3(z)\theta_2(w)) \\ &= ak_a(z,w)\theta_3(z)\theta_2(w). \end{aligned}$$

On the other hand,

$$\begin{aligned} Q_1Q_2Q_2^*f(z,w) &= Q_1Q_2Q_2^*(k_a(z,w)\theta_3(z)\theta_2(w)) \\ &= Q_1(k_a(z,w)\theta_3(z)\theta_2(w) - k_a(z,w)\theta_3(z)\theta_2(0)) \\ &= P_{\mathfrak{Q}}(zk_a(z,w)\theta_3(z)\theta_2(w) - zk_a(z,w)\theta_3(z)\theta_2(0)) \\ &= ak_a(z,w)\theta_3(z)\theta_2(w) - P_{\mathfrak{Q}}(zk_a(z,w)\theta_3(z)\theta_2(0)). \end{aligned}$$

Thus

$$Q_2^*Q_1Q_2f(z,w) - Q_1Q_2Q_2^*f(z,w) = \theta_2(0)P_{\mathfrak{Q}}(zk_a(z,w)\theta_3(z)).$$

Since $\theta_2(0) \neq 0$ and $zk_a(z,w)\theta_3(z) \notin \theta_1(z)\theta_2(w)H^2(\mathbb{D}^2)$, the quotient module $\mathfrak{Q}$ is not φ-doubly commuting.

Case (iii): *θ_2 is nonconstant and there exist $n \in \mathbb{N}$ and an inner function θ_4 on $\mathbb{D}$ such that $\theta_2(w) = w^n\theta_4(w)$ and $\theta_4(0) \neq 0$.* It is easy to see that $k_a(z,w)\theta_3(z)\theta_4(w), wk_a(z,w)\theta_3(z)\theta_4(w)$ are in $\mathfrak{Q}$. Now,

$$Q_2^*Q_1Q_2k_a(z,w)\theta_3(z)\theta_4(w) = \begin{cases} ak_a(z,w)\theta_3(z)\theta_4(w) & \text{if } n = 1, \\ zk_a(z,w)\theta_3(z)\theta_4(w) & \text{if } n \geq 2. \end{cases} \tag{3.11}$$

We have

$$\begin{aligned} &Q_1Q_2Q_2^*k_a(z,w)\theta_3(z)\theta_4(w) \\ &\qquad = Q_1P_{\mathfrak{Q}}(k_a(z,w)\theta_3(z)\theta_4(w) - k_a(z,w)\theta_3(z)\theta_4(0)) \\ &\qquad = zk_a(z,w)\theta_3(z)\theta_4(w) - zk_a(z,w)\theta_3(z)\theta_4(0). \end{aligned}$$

This, together with (3.11) and the fact that $\theta_4(0) \neq 0$, shows that $\mathfrak{Q}$ is not φ-doubly commuting. ∎

Lemma 3.13. *Let θ_1 and θ_2 be inner functions on $\mathbb{D}$ such that $\theta_1(z) = z^nS_1(z)$, where $S_1(z)$ is a singular inner function and $n \in \mathbb{N}$. In addition, for $n = 1$, assume that $S_1(z)$ is nonconstant. Then $\mathfrak{Q} = H^2(\mathbb{D}^2) \ominus \theta_1(z)\theta_2(w)H^2(\mathbb{D}^2)$ is not φ-doubly commuting.*

Proof. The proof is divided into several cases.

Case (i): *θ_2 is constant and $n \geq 2$.* Consider $f(z,w) = S_1(z)$. It is easy to see that $f \in \mathfrak{Q}$ and $Q_1 Q_2 Q_2^* f = 0$, whereas

$$Q_2^* Q_1 Q_2 f(z,w) = Q_2^* Q_1(w S_1(z)) = Q_2^* P_{\mathfrak{Q}}(zw S_1(z)) = z S_1(z).$$

Hence $\mathfrak{Q}$ is not φ-doubly commuting.

Case (ii): *$n = 1$ and either θ_2 is constant or $\theta_2(w) = w$.* Consider $f(z,w) = 1$. Then $f \in \mathfrak{Q}$ and $Q_1 Q_2 Q_2^* f(z,w) = 0$, whereas

$$Q_2^* Q_1 Q_2 f(z,w) = Q_2^* Q_1(w) = Q_2^* P_{\mathfrak{Q}}(zw) \neq 0,$$

since $zw \notin \mathfrak{Q}$ and $zw \notin \theta_1(z)\theta_2(w) H^2(\mathbb{D}^2)$. So $\mathfrak{Q}$ is not φ-doubly commuting.

Case (iii): *θ_2 is nonconstant and $\theta_2(0) \neq 0$.* Consider the function $f(z,w) = z^{n-1} S_1(z)\theta_2(w) \in \mathfrak{Q}$. It is evident that $Q_2^* Q_1 Q_2 f = 0$. On the other hand,

$$\begin{aligned} Q_1 Q_2 Q_2^* f(z,w) &= Q_1 Q_2 Q_2^* (z^{n-1} S_1(z)\theta_2(w)) \\ &= Q_1 P_{\mathfrak{Q}}\big(z^{n-1} S_1(z)\theta_2(w) - z^{n-1} S_1(z)\theta_2(0)\big) \\ &= -P_{\mathfrak{Q}}(z^n S_1(z)\theta_2(0)), \end{aligned}$$

which is nonzero as $\theta_2(0) \neq 0$ and $z^n S_1(z) \notin \theta_1(z)\theta_2(w) H^2(\mathbb{D}^2)$. It follows that $\mathfrak{Q}$ is not φ-doubly commuting.

Case (iv): *Either $\theta_2(w) = w^m$, where $m \geq 2$, or $\theta_2(w) = w$ with $n \geq 2$.* In this case, $1, z, w, zw \in \mathfrak{Q}$. Therefore, $\mathfrak{Q}$ is not φ-doubly commuting.

Case (v): *θ_2 is nonconstant and there exists a nonconstant inner function θ_3 such that $\theta_2(w) = w^m \theta_3(w)$, where $\theta_3(0) \neq 0$ and $m \in \mathbb{N}$.* If $m > 1$ or $n > 1$, then $\mathfrak{Q}$ fails to be φ-doubly commuting as $zw \in \mathfrak{Q}$. So let $m = n = 1$. Then it is evident that $S_1(z)\theta_3(w), w S_1(z)\theta_3(w) \in \mathfrak{Q}$. Moreover, $Q_2^* Q_1 Q_2 S_1(z)\theta_3(w) = 0$, while

$$\begin{aligned} Q_1 Q_2 Q_2^* S_1(z)\theta_3(w) &= Q_1 P_{\mathfrak{Q}}(S_1(z)\theta_3(w) - S_1(z)\theta_3(0)) \\ &= P_{\mathfrak{Q}}(z S_1(z)\theta_3(w) - z S_1(z)\theta_3(0)) \\ &= z S_1(z)\theta_3(w) - z S_1(z)\theta_3(0). \end{aligned}$$

Since θ_3 is nonconstant, it follows that $\mathfrak{Q}$ is not φ-doubly commuting. ∎

Lemma 3.14. *Let S_1 be a nonconstant singular inner function and θ_2 be an inner function such that $\theta_2(b) = 0$ for some $b \in \mathbb{D}$. Then $\mathfrak{Q} = H^2(\mathbb{D}^2) \ominus S_1(z)\theta_2(w) H^2(\mathbb{D}^2)$ is not φ-doubly commuting.*

Proof. Since $\theta_2(b) = 0$, there exists an inner function θ_3 such that $\theta_2(w) = f_b(w)\theta_3(w)$ (see (3.7)). We now consider several cases.

Case (i): *θ_3 is constant and $b = 0$.* Note that $Q_1 Q_2 Q_2^*(1) = 0$, whereas

$$Q_2^* Q_2 Q_1(1) = Q_2^* Q_2(z) = Q_2^* P_{\mathfrak{Q}}(zw) \neq 0,$$

since S_1 is nonconstant. Thus $\mathfrak{Q}$ is not φ-doubly commuting.

Case (ii): *$b \neq 0$ and θ_3 has a zero other than b.* Consider $f(z,w) = m_b(z,w)S_1(z)\theta_3(w)$ (see (3.8)). Clearly, $f \in \mathfrak{Q}$. Now

$$Q_2^*Q_2Q_1f(z,w) = Q_2^*Q_2Q_1(m_b(z,w)S_1(z)\theta_3(w)) \tag{3.12}$$
$$= Q_2^*P_{\mathfrak{Q}}(zwm_b(z,w)S_1(z)\theta_3(w))$$
$$= \frac{bzS_1(z)}{w}[m_b(z,w)\theta_3(w) - \theta_3(0)].$$

On the other hand,

$$Q_1Q_2Q_2^*f(z,w) = Q_1Q_2Q_2^*(m_b(z,w)S_1(z)\theta_3(w)) \tag{3.13}$$
$$= Q_1(S_1(z)m_b(z,w)\theta_3(w) - S_1(z)\theta_3(0))$$
$$= zS_1(z)m_b(z,w)\theta_3(w) - P_{\mathfrak{Q}}(zS_1(z)\theta_3(0)).$$

Subcase (a): $\theta_3(0) = 0$. Then from (3.12) and (3.13), we have

$$Q_2^*Q_2Q_1f(z,w) = bzS_1(z)\frac{m_b(z,w)\theta_3(w)}{w}$$

and $Q_1Q_2Q_2^*f(z,w) = zS_1(z)m_b(z,w)\theta_3(w)$. Thus $\mathfrak{Q}$ is not φ-doubly commuting.

Subcase (b): *$\theta_3(0) \neq 0$ and there exists b' such that $b' \neq b$ and $\theta_3(b')$ $= 0$.* Suppose, for the sake of contradiction, that $\mathfrak{Q}$ is φ-doubly commuting. Then from (3.12) and (3.13), we have

$$P_{\mathfrak{Q}}\bigg(\frac{bzS_1(z)}{w}[m_b(z,w)\theta_3(w) - \theta_3(0)] - zS_1(z)m_b(z,w)\theta_3(w) + zS_1(z)\theta_3(0)\bigg) = 0,$$

which implies

$$\frac{bz}{w}[m_b(z,w)\theta_3(w) - \theta_3(0)] - zm_b(z,w)\theta_3(w) + z\theta_3(0) = \theta_3(w)f_b(w)g(z,w)$$

for some $g \in H^2(\mathbb{D}^2)$. At $w = b'$, the right-hand side vanishes as $\theta_3(b') = 0$, while the left-hand side becomes

$$\frac{(b'-b)\theta_3(0)z}{b'},$$

yielding a contradiction.

Case (iii): *There exists a singular inner function S_2 such that $\theta_2(w) = w^nS_2(w)$, where $n \in \mathbb{N}$. In addition, S_2 is nonconstant when $n = 1$.* Consider $f(z,w) = w^{n-1}S_1(z)S_2(w)$. Then $f \in \mathfrak{Q}$. We have $Q_2^*Q_1Q_2f(z,w) = 0$. On the other hand,

$$\begin{aligned}Q_1Q_2Q_2^*f(z,w) &= Q_1Q_2Q_2^*(w^{n-1}S_1(z)S_2(w))\\ &= \begin{cases} Q_1(S_1(z)S_2(w)-S_1(z)S_2(0)) & \text{if } n=1\\ Q_1(w^{n-1}S_1(z)S_2(w)) & \text{if } n\geq 2\end{cases}\\ &= \begin{cases} zS_1(z)S_2(w)-zS_1(z)S_2(0) & \text{if } n=1\\ zw^{n-1}S_1(z)S_2(w) & \text{if } n\geq 2.\end{cases}\end{aligned}$$

Hence, $\mathfrak{Q}$ is not φ-doubly commuting.

CASE (iv): *For some $b\neq 0$, $\theta_2(w)=f_b(w)^nS_2(w)$, where S_2 is a singular inner function, nonconstant when $n=1$.* Consider the function $f(z,w)=m_b(z,w)f_b(w)^{n-1}S_1(z)S_2(w)$. Now

$$\begin{aligned}(3.14)\qquad & Q_2^*Q_1Q_2f(z,w)\\ &= Q_2^*Q_1Q_2(m_b(z,w)f_b(w)^{n-1}S_1(z)S_2(w))\\ &= Q_2^*Q_1(bm_b(z,w)f_b(w)^{n-1}S_1(z)S_2(w))\\ &= Q_2^*(zbm_b(z,w)f_b(w)^{n-1}S_1(z)S_2(w))\\ &= \frac{bzS_1(z)}{w}[m_b(z,w)f_b(w)^{n-1}S_2(w)-(-bc)^{n-1}S_2(0)].\end{aligned}$$

In (3.14), we have taken f_b as in (3.7), so $f_b(0)=-bc$. On the other hand,

$$\begin{aligned}&Q_1Q_2Q_2^*f(z,w)\\ &\quad= Q_1Q_2Q_2^*(m_b(z,w)f_b(w)^{n-1}S_1(z)S_2(w))\\ &\quad= Q_1P_{\mathfrak{Q}}(m_b(z,w)f_b(w)^{n-1}S_1(z)S_2(w)-(-bc)^{n-1}S_1(z)S_2(0))\\ &\quad= zm_b(z,w)f_b(w)^{n-1}S_1(z)S_2(w)-S_2(0)(-bc)^{n-1}Q_1P_{\mathfrak{Q}}(S_1(z))\\ &\quad= zm_b(z,w)f_b(w)^{n-1}S_1(z)S_2(w)\\ &\qquad - S_2(0)(-bc)^{n-1}zS_1(z)[1-(-\bar{b}\bar{c})^n(f_b(w))^nS_2(w)\overline{S_2(0)}].\end{aligned}$$

This together with (3.14) implies that $\mathfrak{Q}$ is not φ-doubly commuting.

CASE (v): *$\theta_2(w)=f_b(w)$ where $b\neq 0$.* Take $f(z,w)=Q_1^*(S_1(z)-S_1(0))\in\mathfrak{Q}$. It is easy to see that $Q_1Q_2Q_2^*f=0$, whereas

$$\begin{aligned}Q_2^*Q_2Q_1f(z,w) &= Q_2^*Q_2Q_1Q_1^*(S_1(z)-S_1(0)) = Q_2^*Q_2(S_1(z)-S_1(0))\\ &= Q_2^*P_{\mathfrak{Q}}(wS_1(z)-wS_1(0)) = S_1(z)-S_1(0)\neq 0\end{aligned}$$

as S_1 is nonconstant. ∎

We next provide a characterization of φ-doubly commuting quotient modules for inner functions of the form $\theta_1(z)\theta_2(w)$, under specific assumptions on their zeros.

THEOREM 3.15. *Let θ_1 and θ_2 be inner functions on $\mathbb{D}$ such that $\theta_1(a)=\theta_2(b)=0$ for some $a,b\in\mathbb{D}$. Then $\mathfrak{Q}=H^2(\mathbb{D}^2)\ominus\theta_1(z)\theta_2(w)H^2(\mathbb{D}^2)$ is*

φ-doubly commuting if and only if $\theta_1(z) = c_1 z$ and $\theta_2(w) = c_2 w$ for some $c_1, c_2 \in \mathbb{T}$.

Proof. Let $\mathfrak{Q}$ be φ-doubly commuting. Lemmas 3.12 and 3.13 together imply that $\theta_1(z) = c_1 z$ for some $c_1 \in \mathbb{T}$. This, in turn, yields $\mathfrak{Q} = H^2(\mathbb{D}^2) \ominus c_1 z\theta_2(w)H^2(\mathbb{D}^2)$. We now claim that 0 is the only possible zero of θ_2. Let, if possible, $\theta_2(0) \neq 0$. Consider $f(z,w) = \theta_2(w) \in \mathfrak{Q}$. Then, $Q_2^* Q_1 Q_2 f(z,w) = 0$, whereas

$$Q_1 Q_2 Q_2^* f(z,w) = Q_1(\theta_2(w) - \theta_2(0)) = -P_{\mathfrak{Q}}(z\theta_2(0)) \neq 0,$$

yielding a contradiction. Hence $\theta_2(w) = w^m \theta_3(w)$, where θ_3 is an inner function. If $m > 1$, this is not possible, because in that case $1, z, w, zw \in \mathfrak{Q}$. Therefore, $\theta_2(w) = w\theta_3(w)$. Consider $f(z,w) = \theta_3(w)$, which belongs to $\mathfrak{Q}$. Then $Q_2^* Q_1 Q_2 f(z,w) = 0$, while $Q_1 Q_2 Q_2^* f(z,w) = z\theta_3(w) - z\theta_3(0)$. Since $\mathfrak{Q}$ is φ-doubly commuting, it follows that θ_3 must be a constant inner function. The converse is immediate. ∎

We are now ready to prove Theorems 3.11 and 1.3.

Proof of Theorem 3.11. Let $\mathfrak{Q}$ be φ-doubly commuting.

(i) If both θ_1 and θ_2 are singular, then according to Corollary 3.9, both are constant inner functions.

(ii) Let $\theta_2(b) = 0$ for some $b \in \mathbb{D}$, and suppose that θ_1 is singular. If θ_1 is nonconstant, then by Lemma 3.14, we obtain a contradiction. Hence θ_1 must be constant. This further implies that $\mathfrak{Q} = H^2(\mathbb{D}^2) \ominus \theta_2(w)H^2(\mathbb{D}^2)$, which is doubly commuting. Therefore, by Theorem 3.8, $\theta_2 = f_b$ for some $b \in \mathbb{D}$ (see (3.7)).

(iii) Let $\theta_1(a) = 0$ for some $a \in \mathbb{D}$, and let θ_2 be singular. Then, by Lemmas 3.12 and 3.13, $\theta_1(z) = c_1 z$ for some $c_1 \in \mathbb{T}$. We claim that θ_2 must be constant. To see this, consider $f(z,w) = \theta_2(w)$, which belongs to $\mathfrak{Q}$. Then $Q_2^* Q_1 Q_2 f(z,w) = 0$, while

$$Q_1 Q_2 Q_2^* f(z,w) = Q_1(\theta_2(w) - \theta_2(0)) = -P_{\mathfrak{Q}}(z\theta_2(0)) = z\theta_2(w) - z\theta_2(0).$$

Now the φ-double commutativity forces that θ_2 is constant.

(iv) If θ_1 and θ_2 both have zeros, then by Theorem 3.15, $\theta_1(z) = c_1 z$ and $\theta_2(w) = c_2 w$ for some $c_1, c_2 \in \mathbb{T}$.

The converse is immediate. ∎

Proof of Theorem 1.3. This follows from Theorem 3.11, along with (3.3) and (3.1). ∎

Finally, we conclude this section with a φ-doubly commuting quotient module $H^2(\mathbb{D}^2) \ominus \theta H^2(\mathbb{D}^2)$ such that there do not exist inner functions θ_1 and θ_2 on $\mathbb{D}$ satisfying

$$\theta(z,w) = \theta_1(z)\theta_2(w).$$

PROPOSITION 3.16. *Let $\theta_a(z,w) = \frac{zw-a}{1-azw}$, where $0 < a < 1$. Then the quotient module $\mathfrak{Q}_a = H^2(\mathbb{D}^2) \ominus \theta_a H^2(\mathbb{D}^2)$ is φ-doubly commuting.*

Proof. It follows from [14, p. 3225] that θ_a is not separating for any $a \in (0,1)$. By [14, Lemma 3.3], the set

$$B_a = \{(1+a\theta_a)w^j, (1+a\theta_a), (1+a\theta_a)z^i : i,j \geq 1\}$$

forms an orthogonal basis for $\mathfrak{Q}_a$. Let $P_a : H^2(\mathbb{D}^2) \to \mathfrak{Q}_a$ denote the orthogonal projection. We observe that for each $i \geq 1$,

$$\begin{aligned} Q_2(1+a\theta_a)z^i &= P_a[(1+a\theta_a)z^i w] = P_a[z^{i-1}(1+a\theta_a)zw] \\ &= P_a[z^{i-1}(1+a\theta_a)((1-azw)\theta_a + a)] \\ &= az^{i-1}(1+a\theta_a). \end{aligned} \tag{3.15}$$

The last equality holds because $z^{i-1}(1+a\theta_a)(1-azw)\theta_a \in \theta_a H^2(\mathbb{D}^2)$. We also obtain

$$Q_2(1+a\theta_a)w^j = (1+a\theta_a)w^{j+1}, \quad j \geq 0. \tag{3.16}$$

Similarly,

$$Q_1(1+a\theta_a)z^i = (1+a\theta_a)z^{i+1}, \quad i \geq 0, \tag{3.17}$$

$$Q_1(1+a\theta_a)w^j = P_a[w^{j-1}(1+a\theta_a)zw] = aw^{j-1}(1+a\theta_a), \quad j \geq 1. \tag{3.18}$$

Let $i \geq 0$ be fixed. Clearly, for $j \geq 0$,

$$\langle Q_2^*(1+a\theta_a)z^i, (1+a\theta_a)w^j\rangle = \langle (1+a\theta_a)z^i, (1+a\theta_a)w^{j+1}\rangle = 0. \tag{3.19}$$

Further, for $j \geq 1$ we calculate

$$\begin{aligned} \langle Q_2^*(1+a\theta_a)z^i, (1+a\theta_a)z^j\rangle &= \langle (1+a\theta_a)z^i, Q_2[(1+a\theta_a)z^j]\rangle \\ &\overset{(3.15)}{=} \langle (1+a\theta_a)z^i, az^{j-1}(1+a\theta_a)\rangle \\ &= \begin{cases} a\|(1+a\theta_a)z^i\|^2 & \text{if } j = i+1, \\ 0 & \text{elsewhere.} \end{cases} \end{aligned} \tag{3.20}$$

Thus, by (3.19) and (3.20) we have

$$Q_2^*(1+a\theta_a)z^i = a\|(1+a\theta_a)z^i\|^2(1+a\theta_a)z^{i+1} \quad \text{for } i \geq 0. \tag{3.21}$$

Using (3.15) and (3.16) we similarly calculate

$$Q_2^*(1+a\theta_a)w^j = \|(1+a\theta_a)w^j\|^2(1+a\theta_a)w^{j-1} \quad \text{for } j \geq 1. \tag{3.22}$$

Now, from (3.15)–(3.18), (3.21) and (3.22) we finally have

$$[Q_1Q_2, Q_2^*]f = 0 \quad \text{for all } f \in B_a. \ \blacksquare$$

4. Essential normality and double commutativity of quotient modules induced by polynomials. In this section, we study the quotient modules of the form $\mathcal{Q}_p := H^2(\triangle_H) \ominus pH^2(\triangle_H)$, where $p \in \mathbb{C}[z,w]$. For each

$m \in \mathbb{Z}_+$, let $\mathfrak{F}_m$ denote the finite-dimensional closed subspace of $H^2(\triangle_H)$ spanned by F_m, where

$$F_m = \left\{ \frac{1}{w} \left(\frac{z}{w} \right)^{m-j} w^j \right\}_{j=0}^{m}.$$

It follows that

$$H^2(\triangle_H) = \bigoplus_{m=0}^{\infty} \mathfrak{F}_m.$$

DEFINITION 4.1. Given a monomial $z^i w^j$ with nonnegative integer exponents i, j, we define its *Hartogs degree* to be $2i + j + 1$.

It is worth noting that two monomials of different usual degrees may have the same Hartogs degree. For instance, let $p(z,w) = z$ and $q(z,w) = w^2$. While their degrees differ, the Hartogs degree of both is 3.

REMARK 4.2. A function $f(z,w) = \sum_{i=0}^{m} c_i \frac{1}{w} \left(\frac{z}{w}\right)^{m-i} w^i$ is in $\mathcal{Q}_p \cap \mathfrak{F}_m$ for some scalars $c_0, c_1, \ldots, c_m$ if and only if

$$\sum_{i=0}^{m} c_i \left\langle \frac{1}{w} \left(\frac{z}{w} \right)^{m-i} w^i, p(z,w) g(z,w) \right\rangle = 0, \quad g \in H^2(\triangle_H),\ m \in \mathbb{Z}_+. \tag{4.1}$$

4.1. Dimensional analysis. In this subsection, we determine the dimension of $\mathcal{Q}_p \cap \mathfrak{F}_m$ in terms of p and m. This will provide the necessary foundation for establishing the essential normality and double commutativity of $\mathcal{Q}_p$.

THEOREM 4.3. *Let $m \in \mathbb{Z}_+$. Then the following statements are valid:*

(i) *Let $p(z,w) = z^q w^n$, where $q, n \in \mathbb{Z}_+$. Then*

$$\dim(\mathcal{Q}_p \cap \mathfrak{F}_m) = \begin{cases} m+1 & \text{if } m < 2q+n, \\ 2q+n & \text{if } m \geq 2q+n. \end{cases}$$

(ii) *Let $p(z,w) = \sum_{i+j \geq 1} a_{i,j} z^i w^j$ be a polynomial in which no two summands have the same Hartogs degree. Then there exists $m_0 \in \mathbb{N}$ such that*

$$\dim(\mathcal{Q}_p \cap \mathfrak{F}_m) = \min\{i+j : a_{i,j} \neq 0 \text{ in } p\} + \min\{i : a_{i,j} \neq 0 \text{ in } p\}$$

for all $m \geq m_0$.

(iii) *Let $b_k \in \mathbb{C} \setminus \{0\}$ for $0 \leq k \leq t$ for some $t \in \mathbb{Z}_+$. If $p(z,w) = \sum_{k=0}^{t} b_k z^{q_k} w^{n_k}$, where $q_k \leq q_{k+1}$ for all $0 \leq k \leq t-1$, is a polynomial with each summand having the same Hartogs degree, then*

$$\dim(\mathcal{Q}_p \cap \mathfrak{F}_m) = \begin{cases} m+1 & \text{if } m < 2q_0 + n_0, \\ 2q_0 + n_0 & \text{if } m \geq q_t + q_0 + n_0. \end{cases}$$

Proof. (i) Note that p has Hartogs degree $2q+n+1$ and

$$p(z,w) = z^q w^n = \frac{1}{w}\left(\frac{z}{w}\right)^q w^{q+n+1}, \qquad q,n \in \mathbb{Z}_+.$$

For $m < 2q+n$, it is easy to see that $\mathfrak{F}_m \subseteq \mathcal{Q}_p$. This yields

$$\dim(\mathcal{Q}_p \cap \mathfrak{F}_m) = m+1.$$

Assume that $m \geq 2q+n$. Let $f(z,w) = \sum_{j=0}^m c_j \frac{1}{w}\left(\frac{z}{w}\right)^{m-j} w^j \in \mathfrak{F}_m \cap \mathcal{Q}_p$. For $q+n \leq j \leq m-q$, the function $g(z,w) = g_j(z,w) = \left(\frac{z}{w}\right)^{m-q-j} w^{j-q-n-1}$ and $p(z,w) = z^q w^n$ in (4.1) yields $c_j = 0$ (since $f \in \mathfrak{F}_m \cap \mathcal{Q}_p$). Therefore, the dimension of $\mathcal{Q}_p \cap \mathfrak{F}_m$ is at most $2q+n$. Note that the smallest possible power of $\frac{z}{w}$ and w in any summand of

$$z^q w^n g(z,w) = \frac{1}{w}\left(\frac{z}{w}\right)^q w^{q+n+1} g(z,w)$$

is q and $q+n$, respectively, for any $g \in H^2(\triangle_H)$. In contrast, for $j > m-q$, the largest possible power of $\frac{z}{w}$ in any summand of $f(z,w)$ is $q-1$, while for $j < q+n$, the largest possible power of w in any summand of $f(z,w)$ is $q+n-1$. This yields

$$\frac{1}{w}\left(\frac{z}{w}\right)^{m-j} w^j \in \mathcal{Q}_p \cap \mathfrak{F}_m \quad \text{ for all } j = 0,\ldots,q+n-1, m-q+1,\ldots,m.$$

Therefore, the dimension of $\mathcal{Q}_p \cap \mathfrak{F}_m$ is exactly $2q+n$.

(ii) Let i_1 and j_1 be integers such that

$$i_1 + j_1 = \min\{i+j : a_{i,j} \neq 0 \text{ in } p\}.$$

Choose $m \geq 2i_1 + j_1$. Let $f(z,w) = \sum_{k=0}^m c_k \frac{1}{w}\left(\frac{z}{w}\right)^{m-k} w^k \in \mathfrak{F}_m \cap \mathcal{Q}_p$. Then for $i_1+j_1 \leq k \leq m-i_1$, choosing $g(z,w) = g_k(z,w) = \left(\frac{z}{w}\right)^{m-i_1-k} w^{k-i_1-j_1-1}$ and $p(z,w) = \sum_{i+j\geq 1} a_{i,j} z^i w^j$ in (4.1), we obtain $c_k = 0$ by using the fact no two summands in p have the same Hartogs degree.

Furthermore, there exists $i_2 \geq 0$ such that

$$i_2 = \min\{i : a_{i,j} \neq 0 \text{ in } p\}.$$

Clearly, $i_2 \leq i_1$, and it follows that $c_k = 0$ for $m - i_1 \leq k \leq m - i_2$ for a suitable choice of g in (4.1) and for sufficiently large m. The conclusion can now be drawn using an argument analogous to that in (i).

(iii) Let

$$p(z,w) = \sum_{k=0}^t b_k z^{q_k} w^{n_k}$$

be a polynomial such that $2q_k + n_k = a$ for all $0 \leq k \leq t$ and for some $a \in \mathbb{N}$. It is easy to see that $\mathfrak{F}_m \subseteq \mathcal{Q}_p$ for $m < a$. We claim that the dimension of $\mathcal{Q}_p \cap \mathfrak{F}_m$ is $2q_0+n_0$ when $m \geq q_t+q_0+n_0$. Since $q_k \leq q_{k+1}$ for all $0 \leq k \leq t-1$, we have $n_{k+1} \leq n_k$ and $q_{k+1} + n_{k+1} \leq q_k + n_k$ for all $0 \leq k \leq t-1$. Then for any $g \in H^2(\triangle_H)$ and $0 \leq i \leq q_t + n_t - 1$,

$$\left\langle \frac{1}{w}\left(\frac{z}{w}\right)^{m-i} w^i, p(z,w)g(z,w) \right\rangle = 0. \tag{4.2}$$

Let $f(z,w) = \sum_{k=q_t+n_t}^{m} a_k \frac{1}{w}\left(\frac{z}{w}\right)^{m-k} w^k \in \mathfrak{F}_m \cap \mathcal{Q}_p$. For $q_t + n_t \leq j \leq m - q_t$, the function $g(z,w) = g_j(z,w) = \left(\frac{z}{w}\right)^{m-q_t-j} w^{j-q_t-n_t-1}$ and $p(z,w) = \sum_{k=0}^{t} b_k z^{q_k} w^{n_k}$ in (4.1) gives

$$\sum_{k=0}^{t} \overline{b_{t-k}}\, a_{q_t - q_{t-k} + j} = 0 \qquad \text{for } q_t + n_t \leq j \leq m - q_t, \tag{4.3}$$

where we have used the fact that $q_k + n_k - q_t - n_t = q_t - q_k$ for all $k = 0, \ldots, t$. Let $0 \leq l \leq t-1$. Additionally, for $q_l + n_l \leq j \leq m - q_l$ considering $g(z,w) = g_j^l(z,w) = \left(\frac{z}{w}\right)^{m-q_l-j} w^{j-q_l-n_l-1}$ in (4.1) further gives

$$\sum_{k=0}^{t} \overline{b_{t-k}}\, a_{q_l - q_{t-k} + j} = 0 \qquad \text{for } q_l + n_l \leq j \leq m - q_l. \tag{4.4}$$

Since $2q_l + n_l = a$ for all $0 \leq l \leq t$, it follows that the system of equations (4.4) is part of (4.3) for each $0 \leq l \leq t$. Therefore, it is enough to consider (4.3). Note that the number of unknowns in (4.3) is $m + 1 - q_t - q_0 - n_t$; the condition $m \geq q_t + q_0 + n_0$ justifies the last argument. If A is the coefficient matrix of (4.3), then A is already in echelon form of rows. Furthermore, since $b_k \neq 0$ for all $0 \leq k \leq t$, we have $m + 1 - 2q_t - n_t$ pivot entries in A. Hence, the dimension of the solution space will be $q_t - q_0$.

Finally, for $m - q_0 < j \leq m$ and $g \in H^2(\triangle_H)$,

$$\left\langle \frac{1}{w}\left(\frac{z}{w}\right)^{m-j} w^j, p(z,w)g(z,w) \right\rangle = 0. \tag{4.5}$$

Thus, from (4.2), (4.3) and (4.5), we conclude that $\dim(\mathcal{Q}_p \cap \mathfrak{F}_m) = 2q_0 + n_0$. This proves the claim. ∎

Here is an immediate corollary.

Corollary 4.4. *The following statements are valid:*

(i) *If $p(z) = \sum_{i=1}^{k} a_i z^i$ is a polynomial then*

$$\dim(\mathcal{Q}_p \cap \mathfrak{F}_m) = 2\min\{i : a_i \neq 0 \text{ in } p\}$$

for $m \geq 2\min\{i : a_i \neq 0 \text{ in } p\}$.

(ii) *If $p(w) = \sum_{i=1}^{k} a_i w^i$ is a polynomial then*

$$\dim(\mathcal{Q}_p \cap \mathfrak{F}_m) = \min\{i : a_i \neq 0 \text{ in } p\}$$

for $m \geq \min\{i : a_i \neq 0 \text{ in } p\}$.

(iii) *Let* $p(z,w) = \sum_{i=1}^{k} a_i z^i + \sum_{j=1}^{l} b_j w^j$ *be a polynomial with each summand having a different Hartogs degree and* $a_i b_j \neq 0$ *for some* i *and* j*. Then*

$$\dim(\mathfrak{Q}_p \cap \mathfrak{F}_m) = \min\big\{\min\{i : a_i \neq 0 \text{ in } p\}, \min\{j : b_j \neq 0 \text{ in } p\}\big\}$$

for $m \geq 2\min\{i : a_i \neq 0 \text{ in } p\} + \min\{j : b_j \neq 0 \text{ in } p\}$.

It is natural to consider a general polynomial of the form $p(z,w) = \sum_{j=1}^{k} p_j(z,w)$, where each summand p_j is a polynomial consisting of terms with the same Hartogs degree for every $1 \leq j \leq k$. However, obtaining an explicit formula, as in the previous cases, appears to be challenging. The following example illustrates this difficulty.

EXAMPLE 4.5. Consider the polynomial

$$p(z,w) = zw^5 + z^2w^3 + z^3w^5 + z^5w \tag{4.6}$$

with components $p_1(z,w) = zw^5 + z^2w^3$ and $p_2(z,w) = z^3w^5 + z^5w$. The Hartogs degrees of p_1 and p_2 are 8 and 12, respectively.

Assume that $m \geq 30$. Let $f(z,w) = \sum_{j=0}^{m} a_j \frac{1}{w}\left(\frac{z}{w}\right)^{m-j} w^j \in \mathfrak{F}_m \cap \mathfrak{Q}_p$. The function

$$g_j(z,w) = \left(\frac{z}{w}\right)^{m-2-j} w^{j-6}$$

in (4.1) with p given by (4.6) leads to the recurrence relation

$$a_j + a_{j+1} = 0, \qquad 5 \leq j \leq m-2. \tag{4.7}$$

Similarly, the function

$$g_j(z,w) = \left(\frac{z}{w}\right)^{m-5-j} w^{j-7}$$

in (4.1) with p given by (4.6) yields

$$a_j + a_{j+2} = 0, \qquad 6 \leq j \leq m-5. \tag{4.8}$$

From (4.7), we obtain $a_j = (-1)^{j+1} a_5$ for $5 \leq j \leq m-1$. Setting $j = 6$ in (4.8) gives $a_5 = 0$, leading to $\dim(\mathfrak{Q}_p \cap \mathfrak{F}_m) = 6$.

Next, consider $q(z,w) = q_1(z,w) + q_2(z,w)$, where $q_1(z,w) = zw^5 + z^2w^3$ and $q_2(z,w) = z^5w^6 + z^8$. Here, the Hartogs degrees of q_1 and q_2 are 8 and 17, respectively.

Assume that $m \geq 30$. Let $f(z,w) = \sum_{j=0}^{m} a_j \frac{1}{w}\left(\frac{z}{w}\right)^{m-j} w^j \in \mathfrak{F}_m \cap \mathfrak{Q}_q$. Choosing the functions

$$g_j(z,w) = \left(\frac{z}{w}\right)^{m-2-j} w^{j-6} \quad \text{for } 5 \leq j \leq m-2,$$

$$g_j(z,w) = \left(\frac{z}{w}\right)^{m-8-j} w^{j-9} \quad \text{for } 8 \leq j \leq m-8$$

in (4.1), we have the relations

$$a_j + a_{j+1} = 0, \quad 5 \le j \le m-2, \tag{4.9}$$

$$a_j + a_{j+3} = 0, \quad 8 \le j \le m-8, \tag{4.10}$$

respectively. Finally, (4.9) and (4.10) together imply that $\dim(\mathcal{Q}_q \cap \mathfrak{F}_m) = 7$.

4.2. Essential normality of $\mathcal{Q}_p$. [15, Theorem 1.1] provides a characterization of essentially normal quotient modules in $H^2(\mathbb{D}^2)$ for a certain class of polynomials. Here, we present a characterization of essentially normal quotient modules $\mathcal{Q}_p$ of $H^2(\triangle_H)$ under mild assumptions on the polynomial $p \in \mathbb{C}[z,w]$. In fact, we consider those polynomials p for which

$$\mathcal{Q}_p = \bigoplus_{m=0}^{\infty} (\mathfrak{F}_m \cap \mathcal{Q}_p). \tag{4.11}$$

The motivation for introducing (4.11) is drawn from [8, Section 5]. For a homogeneous polynomial $p(z,w)$ of degree n, if $p(z,1)H^2(\mathbb{D}^2)$ is dense in $H^2(\mathbb{D}^2)$ then $\mathcal{Q}_p$ satisfies (4.11). Indeed,

$$\Psi(p(z,w)H^2(\triangle_H)) \overset{(1.4)}{=} p(zw,w)H^2(\mathbb{D}^2) = w^n p(z,1)H^2(\mathbb{D}^2).$$

Therefore, since $p(z,1)H^2(\mathbb{D}^2)$ is dense in $H^2(\mathbb{D}^2)$, it follows from (3.1) that $\mathcal{Q}_p = H^2(\triangle_H) \ominus w^n H^2(\triangle_H)$. It is immediate that this yields (4.11). Hence, for each $n \in \mathbb{N}$, $\mathcal{Q}_p$ satisfies (4.11) when $p(z,w) = (z-w)^n$.

We now present a few other explicit examples where (4.11) holds, as well as examples where it fails.

Example 4.6.

(i) Consider the polynomials z^m, w^n and $z^m w^n$ where $m, n \in \mathbb{N}$. Then $\mathcal{Q}_p$ satisfies (4.11).
(ii) Consider $p(z,w) = w - \frac{1}{2}$. The Hartogs degree of the constant term is 1, which differs from the Hartogs degree of w. This in turn implies that $\mathcal{Q}_p \cap \mathfrak{F}_m = \{0\}$ for all $m \ge 0$. However, $\sum_{i=0}^{\infty} \frac{1}{2^i}\frac{1}{w} w^i \in \mathcal{Q}_p$.

For $m \in \mathbb{Z}_+$, let

$$E_{m-j}^m := \frac{1}{w}\left(\frac{z}{w}\right)^{m-j} w^j, \qquad j = 0, 1, \dots, m. \tag{4.12}$$

We begin with the following elementary observation.

Lemma 4.7. *Let p be a polynomial and $l \in \mathbb{N}$ such that E_k^k and E_{k-1}^k are in $\mathcal{Q}_p$ for all $k \ge l$ (see (4.12)). Then $\mathcal{Q}_p$ is not essentially normal.*

Proof. Let $k \ge l$. First, note that

$$Q_z E_k^k = P_{\mathcal{Q}_p} M_z \left[\frac{1}{w}\left(\frac{z}{w}\right)^k\right] = P_{\mathcal{Q}_p}\left[\frac{1}{w}\left(\frac{z}{w}\right)^{k+1} w\right] = E_{k+1}^{k+2}.$$

Similarly,

$$Q_z^* E_k^k = P_{\mathcal{Q}_p} M_z^* \left[\frac{1}{w} \left(\frac{z}{w} \right)^k \right] = 0.$$

Also,

$$Q_z^* E_{k+1}^{k+2} = P_{\mathcal{Q}_p} M_z^* \left[\frac{1}{w} \left(\frac{z}{w} \right)^{k+1} w \right] = E_k^k.$$

It follows that

$$[Q_z^*, Q_z] E_k^k = Q_z^* Q_z E_k^k - Q_z Q_z^* E_k^k = Q_z^* E_{k+1}^{k+2} = E_k^k.$$

The orthogonality of the family $\{E_k^k\}_{k \geq l}$ implies that the operator $[Q_z^*, Q_z]$ is not compact. Note that $Q_w E_k^k = E_k^{k+1}$, $Q_w^* E_k^k = 0$. It is easy to see that

$$(Q_w^* Q_z - Q_z Q_w^*) E_k^k = Q_w^* E_{k+1}^{k+2} - 0 = E_{k+1}^{k+1}.$$

Hence, $[Q_w^*, Q_z]$ is not compact either. Therefore, $\mathcal{Q}_p$ is not essentially normal. ∎

Remark 4.8. From the proof of Theorem 4.3, it follows that for the monomials $p(z,w) = z^n$ (for $n \geq 2$), $p(z,w) = w^m$ (for $m \geq 2$), and $p(z,w) = z^m w^n$ (for $m, n \geq 1$), we have $E_k^k, E_{k-1}^k \in \mathcal{Q}_p$ for $k \geq 1$. Hence by Lemma 4.7, $\mathcal{Q}_p$ is not essentially normal.

As a consequence, we get the following.

Proposition 4.9. *Let p be a polynomial in z and w represented by*

$$p(z,w) = az + bw + cw^2 + w^3 q(w) + z^2 r(z) + \sum_{i,j \geq 1} a_{i,j} z^i w^j, \tag{4.13}$$

where a, b, c are scalars, and $q(w)$ and $r(z)$ are single variable polynomials. If $a = b = 0$, then the quotient module $\mathcal{Q}_p$ is not essentially normal.

Proof. If $a = b = 0$, then

$$p(z,w) = \sum_{i+j \geq 2} a_{i,j} z^i w^j.$$

It is easy to see that for $k \geq 1$,

$$\langle E_k^k, p(z,w) g(z,w) \rangle = \langle E_{k-1}^k, p(z,w) g(z,w) \rangle = 0 \quad \forall g \in H^2(\triangle_H).$$

Hence, by Lemma 4.7, $\mathcal{Q}_p$ is not essentially normal. ∎

We now turn to the case of quotient modules $\mathcal{Q}_p$ that are essentially normal.

Theorem 4.10. *Let p be a polynomial in z and w given by* (4.13) *such that $\mathcal{Q}_p$ satisfies* (4.11). *Then the following statements are valid:*

(i) *If $b = 1$ then $\mathcal{Q}_p$ is essentially normal.*
(ii) *If $a = 1$, $b = c = 0$ then $\mathcal{Q}_p$ is essentially normal.*
(iii) *If $a = c = 1$, $b = 0$ then $\mathcal{Q}_p$ is essentially normal.*

Proof. (i) Let $b = 1$ in (4.13). Then

$$p(z,w) = az + w + \sum_{i+j\geq 2} a_{i,j} z^i w^j. \tag{4.14}$$

Note that the Hartogs degree of w will never be equal to the Hartogs degree of any summand in $az + \sum_{i+j\geq 2} a_{i,j} z^i w^j$. Let $f(z,w) = \sum_{k=0}^{m} a_k \frac{1}{w}\left(\frac{z}{w}\right)^{m-k} w^k \in \mathcal{Q}_p \cap \mathfrak{F}_m$. For $1 \leq k \leq m$, selecting $g_k(z,w) = \left(\frac{z}{w}\right)^{m-k} w^{k-2}$ and $p(z,w)$ (as given in (4.14)) in (4.1) implies that $a_k = 0$. On the other hand, for any $g \in H^2(\triangle_H)$,

$$\left\langle \frac{1}{w}\left(\frac{z}{w}\right)^m, \Big(az + w + \sum_{i+j\geq 2} a_{i,j} z^i w^j\Big) g(z,w) \right\rangle = 0.$$

This further implies

$$\mathcal{Q}_p \cap \mathfrak{F}_m = \overline{\operatorname{span}}\left\{\frac{1}{w}\left(\frac{z}{w}\right)^m\right\} = \overline{\operatorname{span}}\{E_m^m\} \quad \text{for } m \geq 0.$$

It is now easy to see that $\mathcal{Q}_p$ is essentially normal.

(ii) Let $a = 1$ and $b = c = 0$ in (4.13). Then

$$p(z,w) = z + w^3 q(w) + z^2 r(z) + \sum_{i,j\geq 1} a_{i,j} z^i w^j. \tag{4.15}$$

For all $m \geq 0$ and for any $g \in H^2(\triangle_H)$, we have

$$\langle E_m^m, p(z,w) g(z,w)\rangle = 0,$$

so that $E_m^m \in \mathcal{Q}_p \cap \mathfrak{F}_m$ for all $m \geq 0$. Moreover, $E_0^1 \in \mathcal{Q}_p \cap \mathfrak{F}_1$. We observe that the Hartogs degree of z will never be equal to the Hartogs degree of any summand in $w^3 q(w) + z^2 r(z) + \sum_{i,j\geq 1} a_{i,j} z^i w^j$. Let $f(z,w) = \sum_{k=0}^{m} a_k \frac{1}{w}\left(\frac{z}{w}\right)^{m-k} w^k \in \mathcal{Q}_p \cap \mathfrak{F}_m$ and $m \geq 2$. For $1 \leq k \leq m-1$, choosing $g_k(z,w) = \left(\frac{z}{w}\right)^{m-k-1} w^{k-2}$ and $p(z,w)$ (as in (4.15)) in (4.1) we obtain $a_k = 0$. It now remains to verify whether $E_0^m \in \mathcal{Q}_p \cap \mathfrak{F}_m$ or not for $m \geq 2$. Let $k \in \mathbb{N} \cup \{0\}$ be such that

$$q(w) = \sum_{j=k}^{\deg q} c_j w^j,$$

where $c_k \neq 0$. Therefore, if $m < k+3$, we obtain

$$\langle E_0^m, p(z,w) g(z,w)\rangle = 0$$

for all $g \in H^2(\triangle_H)$. For $m \geq k+3$, we obtain $a_m = 0$. Consequently,

$$\mathcal{Q}_p \cap \mathfrak{F}_m = \begin{cases} \overline{\operatorname{span}}\{E_m^m, E_0^m\} & \text{for } 0 \leq m < k+3, \\ \overline{\operatorname{span}}\{E_m^m\} & \text{for } m \geq k+3. \end{cases} \tag{4.16}$$

It is now easy to see that $\mathcal{Q}_p$ is essentially normal.

(iii) Let $a = c = 1$ and $b = 0$. Then

$$p(z,w) = z + w^2 + w^3 q(w) + z^2 r(z) + \sum_{i,j\geq 1} a_{i,j} z^i w^j. \tag{4.17}$$

Note that the Hartogs degree of z and w^2 is 3. However, the Hartogs degree of any summand in $w^3 q(w) + z^2 r(z) + \sum_{i,j\geq 1} a_{i,j} z^i w^j$ is at least 4.

For $m \geq 0$ and any $g \in H^2(\triangle_H)$, we have

$$\langle E_m^m, p(z,w) g(z,w)\rangle = 0.$$

Thus $E_m^m \in \mathcal{Q}_p \cap \mathfrak{F}_m$ for all $m \geq 0$. Moreover, $E_0^1 \in \mathcal{Q}_p \cap \mathfrak{F}_1$.

Let $m \geq 2$, and suppose

$$f(z,w) = \sum_{i=1}^{m} a_i \frac{1}{w}\left(\frac{z}{w}\right)^{m-i} w^i \in \mathcal{Q}_p \cap \mathfrak{F}_m.$$

For $1 \leq k \leq m-1$, take $g_k(z,w) = \left(\frac{z}{w}\right)^{m-k-1} w^{k-2}$ and p (as in (4.17)) in (4.1), we obtain

$$a_k + a_{k+1} = 0,$$

which further yields $a_k = (-1)^{k+1} a_1$ for $k = 1, \ldots, m$.

Let $k \in \mathbb{N} \cup \{0\}$ be such that $q(w) = \sum_{j=k}^{\deg q} c_j w^j$, where $c_k \neq 0$. Then $a_m = 0$ for $m \geq k+3$, which implies

$$\mathcal{Q}_p \cap \mathfrak{F}_m = \overline{\operatorname{span}}\,\{E_m^m\} \quad \text{for } m \geq k+3.$$

For $0 \leq m < k+3$, it is possible that $a_1 = 0$ or $a_1 \neq 0$, depending on the choice of $r(z)$ and $a_{i,j}$. Therefore,

$$\mathcal{Q}_p \cap \mathfrak{F}_m \subseteq \overline{\operatorname{span}}\left\{E_m^m, a_1 \sum_{k=1}^{m} (-1)^{k+1} E_{m-k}^m\right\} \quad \text{for } 0 \leq m < k+3.$$

It is now easy to see that $\mathcal{Q}_p$ is essentially normal. ∎

As a consequence, we establish Theorem 1.4.

Proof of Theorem 1.4. Let p be as given in (4.13). Then it follows from Theorem 4.10 and Proposition 4.9 that $\mathcal{Q}_p$ is not essentially normal if and only if $a = b = 0$. ∎

REMARK 4.11. Interestingly, when $p(z,w) = (z-w)^2$, the quotient module $H^2(\mathbb{D}^2) \ominus pH^2(\mathbb{D}^2)$ is essentially normal in the Hardy space of the bidisc (see [8, Section 5]). However, $H^2(\triangle_H) \ominus pH^2(\triangle_H)$ is not. This highlights a key difference between these two function spaces.

We end this section with an application of Theorem 4.10. Recall that a quotient module $H^2(\mathbb{D}^2) \ominus pH^2(\mathbb{D}^2)$ is doubly commuting for a polynomial p if and only if $pH^2(\mathbb{D}^2) = GH^2(\mathbb{D}^2)$ for some finite Blaschke product G depending on a single variable (see [10, Corollary 4.3]). In contrast, in $H^2(\triangle_H)$, there exist doubly commuting quotient modules $\mathcal{Q}_p = H^2(\triangle_H) \ominus pH^2(\triangle_H)$ for which p is not necessarily an inner function.

PROPOSITION 4.12. *Let $p(z,w) = az + bw + cw^2 + w^3q(w) + z^2r(z) + \sum_{i,j\geq 1} a_{i,j}z^iw^j$ be such that $\mathcal{Q}_p$ satisfies* (4.11). *If either $b \neq 0$ or $a \neq 0$ and $b = c = 0$, then the quotient module $\mathcal{Q}_p$ is doubly commuting.*

Proof. Consider the case when $b \neq 0$. Then, by Theorem 4.10(i), we have $\mathcal{Q}_p \cap \mathfrak{F}_m = \overline{\text{span}}\,\{E_m^m\}$ for $m \geq 0$. This trivially implies double commutativity.

Now consider the case $b = 0$, $a \neq 0$ and $c = 0$. Then by Theorem 4.10(ii) and (4.16), we again have double commutativity. ∎

5. Concluding remarks. In this article, we introduced the concept of φ-doubly commuting quotient modules of $H^2(\mathbb{D}^2)$ (see Definition 3.4) to classify the doubly commuting quotient modules of $H^2(\triangle_H)$ (see Theorem 1.3). Although the present approach does not provide a complete classification of doubly commuting quotient modules of $H^2(\triangle_H)$, it establishes a connection between the doubly commuting quotient modules of $H^2(\mathbb{D}^2)$ and those of $H^2(\triangle_H)$.

We wrap up this paper with the following natural question for future investigation:

Characterize all doubly commuting quotient modules of $H^2(\triangle_H)$.

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