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Indices of nilpotency in certain spaces of modular forms

by

Matthew Boylan and Swati

Abstract. We study the index of nilpotency relative to certain Hecke operators in spaces of modular forms with integer weight and level N with integer coefficients modulo primes p for $(p, N) \in \{(3, 1), (5, 1), (7, 1), (3, 4)\}$. In these settings, we prove upper bounds on certain indices of nilpotency. As an application of our bounds, we prove infinite families of congruences for p^t-core partition functions modulo p for $p \in \{3, 5, 7\}$ and $t \geq 1$, and we prove an infinite family of congruences modulo 3 for the rth power partition function, $p_r(n)$, when $r = 12k$ with $\gcd(k, 6) = 1$. We also include conjectures on a function which quantifies degree lowering on powers of the Delta-function by the relevant Hecke operators in these settings, and on the index of nilpotency relative to a modification of this degree-lowering function.

1. Introduction and statement of results

1.1. Introduction. Let p be prime, let $\mathbb{Q}_{(p)}$ denote the localization of $\mathbb{Q}$ at p, let $N \geq 1$, let $k \geq 0$, and let

$$M_k(\Gamma_0(N))_{(p)} = M_k(\Gamma_0(N)) \cap \mathbb{Q}_{(p)}[\![q]\!],$$

denote the space of weight k modular forms on $\Gamma_0(N)$ with coefficients in $\mathbb{Q}_{(p)}$. If $f = \sum a(n)q^n \in \mathbb{Q}_{(p)}[\![q]\!]$ then we define

$$\widetilde{f} = \sum \widetilde{a(n)}\, q^n \in \mathbb{F}_p[\![q]\!],$$

to be its coefficientwise reduction modulo p, and we define

$$\widetilde{M}^{(p)}(\Gamma_0(N)) = \{\widetilde{f} : f \in M_k(\Gamma_0(N))_{(p)}\} \subseteq \mathbb{F}_p[\![q]\!],$$

the algebra of holomorphic integer weight modular forms on $\Gamma_0(N)$ with coefficients in $\mathbb{Q}_{(p)}$ reduced modulo p. For relevant details on these spaces, see Section 2. Theorem 1.1 below gives instances where Hecke operators act locally nilpotently on subspaces of $\widetilde{M}^{(p)}(\Gamma_0(N))$. When a Hecke operator T

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acts locally nilpotently on a subspace $\widetilde{M} \subseteq \widetilde{M}^{(p)}(\Gamma_0(N))$, we define, for every nonzero $\widetilde{f} \in \widetilde{M}$, the *index of nilpotency* of $\widetilde{f}$ with respect to T by

$$N_T(\widetilde{f}, p) := \min\{u \geq 1 : \widetilde{f}|T^u = 0 \text{ in } \mathbb{F}_p[\![q]\!]\}. \tag{1.1}$$

In this paper, we study $N_T(\widetilde{f}, p)$ in certain settings, and we give applications of our results. Some recent works on local nilpotency include [CZ23, GG16, Me15, Me18, Mo16, MT06, NS12a, NS12b, OT05]; some recent applications of nilpotency to congruence properties of arithmetic functions include [B02, B06, BO01, C09, C13, D15, M05, SY15].

Before we describe our work, we give some of the benchmark results on local nilpotency of Hecke operators on spaces $\widetilde{M}^{(p)}(\Gamma_0(N))$. We require some notation. We consider the set of primes

$$L_{p,N} = \{\ell : \ell \equiv \pm 1 \pmod{p} \text{ is prime and } \ell \nmid N\},$$

and for all primes $\ell \in L_{p,N}$, we consider the modified Hecke operators

$$T'_\ell = \begin{cases} T_\ell & \text{if } \ell \equiv -1 \pmod{p}, \\ T_\ell - 2 & \text{if } \ell \equiv 1 \pmod{p}. \end{cases} \tag{1.2}$$

We let $\widetilde{M}^{0,(p)}(\Gamma_0(N))$ denote the subspace of $\widetilde{M}^{(p)}(\Gamma_0(N))$ of forms with weights congruent to zero modulo $p-1$. Theorem 3 of [SD73] implies, for $p \in \{2,3\}$, that $\widetilde{M}^{0,(p)}(\Gamma_0(1)) = \widetilde{M}^{(p)}(\Gamma_0(1)) = \mathbb{F}_p[\widetilde{\Delta}]$. Further, Theorem 2(iv) of [SD73] implies that $\widetilde{M}^{(5)}(\Gamma_0(1)) = \mathbb{F}_5[\widetilde{E_6}]$. Since $\widetilde{\Delta} = 2 - 2\widetilde{E}_6^2$, we have $\widetilde{M}^{0,(5)}(\Gamma_0(1)) = \mathbb{F}_5[\widetilde{E}_6^2] = \mathbb{F}_5[\widetilde{\Delta}]$. Similarly, we observe that $\widetilde{M}^{(7)}(\Gamma_0(1)) = \mathbb{F}_7[\widetilde{E_4}]$, and hence $\widetilde{\Delta} = 1 - \widetilde{E}_4^3$ gives $\widetilde{M}^{0,(7)}(\Gamma_0(1)) = \mathbb{F}_7[\widetilde{E}_4^3] = \mathbb{F}_7[\widetilde{\Delta}]$.

The following theorem captures some instances where Hecke operators act locally nilpotently on spaces of modular forms modulo p.

THEOREM 1.1. *In the notation above, we have:*

(i) (Serre, Tate [Se87, Ta94]) *Let $p \in \{2,3,5,7\}$. For all $\ell \in L_{p,1}$, the operator T'_ℓ acts locally nilpotently on $\widetilde{M}^{0,(p)}(\Gamma_0(1)) = \mathbb{F}_p[\widetilde{\Delta}]$.*
(ii) (Ono, Taguchi [OT05]) *Let $1 \leq N \leq 17$ be odd, and let $a \geq 0$. For all primes $\ell \nmid 2N$, the operator T'_ℓ acts locally nilpotently on $\widetilde{M}^{(2)}(\Gamma_0(2^a N))$.*
(iii) (Moon, Taguchi [MT06]) *Let $(p,N) = \{(3,4),(5,2)\}$, and let $a \geq 0$. For all $\ell \in L_{p,N}$, the operator T'_ℓ acts locally nilpotently on $\widetilde{M}^{(p)}(\Gamma_0(p^a N))$.*

We now give theorems due to Nicolas and Serre [NS12b] and Medvedovsky [Me15] on indices of nilpotency. To state the theorem of Nicolas and Serre, we need to extend Definition 1.1. When an algebra A of Hecke operators acts locally nilpotently on a subspace $\widetilde{M} \subseteq \widetilde{M}^{(p)}(\Gamma_0(N))$, we define $N_A(\widetilde{f}, p)$, the index of nilpotency of a nonzero $\widetilde{f} \in \widetilde{M}$ with respect to A, to be the integer $n \geq 1$ such that

- for all $R_1, \ldots, R_n \in A$, we have
$$\widetilde{f}\,|R_1|\,\cdots\,|R_n = 0,$$
- there exist $S_1, \ldots, S_{n-1} \in A$ with
$$\widetilde{f}\,|S_1|\,\cdots\,|S_{n-1} \neq 0.$$

For all $T \in A$ and nonzero $\widetilde{f} \in \widetilde{M}$, we note that $N_T(\widetilde{f}, p) \leq N_A(\widetilde{f}, p)$. To simplify notation, when $\ell \in L_{p,N}$ and $\widetilde{f}$ is in one of the spaces in Theorem 1.1, we set $N_\ell(\widetilde{f}, p) = N_{T'_\ell}(\widetilde{f}, p)$. The following theorem gives bounds on indices of nilpotency modulo $p \in \{2, 3, 5, 7\}$ in level 1.

THEOREM 1.2. *Let $k \geq 1$.*

(i) (Nicolas–Serre (2012) [NS12b]) *Let k be odd with base 2 expansion $k = \sum \beta_i(k) 2^i$, and define*
$$n_3(k) = \sum \beta_{2i+1}(k) 2^i, \quad n_5(k) = \sum \beta_{2i+2}(k) 2^i,$$
and $A = \langle T_\ell : \ell \neq 2 \rangle$. Then
$$\tfrac{1}{2}\sqrt{k} < N_A(\widetilde{\Delta}^k, 2) = 1 + n_3(k) + n_5(k) < \tfrac{3}{2}\sqrt{k}.$$

(ii) (Medvedovsky (2015) [Me15]) *For all (p, ℓ) listed in the table below, the explicit constants $c > 0$ and $0 < \alpha < 1$ given there satisfy $N_\ell(\widetilde{\Delta}^k, p) < c \cdot k^\alpha$.*

Table 1. Explicit values of c and α for different p and ℓ

(p, ℓ)	c	α	(p, ℓ)	c	α
(2, 3)	3	0.5	(2, 5)	$7/3$	$2/3$
(3, 2)	4	$\log_3(2)$	(3, 7)	3.2	$\log_9(6)$
(5, 19)	$138/11$	$\log_{25}(23)$	(5, 11)	6.3	$\log_{25}(21)$
(7, 13)	$564/23$	$\log_{49}(47)$	(7, 29)	$564/23$	$\log_{49}(47)$

1.2. Statement of results. We now turn to our results. For details on modular forms, see Section 2. Our first theorem gives upper bounds on $N_\ell(f^k, p)$ in certain cases.

THEOREM 1.3. *Let $k \geq 1$.*

(i) *For all primes $\ell \neq 3$, we have*
$$N_\ell(\widetilde{\Delta}^k, 3) \leq \begin{cases} 1 + \lfloor k/3 \rfloor, & \ell \equiv 1 \pmod 3, \\ 1 + \lfloor 2k/3 \rfloor, & \ell \equiv 2 \pmod 3. \end{cases}$$

(ii) *Let* $p \in \{5, 7\}$. *For all primes* $\ell \equiv \pm 1 \pmod{p}$, *we have*

$$N_\ell(\widetilde{\Delta}^k, p) \leq \begin{cases} 1 + \lfloor 2k/5 \rfloor, & p = 5,\ p \nmid k, \\ 1 + \lfloor 3k/7 \rfloor, & p = 7,\ k \in \{1, 3, 5\} \pmod{7}, \\ 2 + \lfloor 3k/7 \rfloor, & p = 7,\ k \in \{2, 4, 6\} \pmod{7}, \\ N_\ell(\widetilde{\Delta}^{k/p}, p), & p \mid k. \end{cases}$$

(iii) *Let* $D_2(z) = \eta(2z)^{12} \in S_6(\Gamma_0(4))$, *where* $\eta(z)$ *is the Dedekind eta-function as in* (2.2), *and let* $\ell \notin \{2, 3\}$ *be prime.*

(a) *Suppose that* $\gcd(k, 6) = 1$. *We have*

$$N_\ell(\widetilde{D}_2^k, 3) \leq 1 + \left\lfloor \frac{k}{3} \right\rfloor.$$

(b) *We also have*

$$N_\ell(\widetilde{D}_2^k, 3) \leq \begin{cases} N_\ell(\widetilde{D}_2^{k/3}, 3), & 3 \mid k, \\ N_\ell(\widetilde{\Delta}^{k/2}, 3), & 2 \mid k. \end{cases}$$

Remarks. (1) An important feature of Medvedovsky's bounds on $N_\ell(\widetilde{\Delta}^k, p)$ in part two of Theorem 1.2 is that they are *sublinear*. While ours are linear, we note that they are stronger than the bounds in Table 1 for a large—but finite—number of values of k, as follows:

- for $\ell \equiv 1 \pmod{3}$ and $k \leq 2.1 \times 10^5$,
- for $\ell \equiv -1 \pmod{5}$ and $k \leq 5.86 \times 10^{57}$,
- for $\ell \equiv 1 \pmod{5}$ and $k \leq 1.27 \times 10^{22}$,
- for $\ell \equiv \pm 1 \pmod{7}$ and $k \leq 1.36 \times 10^{164}$.

(2) Let $p \in \{2, 3, 5, 7\}$ and let $\ell \in \{\pm 1 \pmod{p}\}$ be prime. By [NS12b, Théorème 23] and [Me15, Propositions 4.21 and 6.2], the sequence $\{\widetilde{\Delta}^k | T'_\ell : k \geq 1\}$ satisfies a recurrence

$$\widetilde{\Delta}^k | T'_\ell = \sum_{1 \leq m \leq t} b_{m,\ell}(\widetilde{\Delta}) \widetilde{\Delta}^{k-m} | T'_\ell, \tag{1.3}$$

where

$$t = \begin{cases} \ell + 1, & \ell \equiv -1 \pmod{p}, \\ \ell + 2, & \ell \equiv 1 \pmod{p}, \end{cases}$$

is the order of the recurrence and $b_{m,\ell} \in \mathbb{F}_p[Y]$ has degree at most m. For all $\widetilde{f} \in \mathbb{F}_p[\widetilde{\Delta}]$, we let $\deg_{\widetilde{\Delta}}(\widetilde{f})$ denote the degree of $\widetilde{f}$ in $\widetilde{\Delta}$. In the terminology of [Me15, Chapter 4], the sequence is i-filtered for some $i \geq 0$ precisely when we have $\deg_{\widetilde{\Delta}}(\widetilde{\Delta}^n | T'_\ell) \leq n - i$ for all $n \geq 1$. Part (i) of Theorem 1.1 implies that the sequences $\{\widetilde{\Delta}^k | T'_\ell : k \geq 1\}$ are 1-filtered. Parts (i) and (ii) of Theorem 1.3 follow by showing that the sequences are 2-filtered when $p \in \{3, 5, 7\}$.

When $p \in \{3,5,7\}$, we note that for certain primes ℓ, the sequence $\{\widetilde{\Delta}^k | T'_\ell : k \geq 1\}$ may be i-filtered for some $i > 2$, which would yield better bounds on $N_\ell(\widetilde{\Delta}^k, p)$ than in the theorem. For example, when $p = 5$, computations show that $\deg_{\widetilde{\Delta}}(\widetilde{\Delta}^n | T'_{11}) \leq n - 4$ for $1 \leq n \leq 13$. An induction argument using (1.3) shows that the sequence $\{\widetilde{\Delta}^k | T'_{11} : k \geq 1\}$ is 4-filtered. An argument as in the proof of the theorem then gives $N_{11}(\widetilde{\Delta}^k, 5) \leq 1 + \lfloor k/4 \rfloor$.

Medvedovsky's computations and ours suggest that $N(\widetilde{\Delta}^k, 3) = O(k^{1/2})$. Our computations also suggest that $N_{11}(\widetilde{\Delta}^k, 5)$ and $N_{19}(\widetilde{\Delta}^k, 5) = O(k^{1/2})$.

Next, we let $\delta \mid 24$ and $D_\delta(z) = \eta(\delta z)^{24/\delta}$, and we note that $D_1(z) = \Delta(z)$. By [CS17, Propositions 5.9.2 and 5.9.3], we have

$$D_\delta(z) \in \begin{cases} S_{1/2}\big(\Gamma_0(576), \big(\tfrac{12}{\bullet}\big)\big), & \delta = 24, \\ S_{3/2}\big(\Gamma_0(64), \big(\tfrac{-4}{\bullet}\big)\big), & \delta = 8, \\ S_{12/\delta}\big(\Gamma_0(\delta^2), \big(\tfrac{(-1)^{12/\delta}\delta^{24/\delta}}{\bullet}\big)\big), & \delta \notin \{8, 24\}, \end{cases}$$

where the forms with half-integer weight transform with respect to the theta-multiplier system as in Shimura's theory. In the following proposition, we identify some primes p and ℓ and exponents k such that $\widetilde{D}_\delta^k(z) | T_\ell = 0$ in $\mathbb{F}_p[\![q]\!]$. In such cases, when T_ℓ acts nilpotently on an algebra containing $\widetilde{D}_\delta$ as in Theorem 1.1, we have $N_\ell(\widetilde{D}_\delta^k(z), p) = 1$.

PROPOSITION 1.4. *Let $m \geq 1$, let $p \neq \ell$ be primes, and let $\big(\frac{\bullet}{\ell}\big)$ denote the Legendre symbol.*

(1) *Let $\delta \mid 24$, let m be odd, let p satisfy*

   (a) $p + 1 \mid 24/\delta$,
   (b) $\frac{24/\delta}{p+1} = p^t$ *or* $3p^t$ *for some* $t \geq 0$,
   (c) $p \nmid \delta$,

   and let $r_{p,m} = \frac{p^m+1}{p+1} \in \mathbb{Z}$. Then for all primes $\ell \neq p$ with $\ell \nmid \delta$ and $\big(\frac{-p}{\ell}\big) = -1$, we have $\widetilde{D}_\delta(z)^{r_{p,m}} | T_\ell = 0$ in $\mathbb{F}_p[\![q]\!]$. The set of (δ, p) satisfying conditions (a)–(c) *is*

$$\{(1,2), (1,7), (1,23), (2,3), (2,11), (3,7), (4,5)\}.$$

(2) *Let $d \mid 24$ and let p satisfy*

   (a) $24/d = p^t$ *or* $3p^t$ *for some* $t \geq 0$,
   (b) $p \nmid d$.

   Then for all primes $\ell \neq p$ with $\ell \nmid d$ and

$$-1 = \left(\frac{-p^m}{\ell}\right) = \begin{cases} \big(\frac{-p}{\ell}\big), & m \text{ is odd}, \\ \big(\frac{-1}{\ell}\big), & m \text{ is even}, \end{cases}$$

we have $\widetilde{D}_d(z)^{p^m+1}|T_\ell = 0$ *in* $\mathbb{F}_p[\![q]\!]$. *The set of* (d,p) *satisfying conditions* (a) *and* (b) *is*

$$\{(1,2),(3,2)\} \cup \{(8,p) : p \geq 3\} \cup \{(24,p) : p \geq 5\}.$$

Remark. The primes $p \neq 2$ in part (1) of the proposition are precisely those with $\frac{24/\delta}{p+1} \in \{1,3\}$, in which cases we find that $D_\delta(z)^{r_{p,m}}$ is a product of two theta-series modulo p:

$$D_\delta(z)^{r_{p,m}} \equiv \eta(\delta z)^{\frac{24/\delta}{p+1}} \eta(\delta p^m z)^{\frac{24/\delta}{p+1}} \pmod{p}.$$

Our next theorems are applications of Theorem 1.3 to congruences for arithmetic functions. Let $t \geq 1$. We recall that a partition λ of a positive integer n is t-core precisely when none of the hook numbers associated to its Ferrers–Young diagram are divisible by t. The notion of a t-core partition plays a role in modular group representation theory [JK81]. We denote the number of t-core partitions of n by $a_t(n)$. See [CKNS21] for a survey of results on t-core partitions and t-core partition functions. For $p \in \{3,5,7\}$, our first application of Theorem 1.3 yields congruences for p^t-core partition functions $a_{p^t}(n)$ modulo p.

Theorem 1.5. *Let* $t \geq 1$, *let* $p \in \{3,5,7\}$, *let*

$$k_{p,t} = \begin{cases} \frac{9^t-1}{8}, & p=3, \\ \frac{p^{2t}-1}{24}, & p \in \{5,7\}, \end{cases}$$

and let

$$m_{p,t} = \begin{cases} 1 + \lfloor \frac{2k_{3,t}}{3} \rfloor = 1 + 3\left(\frac{9^{t-1}-1}{4}\right), & p = 3 \text{ and } \ell \equiv 2 \pmod{3}, \\ 1 + \lfloor \frac{k_{3,t}}{3} \rfloor = 1 + 3\left(\frac{9^{t-1}-1}{8}\right), & p = 3 \text{ and } \ell \equiv 1 \pmod{3}, \\ 1 + \lfloor \frac{2k_{5,t}}{5} \rfloor = 1 + 5\left(\frac{25^{t-1}-1}{12}\right), & p = 5, \\ 2 + \lfloor \frac{3k_{7,t}}{7} \rfloor = 2 + 7\left(\frac{49^{t-1}-1}{8}\right), & p = 7. \end{cases}$$

(i) *Let* $p \in \{3,5,7\}$, *and let* $\ell \equiv -1 \pmod{p}$ *be prime. Then for all* r *with* $p^r \geq m_{p,t}$, *and for all* $n \geq 1$ *with* $\ell \nmid n$, *we have*

$$a_{p^t}(\ell^{p^r} n - k_{p,t}) \equiv -a_{p^t}(\ell^{p^r-2} n - k_{p,t}) \pmod{p}.$$

(ii) *Let* $p \in \{5,7\}$, *and let* $\ell_1, \ldots, \ell_{m_{p,t}} \equiv -1 \pmod{p}$ *be distinct primes. Then for all* $n \geq 1$ *with* $\gcd(\ell_1 \cdots \ell_{m_{p,t}}, n) = 1$, *we have*

$$a_{p^t}(\ell_1 \cdots \ell_{m_{p,t}} n - k_{p,t}) \equiv 0 \pmod{p}.$$

(iii) *Let* $p \in \{3,5,7\}$, *and let* $\ell \equiv 1 \pmod{p}$. *Then for all* r *with* $p^r \geq m_{p,t}$, *and for all* $n \geq 1$ *with* $\ell \nmid n$, *we have*

$$a_{p^t}(\ell^{p^r} n - k_{p,t}) - a_{p^t}(\ell^{p^r-2} n - k_{p,t}) \equiv 2a_{p^t}(n - k_{p,t}) \pmod{p}.$$

REMARKS. (i) The integers $m_{p,t}$ in the theorem are bounds on $N_\ell(\Delta^{k_{p,t}}, p)$ from parts (i) and (ii) of Theorem 1.3 when $p = 3$ and $p \in \{5, 7\}$, respectively.

(ii) Analogues of parts (i) and (ii) of the theorem hold for $p \in \{2, 3\}$. Using part (i) of Theorem 1.1, the first author [B02] proved congruences of these types for $p = 2$ prior to the result of Nicolas and Serre in part (1) of Theorem 1.2. Later, Chen [C13] used part (i) of Theorem 1.2 to sharpen the results in [B02], settling a conjecture from [HS99]. Dai [D15] used part (ii) of Theorem 1.2 to prove the analogue of part (ii) for $p = 3$.

We now let $r \geq 1$, and define functions $p_r(n)$ by

$$\sum_{n\geq 0} p_r(n)q^n = \prod_{n\geq 1} (1 - q^n)^r. \tag{1.4}$$

We let $p_{e,r}(n)$ and $p_{o,r}(n)$ denote the number of r-colored partitions of $n \geq 1$ into even and odd numbers of parts, respectively, and we note that $p_r(n) = p_{e,r}(n) - p_{o,r}(n)$. Our second application of Theorem 1.3 gives congruences for $p_{12r}(n) \pmod 3$ when $\gcd(r, 6) = 1$.

THEOREM 1.6. *Let $r \geq 1$ with $\gcd(r, 6) = 1$, let $u_r = 1 + \lfloor r/3 \rfloor$, and let $\ell \neq 2$ be prime.*

(i) *Let $\ell \equiv -1 \pmod 3$. Then for all j with $3^j \geq u_r$ and for all $n \geq 1$ with $\ell \nmid 2n$, we have*

$$p_{12r}\left(\frac{\ell^{u_r} n - r}{2}\right) \equiv 2p_{12r}\left(\frac{\ell^{u_r - 2} n - r}{2}\right) \pmod 3.$$

(ii) *Let $\ell_1, \ldots, \ell_{u_r} \equiv -1 \pmod 3$ be distinct odd primes. Then for all $n \geq 1$ relatively prime to $2\ell_1 \cdots \ell_{u_r}$, we have*

$$p_{12r}\left(\frac{\ell_1 \cdots \ell_{u_r} n - r}{2}\right) \equiv 0 \pmod 3.$$

(iii) *Let $\ell \equiv 1 \pmod 3$. Then for all j with $3^j \geq u_r$ and for all $n \geq 1$ with $\ell \nmid 2n$, we have*

$$p_{12r}\left(\frac{\ell^{3^j} n - r}{2}\right) - p_{12r}\left(\frac{\ell^{3^j - 2} n - r}{2}\right) \equiv 2p_{12r}\left(\frac{n - r}{2}\right) \pmod 3.$$

1.3. Computations and conjectures on some related quantities. Here we discuss quantities of interest that we encountered in our study of nilpotency, and we make some conjectures on these quantities. We let $p \in \{5, 7\}$, and for all $k \geq 1$ and all primes $\ell \equiv \pm 1 \pmod p$, we define

$$D_\ell(k, p) = \deg_{\widetilde{\Delta}}(\widetilde{\Delta}^k | T'_\ell), \tag{1.5}$$

where T'_ℓ denotes the modified Hecke operator as in (1.2), and $\deg_{\widetilde{\Delta}}(f)$ denotes the degree of $\widetilde{f} \in \mathbb{F}_p[\widetilde{\Delta}]$ in $\widetilde{\Delta}$.

1.3.1. *Some conjectures modulo* 5. When $\ell = 19$ and $p = 5$, we define the modified degree-lowering function

$$D'_{19}(k,5) = \begin{cases} D_{19}(k,5) - 1, & 5 \mid D_{19}(k,5), \\ D_{19}(k,5), & \text{otherwise.} \end{cases}$$

For all $t \geq 1$, we denote by $D'^{(t)}_{\ell}(k,p)$ the tth iteration of $D_\ell(k,p)$. Theorem 1.1(i) implies that $D'_\ell(k,p) < k$ and that there exists a least $j \geq 1$ with $D'^{(j)}_{\ell}(k,p) = -\infty$. Our computations suggest precise conjectures on $D_{19}(k,5)$ and on a second quantity, $S'_{19}(k,5)$, the nilpotency index of $D'_{19}(k,5)$:

$$S'_{19}(k,5) = \min\{t \geq 1 : D'^{(t)}_{19}(k,5) = -\infty\}.$$

Furthermore, our computations suggest that $N_{19}(\widetilde{\Delta}^k, 5) \leq S'_{19}(k,5)$. For $n \geq 1$, we define $v_5(n)$ to be the largest power of 5 dividing n.

Conjecture 1.7. *Assume the notation above. Let $k \geq 1$ with $5 \nmid k$. Then the following formulas hold for $D_{19}(k,5)$:*

Table 2. Degree $D_{19}(k,5)$ of $\widetilde{\Delta}^k | T_{19}$ in $\mathbb{F}_5[\![q]\!]$

k	$D_{19}(k,5)$
$k \equiv j \pmod 5, j \in \{1,2\}$	$k - \left(\frac{5^{v_5(k-j)}+1}{3}\right), v_5(k-j) \equiv 1 \pmod 2$
	$k - \left(\frac{5^{v_5(k-j)}+5}{3}\right), v_5(k-j) \equiv 0 \pmod 2$
$k \equiv 3,4 \pmod 5$, $k \notin \{13,14\} \pmod{25}$	$k-2$
$k \equiv 13 \pmod{25}$	$k-3$
$k \equiv 14 \pmod{25}$	$k-4$

(i) *Suppose that k has base 5 expansion $k = \sum a_i \cdot 5^i$, and that $s = a_0 + a_1 \cdot 5$. For all $t \geq 0$, let $c_t = S'_{19}(5^t + 1, 5) - 1$. Then*
- *$c_0 = 0$, $c_1 = 2$, and $c_i = 3c_{i-1} + 2c_{i-2}$ for all $i \geq 2$,*
- *$S'_{19}(k,5) = S'_{19}(s,5) + \sum_{i \geq 2} a_i c_i$.*

(ii) *$S'_{19}(k,5) = O(k^\alpha)$, where $\alpha = \log_5\left(\frac{3+\sqrt{17}}{2}\right) \approx 0.7892$.*

(iii) *$N_{19}(\widetilde{\Delta}^k, 5) \leq S'_{19}(k,5)$.*

Remarks. (i) Using PARI/GP [PARI23], we verified the conjecture for $k \leq 8 \times 10^4$.

(ii) The quadratic irrational $\frac{3+\sqrt{17}}{2}$ in the argument of the logarithm in part (3) is the dominant root of the polynomial $x^2 - 3x - 2$, which is the characteristic polynomial of the recurrence in part (2).

(iii) Following Nicolas and Serre [NS12b], one approach to proving the formulas in Table 2 might be to argue by induction using the recurrence satisfied by $\widetilde{\Delta}^k \mid T_{19}$ as in (1.3). As in Chapter 6 of [Me15], one can explicitly write down this order 20 recurrence and its associated characteristic polynomial. However, its relatively large number of terms and the complexity of its coefficients prevented us from effectively pursuing this approach to prove the conjectured formulas.

(iv) We conjecture a more complicated formula for $D_{11}(k,5)$, which we omit for brevity.

1.3.2. *Computations modulo* 7. Our computations suggest a similar formula for the nilpotency index of a modified degree-lowering function in the mod 7 setting, which we briefly describe. With $D_{29}(k,7)$ as in (1.5), we define $D''_{29}(k,7)$ to be the degree of the second highest nonzero term in $\widetilde{\Delta}^k|T'_{29} \in \mathbb{F}_7[\widetilde{\Delta}]$ if $7 \mid D_{29}(k,7)$ or if $D_{29}(k,7) \equiv 5 \pmod{98}$; otherwise, we define $D''_{29}(k,7)$ to be $D_{29}(k,7)$. We also define

$$S''_{29}(k,7) = \min\{s : D''^{(s)}_{29}(k,7) = -\infty\}.$$

We suppose that $7 \nmid k$, that k has base 7 expansion $k = \sum a_i \cdot 7^i$, and that $s \equiv k \pmod{98}$ with $1 \leq s \leq 97$. For all $t \geq 1$, we let $y_t = S''_{29}(2\cdot 7^t+1,7)-1$. Then for $1 \leq k \leq 3.5 \times 10^4$ with certain exceptions depending on the residue class of $k \pmod{2 \cdot 7^j}$ for $j \geq 2$, our computations give

$$S''_{29}(k,7) = S''_{29}(s,7) + \sum_{i \geq 2} x_i\, y_i,$$

where the x_i are linear combinations of $\left\lfloor \frac{a_i}{2} \right\rfloor$ and $\left\lfloor \frac{a_i+7}{2} \right\rfloor$ depending on the parity of a_i. For brevity, we give one example of an exception. When $k \equiv 5 \pmod{98}$, we have

$$S''_{29}(\Delta^k,7) = 1 + S''_{29}(\Delta^s,7) + \sum_{i \geq 2} x_i y_i.$$

Our computations also suggest that y_i satisfies the recurrence relation

$$y_1 = 3,\ y_2 = 16,\ \text{and } y_i = 5\, y_{i-1} + 2\, y_{i-2} \text{ for } i \geq 3;$$

that $S''_{29}(k,7) = O(k^\alpha)$, where $\alpha = \log_7\left(\frac{5+\sqrt{33}}{2}\right) \approx 0.864$; and that $N_{29}(\widetilde{\Delta}^k,7) \leq S''_{29}(k,7)$. The quadratic irrational argument in the logarithm arises as the dominant root of the characteristic polynomial for the recurrence satisfied by $\{y_i\}$. We note that a similar recurrence relation can be conjectured for $\ell = 13$, but we omit it due to the complicated definition of $D''_{13}(k,7)$.

1.3.3. *Computations modulo* 3. We recall that $D_2(z) = \eta(2z)^{12} \in S_6(\Gamma_0(4))$. Theorem 1.3(iii) gives bounds on $N_\ell(\widetilde{D}_2^k,3)$ for all $k \geq 1$ and for primes $\ell \notin \{2,3\}$. While our proof does not require $\langle T_\ell : \ell \geq 5\rangle$-invariance of the $\mathbb{F}_3$-module generated by $\{D_2^k : \gcd(k,6) = 1\}$, computations with $k \leq 4 \times 10^4$ and small ℓ suggest that it is indeed invariant. Under this assumption, for all k and for all primes $\ell \geq 5$, we define

$$E_\ell(k,3) = \deg_{\widetilde{D}_2}(\widetilde{D}_2^k | T'_\ell)$$

and its modification

$$E'_\ell(k,3) := \begin{cases} E_\ell(k,3) - 1, & 3 \mid E_\ell(k,3), \\ E_\ell(k,3), & \text{otherwise}, \end{cases}$$

and we define $S'''_\ell(k,3)$ to be the nilpotency index of $E'_\ell(k,3)$. We suppose that $\gcd(k,6) = 1$, that $k = \sum a_i \cdot 3^i$, and that $s \equiv k \pmod{54}$ with $1 \leq s \leq 53$. For $\ell \in \{7,11\}$ and for all $t \geq 1$, we let $z_{\ell,t} = S'''_\ell(2\cdot 3^t + 1, p) - 1$. For $k \leq 4 \cdot 10^4$ and $\ell = 7$, our computations show that

$$S'''_\ell(k,p) = S'''_\ell(s,p) + \sum_{i \geq 3} w_{\ell,i}\, z_{\ell,i},$$

where the $w_{\ell,i}$ are linear combinations of the coefficients a_i which do not appear to satisfy obvious general formulas. When $\ell = 11$, the formula holds for all $k \not\equiv 7, 11 \pmod{54}$ and for k not of the form $\frac{3^m+1}{4}$ where $m \geq 1$ is odd. Our computations also suggest that

$$z_{7,1} = 1, z_{7,2} = 2, \text{ and } z_{7,i} = z_{7,i-1} + 2z_{7,i-2} \text{ for } i \geq 3;$$

$$z_{11,2} = 2, \text{ and } z_{11,i} = 2z_{11,i-1} \text{ for } i \geq 3;$$

for $\ell \in \{7,11\}$, we have $S'''_\ell(k,p) = O(k^\alpha)$, where $\alpha = \log_3 2 \approx 0.631$; and $N_\ell(\widetilde{D}_2^k, p) \leq S'''_\ell(k,p)$.

1.4. Plan for the paper. We structure the rest of the paper as follows. In Section 2, we give necessary background. In Section 3, we prove Theorem 1.3, Proposition 1.4, and Theorems 1.5 and 1.6.

Before we proceed, we give a brief overview of the proof of Theorem 1.3. When T'_ℓ acts nilpotently mod ℓ on a subspace $\mathbb{F}_p[\widetilde{f}] \subseteq \widetilde{M^{(p)}}(\Gamma_0(N))$, it lowers degrees in $\widetilde{f}$. To prove the theorem, we show that T'_ℓ lowers degrees by at least two, uniformly for all $\ell \neq p$, when $(p,N) \in \{(3,1),(5,1),(7,1),(3,4)\}$. When $N = 1$, we decompose $\widetilde{M^{0,(p)}}(\Gamma_0(1)) = \mathbb{F}_p[\widetilde{\Delta}]$ into subspaces of forms whose expansions in $\mathbb{F}_p[\![q]\!]$ have support on squares mod p, nonsquares mod p, and multiples of p. The residue class of ℓ modulo p determines how T'_ℓ maps between these subspaces.

When $(p,N) = (3,4)$, with $F(z)$ as in (3.7), we note that $\widetilde{M^{(3)}}(\Gamma_0(4)) = \mathbb{F}_3[\widetilde{F}]$ as in (3.22), and that $\widetilde{D}_2(z) = \widetilde{F}(z) - \widetilde{F}^3(z)$ in $\mathbb{F}_3[\![q]\!]$ as in (3.9). We adapt arguments of Monsky [Mo16] to study how T'_ℓ maps between the subspace W_1 generated by $\widetilde{D}_2^k$ for $k \equiv 1 \pmod 6$ and $\widetilde{D}_2^i\widetilde{F}^3$ for $i \equiv 4 \pmod 6$ and the subspace W_5 generated by $\widetilde{D}_2^k$ for $k \equiv 5 \pmod 6$ and $\widetilde{D}_2^i\widetilde{F}^3$ for $i \equiv 2 \pmod 6$. In both the $N = 1$ and $N = 4$ settings, the mapping properties of Hecke operators on our subspaces allow us to prove our degree-lowering results.

2. Background. For details on modular forms, one may consult [CS17], for example. When N and k are integers with $N \geq 1$, we denote the space of weight k holomorphic modular forms on $\Gamma_0(N)$ by $M_k(\Gamma_0(N))$, and we denote its subspace of cusp forms by $S_k(\Gamma_0(N))$. We introduce here some of the forms in these spaces that we require. We let $z \in \mathbb{H}$, the complex upper half-plane, and we let $q = e^{2\pi i z}$. For even $k \geq 2$, the level 1 Eisenstein series in weight k is

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sum_{d \mid n} d^{k-1}\, q^n,$$

where B_k denotes the kth Bernoulli number. When $k \geq 4$, we have $E_k \in M_k(\Gamma_0(1))$; when $k = 2$ and $N \geq 2$, we have

$$E_{2,N}(z) = NE_2(Nz) - E_2(z) \in M_2(\Gamma_0(N)). \tag{2.1}$$

The Dedekind eta-function is defined by

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{2.2}$$

The Delta-function

$$\Delta(z) = \eta(z)^{24} = q \prod_{n=1}^{\infty} (1 - q^n) \in S_{12}(\Gamma_0(1)) \tag{2.3}$$

plays an important role in our work. When D is the discriminant of a quadratic number field, we let χ_D denote the corresponding Kronecker character. Some classical identities relating the eta-function to theta functions are

$$\eta(z) = \sum_{\substack{n \in \mathbb{Z} \\ n \equiv 1 \pmod 6}} \chi_{12}(n) q^{n^2/24}, \tag{2.4}$$

$$\eta(z)^3 = \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \chi_{-4}(n) n q^{n^2/8}, \tag{2.5}$$

$$\Theta(z) = \frac{\eta(2z)^5}{\eta(z)^2 \eta(4z)^2} = \sum_{n \in \mathbb{Z}} q^{n^2}. \tag{2.6}$$

We note that $\Theta(z)$ is a modular form of weight $1/2$ and level 4 in Shimura's theory.

When $p \geq 5$ is prime, we recall some facts in the $N = 1$ setting following [Se73] and [SD73]. Since $\widetilde{E}_{p-1} = 1$, we have $\widetilde{M}_k^{(p)}(\Gamma_0(1)) \subseteq \widetilde{M}_{k+p-1}^{(p)}(\Gamma_0(1))$

for all even $k \geq 0$, and we define

$$\widetilde{M}^{\alpha,(p)}(\Gamma_0(1)) = \bigcup_{k \equiv \alpha \pmod{p-1}} \widetilde{M}_k^{(p)}(\Gamma_0(1)).$$

Lemma 5(ii) of [SD73] implies that

$$\theta = q\frac{d}{dq} \,:\, \widetilde{M}_k^{(p)}(\Gamma_0(1)) \to \widetilde{M}_{k+p+1}^{(p)}(\Gamma_0(1)). \tag{2.7}$$

We next discuss Hecke operators. We let $\ell \nmid N$ be prime, and we let $f(z) = \sum c(n)q^n \in M_k(\Gamma_0(N))$. We define Hecke operators T_ℓ on $M_k(\Gamma_0(N))$ by

$$f|T_\ell = \sum_n (c(\ell n) + \ell^{k-1}c(n/\ell))q^n, \tag{2.8}$$

where $c(n/\ell) = 0$ if $\ell \nmid n$. The Hecke operator T_ℓ maps $M_k(\Gamma_0(N))$ to itself. We let $r \geq 0$, and for all $n \geq 0$, we define $c_r(n)$ by

$$f(z)|T_\ell^r = \sum_n c_r(n)q^n, \tag{2.9}$$

where T_ℓ^r denotes the rth iteration of T_ℓ. For all integers m, we let $v_\ell(m)$ denote the ℓ-adic valuation of m. For our applications, we require formulas for $c_r(n)$. The following lemma gives such formulas depending on $v_\ell(n)$.

LEMMA 2.1. *Let $\ell \nmid N$ be prime, let $r, s \geq 0$; for all $n \geq 0$, let $c_r(n)$ be as in* (2.9)*; and suppose that $v_\ell(n) = s$. Then*

$$\begin{aligned} c_r(n) &= \sum_{i=0}^{s} \binom{r}{i} \ell^{i(k-1)} c(\ell^{r-2i}n) \\ &\quad + \sum_{j=s+1}^{\lfloor \frac{r+s}{2} \rfloor} \left(\binom{r}{j} - \binom{r}{j-s-1} \right) \ell^{j(k-1)} c(\ell^{r-2j}n), \end{aligned}$$

where the second sum is zero when it is undefined (*i.e., when $s+1 > \lfloor \frac{r+s}{2} \rfloor$*).

Proof. We induct on r. The base case follows directly from (2.8). We let $t \geq 1$, and we suppose that the statement holds for $0 \leq r \leq t$. We fix $s \geq 0$. For $n \geq 0$ with $v_\ell(n) = s \geq 0$, using (2.8) and the induction hypothesis, we write

$$c_{t+1}(n) = c_t(\ell n) + \ell^{k-1} c_t\left(\frac{n}{\ell}\right)$$

in terms of the coefficients $c(n)$. Since $v_\ell(\ell n) = s+1$ and $v_\ell\left(\frac{n}{\ell}\right) = s-1$, the induction hypothesis gives

$$\begin{aligned} c_t(\ell n) &= \sum_{i=0}^{s+1} \binom{t}{i} \ell^{i(k-1)} c(\ell^{t+1-2i} n) \\ &\quad + \sum_{j=s+2}^{\lfloor \frac{t+s+1}{2} \rfloor} \left(\binom{t}{j} - \binom{t}{j-s-2} \right) \ell^{j(k-1)} c(\ell^{t+1-2j} n), \\ c_t\left(\frac{n}{\ell}\right) &= \sum_{i=0}^{s-1} \binom{t}{i} \ell^{i(k-1)} c(\ell^{t-1-2i} n) \\ &\quad + \sum_{j=s}^{\lfloor \frac{t+s-1}{2} \rfloor} \left(\binom{t}{j} - \binom{t}{j-s} \right) \ell^{j(k-1)} c(\ell^{t-1-2j} n). \end{aligned}$$

It follows that

$$\begin{aligned} &c_{t+1}(n) = c_t(\ell n) + \ell^{k-1} c_t\left(\frac{n}{\ell}\right) \\ &= \sum_{i=0}^{s+1} \binom{t}{i} \ell^{i(k-1)} c(\ell^{t+1-2i} n) + \sum_{j=s+2}^{\lfloor \frac{t+s+1}{2} \rfloor} \left(\binom{t}{j} - \binom{t}{j-s-2} \right) \\ &\qquad\qquad \times \ell^{j(k-1)} c(\ell^{t+1-2j} n) \\ &\quad + \sum_{i=0}^{s-1} \binom{t}{i} \ell^{(i+1)(k-1)} c(\ell^{t-1-2i} n) \\ &\quad + \sum_{j=s}^{\lfloor \frac{t+s-1}{2} \rfloor} \left(\binom{t}{j} - \binom{t}{j-s} \right) \ell^{(j+1)(k-1)} c(\ell^{t-1-2j} n) \\ &= c(\ell^{t+1} n) + \sum_{i=1}^{s} \left(\binom{t}{i} + \binom{t}{i-1} \right) \ell^{i(k-1)} c(\ell^{t+1-2i} n) \\ &\quad + \left(\binom{t}{s+1} + \binom{t}{s} - \binom{t}{0} \right) \ell^{(s+1)(k-1)} c(\ell^{t+1-2(s+1)} n) \\ &\quad + \sum_{j=s+2}^{\lfloor \frac{t+s+1}{2} \rfloor} \left(\left(\binom{t}{j} + \binom{t}{j-1} \right) - \left(\binom{t}{j-s-1} + \binom{t}{j-s-2} \right) \right) \\ &\qquad\qquad \times \ell^{j(k-1)} c(\ell^{t+1-2j} n) \\ &= \sum_{i=0}^{s} \binom{t+1}{i} \ell^{i(k-1)} c(\ell^{t+1-2i} n) + \sum_{j=s+1}^{\lfloor \frac{t+s+1}{2} \rfloor} \left(\binom{t+1}{j} - \binom{t+1}{j-s-1} \right) \\ &\qquad\qquad \times \ell^{j(k-1)} c(\ell^{r-2i} n), \end{aligned}$$

where the third equality follows from shifting j to $j-1$ in the second sum and combining terms, and the final equality uses Pascal's identity. ∎

We have a particular interest in the following special case of the lemma.

COROLLARY 2.2. *Assume the notation in Lemma 2.1. Let $r \geq 1$, let ℓ and p be distinct primes with $p\ell \nmid N$, and suppose that $f(z) = \sum c(n)q^n \in M_k(\Gamma_0(N))_{(p)}$. Then for all $n \geq 0$ with $\ell \nmid n$, we have*

$$c_{p^r}(n) \equiv c(\ell^{p^r} n) - \ell^{k-1} c(\ell^{p^r-2} n) \pmod{p}.$$

Proof. Letting $s = 0$ and replacing r by p^r in Lemma 2.1, it follows for all $n \geq 0$ with $\ell \nmid n$ that

$$\begin{aligned} c_{p^r}(n) &= c(\ell^{p^r} n) + \sum_{j=1}^{\frac{p^r-1}{2}} \left(\binom{p^r}{j} - \binom{p^r}{j-1} \right) \ell^{j(k-1)} c(\ell^{p^r-2j} n) \\ &\equiv c(\ell^{p^r} n) - \ell^{k-1} c(\ell^{p^r-2} n) \pmod{p}, \end{aligned}$$

where the congruence follows since $\binom{p^r}{m} \equiv 0 \pmod{p}$ for $1 \leq m \leq p^r - 1$ and $\binom{p^r}{0} = 1$. ∎

Lastly, for $m \geq 1$, we recall the U_m-operator on $f(z) = \sum c(n)q^n \in M_k(\Gamma_0(N))$. We have

$$f(z)|U_m = \sum c(mn)q^n \tag{2.10}$$

and

$$U_m : M_k(\Gamma_0(N)) \to \begin{cases} M_k(\Gamma_0(N)), & m \mid N, \\ M_k(\Gamma_0(Nm)), & m \nmid N. \end{cases}$$

3. Proofs of Theorem 1.3, Proposition 1.4, and Theorems 1.5 and 1.6

3.1. Proof of Theorem 1.3(i)–(ii). Since the proofs of Theorem 1.3(i)–(ii) are similar for $p \in \{3, 5, 7\}$, it suffices to give details for the $p = 5$ case. To begin, we require a lemma. We recall from Theorem 1.1(i) that when $\ell \equiv \pm 1 \pmod{5}$, the Hecke operator T'_ℓ acts locally nilpotently on $\widetilde{M}^{0,(5)}(\Gamma_0(1)) = \mathbb{F}_5[\widetilde{\Delta}]$.

LEMMA 3.1. *The space $\widetilde{M}^{0,(5)}(\Gamma_0(1))$ has a basis $B = \{\widetilde{f}_{5i+j} : i \geq 0,\ 0 \leq j \leq 4\}$, where*

$$\widetilde{f}_{5i+j} = q^{5i+j} + \cdots = \begin{cases} \displaystyle\sum_{n \equiv 1,4 \,(\mathrm{mod}\, 5)} \widetilde{a_{i,j}(n)} q^n, & j \in \{1, 4\}, \\ \displaystyle\sum_{n \equiv 2,3 \,(\mathrm{mod}\, 5)} \widetilde{a_{i,j}(n)} q^n, & j \in \{2, 3\}. \end{cases}$$

Proof. We define

$$\begin{aligned}
\widetilde{f}_0 &= 1,\\
\widetilde{f}_1 &= \widetilde{\Delta} + 4\widetilde{\Delta}^2 = q + \cdots,\\
\widetilde{f}_2 &= \widetilde{\Delta}^2 = q^2 + \cdots,\\
\widetilde{f}_3 &= \widetilde{\Delta}^3 + 2\widetilde{\Delta}^4 + 3\widetilde{\Delta}^5 = q^3 + \cdots,\\
\widetilde{f}_4 &= \widetilde{\Delta}^4 + \widetilde{\Delta}^5 = q^4 + \cdots.
\end{aligned}$$

For all $i \geq 0$ and $0 \leq j \leq 4$, we let $\widetilde{f}_{5i+j} = \widetilde{\Delta}^{5i}\widetilde{f}_j = q^{5i+j} + \cdots$. With $\theta = q\frac{d}{dq}$ as in (2.7), we claim that

$$\theta^2(\widetilde{f}_{5i+j}) = \left(\frac{j}{5}\right)\widetilde{f}_{5i+j},$$

where $\left(\frac{\bullet}{5}\right)$ is the Legendre symbol. We compute

$$\theta^2(\widetilde{f}_j) = \widetilde{f}_j \text{ for } j \in \{1,4\} \quad \text{and} \quad \theta^2(\widetilde{f}_j) = -\widetilde{f}_j \text{ for } j \in \{2,3\}.$$

Furthermore, for all $i \geq 0$ and $1 \leq j \leq 4$, we have

$$\begin{aligned}
\theta^2(\widetilde{f}_{5i+j}) &= \theta(\widetilde{\Delta}^{5i}\theta(\widetilde{f}_j)) = \widetilde{\Delta}^{5i}\theta^2(\widetilde{f}_j) + \theta(\widetilde{f}_j)\theta(\widetilde{\Delta}^{5i})\\
&= \widetilde{\Delta}^{5i}\theta^2(\widetilde{f}_j) = \begin{cases} \widetilde{f}_{5i+j}, & j \in \{1,4\},\\ -\widetilde{f}_{5i+j}, & j \in \{2,3\}. \end{cases}
\end{aligned}$$

To conclude, we observe that $\theta^2(\widetilde{f}_{5i+j}) = \widetilde{f}_{5i+j}$ for $j \in \{1,4\}$ implies that $\widetilde{f}_{5i+j}$ is supported on squares modulo 5, and for $j \in \{2,3\}$ that $\theta^2(\widetilde{f}_{5i+j}) = -\widetilde{f}_{5i+j}$ implies that $\widetilde{f}_{5i+j}$ is supported on nonsquares modulo 5. ∎

We also require explicit representations of powers of $\widetilde{\Delta}$ in $\mathbb{F}_5[[q]]$ in the basis B:

$$\widetilde{\Delta}^{5i+j} \equiv \begin{cases} \widetilde{f}_{5i} & \text{if } j = 0,\\ \widetilde{f}_{5i+1} + \widetilde{f}_{5i+2} & \text{if } j = 1,\\ \widetilde{f}_{5i+2} & \text{if } j = 2,\\ \widetilde{f}_{5i+3} + 3\widetilde{f}_{5i+4} + 4\widetilde{f}_{5(i+1)} & \text{if } j = 3,\\ \widetilde{f}_{5i+4} + 4\widetilde{f}_{5(i+1)} & \text{if } j = 4. \end{cases} \tag{3.1}$$

We denote the subspace of $\widetilde{M}^{0,(5)}(\Gamma_0(1))$ of cusp forms by $\widetilde{S}^{0,(5)}(\Gamma_0(1))$, and we note that $\widetilde{S}^{0,(5)}(\Gamma_0(1)) = \widetilde{\Delta}\mathbb{F}_5[\widetilde{\Delta}]$. We consider the following subspaces of $\widetilde{S}^0 = \widetilde{S}^{0,(5)}(\Gamma_0(1))$:

$$\begin{aligned}
\widetilde{S}^{0,1} &= \{\widetilde{f} \in \widetilde{S}^0 : \theta^2(\widetilde{f}) = \widetilde{f}\} = \mathrm{Span}_{\mathbb{F}_5}\{\widetilde{f}_{5i+j} : j \in \{1,4\}\},\\
\widetilde{S}^{0,-1} &= \{f \in \widetilde{S}^0 : \theta^2(\widetilde{f}) = -\widetilde{f}\} = \mathrm{Span}_{\mathbb{F}_5}\{\widetilde{f}_{5i+j} : j \in \{2,3\}\},\\
\widetilde{S}^{0,0} &= \{\widetilde{f} \in \widetilde{S}^0 : \theta^2(\widetilde{f}) = 0\} = \mathrm{Span}_{\mathbb{F}_5}\{\widetilde{f}_{5i} : i \geq 1\}.
\end{aligned}$$

Let $\ell \neq 5$ be prime. It follows from (2.8) that

$$\begin{aligned} &T'_\ell : \widetilde{S}^{0,j} \to \widetilde{S}^{0,j} \text{ for } j \in \{-1,0,1\} \text{ and } \ell \equiv \pm 1 \pmod 5; \\ &T'_\ell : \widetilde{S}^{0,1} \to \widetilde{S}^{0,-1},\, \widetilde{S}^{0,-1} \to \widetilde{S}^{0,1}, \widetilde{S}^{0,0} \to \widetilde{S}^{0,0} \text{ for } \ell \equiv \pm 2 \pmod 5. \end{aligned} \tag{3.2}$$

We also observe that $\widetilde{S}^0 = \widetilde{S}^{0,1} \oplus \widetilde{S}^{0,-1} \oplus \widetilde{S}^{0,0}$. Let $\ell \equiv \pm 1 \pmod 5$ be prime, let $i \geq 0$, and let $0 \leq j \leq 4$. We prove Theorem 1.3(ii) for $p = 5$ of the following form:

$$N_\ell(\widetilde{\Delta}^{5i+j}, 5) \leq \begin{cases} 2i+1, & j \in \{1,2\}, \\ 2i+2, & j \in \{3,4\}, \\ i, & j = 0. \end{cases} \tag{3.3}$$

We first claim that in order to prove (3.3), it suffices to show that

$$N_\ell(\widetilde{f}_{5i+j}, 5) \leq \begin{cases} 2i+1, & j \in \{1,2\}, \\ 2i+2, & j \in \{3,4\}, \\ i, & j = 0. \end{cases} \tag{3.4}$$

To see this, we let $\ell \equiv \pm 1 \pmod 5$ and $j = 3$, and we assume that (3.4) holds. For all $i \geq 0$, we use (3.1) to obtain

$$\widetilde{\Delta}^{5i+3} | T'_\ell = (\widetilde{f}_{5i+3} + 3\widetilde{f}_{5i+4} + 4\widetilde{f}_{5(i+1)}) | T'_\ell$$

in $\mathbb{F}_5[[q]]$. Using (3.4), it follows that

$$N_\ell(\widetilde{\Delta}^{5i+3}, 5) \leq \max\{N_\ell(\widetilde{f}_{5i+3}, 5), N_\ell(\widetilde{f}_{5i+4}, 5), N_\ell(\widetilde{f}_{5(i+1)}, 5)\} \leq 2i+2.$$

The arguments for $j \neq 3$ are similar.

We now prove (3.4). We will show, for all $i \geq 1$, that

$$N_\ell(\widetilde{f}_{5i+j}, 5) \leq 1 + \begin{cases} \max\{N_\ell(\widetilde{f}_k, 5) : 1 \leq k \leq 5(i-1),\ 5 \mid k\}, & j = 0, \\ \max\{N_\ell(\widetilde{f}_k, 5) : 1 \leq k \leq 5i-1,\ \left(\frac{k}{5}\right) = 1\}, & j = 1, \\ \max\{N_\ell(\widetilde{f}_k, 5) : 1 \leq k \leq 5i-2,\ \left(\frac{k}{5}\right) = -1\}, & j = 2, \\ \max\{N_\ell(\widetilde{f}_k, 5) : 1 \leq k \leq 5i+2,\ \left(\frac{k}{5}\right) = -1\}, & j = 3, \\ \max\{N_\ell(\widetilde{f}_k, 5) : 1 \leq k \leq 5i+1,\ \left(\frac{k}{5}\right) = 1\}, & j = 4. \end{cases} \tag{3.5}$$

To illustrate, we prove (3.5) for $j = 1$. We observe that

$$\widetilde{f}_{5i+1} \in \widetilde{S}^{0,(5)}_{12(5i+2)} \cap \widetilde{S}^{0,1} = \mathrm{Span}_{\mathbb{F}_5}\{\widetilde{f}_{5a+b} : a \geq 0, b \in \{1,4\}, 5a+b \leq 5i+1\}.$$

When $\ell \equiv \pm 1 \pmod 5$, we see from (3.2) that T'_ℓ maps this space to itself. It follows that

$$\widetilde{f}_{5i+1} | T'_\ell = \sum_{r=0}^{i} \widetilde{c}_r \widetilde{f}_{5r+1} + \sum_{s=0}^{i-1} \widetilde{d}_s \widetilde{f}_{5s+4}.$$

We claim that $\widetilde{c}_i = 0$ in $\mathbb{F}_5$. We suppose that $\widetilde{c}_i \neq 0$. We compute

$$\begin{aligned}\deg_{\widetilde{\Delta}}(\widetilde{f}_{5i+1}) = 5i + 2 > 5i &= \max_{0\le r,s\le i-1} \{\deg_{\widetilde{\Delta}}(\widetilde{f}_{5r+1}), \deg_{\widetilde{\Delta}}(\widetilde{f}_{5s+4})\} \\ &= \max_{0\le r,s\le i-1} \{5r+2, 5(s+1)\} = \deg_{\widetilde{\Delta}}(\widetilde{f}_{5i-1}).\end{aligned}$$

Therefore, $\widetilde{c}_i \neq 0$ implies that $\deg_{\widetilde{\Delta}}(\widetilde{f}_{5i+1}|T'_\ell) = 5i+2 = \deg_{\widetilde{\Delta}}(\widetilde{f}_{5i+1})$, which contradicts nilpotency of T'_ℓ. We conclude that

$$\widetilde{f}_{5i+1}|T'_\ell = \sum_{r=0}^{i-1} \widetilde{c}_r \widetilde{f}_{5r+1} + \sum_{s=0}^{i-1} \widetilde{d}_s \widetilde{f}_{5s+4}, \tag{3.6}$$

which gives (3.5) for $j = 1$. The proof of the remaining cases follows similarly. We now use induction to prove (3.4), and with it, the $p = 5$ case of Theorem 1.3(ii). The base cases are

$$\begin{aligned} N_\ell(\widetilde{f}_1, 5) = N_\ell(\widetilde{f}_2, 5) = 1, \qquad & N_\ell(\widetilde{f}_3, 5) \le 1 + N_\ell(\widetilde{f}_2, 5) = 2, \\ & N_\ell(\widetilde{f}_4, 5) \le 1 + N_\ell(\widetilde{f}_2, 5) = 2. \end{aligned}$$

We let $i \ge 1$ and $0 \le b \le 4$, and we suppose for all $0 \le a \le i-1$ that the statement holds for $5a + b$. To illustrate the general argument, we prove the statement for $5i + j$ when $j = 1$. Using the induction hypothesis (3.4) and (3.6), we find that

$$\begin{aligned} N_\ell(\widetilde{f}_{5i+1}, 5) &\le 1 + \max\{N_\ell(\widetilde{f}_k, 5) : 1 \le k \le 5(i-1)+4,\ k \equiv 1, 4 \pmod 5\} \\ &\le 1 + \max\{N_\ell(\widetilde{f}_1, 5), N_\ell(\widetilde{f}_4, 5), \ldots, N_\ell(\widetilde{f}_{5(i-1)+1}, 5), N_\ell(\widetilde{f}_{5(i-1)+4}, 5)\} \\ &\le 1 + \max\{1, 2, \ldots, 2(i-1)+1, 2(i-1)+2\} = 2i+1, \end{aligned}$$

which proves (3.4) for $j = 1$.

3.2. Proof of Theorem 1.3(iii). We shift our attention to $\widetilde{M}^{(3)}(\Gamma_0(4))$. We let

$$\begin{aligned} P(z) &= E_{2,2}(z) = 2E_2(2z) - E_2(z) \in M_2(\Gamma_0(2)), \\ A(z) &= \frac{\eta(2z)^{16}}{\eta(z)^8} \in M_4(\Gamma_0(2)), \\ F(z) &= \frac{\eta(4z)^8}{\eta(2z)^4} \in M_2(\Gamma_0(4)), \qquad G(z) = \Delta(2z) \in S_{12}(\Gamma_0(2)), \end{aligned} \tag{3.7}$$

where $E_{2,2}(z)$ is an Eisenstein series as in (2.1). We recall from Section 1 that $D_2(z) = \eta(2z)^{12} \in S_6(\Gamma_0(4))$, and we note that

$$D_2(z)^2 = G(z), \tag{3.8}$$

$$\widetilde{D}_2(z) = \widetilde{F}(z) - \widetilde{F}^3(z) \quad \text{in } \mathbb{F}_3[[q]], \tag{3.9}$$

$$\widetilde{P}(z) = 1 \quad \text{in } \mathbb{F}_3[[q]]. \tag{3.10}$$

We use the next theorem to prove bounds on $N_\ell(\widetilde{D}_2^k, 3)$ for primes $\ell \notin \{2,3\}$ and $k \geq 1$. Its proof is an adaptation of Monsky's work [Mo16] in $\widetilde{M}^{(2)}(\Gamma_0(3))$.

THEOREM 3.2. *Consider the following subspaces of $\widetilde{M}^{(3)}(\Gamma_0(4))$:*

$$W_1 = \mathrm{Span}_{\mathbb{F}_3}\{\widetilde{D}_2^k, \widetilde{D}_2^i\widetilde{F}^3 : k \equiv 1 \pmod 6 \text{ and } i \equiv 4 \pmod 6\},$$
$$W_5 = \mathrm{Span}_{\mathbb{F}_3}\{\widetilde{D}_2^k, \widetilde{D}_2^i\widetilde{F}^3 : k \equiv 5 \pmod 6 \text{ and } i \equiv 2 \pmod 6\}.$$

We have

$$T'_\ell : \begin{cases} W_1 \to W_1, W_5 \to W_5 & \text{if } \ell \equiv 1 \pmod 6,\\ W_1 \to W_5, W_5 \to W_1 & \text{if } \ell \equiv -1 \pmod 6.\end{cases}$$

Our proof of Theorem 3.2 requires studying forms on $\Gamma_0(2)$. We recall, for all even $k \geq 0$, that $\dim(M_k(\Gamma_0(2))) = 1 + \lfloor k/4 \rfloor$. Thus, for all $j \geq 1$, we have $\dim(\widetilde{M}_{8j}^{(3)}(\Gamma_0(2))) \leq 2j+1$. Since

$$B_j = \{\widetilde{A}^i : 0 \leq i \leq 2j\} \tag{3.11}$$

is linearly independent, it is a basis for $\widetilde{M}_{8j}^{(3)}(\Gamma_0(2))$. For all even $t \geq 0$, we define

$$\widetilde{K}_t^{(3)}(\Gamma_0(2)) = \mathrm{Ker}(U_3) \quad \text{on } \widetilde{M}_t^{(3)}(\Gamma_0(2)),$$
$$\widetilde{K}^{(3)}(\Gamma_0(2)) = \bigcup_{t\geq 0,\, t \text{ even}} \widetilde{K}_t^{(3)}(\Gamma_0(2)) \subseteq \widetilde{M}^{(3)}(\Gamma_0(2)),$$

where U_3 is the U-operator as in (2.10). We investigate properties of $\widetilde{K}^{(3)}(\Gamma_0(2))$ in the following lemma.

LEMMA 3.3. *Assume the notation above. For all $i \geq 0$, let*

$$\widetilde{g}_i(z) = \widetilde{A}^i(\widetilde{A} - \widetilde{A}^2). \tag{3.12}$$

A basis for $\widetilde{K}^{(3)}(\Gamma_0(2))$ is

$$B' = \{\widetilde{g}_i : i \geq 0 \text{ and } i \not\equiv 2 \pmod 3\}.$$

Proof. Since $P(z)$ has weight 2 and $\widetilde{P}(z) = 1$ as in (3.10), it suffices to show, for all $j \geq 1$, that

$$B'_j = \{\widetilde{g}_i(z) = q^{i+1} + \cdots : 0 \leq i \leq 2j-2 \text{ and } i \not\equiv 2 \pmod 3\}$$

is a basis for $\widetilde{K}_{8j}^{(3)}(\Gamma_0(2))$. We note that B'_j is linearly independent. To show that it spans, we let $f(z) \in \widetilde{K}_{8j}^{(3)}(\Gamma_0(2))$. Since $B'_j \subseteq \widetilde{K}_{8j}^{(3)}(\Gamma_0(2))$, for all $0 \leq i \leq 2j-2$ with $i \neq 2 \pmod 3$ there exists $\widetilde{\alpha}_i \in \mathbb{F}_3$ such that

$$\widetilde{f}(z) - \sum_{i=0}^{2j-2} \widetilde{\alpha}_i \widetilde{g}_i(z) = \widetilde{c}q^{2j} + \cdots \in \widetilde{K}_{8j}^{(3)}(\Gamma_0(2)) \subseteq \widetilde{M}_{8j}^{(3)}(\Gamma_0(2))$$

for some $\widetilde{c} \in \mathbb{F}_3$. Since $\widetilde{M}_{8j}^{(3)}(\Gamma_0(2))$ has basis B_j in (3.11), we must have

$$\widetilde{f} - \sum_{i=0}^{2j-2} \widetilde{\alpha}_i \widetilde{g}_i(z) = \widetilde{c}\widetilde{A}^{2j}.$$

It follows that $\widetilde{c}\widetilde{A}^{2j} \in \widetilde{K}_{8j}^{(3)}(\Gamma_0(2))$. However, for all $t \geq 0$, the coefficient of q^{3t+3} in $\widetilde{A}^{3t}, \widetilde{A}^{3t+1}$, and $\widetilde{A}^{3t+2}$ is nonzero in $\mathbb{F}_3[[q]]$. We conclude that $\widetilde{c} = 0$, and therefore $\widetilde{M}_{8j}^{(3)}(\Gamma_0(2)) \subseteq \mathrm{Span}_{\mathbb{F}_3}(B'_j)$. ∎

LEMMA 3.4. *With $G(z)$ as in* (3.7), *we have* $\widetilde{K}^{(3)}(\Gamma_0(2)) \subseteq \mathbb{F}_3[\widetilde{\Delta}, \widetilde{G}]$.

Proof. In view of Lemma 3.3, it suffices to show, for all $i \not\equiv 2 \pmod 3$, that $\widetilde{g}_i$ as in (3.12) lies in $\mathbb{F}_3[\widetilde{\Delta}, \widetilde{G}]$. We note that

$$\begin{aligned}
\widetilde{g}_0 &= \widetilde{A} - \widetilde{A}^2 = \widetilde{\Delta} + \widetilde{G}, \\
\widetilde{g}_1 &= \widetilde{A}(\widetilde{A} - \widetilde{A}^2) = \widetilde{G}, \\
\widetilde{g}_3 &= \widetilde{A}^3(\widetilde{A} - \widetilde{A}^2) = -\widetilde{\Delta}^2 + \widetilde{G} + \widetilde{G}^2, \\
\widetilde{g}_4 &= \widetilde{A}^4(\widetilde{A} - \widetilde{A}^2) = -\widetilde{\Delta}^2 + \widetilde{G}.
\end{aligned}$$

Let $n \geq 1$. We have

$$\begin{aligned}
\widetilde{g}_{3n} - \widetilde{g}_{3(n+1)} &= \widetilde{A}^{3n}(\widetilde{A} - \widetilde{A}^2) - \widetilde{A}^{3(n+1)}(\widetilde{A} - \widetilde{A}^2) = (\widetilde{A}^{3n} - \widetilde{A}^{3n+1})\widetilde{g}_0 \\
&= (\widetilde{A}^{n} - \widetilde{A}^{n+1})^3\widetilde{g}_0 = \widetilde{A}^{3(n-1)}(\widetilde{A} - \widetilde{A}^2)\widetilde{g}_0^3 = \widetilde{g}_{3(n-1)}\widetilde{g}_0^3.
\end{aligned}$$

Similarly, we find that $\widetilde{g}_{3n+1} - \widetilde{g}_{3(n+1)+1} = \widetilde{g}_{3(n-1)+1}\widetilde{g}_0^3$. Therefore, for all $i \geq 5$ with $i \not\equiv 2 \pmod 3$, we have $\widetilde{g}_i = \widetilde{g}_{i-3} - \widetilde{g}_{i-6}\widetilde{g}_0^3$. The claim follows by induction on i. ∎

LEMMA 3.5. *The space $\widetilde{K}^{(3)}(\Gamma_0(2))$ is a free $\mathbb{F}_3[\widetilde{G}^3]$-module with basis*

$$\{\widetilde{\Delta}, \widetilde{\Delta}^2, \widetilde{\Delta}\widetilde{G}^2, \widetilde{\Delta}^2\widetilde{G}, \widetilde{G}, \widetilde{G}^2\}.$$

Proof. Since $\widetilde{G}(z)^3 \in q^3\mathbb{F}_3[q^3]$, the space $\widetilde{K}^{(3)}(\Gamma_0(2))$ is an $\mathbb{F}_3[\widetilde{G}^3]$-module. With $h(x) = x^3 - \widetilde{G}x + \widetilde{G}^3 \in (\mathbb{F}_3[\widetilde{G}])[x]$, we observe that $h(\widetilde{\Delta}) = \widetilde{\Delta}^3 - \widetilde{\Delta}\widetilde{G} + \widetilde{G}^3 = 0$. It follows that $[\mathbb{F}_3(\widetilde{\Delta}, \widetilde{G}) : \mathbb{F}_3(\widetilde{G})] = 3$. Therefore, $\mathbb{F}_3[\widetilde{\Delta}, \widetilde{G}]$ is a free $\mathbb{F}_3[\widetilde{G}^3]$-module with basis

$$\{1, \widetilde{\Delta}, \widetilde{\Delta}^2, \widetilde{G}, \widetilde{\Delta}\widetilde{G}, \widetilde{\Delta}^2\widetilde{G}, \widetilde{G}^2, \widetilde{\Delta}\widetilde{G}^2, \widetilde{\Delta}^2\widetilde{G}^2\}.$$

We note that $1, \widetilde{\Delta}\widetilde{G}, \widetilde{\Delta}^2\widetilde{G}^2 \in q^3\mathbb{F}_3[[q^3]]$, while the other forms in the basis lie in $\widetilde{K}^{(3)}(\Gamma_0(2))$. It follows that $\widetilde{K}^{(3)}(\Gamma_0(2))$ is a free $\mathbb{F}_3[\widetilde{G}^3]$-module with basis $\{\widetilde{\Delta}, \widetilde{\Delta}^2, \widetilde{\Delta}\widetilde{G}^2, \widetilde{\Delta}^2\widetilde{G}, \widetilde{G}, \widetilde{G}^2\}$. ∎

Let $i \in \{1, 5\}$. We consider the map $\rho_{6,i}$ on $\mathbb{F}_3[[q]]$ defined by

$$\rho_{6,i}\Big(\sum_n c(n)q^n\Big) = \sum_{n \equiv i \pmod{6}} c(n)q^n. \tag{3.13}$$

We note that $\rho_{6,i}$ is $\mathbb{F}_3[\widetilde{G}^3]$-linear. For primes $\ell \notin \{2, 3\}$, the definitions (2.8) and (3.13) imply that

$$\rho_{6,i}(\widetilde{f})|T_\ell = \rho_{6,i\ell^{-1}}(\widetilde{f}|T_\ell). \tag{3.14}$$

We also note that the $\mathbb{F}_3[\widetilde{G}^3]$-submodules $V_1 = \operatorname{Ker}(\rho_{6,1})$ and $V_5 = \operatorname{Ker}(\rho_{6,5})$ have bases

$$\{\widetilde{\Delta}^2, \widetilde{G}, \widetilde{G}^2, \widetilde{\Delta}\widetilde{G}^2\} \quad \text{and} \quad \{\widetilde{\Delta}, \widetilde{G}, \widetilde{\Delta}^2\widetilde{G}, \widetilde{G}^2\},$$

respectively. From (2.8), we observe that

$$T'_\ell : \begin{cases} V_1 \to V_1,\ V_5 \to V_5 & \text{if } \ell \equiv 1 \pmod{6}, \\ V_1 \to V_5,\ V_5 \to V_1 & \text{if } \ell \equiv -1 \pmod{6}. \end{cases} \tag{3.15}$$

We require

$$\rho_{6,1}(\widetilde{\Delta}) = \widetilde{D}_2, \quad \rho_{6,1}(\widetilde{\Delta}^2\widetilde{G}) = -\widetilde{D}_2^4\widetilde{F}^3, \quad \rho_{6,1}(\widetilde{G}) = \rho_{6,1}(\widetilde{G}^2) = 0, \tag{3.16}$$

$$\rho_{6,5}(\widetilde{\Delta}^2) = -\widetilde{D}_2^2\widetilde{F}^3, \quad \rho_{6,5}(\widetilde{\Delta}\widetilde{G}^2) = \widetilde{D}_2^5, \quad \rho_{6,5}(\widetilde{G}) = \rho_{6,5}(\widetilde{G}^2) = 0. \tag{3.17}$$

We turn to the proof of Theorem 3.2.

3.2.1. *Proof of Theorem 3.2.* We first observe, using (3.8) and (3.16), that $\rho_{6,1}$ maps V_5 onto W_1 via

$$\rho_{6,1}(\widetilde{\Delta} \cdot \widetilde{G}^{3m}) = \widetilde{D}_2^{6m+1}, \tag{3.18}$$

$$\rho_{6,1}(\widetilde{\Delta}^2\widetilde{G} \cdot G^{3m}) = -\widetilde{D}_2^{6m+4}\widetilde{F}^3. \tag{3.19}$$

Similarly, using (3.8) and (3.17), we find that $\rho_{6,5}$ maps V_1 onto W_5 via

$$\rho_{6,5}(\widetilde{\Delta}^2 \cdot \widetilde{G}^{3m}) = -\widetilde{D}_2^{6m+2}\widetilde{F}^3, \tag{3.20}$$

$$\rho_{6,5}(\widetilde{\Delta}\widetilde{G}^2 \cdot \widetilde{G}^{3m}) = \widetilde{D}_2^{6m+5}. \tag{3.21}$$

Let $\ell \equiv 1 \pmod{6}$, and let $m \geq 0$. We use (3.14), (3.15), (3.18) and (3.19) to compute

$$\begin{aligned} \widetilde{D}_2^{6m+1}|T'_\ell &= \rho_{6,1}(\widetilde{\Delta} \cdot \widetilde{G}^{3m})|T'_\ell \equiv \rho_{6,1}((\widetilde{\Delta} \cdot \widetilde{G}^{3m})|T'_\ell) \\ &= \rho_{6,1}\Big(\sum_{r \geq 0} a_r \widetilde{\Delta} \cdot \widetilde{G}^{3r} + \sum_{s \geq 0} b_s \widetilde{G} \cdot \widetilde{G}^{3s} \\ &\qquad + \sum_{t \geq 0} c_t \widetilde{G}^2 \cdot \widetilde{G}^{3t} + \sum_{u \geq 0} d_u (\widetilde{\Delta}^2\widetilde{G}) \cdot \widetilde{G}^{3u}\Big) \\ &= \sum_{r \geq 0} a_r \widetilde{D}_2^{6r+1} - \sum_{u \geq 0} d_u \widetilde{D}_2^{6u+4}\widetilde{F}^3 \quad \text{in } \mathbb{F}_3[\![q]\!], \end{aligned}$$

which gives $\widetilde{D}_2^{6m+1}|T'_\ell \in W_1$. We argue similarly using (3.14), (3.15), (3.18), (3.19)–(3.21) to complete the proof that T'_ℓ preserves W_1 and W_5 when $\ell \equiv 1 \pmod 6$ and swaps W_1 and W_5 when $\ell \equiv 5 \pmod 6$.

3.2.2. *Conclusion of proof of Theorem 1.3(iii).* With $\Theta(z)$ as in (2.6), we note that [CS17, Proposition 15.1.2] implies that $M(\Gamma_0(4)) = \mathbb{C}[\Theta^4, F]$. We also observe that $\widetilde{\Theta}^4 = 1 + 2\widetilde{F}$ in $\mathbb{F}_3[[q]]$. It follows that

$$\widetilde{M}^{(3)}(\Gamma_0(4)) = \mathbb{F}_3[\widetilde{F}]. \tag{3.22}$$

Hence, from (3.9) we see that $\widetilde{f} \in W_1$ has $\deg_{\widetilde{F}}(\widetilde{f}) \in \{3, 15\} \pmod{18}$, while $\widetilde{f} \in W_5$ has $\deg_{\widetilde{F}}(\widetilde{f}) \in \{9, 15\} \pmod{18}$. We conclude Theorem 1.3(iii) by applying Theorem 3.2 while observing from Theorem 1.1(iii) that T'_ℓ lowers degrees in $\widetilde{F}$ for primes $\ell \notin \{2, 3\}$.

3.3. Proof of Proposition 1.4. We use the theta-series expansion for $\eta(z)$ in (2.4) to prove part (1) of the proposition for $(\delta, p) = (1, 2)$. The proofs for

$$(\delta, p) \in \{(1, 23), (2, 11), (3, 7), (4, 5)\}$$

are similar, and are adaptations of an argument of Elkies [E13].

Let m be odd, and let $B(z) = \eta(8z)\eta(8\cdot 2^m z)$. We observe that $\widetilde{\Delta}(z)^{\frac{1+2^m}{3}} = \widetilde{B}(z)$ in $\mathbb{F}_2[[q]]$. Using (2.4), we obtain

$$\widetilde{\Delta}(z)^{\frac{1+2^m}{3}} = \sum_{\substack{a,b \in \mathbb{Z} \\ a,b \equiv 1 \pmod 6}} \chi_{12}(ab) q^{\frac{a^2+2^m b^2}{3}}.$$

Let ℓ be prime with $\left(\frac{-2}{\ell}\right) = -1$. We claim that $\ell \mid (a^2 + 2^m b^2)$ implies that $a, b \equiv 0 \pmod \ell$. We suppose that $\ell \nmid b$. Then we have $(ab^{-1})^2 \equiv -2^m \pmod \ell$. Since m is odd, we find that $\left(\frac{-2^m}{\ell}\right) = \left(\frac{-2}{\ell}\right) = -1$, which is a contradiction. Let $\widetilde{\Delta}(z)^{\frac{1+2^m}{3}} = \sum \widetilde{a(n)} q^n$. Since $\widetilde{\Delta}(z)^{\frac{1+2^m}{3}}$ has even weight, (2.8) implies that

$$\widetilde{\Delta}(z)^{\frac{1+2^m}{3}} | T_\ell = \sum \left(\widetilde{a(\ell n)} - \widetilde{a\left(\frac{n}{\ell}\right)} \right) q^n. \tag{3.23}$$

We suppose that $\ell \nmid n$. Then we have $\widetilde{a\left(\frac{n}{\ell}\right)} = 0$. We also claim that $\widetilde{a(\ell n)} = 0$. If $\widetilde{a(\ell n)} \neq 0$, then there exist $a, b \equiv 1 \pmod 6$ such that $\ell n = \frac{a^2+2^m b^2}{3}$. The above argument shows that $\ell \mid a, b$, and hence that $\ell \mid n$, which is a contradiction. When $\ell \mid n$, we compute

$$\widetilde{a(\ell n)} = \sum_{\substack{a,b\equiv 1\,(\mathrm{mod}\,6)\\ \frac{a^2+2^m b^2}{3}=\ell n}} \chi_{12}(ab) = \sum_{\substack{r,s\equiv 1\,(\mathrm{mod}\,6)\\ \frac{\ell^2 r^2+2^m \ell^2 s^2}{3}=\ell n}} \chi_{12}((\ell r)(\ell s))$$

$$= \sum_{\substack{r,s\equiv 1\,(\mathrm{mod}\,6)\\ \frac{r^2+2^m s^2}{3}=\frac{n}{\ell}}} \chi_{12}(rs) = \widetilde{a\left(\frac{n}{\ell}\right)}.$$

Using (3.23), we conclude that $\widetilde{\Delta}(z)^{\frac{1+2^m}{3}}|T_\ell = 0$. One can argue similarly to prove the remaining cases in part (1) of the proposition using the theta-series expansion for $\eta(z)^3$ as in (2.5). The proof of part (2) of the proposition uses the same arguments.

3.4. Proofs of Theorems 1.5 and 1.6. We prove Theorem 1.5 for $p = 5$. One can argue similarly to prove the theorem for $p \in \{3, 7\}$.

The generating function for $a_t(n)$ is

$$\sum_{n\geq 0} a_t(n) q^n = \prod_{n\geq 1} \frac{(1-q^{nt})^t}{1-q^n}. \tag{3.24}$$

We recall that $k_{5,t} = \frac{5^{2t}-1}{24}$. Using (2.2), (2.3), and (3.24), we deduce that

$$\sum_{n\geq 0} \widetilde{a_{5^t}(n-k_{5,t})}\, q^n = \widetilde{\Delta}(z)^{k_{5,t}} \quad \text{in } \mathbb{F}_5[[q]].$$

We let $\ell \equiv \pm 1 \pmod 5$, and $r \geq 1$ be such that $5^r \geq m_{5,t} = 1 + \left\lfloor \frac{2k_{5,t}}{5} \right\rfloor$. Theorem 1.3(i) implies that

$$0 = \widetilde{\Delta}(z)^{k_{5,t}}|(T'_\ell)^{5^r} = \begin{cases} \widetilde{\Delta}(z)^{k_{5,t}}|T_\ell^{5^r} & \text{if } \ell \equiv -1 \pmod 5,\\ \widetilde{\Delta}(z)^{k_{5,t}}|(T_\ell^{5^r}-2) & \text{if } \ell \equiv 1 \pmod 5 \end{cases}$$

since $(T_\ell - 2)^{5^r} \equiv T_\ell^{5^r} - 2 \pmod 5$. Applying Corollary 2.2, we find, for all $n \geq 0$ with $\ell \nmid n$, that

$$a_{5^t}(\ell^{5^r} n - k_{5,t}) - \ell^{12k_{5,t}-1} a_{5^t}(\ell^{5^r-2} n - k_{5,t})$$
$$\equiv \begin{cases} 0 & \text{if } \ell \equiv -1 \pmod 5,\\ 2\widetilde{a_{5^t}(n-k_{5,t})} & \text{if } \ell \equiv 1 \pmod 5, \end{cases}$$

holds modulo 5, which proves Theorem 1.5(i)–(iii) for $p = 5$.

While Theorem 1.3 concerns the iteration of a single T'_ℓ, its proof applies more generally to show that

$$\widetilde{\Delta}(z)^{k_{5,t}}|T'_{\ell_1}|\cdots|T'_{\ell_{m_{5,t}}} = 0. \tag{3.25}$$

Since $\ell_i \equiv -1 \pmod 5$, we have $T'_{\ell_i} = T_{\ell_i}$. By (2.8), for $n \geq 1$ with $\ell_1 \nmid n$, we observe that $\widetilde{\Delta}^{k_{5,t}}|T_{\ell_1}$ has the nth coefficient

$$\widetilde{a_{5^t}(\ell_1 n - k_{5,t})} - \widetilde{a_{5^t}\left(\frac{n}{\ell_1} - k_{5,t}\right)} = \widetilde{a_{5^t}(\ell_1 n - k_{5,t})}.$$

Similarly, for $\ell_1 \neq \ell_2$, and $n \geq 1$ with $\ell_1, \ell_2 \nmid n$, we find that $\widetilde{\Delta}^{k_{5,t}}|T_{\ell_1}|T_{\ell_2}$ has the nth coefficient

$$\widetilde{a_{5^t}(\ell_1 \ell_2 n - k_{5,t})} - \widetilde{a_{5^t}\left(\frac{\ell_1 n}{\ell_2} - k_{5,t}\right)} = \widetilde{a_{5^t}(\ell_1 \ell_2 n - k_{5,t})}.$$

We iterate in this way using (3.25) to prove Theorem 1.5(ii) for $p = 5$.

To prove Theorem 1.6, we recall that $D_2(z) = \eta(2z)^{12}$, and we let $\gcd(r, 6) = 1$. When n is odd, we use (1.4) to deduce that

$$\sum p_{12r}\left(\frac{n-r}{2}\right) q^n = \eta(2z)^{12r} = D_2(z)^r.$$

From here, the proof runs parallel to the proof of Theorem 1.5 for $p = 5$ given above, this time applying Theorem 1.3(iiia).

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Matthew Boylan, Swati
Department of Mathematics
University of South Carolina
Columbia, SC 29208, USA
E-mail: boylan@math.sc.edu
s10@email.sc.edu