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# Automorphism groups of non-Archimedean groups

by

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**Abstract.** Let $\mathrm{Aut}(G)$ denote the group of (bi-)continuous automorphisms of a non-Archimedean Polish group $G$. We show that for any such $G$ with an invariant countable basis of open subgroups, $\mathrm{Aut}(G)$ carries a unique Polish topology that makes its natural action on $G$ continuous. Furthermore, for any class of groups allowing a Borel assignment of such bases, there is a functorial duality to a class of countable groupoids with a meet operation, extending work of the authors with Tent [J. Math. Logic, 2022]. This provides an alternative description of the topology of $\mathrm{Aut}(G)$. The results hold for instance for the class of locally Roelcke precompact non-Archimedean groups, which contains most classes studied previously. We further provide a model-theoretic proof that the outer automorphism group $\mathrm{Out}(G)$ of an oligomorphic group $G$ is locally compact, a result originally due to Paolini and the first author [arXiv:2410.02248].

**1. Introduction.** A Polish group $G$ is called *non-Archimedean* (*nA*) if $G$ has a countable basis $\mathcal{S}_G = \{U_n : n \in \omega\}$ of neighbourhoods of the identity that consists of open subgroups. Such a group $G$ is topologically isomorphic to a closed subgroup of $S_\infty$, the group of permutations of $\mathbb{N}$ with the topology of pointwise convergence. (To see this, one lets $G$ act on the left on the set of left cosets of subgroups in $\mathcal{S}_G$.) Conversely, each closed subgroup of $S_\infty$ is nA, via taking as $U_n$ the permutations in $G$ that fix each $i < n$. One verifies that the closed subgroups of $S_\infty$ are precisely the automorphism groups of structures with domain $\omega$. The class of nA groups enjoys several permanence properties: closed subgroups, quotients by closed normal subgroups, and countable Cartesian products of nA groups are again nA.

Our paper focuses on two groups derived from a nA group $G$: $\mathrm{Aut}(G)$, the group of *continuous* automorphisms of $G$, and $\mathrm{Out}(G)$, the quotient of

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$\mathrm{Aut}(G)$ by its normal subgroup $\mathrm{Inn}(G)$ of inner automorphisms. One says that a closed subgroup $G$ of $S_\infty$ is *oligomorphic* if for each $n \in \omega$, its action on $\omega^n$ has only finitely many orbits. We will in particular address $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ for oligomorphic $G$.

The paper has three interrelated parts.

(A) The natural action of $\mathrm{Aut}(G)$ on $G$ is given by $\alpha \cdot g = \alpha(g)$ for $\alpha \in \mathrm{Aut}(G)$ and $g \in G$. We provide a sufficient criterion when $\mathrm{Aut}(G)$ has a compatible Polish topology that makes this action continuous: *$G$ has a countable basis $\mathcal{S}_G$ of neighbourhoods of the neutral element consisting of open subgroups that can be chosen invariant under this action.* In this case, $\mathrm{Aut}(G)$ is nA itself.

(B) The class of oligomorphic groups satisfies the criterion; also, $\mathrm{Inn}(G)$ is closed in $\mathrm{Aut}(G)$. So $\mathrm{Out}(G)$ is a Polish group. We study it through the model-theoretic notion of bi-interpretations, and use this to give a model-theoretic proof of the result in [12] that $\mathrm{Out}(G)$ is totally disconnected and locally compact.

(C) Given a Borel class $\mathbf{G}$ of nA groups that satisfies a uniform version of the criterion in (A) due to Kechris et al. [7], we provide a Borel equivalence of the category $\mathbf{G}$ with isomorphism, and a category of countable structures with domain the coset of subgroups in $\mathcal{S}_G$. This provides an alternative way to obtain the Polish topology on $\mathrm{Aut}(G)$ making its action on $G$ continuous.

The study of $\mathrm{Out}(G)$ is motivated in part by the question whether the isomorphism relation between oligomorphic groups is smooth in the sense of Borel reducibility. An upper bound on this relation is known by [13]: it is essentially countable. However the precise complexity, first asked in [7], remains unknown. Certain subclasses are known to be smooth [12], such as the automorphism groups of $\omega$-categorical structures without algebraicity.

In contrast, the conjugacy relation on the Borel space of oligomorphic groups is smooth [13]. Towards answering the question, it is thus useful to know whether an isomorphism between two permutation groups $G$ and $H$ is induced by a conjugation with a permutation of $\mathbb{N}$. For a single permutation group $G$, $\mathrm{Out}(G)$ is trivial iff every continuous automorphism of $G$ is given by conjugating with a permutation of $\mathbb{N}$ in $G$.

We discuss the parts (A)–(C) in some detail, with proofs delegated to the main body of the paper. First we state the definition of Roelcke precompact Polish groups in the nA case.

Definition 1.1. A non-Archimedean group $G$ is *Roelcke precompact* if each open subgroup has only finitely many double cosets.

**(A) When is $\mathrm{Aut}(G)$ non-Archimedean?** The main result for this part will be summarised here; the detailed version is Theorem 2.1.

THEOREM 1.2. *Suppose a nA group satisfies the criterion in* (A) *above. Then there is a copy $\widehat{G}$ of $G$ as a closed subgroup of $S_\infty$ such that the group* $\mathrm{Aut}(G)$ *is isomorphic to the Polish group that is given as the normaliser of $\widehat{G}$ in $S_\infty$ by its centraliser in $S_\infty$. The induced topology on* $\mathrm{Aut}(G)$ *is the unique Polish topology that makes the action on $G$ continuous.*

The criterion implies that $\mathrm{Inn}(G)$ is a Borel subgroup of $\mathrm{Aut}(G)$ that is Polishable (Proposition 2.7). As an application of the criterion, $\mathrm{Aut}(G)$ is itself nA for several natural classes of nA groups. This includes the locally Roelcke precompact groups, where the neighbourhood basis in (A) consists of the Roelcke precompact open subgroups of Definition 1.1. We note that if $G$ is Roelcke precompact, then $\mathrm{Inn}(G)$ is in fact a closed subgroup of $\mathrm{Aut}(G)$ [12, Thm. 2.7].

**(B) Oligomorphic groups and bi-interpretations.** Each oligomorphic group is Roelcke precompact by [16, Thm. 2.4]. If $G$ is oligomorphic, the outer automorphism group $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is t.d.l.c. by [12, Thm. 3.10]. We wish to describe $\mathrm{Out}(G)$ based on the theory of any $\omega$-categorical structure that has $G$ as its automorphism group. For an $\omega$-categorical theory $T$, let $\mathcal{B}(T)$ be the group of self-interpretations of $T$ that have an inverse, all up to definable bijections between sorts of $T^{\mathrm{eq}}$ (for formal details see Definition 3.16). The following re-proves the result that $\mathrm{Out}(G)$ is t.d.l.c. using model theory, and gives a model-theoretic description of $\mathrm{Out}(G)$ based only on the theory of the underlying structure.

THEOREM 1.3. *Let $G$ be an oligomorphic group, and let $T$ be the elementary theory of a structure $M$ such that $G = \mathrm{Aut}(M)$.*

(i) *The group $\mathcal{B}(T)$ is totally disconnected, locally compact (t.d.l.c.).*
(ii) *$\mathcal{B}(T)$ is topologically isomorphic to* $\mathrm{Out}(G)$.

REMARK 1.4. Outer automorphism groups were introduced to study finite groups. For countable groups, outer automorphism groups are important in the study of mapping class groups of surfaces. Dehn, Nielsen and Baer (see [4, Theorem 8.1]) showed that the extended mapping class group $\mathrm{MCG}^+(S)$ of a compact closed orientable surface $S$ of genus $g \geq 1$ is isomorphic to $\mathrm{Out}(\pi(S))$. Here $\mathrm{MCG}^+(S)$ denotes the quotient of the group of orientation preserving homeomorphisms of $S$ by the connected component of the identity, and $\pi(S)$ is the fundamental group.

**(C) Borel equivalence of categories G and $\mathcal{M}$.** The closed subgroups of $S_\infty$ form a standard Borel space $\mathcal{U}(S_\infty)$, which is a subspace of the usual Effros space $\mathcal{F}(S_\infty)$ of closed subsets of $S_\infty$. Kechris et al. [7] studied Borel subclasses $\mathbf{G}$ of $\mathcal{U}(S_\infty)$ of groups $G$ for which the assignment of $\mathcal{S}_G$ to $G$ is Borel and isomorphism invariant. To assign $\mathcal{S}_G$ to $G$ in a Borel way

means that the relation

$$\{(G,U)\colon G \in \mathbf{Gr} \wedge U \in \mathcal{N}_G\}$$

is Borel. The invariance condition means that if $f\colon G \to H$ is a topological isomorphism then $U \in \mathcal{N}_G \leftrightarrow f(U) \in \mathcal{S}_H$. In particular, $\mathcal{S}_G$ is closed under the action of $\mathrm{Aut}(G)$. Kechris et al. [7] showed that the topological isomorphism relation on such a class $\mathbf{G}$ is classifiable by countable structures. Equivalence of categories was introduced by Eilenberg and Mac Lane, and will be recalled in Definition 4.1.

We establish an equivalence of

- the category that has as objects the groups in such a class, and as morphisms their topological isomorphisms,
- the category that has as objects a certain Borel class of countable structures for a finite signature, and as morphisms their isomorphisms.

The functors needed to establish this equivalence will be Borel.

**2. Non-Archimedean topology on** $\mathrm{Aut}(G)$**.** Let $H$ be a closed subgroup of $S_\infty$ (denoted $H \leq_c S_\infty$). By $N_{S_\infty}(H)$ we denote the normaliser, and by $C_{S_\infty}(H)$ the centraliser, of $H$ in $S_\infty$. Note that both are closed subgroups of $S_\infty$.

THEOREM 2.1 (Full version of Theorem 1.2). *Suppose $G$ is a Polish group with a countable basis $\mathcal{S}_G$ of neighbourhoods of the neutral element consisting of open subgroups, such that $\mathcal{S}_G$ is invariant under the action of* $\mathrm{Aut}(G)$ *on the open subgroups.*

(i) *There is a group $\widehat{G} \leq_c S_\infty$ and a topological isomorphism $\Theta\colon G \to \widehat{G}$ such that*

$$N_{S_\infty}(\widehat{G})/C_{S_\infty}(\widehat{G}) \cong \mathrm{Aut}(G)$$

*via sending an $\alpha \in N_{S_\infty}(\widehat{G})$ to its conjugation action on $G$. Using this isomorphism,* $\mathrm{Aut}(G)$ *can be topologised as a non-Archimedean group in such a way that its action on $G$ is continuous.*

(ii) *A neighbourhood basis of the identity for this topology on* $\mathrm{Aut}(G)$ *is given by the subgroups of the form*

$$(*) \qquad \Big\{\Phi \in \mathrm{Aut}(G)\colon \bigwedge_{i=1}^{n} [\Phi(A_i) = A_i]\Big\},$$

*where $A_1, \dots, A_n$ are cosets of subgroups in $\mathcal{S}_G$.*

(iii) *This topology on* $\mathrm{Aut}(G)$ *is the unique Polish topology that makes the action of* $\mathrm{Aut}(G)$ *on $G$ continuous.*

The proof uses the basic idea from Kechris et al. [7, proof of Thm. 3.1], removing references to Borelness and uniformity. Note that our $\mathcal{S}_G$ was denoted $\mathcal{N}_G$ there.

*Proof of Theorem 2.1(i).* Let $\mathcal{S}_G^*$ denote the set of left cosets of the subgroups in $\mathcal{S}_G$. Note that $G$ acts on the left on $\mathcal{S}_G^*$. Since $\mathcal{S}_G^*$ is countably infinite, we can fix a bijection $\rho_G\colon \omega \to \mathcal{S}_G^*$, so the left action of an element $g$ on $\mathcal{S}_G^*$ corresponds to a permutation $\Theta(g) \in S_\infty$. We let $\widehat{G}$ denote the subgroup of $S_\infty$ that is the range of $\Theta$, endowed with the topology inherited from $S_\infty$.

Claim 2.2 ([2, Thm. 1.5.1]; see also [7, Claim 3.2]). *The map $\Theta\colon G \to \widehat{G}$ is a topological group isomorphism. In particular, since $G$ is Polish, the group $\widehat{G}$ is a closed subgroup of $S_\infty$.*

For $\alpha \in N_{S_\infty}(\widehat{G})$ let $K_\alpha \in \mathrm{Aut}(\widehat{G})$ denote conjugation by $\alpha$, that is, $K_\alpha(h) = \alpha \circ h \circ \alpha^{-1}$.

Lemma 2.3. *A retraction $\Gamma\colon N_{S_\infty}(\widehat{G}) \to \mathrm{Aut}(G)$ with kernel $C_{S_\infty}(\widehat{G})$ is given by*

$$\Gamma(\alpha) = \Theta^{-1} \circ K_\alpha \circ \Theta.$$

*Proof.* Clearly $\Gamma$ is a group homomorphism. To show that $\Gamma$ is a retraction, we will define a homomorphism $\Delta\colon \mathrm{Aut}(G) \to S_\infty$ such that $\Gamma \circ \Delta$ is the identity on $\mathrm{Aut}(G)$. By our hypothesis that $\mathcal{S}_G$ is invariant under the action of $\mathrm{Aut}(G)$, any $\phi \in \mathrm{Aut}(G)$ induces a bijection $\bar{\phi}\colon \mathcal{S}_G^* \to \mathcal{S}_G^*$ via

$$\bar{\phi}(rU) = \phi(r)\phi(U) \quad \text{for } U \in \mathcal{S}_G \text{ and } r \in G.$$

Let $\Delta(\phi)$ be this bijection viewed as a permutation of $\omega$, that is, $\Delta(\phi) = \rho_G^{-1} \circ \bar{\phi} \circ \rho_G$.

Claim 2.4. *Given $\phi \in \mathrm{Aut}(G)$ let $\alpha := \Delta(\phi)$. For any $g \in G$ we have*

$$\alpha \circ \Theta(g) \circ \alpha^{-1} = \Theta(\phi(g)).$$

*Thus $\alpha \in N_{S_\infty}(\widehat{G})$ and $\Gamma(\alpha) = \phi$.*

To see this, let $A \in \mathcal{S}_G^*$ and write $A = \phi(r)\phi(U)$ for some $r \in G$, and $U \in \mathcal{S}_G$. Then (suppressing $\rho_G$) we have $\alpha^{-1}(A) = rU$, so

$$(\alpha \circ \Theta(g) \circ \alpha^{-1})(A) = \alpha(grU) = \phi(g)\phi(r)\phi(U) = \phi(g)A,$$

as required. ∎

The group $N_{S_\infty}(\widehat{G})/C_{S_\infty}(\widehat{G})$ is algebraically isomorphic to $\mathrm{Aut}(G)$ via the isomorphism induced by the retraction $\Gamma$ that sends $\alpha$ to its conjugation action on $\widehat{G} \cong G$. It is non-Archimedean as a quotient of $N_{S_\infty}(\widehat{G})$ by a closed normal subgroup. We use this isomorphism to transfer its topology to $\mathrm{Aut}(G)$. This makes the action on $G$ continuous, because the action of $N_{S_\infty}(\widehat{G})$ on $\widehat{G}$ by conjugation is continuous, and for each $g \in G$ and $\alpha \in N$, we have $\Theta^{-1}(K_\alpha(\Theta(g))) = \Gamma(\alpha)(g)$. ∎

Remark 2.5. Since $\Delta \circ \Gamma$ is continuous, $\Delta$ is a topological isomorphism between $\mathrm{Aut}(G)$ and a closed subgroup $H$ of $S_\infty$ contained in $N_{S_\infty}(\widehat{G})$. Thus

$N_{S_\infty}(\widehat{G})$ is a topological split extension: $H \cap C_{S_\infty}(\widehat{G}) = \{1\}$ and $HC_{S_\infty}(\widehat{G}) = N_{S_\infty}(\widehat{G})$.

*Proof of Theorem 2.1(ii).* Write $C = C_{S_\infty}(\widehat{G})$. Given a subgroup $\mathcal{A}$ as in $(*)$, let $A_i$ be a left coset of a subgroup $U_i \in \mathcal{S}_G$. Let $\mathcal{U}$ be the pointwise stabiliser of the set $\{U_1, A_1, \dots, U_n, A_n\}$ where we identify the elements of $\mathcal{S}_G^*$ with their code numbers.

We claim that $\alpha \in C\mathcal{U}$ iff $\Gamma(\alpha) \in \mathcal{A}$. This suffices for (ii) since the subgroups of the form $C\mathcal{U}/C$ form a neighbourhood basis of the identity for $N_{S_\infty}(\widehat{G})/C$.

If $\alpha \in \mathcal{U}$ then for each $i$, and for each $g \in G$, writing $p = \Theta(g)$, we have

$$g \in A_i \leftrightarrow p(U_i) = A_i \leftrightarrow \alpha \circ p \circ \alpha^{-1}(U_i) = A_i \leftrightarrow K_\alpha(p)(U_i) = A_i$$
$$\leftrightarrow \Gamma(\alpha)(g) \in A_i.$$

Thus $\Gamma(\alpha) \in \mathcal{A}$.

Now suppose $\phi = \Gamma(\alpha) \in \mathcal{A}$. Since $U_i = A_i^{-1}A_i$, this implies that the automorphism $\phi$ fixes $U_i$. Hence $\Delta(\phi) \in \mathcal{U}$. Since $\alpha(\Delta(\phi))^{-1} \in C$, this verifies the assertion. ∎

We next establish Theorem 2.1(iii), the uniqueness of a Polish topology on $\mathrm{Aut}(G)$ that makes the action on $G$ continuous. We prove a more general result. To obtain Theorem 2.1(iii) from it, one chooses as the basis $\mathcal{B}$ the set of left cosets of subgroups in $\mathcal{S}_G$.

PROPOSITION 2.6. *Let $G$ be a Polish group, and suppose that there is a countable basis $\mathcal{B}$ of clopen sets for the topology of $G$ such that $\mathcal{B}$ is invariant under the natural action of* $\mathrm{Aut}(G)$. *Let $\tau$ be the topology induced on* $\mathrm{Aut}(G)$ *by the neighbourhood basis $(*)$ of the identity, for $A_1, \dots, A_n \in \mathcal{B}$.*

(a) *If $\sigma$ is a Baire group topology on* $\mathrm{Aut}(G)$ *making its action on $G$ continuous, then $\tau \subseteq \sigma$.*
(b) *If $\tau$ is a Polish group topology on* $\mathrm{Aut}(G)$ *making its action on $G$ continuous, then it is uniquely so.*

*Proof.* (a) Every Baire measurable homomorphism from a Baire group $K$ to a separable group $L$ is continuous [6, Thm. 9.10]. (Here, to be Baire measurable means that the pre-image of every open set in $L$ has meagre symmetric difference with some open set in $K$.) We will show that the identity homomorphism $(\mathrm{Aut}(G), \sigma) \to (\mathrm{Aut}(G), \tau)$ is Baire measurable; we conclude that it is continuous and hence $\tau \subseteq \sigma$.

Each Borel set for $\sigma$ has meagre symmetric difference with some open set in the sense of $\sigma$ (i.e., it has the property of Baire for $\sigma$). We verify that the $\tau$-open sets in the neighbourhood basis $(*)$ of the identity automorphism are $G_\delta$ for $\sigma$; this will imply that each $\tau$-open set is Borel for $\sigma$.

To do so, for $A \in \mathcal{B}$, we verify that the set

$$\mathcal{S} = \{\Phi \in \mathrm{Aut}(G) \colon \Phi(A) = A\}$$

is $G_\delta$ with respect to $\sigma$. Since $A$ is closed, let $D$ be a countable dense subset of $A$. Then

$$\mathcal{S} = \bigcap_{g \in D} \{\Phi \in \mathrm{Aut}(G) \colon \Phi(g) \in A \wedge \Phi^{-1}(g) \in A\}.$$

Thus $\mathcal{S}$ is $G_\delta$ for $\sigma$ because $A$ is open and $\sigma$ makes the action of $\mathrm{Aut}(G)$ on $G$ continuous. Now each set in $(*)$ is a finite intersection of such sets, and hence $G_\delta$ for $\sigma$ as well.

(b) If $\sigma$ is Polish, then it is Baire, so $\tau \subseteq \sigma$. This implies $\tau = \sigma$ because no Polish group topology on a group can be properly contained in another (see, e.g., [5, 2.3.4]). ∎

We close this section with some remarks on the group of inner automorphisms of $G$ as a subgroup of $\mathrm{Aut}(G)$. Note that if $\alpha = \Theta(g)$ then $K_\alpha$ defined after Claim 2.2 induces the inner automorphism of $G$ given by conjugation by $g$: for each $h \in G$,

$$\Gamma(\Theta(g))(h) = [\Theta^{-1} \circ K_{\Theta(g)} \circ \Theta](h) = \Theta^{-1}(\Theta(h)^{\Theta(g)}) = h^g.$$

The group $\mathrm{Inn}(G)$ is in general not closed in $\mathrm{Aut}(G)$, even for discrete groups $G$ (see [12, Section 2]). Via the isomorphism $N_{S_\infty}(\widehat{G})/C_{S_\infty}(\widehat{G}) \cong \mathrm{Aut}(G)$ established above, the subgroup $\Theta(G)C_{S_\infty}(\widehat{G})/C_{S_\infty}(\widehat{G})$ corresponds to $\mathrm{Inn}(G)$, so this subgroup can fail to be closed. However, it satisfies a weaker condition. Recall that a Borel subgroup $H$ of a Polish group is *Polishable* if $H$ carries a (unique) Polish group topology such that the $\sigma$-algebra it generates coincides with the Borel sets inherited from $G$.

Proposition 2.7. *Let $G$ be as in Theorem* 2.1. *Then* $\mathrm{Inn}(G)$ *is a Borel subgroup of* $\mathrm{Aut}(G)$ *that is Polishable.*

*Proof.* We verify that the natural map $\Phi \colon G/Z(G) \to \mathrm{Aut}(G)$ that sends a $g \in G$ to its conjugation action on $G$ is continuous. Note that each left coset $gU$ of an open subgroup $U$ of $G$ is also a right coset of the open subgroup $gUg^{-1}$. Given a basic open subgroup $\mathcal{U}$ of $\mathrm{Aut}(G)$ as in $(*)$, suppose $A_i$ is a right coset of $U_i$ and left coset of $V_i$. If $g \in \bigcap_i U_i \cap V_i$ then $A_i^g = A_i$ for each $i$, so $\Phi(g) \in \mathcal{U}$.

Now, since $\Phi$ is injective, by the Luzin–Suslin theorem, $\Phi(A)$ is Borel for each Borel set $A \subseteq G$. So $\mathrm{Inn}(G)$ is Borel in $\mathrm{Aut}(G)$, and the Borel structure on $G/Z(G)$ equals the Borel structure on $\mathrm{Inn}(G)$ inherited from $\mathrm{Aut}(G)$. ∎

Remark 2.8. Proposition 2.7 implies that for $G$ as in Theorem 2.1, the outer automorphism group $\mathrm{Out}(G)$ is a group with a Polish cover in the sense of [3].

**3. Oligomorphic groups and invertible self-interpretations.** This section will define the topological group $\mathcal{B}(T)$ of invertible self-interpretations up to a syntactic notion of homotopy of an $\omega$-categorical theory $T$ (Definition 3.16). Then it establishes in the proof of Theorem 1.3 that $\mathcal{B}(T)$ is t.d.l.c. and isomorphic to $\mathrm{Out}(G)$, the group of outer automorphisms of $G$, whenever $G = \mathrm{Aut}(M)$ for some model $M$ of $T$ with domain $\omega$.

Firstly, we represent $\mathrm{Aut}(G)$ as a topological group $\mathcal{B}(M)$ of self-interpretations of $M$ (Definition 3.7). They are taken modulo an appropriate equivalence relation denoted $\sim$ that equates them up to definable bijection between their domains; granted that, they are invertible.

Secondly, we show that passing from invertible self-interpretations of $M$ to invertible self-interpretations of its theory corresponds to passing from $\mathrm{Aut}(G)$ to $\mathrm{Out}(G)$: by Lemma 3.19, two self-interpretations $\alpha_0, \alpha_1$ of $M$ get identified in this process iff $\pi \circ \alpha_0 \sim \alpha_1$ for some $\pi \in G$, which means for invertible interpretations that the corresponding automorphisms of $G$ are equal up to an inner automorphism of $G$.

### 3.1. Interpretations between $\omega$-categorical structures

Convention 3.1. Throughout, let $M, N$ denote $\omega$-categorical structures, and let $G = \mathrm{Aut}(M)$ and $H = \mathrm{Aut}(N)$.

For interpretations between $\omega$-categorical structures we "import" the terminology and notation of Ahlbrandt and Ziegler [1]. The $\omega$-categorical structures with interpretations form a category; morphisms are denoted $\alpha\colon M \rightsquigarrow N$. This means that there is a dimension $k \geq 1$ and a definable set $D \subseteq M^k$ (always without parameters) such that $\alpha$ is a map $D \to N$ that is onto with definable kernel $E$; furthermore, each $N$-definable relation $R$ has an $M$-definable pre-image under $\alpha$. We will abuse notation by also viewing $\alpha$ as a bijection $D/E \to N$. By $\mathrm{Int}(M, N)$ we denote the set of morphisms $M \rightsquigarrow N$. Composition of morphisms is defined canonically.

The structure $M^{\mathrm{eq}}$ has infinitely many sorts $D/E$ as above. Its elements are called *imaginaries* over $M$; see [15]. A function $\theta\colon D_0/E_0 \to D_1/E_1$ is called *$M$-definable* if the relation $\{\langle d_0, d_1\rangle\colon \theta(d_0/E_0) = d_1/E_1\}$ is $M$-definable.

Ahlbrandt and Ziegler [1] extend the operation $M \mapsto \mathrm{Aut}(M)$ to a functor from the category of $\omega$-categorical structures with interpretations to oligomorphic groups with continuous homomorphisms, as follows.

Definition 3.2 (essentially [1]). For an interpretation $\alpha\colon M \rightsquigarrow N$ let $\mathrm{Aut}(\alpha)(g) = \alpha \circ g^{\mathrm{eq}} \circ \alpha^{-1}$ for $g \in \mathrm{Aut}(M)$, where $g^{\mathrm{eq}}$ is the canonical extension of $g$ to an automorphism of $M^{\mathrm{eq}}$.

For each sort $D/E$ of $M^{\mathrm{eq}}$, the interpretations $\alpha\colon D \to N$ with kernel $E$ can be seen as a closed subspace of the space of functions $D/E \to N$ with the topology of pointwise convergence. So $\mathrm{Int}(M, N)$ is a topological space

which is a disjoint union of clopen subspaces corresponding to the sorts. The $\omega$-categorical structures with interpretations as morphisms thus form a Polish category: each set of morphisms is a Polish space in a Borel way, and the operations of product and inverse are continuous.

LEMMA 3.3. *Let $M, N$ and $G, H$ be as in Convention* 3.1. *The map* $\mathrm{Int}(M,N) \times G \to H$ *given by* $\alpha \cdot g = \mathrm{Aut}(\alpha)(g)$ *is continuous.*

*Proof.* Suppose $\mathrm{Aut}(\alpha)(g)$ is in the subbasic open set of elements of $H$ that send $r$ to $s$. We have $\alpha(\overline{a}) = r$ and $\alpha(\overline{b}) = s$ for some $\overline{a}, \overline{b} \in D$ such that $g(\overline{a}/E) = \overline{b}/E$. If an interpretation $\alpha' \colon D \to N$ with the same kernel $E$ has the same values as $\alpha$ on these two tuples and an element $g' \in G$ also satisfies $g'(\overline{a}/E) = \overline{b}/E$, then $\mathrm{Aut}(\alpha')(g')$ sends $r$ to $s$ as well. The set of such pairs $\alpha', g'$ is open. ∎

Ahlbrandt and Ziegler [1] defined a "homotopy" relation on $\mathrm{Int}(M,N)$:

DEFINITION 3.4 (Homotopy of interpretations). For morphisms $\alpha_i : M \rightsquigarrow N$ $(i = 0, 1)$, where $\alpha_i \colon D_i/E_i \to N$, one says that $\alpha_0$ is *homotopic* to $\alpha_1$, written $\alpha_0 \sim \alpha_1$, if there is an $M$-definable bijection $\theta \colon D_0/E_0 \to D_1/E_1$ such that $\alpha_0 = \alpha_1 \circ \theta$.

Note that for any bijection $\theta \colon D_0/E_0 \to D_1/E_1$, the set of pairs $\langle \alpha_0, \alpha_1 \rangle$ in $\mathrm{Int}(M,N)$ with $\alpha_i \colon D_i/E_i \to N$ and $\alpha_0 = \alpha_1 \circ \theta$ is closed in $\mathrm{Int}(M,N)^2$.

LEMMA 3.5.

(i) *The equivalence relation $\sim$ is closed.*
(ii) *For each open set $U \subseteq \mathrm{Int}(M,N)$, the saturation $[U]_\sim$ is also open.*

*Proof.* (i) holds since there are only finitely many $M$-definable bijections $D_0/E_0 \to D_1/E_1$.

(ii) We may assume that $U$ is of the form $\{\alpha \colon \alpha(d_i) = n_i \ (i = 1, \dots, k)\}$, where $d_i \in D/E$ for some sort $D/E$ of $M^{\mathrm{eq}}$, and $n_i \in N$. Then $[U]_\sim$ is the union of sets $\{\alpha \circ \theta \colon \alpha \in U\}$ over all definable bijections $\theta \colon D'/E' \to D/E$; such a set equals $\{\beta \colon D'/E' \to N \colon \beta(\theta^{-1}(d_i)) = n_i\}$ and hence is open. ∎

Clearly $\sim$ is compatible with composition of morphisms. So, replacing interpretations by their equivalence classes with respect to $\sim$, we obtain a Polish quotient category. In particular, $\mathrm{Int}(M,M)/_\sim$ is a Polish monoid.

REMARK 3.6. For any Polish monoid $(S, \cdot, 1)$, the group $L$ of (two-sided) invertible elements is a Polish group. Indeed: $L$ is algebraically isomorphic to $\{(a,b) \in S \times \check{S} \colon ab = ba = 1\}$, which is a closed subset of the Polish space $S \times \check{S}$ also closed under the monoid operation (here $\check{S}$ is the dual monoid $(S, \cdot', 1)$, where $r \cdot' s = s \cdot r$). The operation of inversion corresponds to exchanging the two components of a pair, which is a continuous operation.

### 3.2. $\mathcal{B}(M)$ is topologically isomorphic to $\mathrm{Aut}(G)$

DEFINITION 3.7. Let $\mathcal{B}(M)$ be the Polish group of invertible elements of the Polish monoid $\mathrm{Int}(M,M)/_\sim$.

Recall the functor Aut of Definition 3.2. Ahlbrandt and Ziegler [1, Thm. 1.2] show that a continuous homomorphism $R\colon \mathrm{Aut}(M) \to \mathrm{Aut}(N)$ is of the form $\mathrm{Aut}(\alpha)$ for some $\alpha\colon M \rightsquigarrow N$ if and only if the action of $\mathrm{Aut}(M)$ on $N$ induced by $R$ has only finitely many 1-orbits. They also show in [1, Thm. 1.3] that $\mathrm{Aut}(\alpha) = \mathrm{Aut}(\beta)$ if and only if $\alpha \sim \beta$. So the functor Aut induces an algebraic isomorphism of groups $\gamma_M\colon \mathcal{B}(M) \to \mathrm{Aut}(G)$ for $G = \mathrm{Aut}(M)$.

LEMMA 3.8. *$\gamma_M\colon \mathcal{B}(M) \to \mathrm{Aut}(G)$ is a topological isomorphism.*

*Proof.* $\mathcal{B}(M)$ is a Polish group by 3.6, and algebraically isomorphic to $\mathrm{Aut}(G)$ via $\gamma_M$. So, by Proposition 2.6(b), it suffices to show that the action of $\mathcal{B}(M)$ on $G$ given by $[\alpha] \cdot g = \gamma_M(\alpha)(g)$ is continuous. By Lemmas 3.3 and 3.5, the action of $S = \mathrm{Int}(M,M)/_\sim$ on $G$ is continuous. So the action of $S \times \check{S}$ on $G$ given by $([\alpha]_\sim, [\beta]_\sim) \cdot g = [\alpha] \cdot g$ is continuous. This implies the statement. ∎

### 3.3. $\mathcal{B}(T)$ is topologically isomorphic to $\mathrm{Out}(G)$

*The Polish category of $\omega$-categorical theories with interpretations.* In the following, all languages are relational, and all theories are assumed to be $\omega$-categorical. We begin by discussing interpretations of theories, using a notation that is coherent with the approach of Ahlbrandt and Ziegler [1] to the semantic setting. This is somewhat more work than in [1], because all the definitions need to be syntactical (though we sometimes use semantic terminology).

DEFINITION 3.9 (Interpretation of a theory $U$ in a theory $T$). An *interpretation* $\alpha\colon T \rightsquigarrow U$ is given by objects (a) and (b):

(a.i) a dimension $k \geq 1$ [notation: for each variable $z$ we have a $k$-tuple of variables $\overline{z} = (z_1, \ldots, z_k)$],

(a.ii) formulas $\phi_D(\overline{x})$ and $\phi_E(\overline{x}, \overline{y})$ in the language of $T$ such that

$$T \vdash \text{“}D \neq \emptyset\text{”} \quad \text{and} \quad T \vdash \text{“}E \text{ is an equivalence relation on } D\text{”},$$

where $D$ denotes the definable set (in some model of $T$) given by $\phi_D$, and $E$ the equivalence relation given by $\phi_E$,

(a.iii) for each atomic formula $R(x_1, \ldots, x_n)$ in the language of $U$, excluding equality, a formula $\alpha.R(\overline{x}_1, \ldots, \overline{x}_n)$ in the language of $T$ such that $T$ proves its invariance under the equivalence relation $E$.

For any formula $\phi$ in the language of $U$, we define $\alpha.\phi$ to be the formula in the language of $T$ obtained as follows:

(b.i) replace each atomic formula $R(y_1, \dots, y_n)$ with $\alpha.R(\overline{y}_1, \dots, \overline{y}_n)$,
(b.ii) replace $x = y$ with $\phi_E(\overline{x}, \overline{y})$,
(b.iii) replace each quantifier $\forall x$ (or $\exists x$) with $k$ quantifiers of the same type over the components of $\overline{x}$.

We require the following condition to hold:

$$C_{T,U}(\alpha): \quad \text{for every sentence } \psi \in U, \text{ we have } \alpha.\psi \in T.$$

Note that if the signature of $U$ is finite, then the set of interpretations $T \rightsquigarrow U$ is countable. Interpretations are composed similar to the case of structures in [1, p. 65]. The trivial self-interpretation of $T$ in itself is the interpretation $1_T\colon T \rightsquigarrow T$ where $\phi_D(x)$ is $x = x$, $\phi_E$ is the identity, and $\phi_R$ is $Rx_1 \dots x_n$ for each $n$-ary relation symbol $R$.

*The interpretations between theories can be seen as a path space of a tree.*

Convention 3.10. We henceforth assume that for each theory $T$, the set of formulas in its language is provided with a wellordering of type $\omega$ such as the length-lexicographical ordering, and that formulas will be least in their class of equivalence under $T$. Then, for each $n$-tuple of variables $x_1, \dots, x_n$, there are only finitely many formulas $\phi(x_1, \dots, x_n)$; recall here that all theories are assumed to be $\omega$-categorical.

Definition 3.11. The set of interpretations $\alpha\colon T \rightsquigarrow U$ is denoted by $\mathrm{Int}(T, U)$.

Remark 3.12. $\mathrm{Int}(T, U)$ can be seen as the set of paths of a subtree $B_{T,U}$ of $\omega^{<\omega}$, as follows. Only the root (the empty string) can be infinitely branching. The first level of the tree encodes the triples $c = \langle k, \phi_D, \phi_E \rangle$ that $T$ allows. Each further level (the set of strings of a length $k > 1$) is dedicated to the $k$th relation symbol $R$ of $L_U$: the value $n$ on level $k$ decides that the atomic formula $Rx_1 \dots x_n$ in $L_U$ is assigned to the $n$th formula in $L_T$ (there are only finitely many choices, since this formula needs to be invariant under $E$ according to $T$). Let $B_{T,U}$ be the tree of strings $\sigma$ such that the condition $C_{T,U}(\sigma)$ in Definition 3.9 holds for each sentence $\psi$ such that all relation symbols of $U$ occurring in it have been assigned by $\sigma$.

Remark 3.13. Whether the condition $C_{T,U}(\alpha)$ *fails* for an interpretation $\alpha$ can be seen from a finite amount of information, so the path space $[B_{T,U}]$ can be identified with the set of interpretations. We have a natural locally compact, totally disconnected topology on this space where the subbasic open sets are of the form $[\sigma]$ for $\sigma \in B_{T,U}$.

*A syntactic version of homotopy, and the quotient category* $\mathbb{T}$. Homotopy is simpler for theories than in the case of models (Definition 3.4). We use a semantic terminology for easier readability.

Definition 3.14 (The notation $\approx_T$). For interpretations $\alpha_0, \alpha_1 \colon T \rightsquigarrow U$, where $\alpha_i$ is based on (formulas defining) sorts $D_i/E_i$ in any model, we write $\alpha_0 \approx_T \alpha_1$ if for some formula $\phi_\Theta$ the theory $T$ contains the sentence expressing that $\phi_\Theta$ defines a bijection $\Theta \colon D_0/E_0 \to D_1/E_1$, and for each $n$-ary relation symbol $R$ of $L_U$ the theory $T$ contains the sentence expressing that $\Theta^n(\alpha_0.R) = \alpha_1.R$.

Theories with composition of interpretations form a category. We obtain a quotient category $\mathbb{T}$. Its objects are the $\omega$-categorical theories in a countable relational language. Its morphisms $T \rightsquigarrow U$ are the equivalence classes $[\alpha]_{\approx_T}$ of interpretations. $\mathrm{Mor}(T, U)$ denotes this set of morphisms from $T$ to $U$ in $\mathbb{T}$.

Fact 3.15. *Composition of morphisms in $\mathbb{T}$ is well-defined and associative.*

*Proof.* Suppose first that we have interpretations $\alpha_0, \alpha_1 \colon T \rightsquigarrow U$ and $\beta \colon U \rightsquigarrow V$ such that $\alpha_0 \approx_T \alpha_1$ via a formula $\theta$ in the language of $T$. Then $\beta \circ \alpha_0 \approx_T \beta \circ \alpha_1$ via the same formula.

Suppose next that we have interpretations $\alpha \colon T \rightsquigarrow U$ and $\beta_0, \beta_1 \colon U \rightsquigarrow V$ such that $\beta_0 \approx_U \beta_1$ via a formula $\xi$ in the language of $U$. Then $\alpha \circ \beta_0 \approx_T \alpha \circ \beta_1$ via the formula $\alpha.\xi$. ∎

Definition 3.16. Let $\mathcal{B}(T)$ be the group of invertible elements of the topological monoid $\mathrm{Mor}(T, T)$ (see Remark 3.6).

The topology on the path space $[B_{T,U}]$ induces a quotient topology on $\mathrm{Mor}(T, U)$.

Lemma 3.17 (Properties of morphism spaces).

(i) *The projection $q \colon [B_{T,U}] \to \mathrm{Mor}(T, U)$ that sends $\alpha$ to its equivalence class $[\alpha]_\approx$ is open.*
(ii) *The space $\mathrm{Mor}(T, U)$ has a basis consisting of clopen sets, and hence is totally disconnected.*
(iii) *The space $\mathrm{Mor}(T, U)$ is locally compact.*
(iv) *$\mathcal{B}(T)$ is canonically a totally disconnected, locally compact (t.d.l.c.) group, which is discrete in the case that the signature of $T$ is finite.*

*Proof.* We write $\approx$ for $\approx_T$.

(i) It suffices to show that $[\sigma]_\approx$ is open in $[B_{T,U}]$ for each $\sigma \in B_{T,U} - \{\emptyset\}$. By $c, d$ etc. we denote sorts of $T$, encoded by triples $\langle k, \phi_D, \phi_E \rangle$ as in Definition 3.9. They can be identified with strings on $B_{T,U}$ of length 1. Given $c_i = \langle k_i, \phi_{D_i}, \phi_{E_i} \rangle$, if $T$ proves that $\phi_\Theta$ defines a bijection $\Theta \colon D_0/E_0 \to D_1/E_1$ such that for each $n$-ary relation symbol $R$ of $L_U$ we have $\Theta^n(\alpha_0.R) = \alpha_1.R$, then $\phi_\Theta$ defines a homeomorphism $h_\theta \colon [c_0] \to [c_1]$. This implies that $[\sigma]_\approx$ is open for each such $\sigma$, as required.

(ii) It suffices to show that the complement of $[\sigma]_\approx$ is open in $[B_{T,U}]$ for each $\sigma \in B_{T,U} - \{\emptyset\}$. Suppose the first entry of $\sigma$ is $c_0$. Clearly $\mathcal{D} = [c] - [\sigma]_\approx$ is open. The complement of $[\sigma]_\approx$ now consists of all the images of $\mathcal{D}$ under homeomorphisms $h_\Theta \colon [c_0] \to [c_1]$ as above, together with all $[d]$ such that there are no such homeomorphisms with $[c]$. Hence the complement of $[\sigma]_\approx$ is open, as required.

(iii) holds because the open set $[c]_\approx$ is compact in $\mathrm{Mor}(T, U)$ for each sort $c$ of $T$, and each $\alpha_\approx$ has such a set as a neighbourhood.

(iv) Recall from Remark 3.6 that the elements of $\mathcal{B}(T)$ are represented by pairs $\langle [\alpha]_\approx, [\beta]_\approx \rangle \in \mathrm{Mor}(T, T)$ such that $\alpha\beta \approx \beta\alpha \approx 1_T$. As such, they form a closed subset of $\mathrm{Mor}(T, T)$, which is therefore t.d.l.c. Clearly the group operations are continuous. ∎

*From interpretations of structures to interpretations of theories*

Definition 3.18 (The forgetful functor Th). Suppose $T = \mathrm{Th}(M)$ and $U = \mathrm{Th}(N)$ for countable structures $M, N$ in relational languages. Let $\alpha \colon M \rightsquigarrow N$ be a $k$-dimensional interpretation of structures. Let $\phi_U$ and $\phi_E$ be formulas defining in $M$ the domain and $\{\langle \overline{x}, \overline{y} \rangle \colon \alpha(\overline{x}) = \alpha(\overline{y})\}$, respectively. Denote by $\widehat{\alpha} = \mathrm{Th}(\alpha)$ the interpretation of the theory $U$ in the theory $T$ given by $\langle k, \phi_D, \phi_E \rangle$ and the assignment that $\widehat{\alpha}.R$ is the formula defining $\alpha^{-1}(R^N)$ in $M$ (which is unique by Convention 3.10).

If $\alpha_0, \alpha_1 \in \mathrm{Int}(M, N)$ and $T = \mathrm{Th}(M)$, we will write $\alpha_0 \approx \alpha_1$ as an abbreviation for $\widehat{\alpha}_0 \approx_T \widehat{\alpha}_1$. Recall the relation of homotopy on $\mathrm{Int}(M, N)$ from Definition 3.4.

Lemma 3.19. *Suppose $\alpha_0, \alpha_1 \in \mathrm{Int}(M, N)$. Then*

$\alpha_0 \approx \alpha_1$ *iff there exists some* $\pi \in \mathrm{Aut}(N)$ *such that* $\pi \circ \alpha_0 \sim \alpha_1$.

*Proof.* Let $\alpha_i \colon D_i/E_i \to N$. Recall that the $\alpha_i$ are onto maps. Suppose the right hand side holds. Clearly $\pi \circ \alpha_0 \approx \alpha_0$. The hypothesis implies $\pi \circ \alpha_0 \approx \alpha_1$. Since the relation $\approx$ on $\mathrm{Int}(M, N)$ is transitive, we conclude that $\alpha_0 \approx \alpha_1$.

Now suppose the left hand side holds via a formula $\theta$. Let $f \colon D_0/E_0 \to D_1/E_1$ be the bijection defined by $\theta$ in $M$. Then $\pi = \alpha_1 \circ f \circ \alpha_0^{-1}$ is well-defined and is a permutation of $N$. It suffices to show that $\pi \in \mathrm{Aut}(N)$. Let $k \geq 1$ and $A \subseteq N^k$ be definable in $N$. Since $\alpha \approx \beta$ via $\theta$, we have $f(\alpha_0^{-1}(A)) = \alpha_1^{-1}(A)$. Therefore $\pi(A) = \alpha_1(f(\alpha_0^{-1}(A))) = A$. ∎

Lemma 3.20.

(i) Th *is a full functor from the category of $\omega$-categorical countable structures to the category of $\omega$-categorical theories.*
(ii) *If $\alpha, \beta \colon M \rightsquigarrow N$ and $\alpha \sim \beta$, then* $\mathrm{Th}(\alpha) \approx \mathrm{Th}(\beta)$.
(iii) Th *induces a surjective group homomorphism $\chi_M \colon \mathcal{B}(M) \to \mathcal{B}(T)$.*
(iv) *$\chi_M$ is continuous.*

*Proof.* (i) Th is clearly a functor. To see that Th is full, suppose that $\alpha \in \mathrm{Mor}(T,T)$. Let $M$ be a countable model of $T$. By $(*)_{T,U}$, $D/E$ is a model of $T$, where $D$ and $E$ are as in Definition 3.9. Since $T$ is $\omega$-categorical, there exists an isomorphism $D/E \to M$. The lifting $\gamma\colon D \to M$ to $D$ then satisfies $\widehat{\gamma} = \alpha$.

(ii) is evident.

(iii) $\chi_M$ is clearly a homomorphism. To show it is onto, suppose that $(\alpha,\beta)$ represents an element of $\mathcal{B}(T)$. Since Th is full by (i), there exist $\gamma,\delta \in \mathrm{Int}(M,M)$ such that $\widehat{\gamma} = \alpha$ and $\widehat{\delta} = \beta$. Since $\gamma \circ \delta \approx_T \mathrm{id}$, by Lemma 3.19 there is $\pi \in \mathrm{Aut}(M)$ such that $(\pi \circ \gamma) \circ \delta \sim \mathrm{id}$. Since $\pi \circ \gamma \in \mathrm{Int}(M,M)$ satisfies $\widehat{\pi \circ \gamma} = \alpha$, $\chi_M$ is surjective.

(iv) is clear by (iii) and by the functor Th being continuous on $\mathrm{Int}(M,N)$. ∎

Recall the topological isomorphism $\gamma_M$ from Lemma 3.8, and the surjective group homomorphism $\chi_M$ from Lemma 3.20(iii):

$$\mathrm{Aut}(G) \xleftarrow{\gamma_M} \mathcal{B}(M) \xrightarrow{\chi_M} \mathcal{B}(T).$$

Our goal is to show that $\chi_M \circ \gamma_M^{-1}$ induces a group homeomorphism $\mathrm{Out}(G) \to \mathcal{B}(T)$ as required for Theorem 1.3(ii).

Lemma 3.21. *Let $\alpha \in \mathrm{Int}(M,M)$. Then*

$$\gamma_M(\alpha) \in \mathrm{Inn}(G) \iff \chi_M([\alpha]_\sim) = 1.$$

*Proof.* First suppose that the left-hand side holds. Thus, in the notation of Definition 3.2, there is $g \in G$ such that $\mathrm{Aut}(\alpha) = \mathrm{Aut}(g)$. Therefore $\alpha \sim g$, which implies $\widehat{\alpha} = \widehat{g}$. Clearly $\widehat{g} \approx 1_T$, whence the right-hand side holds.

Now suppose the right-hand side holds. Then $\widehat{\alpha} \approx \widehat{\mathrm{id}_M}$. So by Lemma 3.19, there is $\pi \in \mathrm{Aut}(M) = G$ such that $\pi \circ \alpha \sim \mathrm{id}_M$, and hence $\alpha \sim h$, where $h = \pi^{-1} \in G$. So $\gamma_M(\alpha) = \gamma_M(h)$, which is in $\mathrm{Inn}(G)$. ∎

*Proof of Theorem 1.3.* $\mathcal{B}(T)$ is totally disconnected and locally compact by Lemma 3.17. It remains to show that it is topologically isomorphic to $\mathrm{Out}(G)$.

By Lemma 3.8 we have a topological isomorphism $\gamma_M^{-1}\colon \mathrm{Aut}(G) \to \mathcal{B}(M)$. By Lemma 3.21, $\gamma_M^{-1}$ sends $\mathrm{Inn}(G)$ to the kernel $K$ of $\chi_M$. So, $\chi_M$ being surjective, $\chi_M \circ \gamma_M^{-1}$ induces a group homeomorphism $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G) \cong \mathcal{B}(M)/K \cong \mathcal{B}(T)$, as required. ∎

## 4. Borel equivalence between categories of groups and of meet-groupoids

### 4.1. Some category-theoretic preliminaries

Definition 4.1 (Mac Lane [8]). (a) Given a category $\mathcal{C}$, a functor $\Upsilon\colon \mathcal{C} \to \mathcal{C}$ is *homotopic to the identity*, written $\Upsilon \sim 1_{\mathcal{C}}$, if the following holds.

For each object $M$ of $\mathcal{C}$, there is an isomorphism $\eta_M \colon M \to \Upsilon(M)$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ p\ } & Y \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ \Upsilon(X) & \xrightarrow{\ \Upsilon(p)\ } & \Upsilon(Y) \end{array}$$

commutes for each morphism $p \colon X \to Y$.

(b) An *equivalence* of categories $\mathcal{C}, \mathcal{D}$ is given by a pair of functors $\Gamma \colon \mathcal{C} \to \mathcal{D}$ and $\Delta \colon \mathcal{D} \to \mathcal{C}$ such that $\Delta \circ \Gamma \sim 1_{\mathcal{C}}$ and $\Gamma \circ \Delta \sim 1_{\mathcal{D}}$.

Definition 4.2. A *Borel category* is a small category that can be seen as a two-sorted Borel structure $(\mathrm{Obj}, \mathrm{Mor}, \circ, \mathrm{d}, \mathrm{c})$ in the sense of [10, Definition 1.1], where Obj denotes the set of objects, Mor the set of morphisms, $\circ$ composition of morphisms, and $\mathrm{d}(f)$ and $\mathrm{c}(f)$ the domain and codomain of a morphism $f$, respectively. In particular, the sets of objects and morphisms are given as quotients of Borel sets by Borel equivalence relations in appropriate Polish spaces. If $\mathcal{C}$ and $\mathcal{D}$ are Borel categories, then a Borel equivalence between $\mathcal{C}$ and $\mathcal{D}$ is an equivalence such that the functors and the assigments $X \mapsto \eta_X$ witnessing the homotopies are Borel.

### 4.2. The Borel equivalence

Definition 4.3 (Kechris et al. [7], in Thm. 3.1). Let $\mathbf{G}$ be a Borel class of closed subgroups of $S_\infty$ closed under conjugation. We say that $\mathbf{G}$ satisfies the *invariant countable basis condition* (ICB) if one can assign to $G \in \mathbf{G}$ in a Borel way a countable set $\mathcal{S}_G$ of open subgroups of $G$ that form a neighbourhood basis of $1_G$, in a way that is invariant under topological isomorphisms of groups in $\mathbf{G}$: if $h \colon G \to H$ is such an isomorphism, then $U \in \mathcal{S}_G$ iff $h(U) \in \mathcal{S}_H$ for each $U$.

The following is the main result of the section.

Theorem 4.4. *Suppose that a Borel class $\mathbf{G}$ of closed subgroups of $S_\infty$ is closed under topological isomorphism and satisfies the invariant countable basis condition. Then $\mathbf{G}$ as a category with topological isomorphism is Borel, and there is a Borel category $\mathcal{M}$ of countable structures in a finite signature with isomorphism, and functors $\mathcal{W} \colon \mathbf{G} \to \mathcal{M}$ and $\mathcal{G} \colon \mathcal{M} \to \mathbf{G}$ that induce a Borel equivalence of categories.*

After discussing in this subsection some consequences of the theorem, we dive into the proof, which will take the rest of the section. For this, we will introduce the notion of a full meet groupoid, which is central in the proof. The objects of $\mathcal{M}$ will comprise a Borel class of countable full meet groupoids that is dual to $\mathbf{G}$. It may help to think of this as an analog of Stone duality, where $\mathbf{G}$ corresponds to the Stone spaces and $\mathcal{M}$ to the Boolean algebras.

However, in our setting the topological objects have an algebraic structure as well.

By Theorem 4.4, $\mathcal{W}$ is a full functor. This yields another (albeit roundabout) proof that $\mathrm{Aut}(G)$, now assuming that $G \in \mathbf{G}$, can be topologised as a non-Archimedean group: $\mathrm{Aut}(G)$ is topologically isomorphic to $\mathrm{Aut}(\mathcal{W}(G))$. One can easily check that when $\mathrm{Aut}(G)$ carries this topology, its action on $G$ is continuous. So by Theorem 2.1(iii) this is the same topology as the one given in Theorem 2.1(i).

Theorem 4.4 applies to the class of locally Roelcke precompact (locally R.p.), non-Archimedean groups, where $G$ is *locally Roelcke precompact* if it has a Roelcke precompact open subgroup (see Definition 1.1). This notion was introduced in [14, 17] for the wider context of Polish groups. We note that each t.d.l.c. group is locally Roelcke precompact; in contrast, the group $\mathrm{Aut}(T_\infty)$ of automorphisms of an infinitely branching unrooted tree is locally R.p., without being Roelcke precompact or t.d.l.c. [17].

The following is easily checked.

PROPOSITION 4.5. *The class of locally R.p. groups $G$ satisfies the invariant countable basis condition, for $\mathcal{S}_G$ the class of R.p. open subgroups of $G$.*

Throughout this section, let $\mathbf{G}$ be a class of closed subgroups of $S_\infty$ that satisfies the invariant countable basis condition as in Definition 4.3. By $G$ we will always denote a group in $\mathbf{G}$. We may suppose without loss of generality that for each $G \in \mathbf{G}$ the neighbourhood basis $\mathcal{S}_G$ of $1_G$ is closed under finite intersections. For, one can replace the given class of open subgroups $\mathcal{S}_G$ by its closure under finite intersections, maintaining the Borel and invariance conditions. We let the domain of $\mathcal{W}(G)$ be the left cosets of subgroups of $G$ in $\mathcal{S}_G$, together with $\emptyset$. It has a groupoid structure given by products of "matching" cosets $A, B$ (namely, $A$ is a left coset and $B$ a right coset of the same subgroup), and a lower semilattice structure given by intersection. $\mathcal{W}(G)$ is called the meet groupoid of $G$ (see Definition 4.13).

**4.3. Full meet groupoids.** This subsection introduces an abstract algebraic framework for the structures $\mathcal{W}(G)$.

DEFINITION 4.6. A *groupoid* consists of a domain $M$, a partial binary operation $\cdot$ and a unary operation $(\cdot)^{-1}$ with the following properties for each $A, B, C \in M$:

(a) $(A \cdot B) \cdot C = A \cdot (B \cdot C)$, with either both sides or no side defined (and so the parentheses can be omitted in products);
(b) $A \cdot A^{-1}$ and $A^{-1} \cdot A$ are always defined;
(c) if $A \cdot B$ is defined then $A \cdot B \cdot B^{-1} = A$ and $A^{-1} \cdot A \cdot B = B$.

Given a groupoid $M$, the letters $A, B, C$ will range over elements of $M$, and the letters $U, V, W$ will range over idempotents.

The following goes back to [9] in the context of t.d.l.c. groups; also see [11, Section 3]. In a structure $(M, \cdot)$ with multiplication, an *idempotent* is an element $U$ of $M$ with $U \cdot U = U$.

DEFINITION 4.7. A *meet groupoid* is a groupoid $(M, \cdot, (\cdot)^{-1})$ that is also a meet semilattice $(M, \cap, \emptyset)$ of which $\emptyset$ is the least element. (Here $\cap$ is an abstract symbol for the meet.) It satisfies the conditions that $\emptyset^{-1} = \emptyset = \emptyset \cdot \emptyset$, that $\emptyset \cdot A$ and $A \cdot \emptyset$ are undefined for each $A \neq \emptyset$, and that $U \cap V \neq \emptyset$ for idempotents $U, V$ with $U, V \neq \emptyset$. Further, if we write $A \subseteq B \Leftrightarrow A \cap B = A$, it satisfies

(d) $A \subseteq B \Leftrightarrow A^{-1} \subseteq B^{-1}$;
(e) if $A_i \cdot B_i$ are defined $(i = 0, 1)$ and $A_0 \cap A_1 \neq \emptyset \neq B_0 \cap B_1$, then

$$(A_0 \cap A_1) \cdot (B_0 \cap B_1) = A_0 \cdot B_0 \cap A_1 \cdot B_1$$

REMARK 4.8. Since inversion is an order isomorphism, if $A \cap B \neq \emptyset$ then $A^{-1} \cap B^{-1} = (A \cap B)^{-1}$. Monotonicity of the groupoid product follows from (e):

(f) if $A_i \cdot B_i$ are defined $(i = 0, 1)$ and $A_0 \subseteq A_1, B_0 \subseteq B_1$, then $A_0 \cdot B_0 \subseteq A_1 \cdot B_1$.

Another consequence of (e) is that the intersection of two idempotents is again an idempotent.

Given meet groupoids $\mathcal{W}_0, \mathcal{W}_1$, a bijection $h \colon M_0 \to M_1$ is an *isomorphism* if it preserves the three operations.

DEFINITION 4.9. Suppose we are given a groupoid as in Definition 4.6. Let $U$ be an idempotent. We say $A$ is a *left $U$ *coset* if $A \cdot U = A$. We say $B$ is a *right $U$ *coset* if $U \cdot B = B$. We write $\mathrm{LC}(U)$ and $\mathrm{RC}(U)$ for the collections of left and right *cosets of $U$, respectively.

REMARK 4.10. Each $A$ is a left $U$ *coset and right $V$ *coset for unique idempotents $U$ and $V$, by cancellation and 4.6(c). Further, $U = A \cdot A^{-1}$ and $V = A^{-1} \cdot A$.

DEFINITION 4.11. A *full meet groupoid* is a meet groupoid $(M, \cdot, {}^{-1}, \cap, \emptyset)$ that additionally satisfies the following conditions for all idempotents $U \subseteq V$:

(g) (Level up) If $A$ is a left (resp. right) $U$ *coset, then there exists a unique left (resp. right) $V$ *coset $B$ with $A \subseteq B$.
(h) (Level down) Suppose that $B$ is a left (resp. right) $V$ *coset. If $U \subseteq V$ but $V \not\subseteq U$, then there exist at least two distinct left (resp. right) $U$ *cosets $A \subseteq B$.

It follows from 4.11(g) that any two distinct left (resp. right) $U$ *cosets are disjoint.

FACT 4.12. *In a full meet groupoid, if $A_i$ is a left $U_i$ *coset for $i = 0, 1$ and $A_0 \cap A_1$ is non-empty, then $A_0 \cap A_1$ is a left $U_0 \cap U_1$ *coset. A similar property holds for right cosets.*

*Proof.* Using 4.7(e) we have

$$\begin{aligned} U_0 \cap U_1 &= (A_0^{-1} \cdot A_0) \cap (A_1^{-1} \cdot A_1) \\ &= (A_0^{-1} \cap A_1^{-1}) \cdot (A_0 \cap A_1) \\ &= (A_0 \cap A_1)^{-1} \cdot (A_0 \cap A_1). \end{aligned}$$

So $A_0 \cap A_1$ is a left $U_0 \cap U_1$ *coset by Remark 4.10. ∎

In particular, letting $U = U_0 = U_1$ and using uniqueness from 4.11(g), we find that distinct left (resp. right) *cosets of $U$ are disjoint.

**4.4. The functor $\mathcal{W}$.** We define a functor $\mathcal{W}$ from the category of groups $\mathbf{G}$ with topological isomorphisms to the category of countable full meet groupoids with isomorphisms.

DEFINITION 4.13 (The functor $\mathcal{W}$). Let $\mathcal{W}(G)$ be the collection of left cosets of subgroups in $\mathcal{S}_G$, together with $\emptyset$ (so that it is closed under $\cap$). Note that $\mathcal{W}(G)$ is a basis that is closed under inverse because $\mathcal{S}_G$ is closed under conjugation. We write $A \cdot B = C$ if $A$ is a left coset and $B$ is a right coset of the same subgroup, and $AB = C$. Clearly $\mathcal{W}(G)$ is closed under this operation. So $\mathcal{W}(G)$ forms a groupoid. We extend the groupoid operation to $\mathcal{W}(G)$ by letting $\emptyset \cdot \emptyset = \emptyset$, and $\emptyset \cdot A$ be undefined for $A \neq \emptyset$. The structure $(\mathcal{W}(G), \cdot, \cap)$ is called the *meet groupoid* of $G$. If $\alpha\colon G \to H$ is a topological isomorphism, we define $\mathcal{W}(\alpha)(A) = \{\alpha(g)\colon\ g \in A\}$ for any $A \in \mathcal{W}(G)$.

We show that $\mathcal{W}(G)$ is indeed a full meet groupoid. Clearly 4.7(d) and monotonicity in 4.7(f) hold. The condition to check is 4.7(e). Let $A_i$ be a right coset of a subgroup $U_i$ and a left coset of subgroup $V_i$, so that $B_i$ is a right coset of $V_i$ by hypothesis. Then $A_0 \cap A_1$ is a left coset of $V_0 \cap V_1$, and $B_0 \cap B_1$ a right coset of $V_0 \cap V_1$, so the left hand side in (e) is defined. Clearly the left hand side is contained in the right hand side by monotonicity. If $A$ is a right coset of $U$ and $A \cdot B$ is defined, then $A \cdot B$ is a right coset of $U$. Since $A_0 \cap A_1$ is a right coset of $U_0 \cap U_1$, $(A_0 \cap A_1) \cdot (B_0 \cap B_1)$ is a right coset of $U_0 \cap U_1$. Since $A_i$ is a right coset of $U_i$, $A_i \cdot B_i$ is a right coset of $U_i$. Hence $A_0 \cdot B_0 \cap A_1 \cdot B_1$ is a right coset of $U_0 \cap U_1$. Since the left hand side is contained in the right hand side and both are cosets of $U_0 \cap U_1$, they are equal.

DEFINITION 4.14 (Full filters, cf. [13, Def. 2.4] for coarse groups). A *filter* $R$ on the partial order $(\mathcal{W}(G), \subseteq)$ is a proper subset that is directed

downward and closed upward. It is called *full* if each subgroup in $\mathcal{W}(G)$ has a left and a right coset in $R$. (Such cosets are necessarily unique.)

LEMMA 4.15. *$\bigcap R$ is a singleton for each full filter $R$.*

*Proof.* The argument is essentially in [13, Prop. 2.13], which in turn goes back to [7, Claims 3.6 and 3.7]. Since $\mathcal{S}_G$ is a neighbourhood basis of $1_G$, the set $\bigcap R$ has at most one element. To show it is non-empty, let $U_n$ be the open subgroup of $G$ consisting of the permutations that fix $0, \dots, n$. Since $R$ is a full filter, there are permutations $r_n, s_n \in G$ such that $r_n U_n \in R$ and $U_n s_n \in R$. Let $g(n) = r_n(n)$ and $g^*(n) = s_n^{-1}(n)$. As in [7] one shows that $g^* = g^{-1}$ using the fact that $R$ is a filter. So $g$ is a permutation, and then clearly $g \in \bigcap R$ since $G$ is closed. ∎

LEMMA 4.16. *There is a canonical bijection $\mathcal{K}$ between the elements of $G$ and the set of full filters on $\mathcal{W}(G)$. It is given by*

$$\mathcal{K}(g) = \{gU \colon U \in \mathcal{S}_G\}.$$

*Its inverse $\mathcal{L}$ is given by $\mathcal{L}(R) = g$, where $\{g\} = \bigcap R$ (see Lemma 4.15).*

*Proof.* $\mathcal{K}(g)$ is indeed a full filter: Firstly, by definition each subgroup $U \in S_G$ has the left coset $gU$ in $\mathcal{K}(g)$. Secondly, given a subgroup $V \in \mathcal{S}_G$, let $U = g^{-1}Vg$; then $U \in \mathcal{S}_G$ by invariance. So $Vg = gU$ is a right coset of $V$ in $\mathcal{K}(g)$.

Clearly we have $\mathcal{L}(\mathcal{K}(g)) = \bigcap_{U \in S_G} gU = \{g\}$ for each $g \in G$. It remains to show that $\mathcal{K}(\mathcal{L}(R)) = R$ for each full filter $R$. Suppose $\mathcal{L}(R) = g$ and let $T = \mathcal{K}(g)$. For $R \subseteq T$, if $A \in R$, then $A = AU$ for some $U \in \mathcal{S}_G$. Since $g \in A$, we have $A = gU \in T$. For $T \subseteq R$, consider a set $gU \in T$ where $U \in \mathcal{S}_G$. Since $R$ is a full filter, some left $^*$coset $A$ of $U$ is in $R$. Then $g \in A$, which implies that $gU = A$. ∎

**4.5. The functor $\mathcal{G}$ on the category of full meet groupoids.** From now one we will assume that *meet groupoids have domain $\omega$.* We tacitly adjust the functor $\mathcal{W}$ in Definition 4.13 in such a way that each object in its range is of this kind. This is a standard reduction using the Luzin–Novikov theorem, as in [7, proof of Th. 3.1]: to $G \in \mathbf{G}$ one assigns in a Borel way a bijection $\omega \to \mathcal{S}_G$, which then yields a bijection $\omega \to \mathcal{W}(G)$ by picking a countable dense subset of $G$ in a Borel way. In this way $\mathcal{W}$ becomes a Borel functor.

We next define a functor $\mathcal{G}$ from the category of countable full meet groupoids (Definition 4.11) to the category of non-Archimedean groups with topological isomorphism. Later on, we will restrict it to the category $\mathcal{M}$. We need the definition of the functor $\mathcal{G}$ on the larger category to define the Borel category $\mathcal{M}$: one of the conditions on a full meet groupoid $M$ for being an object of $\mathcal{M}$ is that $\mathcal{G}(M) \in \mathbf{G}$, which is a Borel condition.

DEFINITION 4.17 (The functor $\mathcal{G}$). For a countable full meet groupoid $M$, let

$$\mathcal{G}(M) = \{p \in \mathrm{Sym}(M) \colon p \text{ is an } (M, \cap) \text{ automorphism} \\ \wedge\, p(A \cdot B) = p(A) \cdot B \text{ whenever } A \cdot B \text{ is defined}\}.$$

If $\theta \colon M \to N$ is an isomorphism of meet groupoids, let

$$\mathcal{G}(\theta)(p) = \theta \circ p \circ \theta^{-1}.$$

REMARK 4.18. Clearly, when $\theta \colon M \to N$ is an isomorphism then $\mathcal{G}(\theta)$ is an isomorphism $\mathcal{G}(M) \to \mathcal{G}(N)$, with inverse $\mathcal{G}(\theta^{-1})$.

$\mathcal{G}(M)$ is the set of automorphisms of the structure $M'$ with the binary function $\cap$ and unary partial functions $f_B$ for each $B \in M$, that send $A$ to $A \cdot B$ when defined. It is thus a closed subgroup of $\mathrm{Sym}(M)$.

REMARK 4.19. The elements of $\mathcal{G}(M)$ correspond to full filters (Definition 4.14), as follows:

(a) Given $p \in \mathcal{G}(M)$, let $R = \{pU \colon U \in \mathcal{N}_G\}$.
(b) Given a full filter $R$, let $p(U) = A$, where $A$ is the left coset of $U$ in $R$; this determines the values on all cosets because $p(B) = p(U) \cdot B$ when $B$ is a right coset of $U$.

**4.6. Towards the homotopies of categories.** We aim to show that after suitably refining the category of full meet groupoids, the functors $\mathcal{W}$ and $\mathcal{G}$ induce an equivalence of categories according to Definition 4.1. For this we need to define two homotopies:

DEFINITION 4.20. Given a full meet groupoid $M$, define a map $\eta_M \colon M \to \mathcal{W}(\mathcal{G}(M))$ as follows. For $A \in M - \{\emptyset\}$ let

$$\eta_M(A) := \{p \in \mathcal{G}(M) \colon p(U) = A\},$$

where $A$ is a left $U$ *coset. Also let $\eta_M(\emptyset) := \emptyset$. If $M$ is understood from the context, we also write $\widehat{A}$ for $\eta_M(A)$.

The finite intersections of sets $\widehat{A}$ form a basis for the topology of $\mathcal{G}(M)$. This is because for $p \in \mathcal{G}(M)$ and $C, D \in M$, $p(C) = D$ is equivalent to $p(C) \cdot C^{-1} = D \cdot C^{-1}$, or again $p(U) = A$ where $U = C \cdot C^{-1}$ and $A = D \cdot C^{-1}$.

DEFINITION 4.21. For $G \in \mathbf{G}$ define a map $\eta_G \colon G \to \mathcal{G}(\mathcal{W}(G))$ by $\eta_G(g)(A) = gA$.

LEMMA 4.22. *$\eta_G \colon G \to \mathcal{G}(\mathcal{W}(G))$ is a topological isomorphism.*

*Proof.* Clearly $\eta_G$ is a group monomorphism. To show that it is onto, given $p \in \mathcal{G}(\mathcal{W}(G))$ let $R = \{pU \colon U \in \mathcal{S}_G\}$. Since $R$ is a full filter on $\mathcal{W}(G)$, by Lemma 4.16 there is a unique $g \in G$ such that $\bigcap R = \{g\}$. Clearly $p = \eta_G(g)$.

To check $\eta_G$ is continuous, take a subbasic subset $\widehat{A}$ of $\mathcal{G}(\mathcal{W}(G))$, where $A \in \mathcal{W}(G)$. Note that $\eta_G^{-1}(\widehat{A}) = A$ is open.

Given that both $G$ and $\mathcal{G}(\mathcal{W}(G))$ are Polish, this shows $\eta_G$ is a topological isomorphism. ∎

The next lemmas will show that the map $\eta_M$ from Definition 4.20 is an embedding of meet groupoids.

Note that $\widehat{U}$ is an open subgroup of $\mathcal{G}(M)$. Furthermore, for any left $U$ *coset $A$, the set $\widehat{A}$ is a left $\widehat{U}$ coset, because $x\widehat{U} = \widehat{A}$ for any $x \in \widehat{A}$. Similarly, $\widehat{B}$ is a right $\widehat{V}$ coset for every right $V$ *coset $B$, since

$$x(B) = x(V \cdot B) = x(V) \cdot B = V \cdot B = B$$

for any $x \in \widehat{V}$ and hence $\widehat{V}y = \widehat{B}$ for any $y \in \widehat{B}$.

We first show that $\eta_M$ sends elements of $M$ to open cosets of $\mathcal{G}(M)$.

LEMMA 4.23. *$\widehat{A} \neq \emptyset$ for any $A \neq \emptyset$.*

*Proof.* First suppose that $M$ has a least idempotent $U \neq \emptyset$. Suppose that $A$ is a left $V$ * coset. Then $A$ contains a left $U$ *coset $B$ by 4.11(h). We define $f \in \mathcal{G}(M)$ as follows. Suppose that $W$ is any idempotent with $U \subseteq W$. There is a unique left $W$ *coset $C$ with $A \subseteq C$ by 4.11(g). Let $f(W) = C$ and $f(B) = C \cdot B$ for any right $W$ *coset $B$. Then $f \in \widehat{B} \subseteq \widehat{A}$.

Now suppose $M$ has no least non-empty idempotent. Suppose that $A$ is a left $U$ *coset. Let $\langle U_n \mid n \in \mathbb{N}\rangle$ enumerate all idempotents of $M$ in such a way that $U_0 = U$. We construct a strictly increasing sequence $\langle n_i \mid i \in \mathbb{N}\rangle$ and an increasing sequence $\langle f_i \mid i \in \mathbb{N}\rangle$ of partial functions on $M$ whose union will be an element of $\widehat{A}$.

Let $n_0 = 0$ and suppose that $V$ is any idempotent with $U_0 \subseteq V$. There is a unique left $V$ *coset $C$ with $A \subseteq C$ by 4.11(g). Let $f_0(B) = C \cdot B$ for any right $V$ *coset $B$; in particular, $f_0(V) = C$. Note that $f_0(B)$ is a right $CVC^{-1}$ coset and $f_0(VC^{-1}) = CVC^{-1}$.

If $f_i$ has been defined, let $n_{i+1} > n_i$ be least such that

$$U_{n_{i+1}} \subseteq U_{n_i} \cap (A_{n_i} U_{n_i} A_{n_i}^{-1}) \cap U_j \quad \text{for all } j < n_i.$$

Note that $n_{i+1}$ exists since $H$ is not discrete. There exists a left $U_{n_{i+1}}$ *coset $A_{n_{i+1}} \subseteq f(U_{n_i})$ by 4.11(h). Suppose that $V$ is any idempotent with $U_{n_{i+1}} \subseteq V$. Let $f_{i+1}(V)$ be the unique left $V$ *coset $C$ with $A_{n_{i+1}} \subseteq C$ by 4.11(g). Let $f_{i+1}(B) = C \cdot B$ for any right $V$ *coset $B$. Note that $f_{i+1}(B)$ is a right $CVC^{-1}$ coset and $f_{i+1}(VC^{-1}) = CVC^{-1}$. It follows that every right *coset of $U_{i+1}$ is in the domain and every right *coset of $U_i$ in the range of $f_{n_{i+1}}$. Since $f_i \subseteq f_{i+1}$ by construction and each $f_i$ is a partial automorphism of $M'$ in the sense of Remark 4.18, their union is an automorphism of $M'$ and hence is in $\widehat{A}$. ∎

We next show that the map $\eta_M$ is compatible with intersections, products and inverses. Let $\widehat{A}\widehat{B}$ denote the setwise product of $\widehat{A}$ and $\widehat{B}$, and $\widehat{A}^{-1}$ the setwise inverse of $\widehat{A}$ in $\mathcal{G}(M)$. Note that for any idempotent $U$, $\widehat{U}$ is an open subgroup of $\mathcal{G}(M)$.

LEMMA 4.24. *The following hold for all *cosets $A$ and $B$:*

(a) $\widehat{A \wedge B} = \widehat{A} \cap \widehat{B}$.
(b) $\widehat{A \cdot B} = \widehat{A}\widehat{B}$ *if* $A \cdot B$ *is defined.*
(c) $\widehat{A^{-1}} = \widehat{A}^{-1}$.

*Proof.* (a) Suppose that $A$ is a left $U$ *coset and $B$ is a left $V$ *coset. If $A \wedge B = \emptyset$, then $\widehat{A} \cap \widehat{B} = \emptyset$. To see this, note that for any $x \in \widehat{A}$ and $y \in \widehat{B}$, we have $x(U \wedge V) \neq y(U \wedge V)$, since $x$ and $y$ preserve order. Now suppose that $A \wedge B \neq \emptyset$. We argued in Fact 4.12 that $A \wedge B$ is a left $U \wedge V$ *coset. By uniqueness in 4.11(g), every element of $\widehat{A \wedge B}$ is an element of both $\widehat{A}$ and $\widehat{B}$. The converse also holds, since every element of $\mathcal{G}(M)$ is an automorphism of the structure $(M, \wedge)$.

(b) Suppose that $z \in \widehat{A \cdot B}$, where $A$ and $B$ are as above. Then $z(V) = A \cdot B$. Pick any $x \in \widehat{A}$ and let $y := x^{-1}z$. Then

$$y(V) = x^{-1}z(V) = x^{-1}(A \cdot B) = U \cdot B = B.$$

Conversely, suppose that $x \in \widehat{A}$ and $y \in \widehat{B}$. Then $xy(V) = x(B) = x(U \cdot B) = x(U) \cdot B = A \cdot B$ and hence $xy \in \widehat{A \cdot B}$.

(c) Note that $\widehat{A}$ is a left $\widehat{U}$ coset, since $x\widehat{U} = \widehat{A}$ for any $x \in \widehat{A}$. Furthermore, $A^{-1}$ is a right $U$ *coset by 4.6, so $\widehat{A^{-1}}$ is a right $\widehat{U}$ *coset. The claim holds since $\widehat{A^{-1}} \cdot \widehat{A} = \widehat{A^{-1} \cdot A} = \widehat{U}$ by (b). ∎

LEMMA 4.25. *The map $\eta_M$ is injective and preserves the order in both directions.*

*Proof.* By Lemma 4.24(a) it suffices to show that the map is injective. Suppose that $A \neq B$ and assume that $A \not\subseteq B$, so that $A \wedge B \neq A$. If $A \wedge B$ is a $U$ *coset, then by 4.11(h), $A$ contains a left $U$ *coset $D$ disjoint from $A \wedge B$ and thus from $B$. Then $\widehat{D} \cap \widehat{B} = \emptyset$ and $\widehat{D} \subseteq \widehat{A}$ by 4.11(g), so that $\widehat{A} \neq \widehat{B}$. ∎

LEMMA 4.26.

(a) *For any left (resp. right) $U$ *coset $A$, $\widehat{A}$ is a left (resp. right) $\widehat{U}$ coset in $\mathcal{G}(M)$.*
(b) *For any left $\widehat{U}$ coset $g\widehat{U}$ in $\mathcal{G}(M)$, there exists a left $U$ *coset $A$ such that $\widehat{A} = g\widehat{U}$. A similar fact holds for right cosets.*

*Proof.* (a) This was shown in the proof of Lemma 4.24(c).

(b) Suppose that $f \in \widehat{A}$. We claim that $f\widehat{V} = \widehat{A}$. We have $f\widehat{V} \subseteq \widehat{A}\widehat{V} = \widehat{A}$, since $\widehat{A}$ is a left coset of $\widehat{V}$ by (a). To see that $\widehat{A} \subseteq f\widehat{V}$, let $g \in \widehat{A}$. Since $f, g \in \widehat{A}$, we have $f^{-1}g \in \widehat{A}^{-1}\widehat{A} = \widehat{A^{-1}A} \subseteq \widehat{V}$ by Lemma 4.24(b, c). Hence $g \in f\widehat{V}$. ∎

In particular, a *coset $A$ is a left $U$ *coset if and only if $\widehat{A}$ is a left $\widehat{U}$ coset. A similar fact holds for right *cosets.

### 4.7. The category $\mathcal{M}$

Definition 4.27. Let $\mathcal{M}$ be the category that has as objects the full meet groupoids $M$ with domain $\omega$ such that

$$\mathcal{G}(M) \in \mathbf{G} \quad \text{and} \quad \forall \mathcal{U} \in \mathcal{S}_{\mathcal{G}(M)} \exists U \in M \; [\widehat{U} = \mathcal{U}].$$

As before, the morphisms are the isomorphisms of meet groupoids.

Lemma 4.28. *For an object $M$ of $\mathcal{M}$, the map $\eta_M$ is an isomorphism of meet groupoids.*

*Proof.* Lemma 4.25 showed that $\eta_M$ is an embedding for each full meet groupoid $M$. Now by the definition of the objects of $\mathcal{M}$, Lemma 4.26(b) shows that $\eta_M$ is onto when $M$ is in $\mathcal{M}$. ∎

Lemma 4.29. *The operation sending a full meet groupoid $M$ with domain $\omega$ to the map $A \mapsto \widehat{A}$ (as an element of $\mathcal{F}(S_\infty)^\omega$) is Borel.*

*Proof.* It suffices to show that the map sending a pair $(M, A)$, where $A \in M$, to $\widehat{A}$ has Borel graph. This is evident because $A$ is a left coset of $U = A^{-1} \cdot A$, and $p \in \widehat{A}$ iff $p(U) = A$. ∎

By the hypothesis on the class $\mathbf{G}$ and Lemma 4.29, the objects of $\mathcal{M}$ form a Borel set. So $\mathcal{M}$ is a Borel category.

### 4.8. Completing the proof of Theorem 4.4

Lemma 4.30. *If $G \in \mathbf{G}$ then $\mathcal{W}(G)$ is an object of $\mathcal{M}$.*

*Proof.* Let $M = \mathcal{W}(G)$. For the first condition in Definition 4.27, note that we have $\mathcal{G}(M) \in \mathbf{G}$ by Lemma 4.22 and by the class $\mathbf{G}$ being isomorphism invariant. For the second condition, let $\mathcal{U} \in \mathcal{S}_{\mathcal{G}(M)}$. Since $\eta_G \colon G \to \mathcal{G}(M)$ is a topological isomorphism and the assignment of $H \in \mathbf{G}$ to $\mathcal{S}_H$ is isomorphism invariant, we have $U := \eta_G^{-1}(\mathcal{U}) \in \mathcal{S}_G$, so that $U \in M$. Then $\widehat{U} = \mathcal{U}$ as required. ∎

Let $\mathcal{G}'$ be the functor $\mathcal{G}$ restricted to the Borel category $\mathcal{M}$. We will show that the diagrams in Definition 4.1 are commutative for the functors $\Upsilon = \mathcal{G}' \circ \mathcal{W}$ and $\Upsilon = \mathcal{W} \circ \mathcal{G}'$ using the assignments $G \mapsto \eta_G$ and $M \mapsto \eta_M$, respectively, and the fact that the maps $\eta_G$ and $\eta_M$ are isomorphisms in their respective categories $\mathbf{G}$ and $\mathcal{M}'$ (recall here that all morphisms in the categories we consider are isomorphisms).

LEMMA 4.31.

(a) $\mathcal{G}' \circ \mathcal{W} \sim 1_{\mathbf{G}}$ *via* $G \mapsto \eta_G$.
(b) $\mathcal{W} \circ \mathcal{G}' \sim 1_{\mathcal{M}}$ *via* $M \mapsto \eta_M$.

*Proof.* (a) Write $\Upsilon = \mathcal{G}' \circ \mathcal{W}$. Clearly $\eta_G \colon G \to \Upsilon(G)$ is in the category $\mathbf{G}$. Let $\alpha \colon G \to H$ be a morphism in $\mathbf{G}$. Note that by Definition 4.17 we have $\Upsilon(\alpha)(p) = \mathcal{W}(\alpha) \circ p \circ \mathcal{W}(\alpha)^{-1}$ (recall that $\mathcal{W}(\alpha)$ is given by the action of $\alpha$ on open cosets of $G$). We need to verify the commutativity of the diagram in Definition 4.1, which here is

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & H \\ {\scriptstyle \eta_G}\downarrow & & \downarrow{\scriptstyle \eta_H} \\ \Upsilon(G) & \xrightarrow{\Upsilon(\alpha)} & \Upsilon(H) \end{array}$$

We need to show, for each $g \in G$, the equality of two maps on $\mathcal{W}(H)$ depending on $g$:

$$\Upsilon(\alpha)(\eta_G(g)) = \eta_H(\alpha(g)).$$

Applying the map on the right hand side to $C \in \mathcal{W}(H)$ yields $\alpha(g)C$ by Definition 4.21 of $\eta_H$. Applying the map on the left hand side yields

$$\mathcal{W}(\alpha)(g(\mathcal{W}(\alpha)^{-1}(C)) = \alpha(g\alpha^{-1}(C)) = \alpha(g)C$$

by Definition 4.13 of the functor $\mathcal{W}$.

(b) Now write $\Upsilon = \mathcal{W} \circ \mathcal{G}'$. By the second condition in Definition 4.27, we have $\eta_M \colon M \to \Upsilon(M)$ in the category $\mathcal{M}$. Let $\theta \colon M \to N$ be a morphism in $\mathcal{M}$. For commutativity of the diagram

$$\begin{array}{ccc} M & \xrightarrow{\theta} & N \\ {\scriptstyle \eta_M}\downarrow & & \downarrow{\scriptstyle \eta_N} \\ \Upsilon(M) & \xrightarrow{\Upsilon(\theta)} & \Upsilon(N) \end{array}$$

we need, for each $A \in M$, the equality of two open subsets of $\Upsilon(N)$:

$$\eta_N(\theta(A)) = \Upsilon(\theta)(\widehat{A}).$$

Suppose $A$ is a left *coset of $U$. Let $q \in \Upsilon(N)$. Note that $q$ is in the left hand side if $q(\theta(U)) = \theta(A)$ by Definition 4.20. For the right-hand side, write $\widehat{A} = \eta_M(A)$. Note that by Definition 4.17 we have

$$\Upsilon(\theta)(\widehat{A}) = \{\mathcal{G}'(\theta)(p) \colon p \in \widehat{A}\} = \{\theta \circ p \circ \theta^{-1} \colon p \in \widehat{A}\}.$$

Therefore, $q$ is in the right-hand side if $p(U) = A$, where $p \in \Upsilon(M)$ is given as $p = \theta^{-1} \circ q \circ \theta$. Since $\theta$ is an isomorphism, this also says that $q(\theta(U)) = \theta(A)$. So the two sides are equal as sets. ∎

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## References

[1] G. Ahlbrandt and M. Ziegler, *Quasi finitely axiomatizable totally categorical theories*, Ann. Pure Appl. Logic 30 (1986), 63–82.

[2] H. Becker and A. S. Kechris, *The Descriptive Set Theory of Polish Group Actions*, London Math. Soc. Lecture Note Ser. 232, Cambridge Univ. Press, 1996.

[3] J. Bergfalk, M. Lupini, and A. Panagiotopoulos, *The definable content of homological invariants I:* Ext *and* $\lim^1$, Proc. London Math. Soc. 129 (2024), art. e12631, 55 pp.

[4] B. Farb and D. Margalit, *A Primer on Mapping Class Groups*, Princeton Math. Ser. 49, Princeton Univ. Press, 2012.

[5] S. Gao, *Invariant Descriptive Set Theory*, Pure Appl. Math. (Boca Raton) 293, CRC Press, 2009.

[6] A. S. Kechris, *Classical Descriptive Set Theory*, Grad. Texts in Math. 156, Springer, New York, 1995.

[7] A. S. Kechris, A. Nies, and K. Tent, *The complexity of topological group isomorphism*, J. Symbolic Logic 83 (2018), 1190–1203.

[8] S. Mac Lane, *Categories for the Working Mathematician*, 2nd ed., Grad. Texts in Math. 5, Springer, New York, 1998.

[9] A. Melnikov and A. Nies, *Computably totally disconnected locally compact groups*, J. Symbolic Logic (online first, 2026).

[10] A. Montalbán and A. Nies, *Borel structures: a brief survey*, in: Effective Mathematics of the Uncountable, Lecture Notes in Logic 41, Assoc. for Symbolic Logic and Cambridge Univ. Press, 2013, 124–134.

[11] A. Nies (ed.), *Logic Blog 2022*, arXiv:2302.11853v1 (2023).

[12] A. Nies and G. Paolini, *Oligomorphic groups, their automorphism groups, and the complexity of their isomorphism*, Beijing J. Pure Appl. Math. 3 (2026), 387–416.

[13] A. Nies, P. Schlicht, and K. Tent, *Coarse groups, and the isomorphism problem for oligomorphic groups*, J. Math. Logic 22 (2022), art. 2150029, 31 pp.

[14] C. Rosendal, *Coarse Geometry of Topological Groups*, Cambridge Tracts in Math. 223, Cambridge Univ. Press, 2021.

[15] K. Tent and M. Ziegler, *A Course in Model Theory*, Lecture Notes in Logic 30, Assoc. for Symbolic Logic and Cambridge Univ. Press, 2012.

[16] T. Tsankov, *Unitary representations of oligomorphic groups*, Geom. Funct. Anal. 22 (2012), 528–555.

[17] J. Zielinski, *Locally Roelcke precompact Polish groups*, Groups Geom. Dynam. 15 (2021), 1175–1196.

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