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Rudyak's conjecture for low-dimensional simply connected manifolds

by

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In memory of Yuli Rudyak

Abstract. Rudyak's conjecture states that for any degree one map $f : M \to N$ between oriented closed manifolds there is the inequality $\mathrm{cat}(M) \geq \mathrm{cat}(N)$ for the Lusternik–Schnirelmann category. We prove this conjecture for n-dimensional simply connected spin manifolds with $n \leq 8$.

1. Introduction. The reduced *Lusternik–Schnirelmann category* (briefly LS-category) $\mathrm{cat}_{\mathrm{LS}}(X)$ of a topological space X is the minimal number n such that there is an open cover $\{U_0, \dots, U_n\}$ of X by $n+1$ sets contractible in X. We note that the LS-category is a homotopy invariant. Most of the deep applications of the LS-category are based on the classical Lusternik and Schnirelmann theorem [CLOT] which states that $\mathrm{cat}_{\mathrm{LS}}(M) + 1$ gives a lower bound for the number of critical points on a manifold M of any smooth not necessarily Morse function. The concept of LS-category and the above theorem were brought forth by Lusternik and Schnirelmann to solve the Poincaré problem on the existence of three closed geodesics on the 2-sphere for any Riemannian metric on it [LS1, LS2]. In modern times Yu. Rudyak applied LS-category in his approach to the Arnold conjecture on the number of fixed points of symplectomorphisms [Ru2].

LS-category is a numerical homotopy invariant which is difficult to compute. Clearly $\mathrm{cat}_{\mathrm{LS}}(S^n) = 1$. The converse is also true in view of the classical results by Smale, Freedman, and Perelman: If $\mathrm{cat}_{\mathrm{LS}}(M) = 1$ then M is homeomorphic to a sphere. It was proven in [DKR] that if $\mathrm{cat}_{\mathrm{LS}}(M) = 2$, then the fundamental group of M is free.

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Moreover, many natural conjectures on LS-category turned out to be false. Perhaps the most famous one was the Ganea Conjecture of the 60s (see [Ga]) which states that $\mathrm{cat}_{\mathrm{LS}}(M \times S^n) = \mathrm{cat}_{\mathrm{LS}}(M) + 1$ for a closed manifold M. On the break of the 20th century N. Iwase [Iw1, Iw2] constructed two simply connected 16-dimensional counterexamples M_2 and M_3 to the Ganea Conjecture. His examples are quite delicate and they are based on some phenomena for primary 2-components and primary 3-components in unstable homotopy groups of spheres.

In the late 90s Rudyak proposed the following [Ru1]:

Conjecture 1.1. For a degree one map $f : M \to N$ between closed orientable manifolds, $\mathrm{cat}_{\mathrm{LS}}(M) \geq \mathrm{cat}_{\mathrm{LS}}(N)$.

In [Ru1] Rudyak proved this conjecture under some restrictions on the dimension, connectivity and LS-category of the domain. It turned out that his conjecture is quite difficult even in the case of simplest maps of degree one, $f : M\#N \to N$, collapsing a summand in a connected sum [DS]. Perhaps the most interesting and still challenging case of Rudyak's conjecture is the case of a birational morphism $f : M' \to M$ between nonsingular projective varieties. We note when M' is a blowup of M at a point, Rudyak's conjecture for f follows from the above formula.

Rudyak's conjecture like many other natural conjectures on LS-category could be false. In [Dr3] there was an attempt to construct a counterexample by means of Iwase's manifolds. Using Iwase's examples the first author constructed a map $f : M \to N$ of degree one between 32-dimensional simply connected manifolds which would be a counter-example to Rudyak's conjecture if $\mathrm{cat}_{\mathrm{LS}}(M_2 \times M_3) \geq 5$. We recall that $\mathrm{cat}_{\mathrm{LS}}(M_2) = \mathrm{cat}_{\mathrm{LS}}(M_3) = 3$. The classical inequality [CLOT]

$$\mathrm{cat}_{\mathrm{LS}}(X \times Y) \leq \mathrm{cat}_{\mathrm{LS}}(X) + \mathrm{cat}_{\mathrm{LS}}(Y)$$

and computations in [Dr3] imply that $4 \leq \mathrm{cat}_{\mathrm{LS}}(M_2 \times M_3) \leq 6$.

A different idea for a counterexample was suggested in [Dr2].

In any case we believe Rudyak's conjecture is valid for lower-dimensional manifolds. In [Ru3] Rudyak verified his conjecture for manifolds of dimension ≤ 4. In this paper we prove Rudyak's conjecture for simply connected spin manifolds of dimension ≤ 8 (Theorem 4.5). At the end of the paper we give an example of a non-spin simply connected 7-dimensional manifold. The existence of such an example does not allow us to extend our approach to non-spin manifolds.

2. Preliminaries

2.1. Properties of maps of degree one. The following is well-known.

PROPOSITION 2.1. *Let $f : M \to N$ be a map of degree one between closed connected orientable n-manifolds. Then the map $f_* : H_*(M;R) \to H_*(N;R)$ is a split epimorphism and $f^* : H^*(N;R) \to H^*(M;R)$ is a split monomorphism, for any coefficient ring R.*

Proof. The splitting map for $f_* : H_*(M;R) \to H_*(N;R)$ is the composition

$$H_*(N;R) \xrightarrow{\text{PD}} H^{n-*}(N;R) \xrightarrow{f^*} H^{n-*}(M;R) \xrightarrow{\text{PD}} H_*(M;R)$$

and for $f^* : H^*(N;R) \to H^*(M;R)$ is the composition

$$H^*(M;R) \xrightarrow{\text{PD}} H_{n-*}(M;R) \xrightarrow{f_*} H_{n-*}(N;R) \xrightarrow{\text{PD}} H^*(N;R). \blacksquare$$

2.2. G. W. Whitehead Theorem. For nice spaces, such as CW complexes, it is an easy observation that $\text{cat}_{\text{LS}}(X) \leq \dim X$. D. P. Grossman [Gro] and independently G. W. Whitehead [Wh], [CLOT] proved that for simply connected CW complexes we have $\text{cat}_{\text{LS}}(X) \leq \dim X/2$. Moreover, the following inequality holds true [CLOT].

THEOREM 2.2. *For k-connected CW complexes,*

$$\text{cat}_{\text{LS}}(X) \leq \dim X/(k+1).$$

2.3. Ganea's approach to LS-category. Recall that an element of an iterated join $X_0 * X_1 * \cdots * X_n$ of topological spaces is a formal linear combination $t_0x_0 + \cdots + t_nx_n$ of points $x_i \in X_i$ with $\sum t_i = 1$, $t_i \geq 0$, in which all terms of the form $0x_i$ are dropped. Given fibrations $f_i : X_i \to Y$ for $i = 0, \ldots, n$, the fiberwise join of the spaces $X_0, \ldots, X_n$ is defined to be

$$X_0 *_Y \cdots *_Y X_n = \{t_0x_0 + \cdots + t_nx_n \in X_0 * \cdots * X_n \mid f_0(x_0) = \cdots = f_n(x_n)\}$$

and the fiberwise join of fibrations $f_0, \ldots, f_n$ is the fibration

$$f_0 *_Y * \cdots *_Y f_n : X_0 *_Y X_1 *_Y \cdots *_Y X_n \to Y$$

defined by taking a point $t_0x_0 + \cdots + t_nx_n$ to $f_i(x_i)$ for any i with $t_i > 0$. As the name 'fiberwise join' suggests, the fiber of the fiberwise join of fibrations is given by the join of fibers of fibrations.

When $X_i = X$ and $f_i = f : X \to Y$ for all i, the fiberwise join of spaces is denoted by $*_Y^{n+1}X$ and the fiberwise join of fibrations is denoted by $*_Y^{n+1}f$.

For a topological space X, we turn an inclusion of a point $* \to X$ into a fibration $p_0^X : G_0(X) \to X$. The *n-th Ganea space* of X is defined to be $G_n(X) = *_X^{n+1}G_0(X)$, while the *n-th Ganea fibration* $p_n^X : G_X^n \to X$ is the fiberwise join of fibrations $p_0^X : G_0(X) \to X$.

The following theorem was proven by Shvarts [Shv] in a more general form:

THEOREM 2.3. *The inequality $\text{cat}_{\text{LS}}(X) \leq n$ holds if and only if the fibration $p_n^X : G_n(X) \to X$ admits a section.*

We note that the fiber of p_0^X is homotopy equivalent to the loop space $\Omega(X)$. Thus, the fiber of p_n^X is homotopy equivalent to the join product $*^{n+1}\Omega(X)$.

We recall that the *LS-category* of a map $f : Y \to X$, denoted by $\mathrm{cat}_{\mathrm{LS}}(f)$, is the least integer k such that Y can be covered by $k+1$ open sets $U_0, \dots, U_k$ such that the restrictions $f|_{U_i}$ are null-homotopic for all i. The above theorem can be extended to maps [Dr2]:

THEOREM 2.4. *For a map $f : Y \to X$ to a CW-space X, $\mathrm{cat}_{\mathrm{LS}}(f) \le n$ if and only if there exists a lift of f with respect to $p_n^X : G_n(X) \to X$.*

We note that $\mathrm{cat}_{\mathrm{LS}}(f) \le \min\{\mathrm{cat}_{\mathrm{LS}}(X), \mathrm{cat}_{\mathrm{LS}}(Y)\}$. We recall that the cohomology cup-length of X is a lower bound for $\mathrm{cat}_{\mathrm{LS}}(X)$. A similar statement holds true for maps: $\mathrm{cat}_{\mathrm{LS}}(f) \ge \text{cup-length}(\mathrm{Im}\, f^*)$ (see e.g. [Sr]).

We call a map $f : X \to Y$ a *k-equivalence* if it induces an isomorphism of homotopy groups in dimension $< k$ and an epimorphism in dimension k.

The following proposition was proven in [DKR] and in [DS]:

PROPOSITION 2.5. *Let $f : X \to Y$ be an $(n-2)$-equivalence of $(r-1)$-connected pointed CW-complexes with $r \ge 0$. Then the map $*^{s+1}f$ is an $(sr+s+n-2)$-equivalence.*

3. Reduction to the 2 vs 3 case. All manifolds in this paper are assumed to be connected, closed, orientable, and smooth.

PROPOSITION 3.1. *Let N be an n-dimensional simply connected manifold with $n \le 5$. Let $f : M \to N$ be a map of degree one. Then $\mathrm{cat}_{\mathrm{LS}}(M) \ge \mathrm{cat}_{\mathrm{LS}}(N)$.*

Proof. By Theorem 2.2 $\mathrm{cat}_{\mathrm{LS}}(N) \le 2$. If $\mathrm{cat}_{\mathrm{LS}}(M) = 1$ then M is a homotopy sphere. By Proposition 2.1 N is a homology sphere. Since N is simply connected, by the Hurewicz theorem it is a homotopy sphere. Then by the J. H. C. Whitehead theorem, N is homotopy equivalent to S^n. Hence $\mathrm{cat}_{\mathrm{LS}}(N) = 1$. So, always $\mathrm{cat}_{\mathrm{LS}}(M) \ge \mathrm{cat}_{\mathrm{LS}}(N)$. ∎

PROPOSITION 3.2. *Let $f : M \to N$ be a map of degree one between simply connected manifolds of dimension ≤ 8. Suppose that $\mathrm{cat}_{\mathrm{LS}}(M) \ge 3$. Then $\mathrm{cat}_{\mathrm{LS}}(M) \ge \mathrm{cat}_{\mathrm{LS}}(N)$.*

Proof. By Theorem 2.2 the LS-category of simply connected manifolds of dimension ≤ 7 does not exceed 3. So, we have to consider only the case of $n = 8$ where $\mathrm{cat}_{\mathrm{LS}}(M), \mathrm{cat}_{\mathrm{LS}}(N) \le 4$. Assume that $\mathrm{cat}_{\mathrm{LS}}(N) = 4$. Then there is no section of the Ganea fibration $p_3 : G_3(N) \to N$. The fiber F of this fibration is homotopy equivalent to a 6-connected space $*^4\Omega(N)$. The primary obstruction $\kappa \in H^8(N; \pi_7(F))$ to a section of p_3 is the only obstruction. Since the primary obstruction is natural, $f^*(\kappa)$ is the primary

obstruction to the lift of f with respect to p_3. By Proposition 2.1 $f^*(\kappa) \neq 0$. Hence by Proposition 2.4 $\mathrm{cat}_{\mathrm{LS}}(f) > 3$. The inequality $\mathrm{cat}_{\mathrm{LS}}(f) \leq \mathrm{cat}_{\mathrm{LS}}(M)$ completes the proof. ∎

We recall that if an n-manifold M' is obtained from M by a surgery then there is a bordism beetween them, i.e. an $(n+1)$-dimensional manifold W with boundary $\partial W = M \sqcup M'$. If the surgery was in dimension k, then W is homotopy equivalent to M with some $(k+1)$-cells attached. If M' is obtained by a k-surgery from M, then M is obtained by an $(n-k-1)$-surgery from M'.

Proposition 3.3.

(a) *Suppose that M' is obtained by a k-surgery, $k \leq 4$, from a simply connected n-manifold M with $\mathrm{cat}_{\mathrm{LS}}(M) \leq 2$. Then $\mathrm{cat}_{\mathrm{LS}}(W) \leq 2$ where W is a bordism of the surgery.*

(b) *Suppose that M' is obtained from a simply connected manifold M with $\mathrm{cat}_{\mathrm{LS}}(M) \leq 2$ by a surgery in dimension 2. Then $\mathrm{cat}_{\mathrm{LS}}(M') \leq 2$.*

Proof. (a) The inequality $\mathrm{cat}_{\mathrm{LS}}(M) \leq 2$ implies that there is a section $\sigma : M \to G_2(M)$ of the Ganea fibration $p_M : G_2(M) \to M$. The bordism W is homotopy equivalent to the space

$$X = M \cup \bigcup_\alpha D^{k+1}_\alpha$$

obtained from M by attaching $(k+1)$-balls along disjoint embedded k-spheres S^k_α. We show that $\mathrm{cat}_{\mathrm{LS}}(X) \leq 2$. Note that the fiber of the Ganea fibration $p_X : G_2(X) \to X$ is 4-connected. Moreover, for any contractible subset $C \subset X$, the preimage $p_X^{-1}(C)$ is 4-connected. Let $p_X^M : G_2^M(X) \to M$ be the restriction of p_X over M. Clearly p_X^M contains p_M as a subfibration. Then the restriction of σ to S_α is a null-homotopic map $\sigma|_\alpha : S^k_\alpha \to p_X^{-1}(D^{k+1})$ for all α. Let

$$\sigma'_\alpha : D^{k+1}_\alpha \to p_X^{-1}(D^{k+1}_\alpha)$$

be an extension. Then $\sigma \cup \bigcup_\alpha \sigma'_\alpha$ is a homotopy section of p_X. Then the Homotopy Lifting Property produces an honest section of p_X.

(b) We note that W is homotopy equivalent to the space

$$X' = M' \cup \bigcup_\alpha D^{n-2}_\alpha.$$

Hence the inclusion $j : M' \to X'$ is an $(n-3)$-equivalence. Then $\Omega(j) : \Omega(M') \to \Omega(X')$ is an $(n-4)$-equivalence. By Proposition 2.5 the induced map

$$*^3\Omega(f) : *^3\Omega(M') \to *^3\Omega X$$

between the fibers of $G_2(M')$ and $G_2(X')$ is a $(2+2+n-4)$-equivalence. This implies that the map $q : G_2(M') \to Z$ to the pull-back in

$$\begin{array}{ccccc} G_2(M') & \xrightarrow{q} & Z & \xrightarrow{j'} & G_2(X) \\ & \searrow^{p_{M'}} & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p_{X'}} \\ & & M' & \xrightarrow{j} & X' \end{array}$$

is an n-equivalence. A section $\sigma : X' \to G_2(X')$ defines a section $s : M' \to Z$. Since q is an n-equivalence, the map s can be lifted with respect to q. Thus, $p_{M'}$ admits a section, and hence, by Theorem 2.3 $\mathrm{cat}_{\mathrm{LS}}(M') \leq 2$. ∎

4. Main result. The following theorem was proven by I. M. James [Ja]:

THEOREM 4.1. *Let $p \geq 2$ and X be a finite $(p-1)$-connected CW complex of dimension $N \leq p(n+1)$. Denote by $\alpha_X \in H^p(X; \pi_p(X))$ the fundamental class of X. Then $\mathrm{cat}_{\mathrm{LS}}(X) = n+1$ if and only if*

$$0 \neq (\alpha_X)^{n+1} \in H^{p(n+1)}(X; \otimes^{n+1}\pi_p(X)),$$

where $(\alpha_X)^{n+1}$ is the $(n+1)$-fold cup product of α_X.

Proof. For the proof see [CLOT, Proposition 2.50]. ∎

THEOREM 4.2. *Let N be a $3k$-dimensional $(k-1)$-connected manifold. Let $f : M \to N$ be a map of degree one. Then $\mathrm{cat}_{\mathrm{LS}}(M) \geq \mathrm{cat}_{\mathrm{LS}}(N)$.*

Proof. By Theorem 2.2, $\mathrm{cat}_{\mathrm{LS}}(N) \leq 3$. Assume that $\mathrm{cat}_{\mathrm{LS}}(N) = 2$. If $\mathrm{cat}_{\mathrm{LS}}(M) = 1$ then M is a homotopy sphere, so N is a homotopy sphere in view of $\deg f = 1$, hence $\mathrm{cat}_{\mathrm{LS}}(N) = 1$, which is a contradiction.

Now let $\mathrm{cat}_{\mathrm{LS}}(N) = 3$. The case $\mathrm{cat}_{\mathrm{LS}}(M) = 1$ is impossible by the previous argument. Suppose that $\mathrm{cat}_{\mathrm{LS}}(M) = 2$. Assume that $\pi_k(N) \neq 0$. Let $u \in H^k(N; \pi_k(N))$ be the fundamental class. Since $\dim N = 3k$, we conclude that $u^3 \neq 0$, by the James Theorem. Since $f^* : N \to M$ is a monomorphism (see Proposition 2.1) by the previous proposition, we conclude that $(f^*(u))^3 \neq 0$. ∎

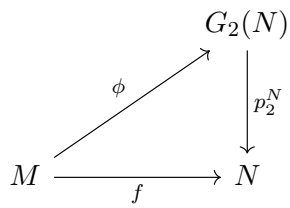
COROLLARY 4.3. *Let N be a 6-dimensional simply connected manifold. Let $f : M \to N$ be a map of degree one. Then $\mathrm{cat}_{\mathrm{LS}}(M) \geq \mathrm{cat}_{\mathrm{LS}}(N)$.*

LEMMA 4.1. *Let $f : M \to N$ be a map of degree one between n-dimensional simply connected manifolds where $n = 7$ or 8 that induces an isomorphism of homotopy groups $f_* : \pi_i(M) \to \pi_i(N)$ for $i \leq n-5$. Then $\max\{\mathrm{cat}_{\mathrm{LS}}(f), 2\} \geq \mathrm{cat}_{\mathrm{LS}}(N)$.*

Proof. We may assume that $\mathrm{cat}_{\mathrm{LS}}(N) = 3$. If $n = 7$ this is a maximal value for $\mathrm{cat}_{\mathrm{LS}}(N)$. If $n = 8$ and $\mathrm{cat}_{\mathrm{LS}}(N) = 4$, by Theorem 4.1 the cup-length of N is 4. Then by Proposition 2.1 the cup-length of $\mathrm{Im}\, f^*$ is 4. Since $\mathrm{cat}_{\mathrm{LS}}(f) \geq \text{cup-length}(\mathrm{Im}\, f^*)$, we may assume that $\mathrm{cat}_{\mathrm{LS}}(f) \leq 2$.

Now, since M and N are simply connected, we see that $\pi_2(M)$ is isomorphic to $H_2(M)$ and $\pi_2(N)$ is isomorphic to $H_2(N)$ by the Hurewicz theorem. Hence, the map f induces an isomorphism $f_* : H_2(M) \to H_2(N)$.

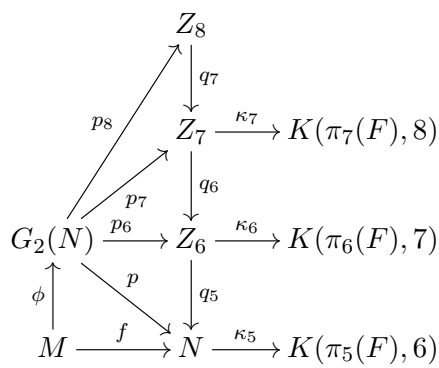
Since $\mathrm{cat}_{\mathrm{LS}}(f) \leq 2$, by Theorem 2.4 there is a lift $\phi : M \to G_2(N)$ of f with respect to the Ganea fibration p_2^N,



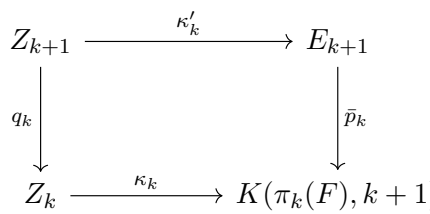
We show that there is a section $s : N \to G_2(N)$.

Since M and N are simply connected, the assumptions of the lemma imply that $f_* : H_i(M) \to H_i(N) = \pi_i(N)$ are isomorphisms for $i \leq n-5$ by the classical Whitehead theorem. Then by Poincaré duality, $f^* : H^i(N) \to H^i(M)$ are isomorphisms for $i \geq 5$.

Note that the fiber $F = *^3\Omega N$ of $p = p_2^N : G_2(N) \to N$ is the join product of three copies of the connected space ΩN and hence it is 4-connected. Therefore in the Moore–Postnikov tower for $p : G_2(N) \to N$ we have $N = Z_1 = Z_2 = Z_3 = Z_4 = Z_5$:



We recall that the maps $q_k : Z_{k+1} \to Z_k$ are principal bundles with fibers $K(\pi_k(F), k)$. Thus, there are pull-back diagrams



where E_n are contractible and $\bar{p}_n$ are principal bundles with fibers topologi-

cal abelian groups having homotopy type of the Eilenberg–MacLane spaces $K(\pi_{n-1}(F), n-1)$ (see [Mc] or [Dr1]).

By commutativity of the above diagrams we obtain

$$\kappa_5 f = \kappa_5 q_5 p_6 \phi = \bar{p}_5 \kappa_5' p_6 \phi.$$

Since $\bar{p}_5 \kappa_5' p_6 \phi$ factors through a contractible space E_6, the map $\kappa_5 f$ is null-homotopic. We recall that f^* is injective on 5-cohomology. Since the map $\kappa_5 f$ represents the image $f^*[\kappa_5]$ of the cohomology class represented by $\kappa_5 : N \to K(\pi_5(F), 6)$, the map κ_5 is null-homotopic. Therefore, κ_5 admits a lift, and hence q_5 admits a section. Since q_5 is a principal bundle that has a section, it is a trivial bundle over N with fiber $K(\pi_5(F), 5)$.

Thus,

$$Z_6 \cong K(\pi_5(F), 5) \times N.$$

The projection $\mathrm{pr} : Z_6 \to K(\pi_5(F), 5)$ to the fiber defines a cohomology class

$$\alpha = [\mathrm{pr} \circ p_6 \circ \phi] \in H^5(M; \pi_5(F)).$$

Since f^* is an isomorphism on 5-cohomology, there is $\beta \in H^5(N; \pi_5(F))$ with $f^*(\beta) = \alpha$. The class β can be represented by a map

$$s : N \to K(\pi_5(F), 5)$$

which defines a section $s_5 : N \to Z_6$ of q_5 such that the map

$$p_6 \phi = (\mathrm{pr} \circ p_6 \circ \phi, f) : M \to K(\pi_5(F), 5) \times N$$

is homotopic to

$$s_5 f = (sf, f) : M \to K(\pi_5(F), 5) \times N.$$

Next we show that the map $s_5 : N \to Z_6$ admits a lift with respect to q_6. This is equivalent to saying that

$$\kappa_6 s_5 : N \to K(\pi_6(F), 7)$$

is null-homotopic. Since $p_7 \phi$ is a lift of $p_6 \phi$ with respect to q_6, the composition

$$\kappa_6 p_6 \phi : M \to K(\pi_6(F), 7)$$

is null-homotopic. Therefore, $\kappa_6 s_5 f$ is null-homotopic. This means that f^* takes the cohomology class $[\kappa_6 s_5]$ to zero. Since by Proposition 2.1 the homomorphism f^* is injective, we find that $[\kappa_6 s_5] = 0$. Thus, the map $\kappa_6 s_5$ is null-homotopic.

Suppose that $n = 7$. Since p_7 is a 7-equivalence, i.e. it is an isomorphism for i-dimensional homotopy groups for $i < 7$ and epimorphism for $i = 7$, there is a homotopy lift $\sigma : N \to G_2(N)$ of the map $s_6 : N \to Z_7$ with respect to p_7. Then σ is a homotopy section of p. Since p is a fibration, it admits a regular section. Therefore, $\mathrm{cat}_{\mathrm{LS}}(N) \leq 2$.

If $n = 8$, we construct a lift $s_7 : N \to Z_8$ of s_5 with respect to q_6q_7 and complete the proof similarly to the case of $n = 7$. For that we modify s_6 to a map such that $s_6 f$ becomes homotopic to $p_7\phi$.

Since $q_6 s_6 = s_5$ is an embedding, the map q_6 is defined by a section of q_6 on $s_5(N)$. Since the map

$$\kappa_6 s_5 : N \to K(\pi_7(F), 7)$$

is null-homotopic, the restriction of q_6 over $s_5(N)$ is a trivial bundle,

$$q_6^{-1}(s_5(N)) \cong s_5(N) \times K(\pi_6(F), 6).$$

Since $p_6\phi$ is homotopic to $s_5 f$ and $q_6 p_7\phi = p_6\phi$, by the homotopy lifting property of q_6, we can homotope $p_7\phi$ to a map $\psi : M \to Z_7$ such that $q_6\psi = s_5 f$. Thus the map

$$\psi : M \to s_5(N) \times K(\pi_6(F), 6)$$

has two components $(s_5 f, \psi_1)$. Since f induces an isomorphism of 6-dimensional cohomology groups, there is a map

$$\xi : N \to K(\pi_6(F), 6)$$

such that f^* takes the cohomology class $[\xi] \in H^6(N; \pi_6(F))$ to the cohomology class $[\psi_1] \in H^6(M; \pi_6(F))$. Thus, ψ_1 is homotopic to ξf. We consider a section

$$s' : s_5(N) \cong N \to q_6^{-1}(s_5(N)) \cong s_5(N) \times K(\pi_6(F), 6)$$

defined by ξ. Then ψ is homotopic to $s's_5 f$. Since $p_8\phi$ is a lift of $p_7\phi$ with respect to q_7, the composition

$$\kappa_7 p_7\phi : M \to K(\pi_6(F), 7)$$

is null-homotopic. Hence $\kappa_7\psi$ is null-homotopic and so is $\kappa_7 s's_5 f$. This means that f^* takes the cohomology class $[\kappa_7 s's_5]$ to zero. Since f^* is injective, we obtain $[\kappa_7 s's_5] = 0$. Thus, the map $\kappa_7 s's_5$ is null-homotopic, and hence there is a lift $s_7 : N \to Z_8$ of $s's_5$ with respect to q_7.

Since p_8 is an 8-equivalence, there is a homotopy lift $\sigma : N \to G_2(N)$ of $s_7 : N \to Z_8$ with respect to p_8. Then σ is a homotopy section of p. Since p is a fibration, it admits a regular section. Therefore, $\mathrm{cat}_{\mathrm{LS}}(N) \leq 2$. ∎

PROPOSITION 4.4. *A degree one map $f : M \to N$ of an n-dimensional, $n = 7, 8$, simply connected spin manifold with $\mathrm{cat}_{\mathrm{LS}}(M) \leq 2$ is bordant by means of surgery in dimension 2 and 3 to a degree one map $f' : M' \to N$ with $\mathrm{cat}_{\mathrm{LS}}(f') \leq 2$ that induces isomorphisms $f' : \pi_i(M') \to \pi_i(N)$ for $i \leq n - 5$.*

Proof. For $n = 7, 8$ by Proposition 2.1 the induced homomorphism $f_* : H_2(M) \to H_2(N)$ is surjective with a finitely generated kernel $K = \ker f_*$ which by the Hurewicz Theorem is isomorphic to $\ker\{f_* : \pi_2(M) \to \pi_2(N)\}$.

Let $S_1, \dots, S_k$ be disjoint 2-spheres smoothly embedded in M that generate K. For each sphere S_i its normal bundle ν_i is classified by a map $\nu_i : S_i \to BSO(n-2)$. The map ν_i is null-homotopic, since otherwise the evaluation of the Stiefel–Whitney class $w_2 \in H^2(BSO(n-2); \mathbb{Z}_2)$ on the image $(\nu_i)_*([S_i]) \in H_2(BSO(n-2); \mathbb{Z}_2)$ would be nonzero. The latter would contradict the spin condition on M. Thus, we can perform a surgery to obtain a bordism W of M to M' together with a map $\bar{f} : W \to N$. The restriction $f' : M' \to N$ of $\bar{f}$ to M' has degree one and f' induces an isomorphism of 2-dimensional homotopy groups. For $n = 7$ by Proposition 3.3(b) we have $\mathrm{cat}_{\mathrm{LS}}(M') \le 2$, and hence $\mathrm{cat}_{\mathrm{LS}}(f') \le 2$.

When $n = 8$, by the above we may assume that $f : M \to N$ induces an isomorphism of 2-dimensional homotopy groups. Since $\pi_3(BSO(n-3)) = 0$, we deduce that the normal bundle of any embedded 3-sphere in M' is trivial. Hence we can perform surgery on generators of $\ker\{f_* : \pi_3(M') \to \pi_3(N)\}$ to get a bordism W between M and M' together with a map $\hat{f} : W \to N$ that agrees with f on M such that the restriction $f' : M' \to N$ of $\hat{f}$ to M' has degree one and it induces isomorphisms of homotopy groups of dimension ≤ 3. By Proposition 3.3(a) we find that $\mathrm{cat}_{\mathrm{LS}}(W) \le 2$. Hence $\mathrm{cat}_{\mathrm{LS}}(\hat{f}) \le 2$. By Theorem 2.4, $\hat{f}$ admits a lift $\hat{\phi} : W \to G_2(N)$ of $\hat{f}$ with respect to p_2^N. The restriction of $\hat{\phi}$ to M' is a lift of $f' : M' \to N$ with respect to p_2^N. Then by Theorem 2.4, $\mathrm{cat}_{\mathrm{LS}}(f') \le 2$. ∎

THEOREM 4.5. *A degree one map $f : M \to N$ between simply connected spin n-manifold for $n \le 8$ satisfies the Rudyak conjecture,*

$$\mathrm{cat}_{\mathrm{LS}}(M) \ge \mathrm{cat}_{\mathrm{LS}}(N).$$

The spin condition can be dropped when $n \le 6$.

Proof. When $n \le 5$ the statement is proven in Proposition 3.1. In view of Proposition 3.2 it suffices to show that there is no degree one map $f : M \to N$ with $\mathrm{cat}_{\mathrm{LS}}(M) \le 2$ and $\mathrm{cat}_{\mathrm{LS}}(N) \ge 3$. The case of $n = 6$ is taken care of in Corollary 4.3.

We consider the case when $n \ge 7$ and M is spin. Let $\mathrm{cat}_{\mathrm{LS}}(M) = 2$. In view of Propositions 4.4 and 3.3 we may assume that f induces isomorphisms of homotopy groups of dimension ≤ 3, and $\mathrm{cat}_{\mathrm{LS}}(f) \le 2$. Then by Lemma 4.1, $\mathrm{cat}_{\mathrm{LS}}(N) \le 2$. ∎

REMARK 4.6. There are nonspin simply connected 7-manifolds with LS-category 2. We recall that the Wu manifold $\mathrm{SU}(3)/\mathrm{SO}(3)$ is simply connected and non-spin. A 7-manifold M constructed from the product of a 2-sphere and the Wu manifold,

$$S^2 \times (\mathrm{SU}(3)/\mathrm{SO}(3)),$$

by a 2-surgery on the factor S^2 is called a *gyration* of the Wu manifold [Xu]. It is simply connected and nonspin. Below we compute its LS-category.

PROPOSITION 4.7. *The LS-category of the gyration of the Wu manifold is* 2.

Proof. Let $Y = \mathrm{SU}(3)/\mathrm{SO}(3)$. By Theorem 2.2, $\mathrm{cat}_{\mathrm{LS}}(Y) = 2$. The bordism X defined by the 2-surgery on $Y \times S^2$ is homotopy equivalent to the union $(Y \times S^2) \cup (\{y_0\} \times D^3)$ which is homotopy equivalent to the half-smash product

$$Y \rtimes S^2 = (Y \times S^2)/(\{y_0\} \times D^3).$$

It is known that for the half-smash product we have [SS]

$$\mathrm{cat}_{\mathrm{LS}}(A \rtimes B) = \mathrm{cat}_{\mathrm{LS}}(A).$$

Hence $\mathrm{cat}_{\mathrm{LS}}(X) = \mathrm{cat}_{\mathrm{LS}}(Y) = 2$. Let M be the gyration of the Wu manifold. Then X is homotopy equivalent to $X' = M \cup D^5$. The rest of the proof coincides with the end of the proof of Proposition 3.3(b). Since the inclusion $j : M \to X'$ is a 4-equivalence, $\Omega(j) : \Omega(M) \to \Omega(X')$ is a 3-equivalence. By Proposition 2.5,

$$*^3\Omega(j) : *^3\Omega(M) \to *^3\Omega(X')$$

is a $(2 + 2 + 3 = 7)$-equivalence. This implies that the map $q : G_2(M) \to Z$ to the pull-back in the diagram

$$\begin{array}{ccccc} G_2(M) & \xrightarrow{q} & Z & \xrightarrow{j'} & G_2(X') \\ & \searrow^{p_M} & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p_{X'}} \\ & & M & \xrightarrow{j} & X' \end{array}$$

is a 7-equivalence. Since $\mathrm{cat}_{\mathrm{LS}}(X') = \mathrm{cat}_{\mathrm{LS}}(X) = 2$, there is a section $\sigma : X' \to G_2(X')$ which defines a section $s : M \to Z$ of p'. Since the map q is a 7-equivalence, s can be lifted with respect to q. Thus, p_M admits a section, and hence $\mathrm{cat}_{\mathrm{LS}}(M) \leq 2$.

Since M is not homeomorphic to a sphere, $\mathrm{cat}_{\mathrm{LS}}(M) > 1$ and we conclude that $\mathrm{cat}_{\mathrm{LS}}(M) = 2$. ∎

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