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# Multifractal analysis of the cusp winding spectrum for Fuchsian groups of the first kind

by

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**Abstract.** We study the cusp winding process of the geodesic flow on a hyperbolic surface with cusps, uniformised by a finitely generated Fuchsian group of the first kind. We establish the exact exponential asymptotics of the cusp winding spectrum at infinity, extending a result of Cesaratto and Vallée for continued fractions.

**1. Introduction.** Let $(\mathbb{D}, d_H)$ denote the Poincaré disk model of two-dimensional hyperbolic space. Throughout, $G$ denotes a non-elementary, finitely generated Fuchsian group [Bea12]. By a fundamental domain $R \subset \mathbb{D}$ for $G$ we mean a convex and locally finite fundamental domain for $G$ that contains 0 in its interior. We say that $R$ is *admissible* if it has even corners [BS79, Ser86] and satisfies a technical condition (see Section 2). We always assume that $G$ contains parabolic elements and that $G$ is *of the first kind*, that is, the limit set of $G$ coincides with $\partial\mathbb{D}$. The *conical limit set* of $G$, denoted by $\Lambda_c(G)$, is the complement of the set of parabolic fixed points of elements of $G$.

Let $(X_i)_{i\geq 1} : \Lambda_c(G) \to \mathbb{N}$ be the cusp winding process associated with $G$ and $R$ (see Section 2). The *cusp winding spectrum* $b : [0,\infty] \to [0,1]$ is defined by

$$b(\alpha) := \dim_H(B(\alpha)), \qquad \text{where}$$
$$B(\alpha) := \left\{ x \in \Lambda_c(G) : \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i(x) = \alpha \right\},$$

and $\dim_H$ denotes the Hausdorff dimension with respect to the Euclidean metric on $\partial\mathbb{D}$. Let $T : \Lambda_c(G) \to \Lambda_c(G)$ be the Bowen–Series map associated

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with $G$ and $R$ (see Section 2). Let $m_1$ denote the normalised Lebesgue measure on $\partial\mathbb{D}$ and let $\mu_1$ denote the unique $T$-invariant measure on $\Lambda(G)$ which is absolutely continuous with respect to $m_1$ and satisfies $\mu_1(\{X_1 = 1\}) = 1$ (see Proposition 2.1). The Lyapunov exponent of $T$ is given by $\lambda(\mu_1) := \int_{\Lambda_c(G)} \log|T'|\, d\mu_1 < +\infty$. Lemma 3.5 allows us to define the following constants depending on $G$ and $R$:

$$\kappa := \lim_{n\to\infty} n^2 \mu_1(\{X_1 = 1, X_2 = n\})$$
$$\gamma := \sum_{n=1}^{\infty} (n^{-1} - \kappa^{-1}\mu_1(\{X_1 = 1,\ X_2 \geq n\})).$$

THEOREM 1.1. *Let $G$ be a finitely generated Fuchsian group of the first kind with parabolic elements, and let $R$ be an admissible fundamental domain for $G$. Then, for the cusp winding spectrum associated with $G$ and $R$, we have*

$$\lim_{\alpha\to\infty} (1 - b(\alpha)) \exp\left(\frac{\alpha}{\kappa}\right) = \frac{\kappa \exp(-\gamma - 1)}{\lambda(\mu_1)}.$$

Theorem 1.1 is motivated by an analogous result for continued fractions due to Cesaratto and Vallée [CV06]. To state the result, let $(a_i(x))_{i\geq 1}$ denote the regular continued fraction expansion of $x \in (0,1)\setminus\mathbb{Q}$. Define the Birkhoff spectrum

$$\alpha \mapsto b_{\mathrm{cf}}(\alpha) := \dim_H\left\{x \in (0,1)\setminus\mathbb{Q} : \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_i(x) = \alpha\right\}, \quad \alpha \geq 1. \tag{1.1}$$

The Euler–Mascheroni constant is given by

$$\tilde{\gamma} := \lim_{N\to\infty}\Big(\sum_{n=1}^{N} n^{-1} - \log N\Big).$$

By combining [IJ15, Lemma 4.4] with [CV06, Theorems 1 and 2] we have

$$\lim_{\alpha\to\infty} (1 - b_{\mathrm{cf}}(\alpha)) 2^{\alpha} = \frac{6\exp(-1-\tilde{\gamma})}{\pi^2}. \tag{1.2}$$

By a result of Series [Ser85] it is well known that continued fraction expansions admit an interpretation in terms of the cusp windings of the geodesic flow on the modular surface $\mathrm{PSL}(2,\mathbb{Z}) \setminus \mathbb{D}$. Theorem 1.1 can therefore be viewed as an extension of (1.2) for $G = \mathrm{PSL}(2,\mathbb{Z})$ to other Fuchsian groups of the first kind. Although the modular surface has no admissible fundamental domain, Theorem 1.1 is in line with (1.2). Indeed, for regular continued fractions, $\mu_1$ is the Gauss measure $1/((1+x)\log 2)\,dx$ on $[0,1]$, and we have $\kappa = 1/\log 2$, $\lambda(\mu_1) = \pi^2/(6\log 2)$. Moreover, an elementary calculation shows that $\gamma$ is the Euler–Mascheroni constant. Finally, we note that Theorem 1.1 is in stark contrast to Fuchsian Schottky groups, for which the convergence rate of the cusp winding spectrum is polynomial [AJ25].

To prove Theorem 1.1, we introduce an induced dynamical system $T_{\mathcal{D}}$, derived from the Bowen–Series map [BS79] associated with an admissible fundamental domain. The map $T_{\mathcal{D}}$ plays a role analogous to that of the Gauss map in the theory of continued fractions. We adapt the arguments of Cesaratto and Vallée [CV06] to our setting. We introduce a Dirichlet series $D(u,q)$ to investigate the dependence of the average number of cusp windings on the thermodynamic functions $b$ and $q$ appearing in Theorem 4.1. It follows from the asymptotic expansion of the polylogarithm in (4.3) that $D(2b(\alpha)-1, q(\alpha)) \sim -\log q(\alpha)$ as $\alpha \to \infty$. To derive Theorem 1.1, a finer asymptotic analysis of $D(u,q)$ is required.

Using the Mellin transform, we establish an explicit error bound in Theorem 4.5. The constant $\gamma$ appearing in Theorem 1.1 arises as the constant term in the Laurent expansion at $z=1$ of a Dirichlet series associated with $D(u,q)$ (see Proposition 4.4).

The structure of the paper is as follows. In Section 2, we describe the Markov structure of the Bowen–Series map associated with Fuchsian groups with even corners. In Section 3, we review several properties of the thermodynamic formalism for countable Markov shifts. In Section 4, we prove our main theorem.

**2. Bowen–Series map.** Let $G$ be a non-elementary, finitely generated Fuchsian group. Let $R$ be a fundamental domain for $G$ and let $G_0$ be the set of side paring transformations of $R$. Note that $G_0$ is a finite generating set of $G$. We denote by $P_0$ the set of parabolic elements in $G_0$ and we always assume that $P_0 \neq \emptyset$. We label the side $s$ common to $R$ and $g(R)$ ($g \in G_0$) on the side of $gR$ by $g$, and on the other side by $g^{-1}$. For $g \in G_0$ we denote by $C(g)$ the complete geodesic that contains the side of $R$ with exterior label $g^{-1}$. For convenience, we assume that the sides of $R$ have exterior labels $g_1, \ldots, g_m$ in anticlockwise order. By a side or vertex of $N := G\partial R$ we mean the $G$-image of a side or vertex of $R$. We say $R$ has *even corners* if $N$ is a union of complete geodesics. We say $R$ is *admissible* if $R$ has even corners with at least four sides and satisfies the following property: if $R$ has precisely four sides with all vertices in $\mathbb{D}$, then at least three geodesics in $N$ meet at each vertex of $R$ [BS79, JT25]. In addition, we always assume that if two sides $s$ and $s'$ of $R$ meet in a point of $\partial\mathbb{D}$, then there exists a side pairing in $G_0$ which maps $s$ onto $s'$. Unless otherwise stated we assume that all geodesics are complete. If the oriented geodesic $\eta$ passes through a vertex $v$ of $N$ in $\mathbb{D}$ then we make the convention that $\eta$ is replaced by a curve deformed to the right around $v$. We shall take as understood that all geodesics have been deformed, where necessary, in this way [Ser86, JT25].

For $g_i \in G_0$ we denote the two endpoints of $C(g_i^{-1})$ by $P_i$ and $Q_{i+1}$ in anticlockwise order. For $j \in \mathbb{Z}$ with $i = j \bmod m$ we set $g_j = g_i$, $P_j = P_i$

and $Q_j = Q_i$. Since $G$ is of the first kind, we have $\bigcup_{i=1}^m [P_i, U_{i+1}) = \partial\mathbb{D}$. The *Bowen–Series map* $T : \partial\mathbb{D} \to \partial\mathbb{D}$ is defined by

$$T|_{[P_i,P_{i+1})} := g_i^{-1}.$$

The $T$-*expansion* of a point $\xi \in \Lambda(G)$ is the one-sided infinite sequence $\xi_T = (g_{i_n})_{n=0}^{\infty} \in G_0^{\mathbb{N}_0}$ satisfying $T^n(\xi) \in [P_{i_n}, P_{i_n+1})$ for all $n \geq 0$. We define

$$\Sigma_T^+ := \{\xi_T : \xi \in \Lambda(G)\} \subset G_0^{\mathbb{N}_0}$$

and the map $\Pi_T : \Lambda(G) \to \Sigma_T^+$ by $\Pi_T(\xi) := \xi_T$ for all $\xi \in \Lambda(G)$. For $\xi \in \Lambda_c(G)$ with the $T$-expansion $\Pi_T(\xi) = g_{i_0} g_{i_1} \dots$ we define a sequence of blocks $B_l(\xi)$ $(l \in \mathbb{N})$ such that $g_{i_0} g_{i_1} \dots = B_1(\xi) B_2(\xi) \dots$, where each $B_l(\xi)$ is either a hyperbolic generator, or a maximal block of consecutive appearances of the same parabolic generator. The *cusp winding process* associated with $G$ and $R$ is given by

$$(X_i)_{i\geq 1} : \Lambda_c(G) \to \mathbb{N}, \qquad X_i(\xi) = \begin{cases} m & \text{if } B_i(\xi) = \gamma^m,\ \gamma \in P_0,\ m \in \mathbb{N}, \\ 1 & \text{otherwise.} \end{cases}$$

Note that this cusp winding process takes values in $\mathbb{N}$, whereas the one in [JKM21] for groups of the second kind takes values in $\mathbb{N}_0$.

We define $\overline{T} : \partial\mathbb{D} \to \partial\mathbb{D}$ by $\overline{T}|_{(Q_i,Q_{i+1}]} := g_i^{-1}$ [Ser86, p. 616]. The $\overline{T}$-expansion of a point $\xi \in \Lambda(G)$ is the one-sided infinite sequence $\xi_{\overline{T}} = (g_{i_n})_{n=0}^{\infty} \in G_0^{\mathbb{N}_0}$ defined by $\overline{T}^n(\xi) \in (Q_{i_n}, Q_{i_n+1}]$ for all $n \geq 0$. We also define

$$\Sigma_{\overline{T}}^+ := \{\xi_{\overline{T}} \in G_0^{\mathbb{N}_0} : \xi \in \Lambda(G)\}, \qquad \Pi_{\overline{T}} : \Lambda(G) \to \Sigma_{\overline{T}}^+, \qquad \Pi_{\overline{T}}(\xi) := \xi_{\overline{T}}.$$

For all $\xi_{\overline{T}} := \{\xi_j\}_{j=0}^{\infty} \in \Sigma_{\overline{T}}^+$ and $\eta_T := \{\eta_j\}_{j=0}^{\infty} \in \Sigma_T^+$ we define the two-sided sequence $\xi_{\overline{T}} * \eta_T := \dots \xi_1^{-1}\xi_0^{-1}\eta_0\eta_1 \dots$ and the map $\Pi_{\overline{T}} * \Pi_T : \Lambda(G) \times \Lambda(G) \to G_0^{\mathbb{Z}}$ by $\Pi_{\overline{T}} * \Pi_T(\xi, \eta) = \Pi_{\overline{T}}(\xi) * \Pi_T(\eta)$.

We denote by $F(\Sigma_T^+)$ the set of all finite sequences which occur in $\Sigma_T^+$. Let $\Sigma$ denote the set of all $(g_{i_j})_{j=-\infty}^{\infty} \in \Pi_{\overline{T}} * \Pi_T(\Lambda(G) \times \Lambda(G))$ such that $g_{i_l} \dots g_{i_m} \in F(\Sigma_T^+)$ for all $l < m$ and $(g_{i_j})$ does not begin or end with an infinite chain of anticlockwise cycles [Ser86, p. 617].

We define the measure $v_1$ on $\mathcal{Y} := (\Pi_{\overline{T}} * \Pi_T)^{-1}(\Sigma)$ by

$$dv_1(\xi, \eta) := \frac{dm_1(\xi) dm_1(\eta)}{|\xi - \eta|^2}.$$

The map $H : \mathcal{Y} \to \mathcal{Y}$ is given by

$$H(\xi, \eta) := ({\eta_0}^{-1}(\xi), {\eta_0}^{-1}(\eta)), \quad \text{where} \quad \Pi_T(\eta) = (\eta_i)_{i=0}^{\infty}.$$

We define the projection map $P : \mathcal{Y} \to \Lambda(G)$ by $P(\xi, \eta) := \eta$. By the definitions of $H$ and $T$ we have $P \circ H = T \circ P$. We also define the measure on $\Lambda(G)$ by

$$\mu_1 := v_1 \circ P^{-1}. \tag{2.1}$$

For $\xi, \eta \in \Lambda(G)$ with $\xi \neq \eta$ let $s_{\xi,\eta}$ denote the oriented geodesic with negative endpoint $\xi$ and positive endpoint $\eta$. We consider

$$Y := \{(\xi, \eta) \in \Lambda_c(G) \times \Lambda_c(G) : \xi \neq \eta,\, s_{\xi,\eta} \cap R \neq \emptyset\}$$

as well as the measure $\nu_1$ on $Y$ given by

$$d\nu_1(\xi, \eta) = \frac{dm_1(\xi) dm_1(\eta)}{|\xi - \eta|^2}.$$

For $(\xi, \eta) \in Y$, the oriented geodesic $s_{\xi,\eta}$ intersects infinitely many tiles $R, g_0(R), g_0g_1(R), \dots$ of $R$, with $g_i \in G_0$ and $i \in \mathbb{N}$. The unique infinite sequence $g_0g_1g_2 \dots \in G_0^{\mathbb{N}_0}$ is called the *cutting sequence of* $s_{\xi,\eta}$. We define the map $S : Y \to Y$ as follows. For each $(\xi, \eta) \in Y$ such that $s_{\xi,\eta}$ has the cutting sequence $g_0g_1 \dots$ we set

$$S(\xi, \eta) = (g_0^{-1}(\xi), g_0^{-1}(\eta)).$$

By [Ser86, Theorem II], there exists an invertible map

$$\iota : Y \to \mathcal{Y}$$

such that $\iota$ is bimeasurable and $\iota \circ S = H \circ \iota$. Clearly, we have $v_1 = \nu_1 \circ \iota^{-1}$. Thus, $(Y, S, v_1)$ is measure-theoretically isomorphic to $(\mathcal{Y}, H, v_1)$.

By the arguments in [SS05, Section 1.2], the map $S$ is measure-preserving, conservative and ergodic with respect to $\nu_1$. Hence, we obtain the following.

PROPOSITION 2.1. *The map $H$ is measure-preserving, conservative and ergodic with respect to $v_1$. Moreover, $T$ is measure-preserving, conservative and ergodic with respect to $\mu_1$, which is absolutely continuous with respect to $m_1$.*

**2.1. Markov partition and induced map of Bowen–Series map.** We recall the construction of the Markov partition of $T$ in [BS79] (see also [JT25]). Let $V$ be the set of vertices of $R$, and $V_c$ be the set of vertices of $R$ in $\partial\mathbb{D}$. Since we assume that $G$ contains parabolic elements, we have $V_c \neq \emptyset$.

For each $v \in V$ we denote by $N(v)$ the set of geodesics in $N$ passing through $v$, and by $W(v)$ the set of endpoints of geodesics in $N(v)$.

Let $v \in V_c$. There exists $i \in \mathbb{Z}$ such that $v$ is in $C(g_{i-1}^{-1}) \cap C(g_i^{-1})$. We denote the arcs of $\partial\mathbb{D}$ cut off by successive points of $W(v)$ in clockwise order from $Q_{i+1}$ to $Q_i = v$ as $L_1(v), L_2(v), \dots$, and anticlockwise from $Q_{i+1}$ to $Q_i$ by $R_1(v), R_2(v), \dots$. For each $v \in V_c$ we define

$$L(v) := \overline{\bigcup_{r \geq 3} L_r(v)} \cup L_2 \setminus g_i([P_{i+1}, Q_{i+1})) \quad \text{and} \quad R(v) := \overline{\bigcup_{r \geq 3} R_r(v)},$$

where the closure is taken with respect to the Euclidean metric on $\mathbb{C}$. For

$v \in V$ we define

$$W'(v) = \begin{cases} W(v) & \text{if } v \notin V_c, \\ \partial L_1(v) \cup \partial L(v) \cup \partial R_2(v) \cup \partial R(v) & \text{if } v \in V_c, \end{cases}$$
$$W' = \bigcup_{v \in V} W'(v).$$

As in [BS79, Lemma 2.3], one verifies that $T(W') \subset W'$ (see also [JT25, Lemma 3.1]). Hence, $W'$ induces the following Markov partition for $T$. Define a partition of $\partial\mathbb{D}$ into arcs with endpoints given by two consecutive points in $W'$. We choose all partition elements to be left-closed and right-open, in anticlockwise order. We label the partition elements by integers in a finite subset $F$ of $\mathbb{N}$, and denote the element labeled with $a \in F$ by $\Delta_a$. We define the incidence matrix $A : F \times F \to \{0,1\}$ by $A_{a,\tilde{a}} = 1$ if $\Delta_{\tilde{a}} \subset T(\Delta_a)$ and $A_{a,\tilde{a}} = 0$ otherwise. For $k \in \mathbb{N}$ we set $F_A^k := \{\tau \in F^k : A_{\tau_{i-1},\tau_i} = 1\}$. For each $\tau \in F_A^k$ we define

$$\Delta_\tau := \bigcap_{i=0}^{k-1} T^{-i}(\Delta_{\tau_i}).$$

Next, we introduce a map induced by $T$ and an induced Markov partition. Let

$$\mathcal{D} := \partial\mathbb{D} \setminus \bigcup_{v \in V_c} (L(v) \cup R(v))$$

and define the first return time function $\rho : \mathcal{D} \to \mathbb{N}$ by $\rho(\xi) = \inf\{n \in \mathbb{N} : T^n(\xi) \in \mathcal{D}\}$. Note that we have $\mathcal{D} \cap \Lambda_c(G) = \{X_1 = 1\}$. We then define the induced Bowen–Series map $T_{\mathcal{D}} : \mathcal{D} \cap \{\rho < \infty\} \to \partial\mathbb{D}$ by

$$T_{\mathcal{D}}(\xi) = T^{\rho(\xi)}(\xi).$$

Note that for each $n \geq 1$ we have

$$\{\rho = n\} = \bigcup_{g \in E_n} \Delta_g,$$

where we have set

$$E_n := \{g \in F_A^{n+2} : \Delta_g \subset \mathcal{D},\ \rho|_{\Delta_g} = n\}.$$

Therefore, with $I := \bigcup_{n=1}^\infty E_n$, $(\Delta_g)_{g \in I}$ forms a countable Markov partition of $T_{\mathcal{D}}$. Note that $\bigcap_{n \geq 0} T_{\mathcal{D}}^{-n}(\mathcal{D}) = \Lambda_c(G) \cap \mathcal{D}$.

We define the incidence matrix $B : I \times I \to \{0,1\}$ by $B_{e,e'} = 1$ if $\Delta_{e'} \subset T_{\mathcal{D}}(\Delta_e)$ and $B_{e,e'} = 0$ otherwise. Define the coding space

$$\Sigma_B := \{\omega \in I^{\mathbb{N}_0} : B_{\omega_{n-1},\omega_n} = 1,\ n \in \mathbb{N}\}.$$

We denote by $I^n$ $(n \in \mathbb{N})$ the set of all admissible words of length $n$ with respect to the incidence matrix $B$ and by $I^*$ the set of all admissible words which have a finite length (i.e. $I^* = \bigcup_{n \in \mathbb{N}} I^n$). For convenience, put $I^0 = \{\emptyset\}$. For $\omega \in I^n$ we define the cylinder set of $\omega$ by $[\omega] := \{\tau \in \Sigma_B : \tau_i = \omega_i, 0 \le i \le n-1\}$. For $n \in \mathbb{N}$, $\omega \in I^n$ and $\tau \in \Sigma_B$ we set $\omega\tau := \omega_0 \dots \omega_{n-1}\tau_0\tau_1 \dots$. It is easy to see that $\Sigma_B$ is finitely primitive [URM22, Definition 17.1.4]. We endow $\Sigma_B$ with the metric $d$ defined by $d(\omega, \omega') = e^{-k}$ if $\omega_i = \omega_i'$ for all $i = 0, \dots, k-1$ and $\omega_k \neq \omega_k'$ and $d(\omega, \omega') = 0$ otherwise. Let $\sigma : \Sigma_B \to \Sigma_B$ be the left-shift map. Since $T_{\mathcal{D}}$ is uniformly expanding (see [JT25, Section 4]) and for each $\omega \in I$ the set $\overline{\Delta_\omega}$ is compact, for any $\omega = \omega_0\omega_1 \ldots \in \Sigma_B$ the set $\bigcap_{j=0}^{\infty} \overline{T_{\mathcal{D}}^{-j}(\Delta_{\omega_j})}$ is a singleton. Thus, we can define the coding map $\pi : \Sigma_B \to \mathcal{D}$ by $\pi(\omega) \in \bigcap_{j=0}^{\infty} \overline{T_{\mathcal{D}}^{-j}(\Delta_{\omega_j})}$. Note that $\pi(\Sigma_B) = \Lambda_c(G) \cap \mathcal{D}$.

**3. Thermodynamic formalism for $(\Sigma_B, \sigma)$.** We summarise results on thermodynamic formalism for the countable Markov shift $(\Sigma_B, \sigma)$. For further details we refer to [MU03, Chapter 2] and [URM22, Chapter 17]. Let $\phi : \Sigma_B \to \mathbb{R}$ be continuous. The topological pressure of $\phi$ [MU03] is given by

$$P(\phi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in I^n} \exp\Big(\sup_{\tau\in[\omega]} \sum_{j=0}^{n-1} \phi(\sigma^j(\tau))\Big).$$

We define the function $p : \mathbb{R}^2 \to \mathbb{R}$ by $p(b, q) := P(-q\rho \circ \pi - b \log |T_{\mathcal{D}}' \circ \pi|)$. We frequently make use of the existence of a constant $D \ge 1$ such that for each $\omega \in I$ and each $\tau \in [\omega]$ we have

$$D^{-1} \le |T_{\mathcal{D}}' \circ \pi(\tau)| \rho^{-2}(\pi(\tau)) \le D. \tag{3.1}$$

By (3.1) and [URM22, Theorem 17.2.8], we have $p(b, q) < \infty$ if and only if $(b, q) \in \mathcal{F} := (\mathbb{R} \times (0, \infty)) \cup ((1/2, \infty) \times \{0\})$.

We say that $\phi : \Sigma_B \to \mathbb{R}$ is *Hölder continuous on cylinders with exponent* $\beta > 0$ if $v_\beta(\phi) := \sup_{n\ge1} \sup_{\omega\in I^n} \sup\{|\phi(\tau) - \phi(\tilde{\tau})|/(d(\tau, \tilde{\tau}))^\beta : \tau, \tilde{\tau} \in [\omega], \tau \neq \tilde{\tau}\} < \infty$. Since $T_{\mathcal{D}}$ is uniformly expanding, by standard arguments on conformal graph-directed Markov systems involving Koebe's distortion theorem [URM22, Section 19.3], there exists $\beta > 0$ such that $\log |T_{\mathcal{D}}' \circ \pi|$ is Hölder continuous on cylinders with exponent $\beta$ and the coding map $\pi : \Sigma_B \to \mathcal{D} \cap \Lambda_c(G)$ is Hölder continuous with exponent $\beta$. From now on, we fix such $\beta > 0$. We denote by $C_b(\Sigma_B)$ the set of bounded continuous functions on $\Sigma_B$ equipped with the supremum norm $\|\cdot\|_\infty$, and let $H(\Sigma_B) := \{\phi \in C_b(\Sigma_B) : v_\beta(\phi) < +\infty\}$ be equipped with the norm $\|\phi\|_\beta := \|\phi\|_\infty + v_\beta(\phi)$ ($\phi \in H(\Sigma_B)$). Note that $C_b(\Sigma_B)$ and $H(\Sigma_B)$ are Banach spaces. Thus, by the same argument as in the proof of [URM22, Lemma 13.6.1], for all $(b, q) \in \mathcal{F}$ the Perron–Frobenius operator $\mathcal{L}_{b,q} : C_b(\Sigma_B) \to C_b(\Sigma_B)$ is for $\psi \in C_b(\Sigma_B)$ and $\omega \in \Sigma_B$ defined by

$$\mathcal{L}_{b,q}(\psi)(\omega) := \sum_{\tau\in\sigma^{-1}(\omega)} \exp\big(-q\rho(\pi(\tau)) - b \log |T_{\mathcal{D}}'(\pi(\tau))|\big)\psi(\tau). \tag{3.2}$$

Note that $\mathcal{L}_{b,q}$ is a bounded linear operator. By [URM22, Lemma 18.1.4], we also have $\mathcal{L}_{b,q}(H(\Sigma_B)) \subset H(\Sigma_B)$ for all $(b,q) \in \mathcal{F}$. For $(b,q) \in \mathcal{F}$ we denote by $\mathcal{L}^*_{b,q}$ the dual operator of $\mathcal{L}_{b,q}$ acting on the dual space $C^*_b(\Sigma_B)$ of $C_b(\Sigma_B)$. Recall that $\Sigma_B$ is finitely primitive. Hence, we have the following fundamental facts.

Theorem 3.1 ([URM22, Corollary 17.7.5]). *Let $(b,q) \in \mathcal{F}$. There exists a unique Borel probability eigenmeasure $\widetilde{m}_{b,q}$ of the dual operator $\mathcal{L}^*_{b,q}$ and its eigenvalue is* $\exp(p(b,q))$.

Assume that $(b,q) \in \mathcal{F}$. Since $\widetilde{m}_{b,q}$ is an eigenmeasure of $\mathcal{L}^*_{b,q}$, for $\omega, \tau \in I$ with $B_{\omega,\tau} = 1$ we have

$$\widetilde{m}_{b,q}([\omega\tau]) = \exp(-p(b,q)) \int_{[\tau]} \exp\big((-q\rho\circ\pi - b\log|T'_{\mathcal{D}}\circ\pi|)(\omega\xi)\big)\, d\widetilde{m}_{b,q}(\xi). \tag{3.3}$$

We refer to the property in (3.3) as the *conformality* of $\widetilde{m}_{b,q}$.

Theorem 3.2 ([URM22, Theorem 18.1.7 and Corollary 18.1.8]). *Let $(b,q) \in \mathcal{F}$. Then* $\exp(p(b,q))$ *is an isolated and dominant eigenvalue of the Perron–Frobenius operator $\mathcal{L}_{b,q} : H(\Sigma_B) \to H(\Sigma_B)$. Moreover, there exists a unique real-valued function $\widetilde{\Phi}_{b,q} \in H(\Sigma_B)$ such that $\widetilde{\Phi}_{b,q}$ is an eigenfunction corresponding to the eigenvalue* $\exp(p(b,q))$ *and $\int \widetilde{\Phi}_{b,q}\, d\widetilde{m}_{b,q} = 1$. Furthermore, there exists a constant $C > 1$ such that $1/C \le \widetilde{\Phi}_{b,q} \le C$.*

For $(b,q) \in \mathcal{F}$ we define

$$\widetilde{\mu}_{b,q} := \widetilde{\Phi}_{b,q}\widetilde{m}_{b,q},$$

with $\widetilde{m}_{b,q}$ and $\widetilde{\Phi}_{b,q}$ given by Theorems 3.1 and 3.2. Note that if

$$\int (-q\rho\circ\pi - b\log|T'_{\mathcal{D}}\circ\pi|)\, d\widetilde{\mu}_{b,q} > -\infty,$$

$\widetilde{\mu}_{b,q}$ is the unique equilibrium state for the potential $-q\rho\circ\pi - b\log|T'_{\mathcal{D}}\circ\pi|$.

We endow the space of bounded linear operators on $H(\Sigma_B)$ with the operator norm denoted by $\|\cdot\|_\beta$. We also endow the dual space of $H(\Sigma_B)$ with the dual norm denoted by $\|\cdot\|^*_\beta$.

The following proposition results from well-known calculations used to prove the Ionescu-Tulcea and Marinescu inequality (for example see [URM22, Lemma 18.1.4]).

Proposition 3.3. *The map $(b,q) \mapsto \mathcal{L}_{b,q}$ is continuous on* $(1/2,1]\times[0,1]$.

This proposition enables us to proceed as in the proof of [CV06, Lemma 18]. We obtain the following.

PROPOSITION 3.4. *For all $\epsilon > 0$ there exists a constant $C > 0$ such that for all $(b,q) \in [1/2+\epsilon, 1] \times (0,1]$ we have*

$$\|\widetilde{\Phi}_{1,0} - \widetilde{\Phi}_{b,q}\|_\beta \le C(|1-b| + |p(1,0) - p(1,q)|) = C(|1-b| + |p(1,q)|),$$
$$\|\widetilde{m}_{1,0} - \widetilde{m}_{b,q}\|_\beta^* \le C(|1-b| + |p(1,0) - p(1,q)|) = C(|1-b| + |p(1,q)|).$$

For each $\gamma \in P_0$ we denote by $i_\gamma \in \{1,\dots,m\}$ the index such that $\gamma = g_{i_\gamma}$. We consider the function $\Phi : \partial\mathbb{D} \to \mathbb{R}$ defined by

$$\Phi(\eta) := \int 1_{\mathcal{Y}}(\xi,\eta)|\xi-\eta|^{-2}\, dm_1(\xi),$$

where $1_{\mathcal{Y}}$ denotes the indicator function of $\mathcal{Y}$. By the definition of $\mu_1$, $\Phi$ is a version of the Radon–Nikodym derivative of $\mu_1$ with respect to $m_1$. Since $G$ is finitely generated, we have $n(G) := \max\{\#N(v) : v \in V \setminus V_c\} < \infty$. Therefore, by [Ser86, Theorem 4.2], for all $\tau \in F_A^{n(G)}$ and $\eta_1, \eta_2 \in \Delta_\tau$ we have

$$\{\xi \in \Lambda(G) : (\xi,\eta_1) \in \mathcal{Y}\} = \{\xi \in \Lambda(G) : (\xi,\eta_2) \in \mathcal{Y}\}. \tag{3.4}$$

In particular, for all $\gamma \in P_0$, $g \in G_0 \setminus \{\gamma^{-1}\}$, $\eta_1, \eta_2 \in [P_{i_\gamma}, P_{i_\gamma+1})$ and $n \ge n(G)$ we have $\{\xi \in \Lambda(G) : (\xi, g\gamma^{n-1}(\eta_1)) \in \mathcal{Y}\} = \{\xi \in \Lambda(G) : (\xi, g\gamma^{n-1}(\eta_2)) \in \mathcal{Y}\}$. For all $\gamma \in P_0$ and $g \in G_0 \setminus \{\gamma^{-1}\}$ we define

$$O_{g\gamma} := \{\xi \in \Lambda(G) : (\xi, g\gamma^{n(G)-1}(\eta)) \in \mathcal{Y}\},$$

where $\eta \in [P_{i_\gamma}, P_{i_\gamma+1})$. By (3.4), the function $\Phi$ is continuous on $\Lambda(G)$ except on the finite set $J := \bigcup_{\tau \in F_A^{n(G)}} \partial\Delta_\tau$ and for each $\eta \in J$ the right limit $\lim_{\tilde\eta \to \eta+0} \Phi(\tilde\eta)$ and the left limit $\lim_{\tilde\eta\to\eta-0}\Phi(\tilde\eta)$ exist.

For each $\gamma \in P_0$ we define

$$G_\gamma := \{g_i \in G_0 : B_1(g_i(\eta)) = g_i \text{ for all } \eta \in [P_{i_\gamma}, P_{i_\gamma+1}) \cap \{X_1 \ge 2\}\}.$$

Then for all $n \ge 2$ we have

$$\{X_1 = 1,\, X_2 = n\} = \bigcup_{\gamma \in P_0} \bigcup_{g \in G_\gamma} g\gamma^{n-1}([P_{i_\gamma}, P_{i_\gamma+1}) \cap \mathcal{D}). \tag{3.5}$$

For $\gamma \in P_0$ we define the constants

$$\tilde\Phi_\gamma := \sum_{g \in G_\gamma} \int_{g^{-1}(O_{g\gamma})} \frac{1}{|\eta - p_\gamma|^2}\, dm_1(\eta),$$
$$\tilde\Phi'_\gamma := \int_{T([P_{i_\gamma}, P_{i_\gamma+1}) \cap \mathcal{D})} \frac{1}{|\eta - p_\gamma|^2}\, dm_1(\eta).$$

We let $w_\gamma$ refer to the width of the cusp associated with $\gamma \in P_0$, that is, $w_\gamma$ is the unique real number such that $\gamma$ can be written as $\gamma = \varphi_\gamma \circ \psi_\gamma \circ \varphi_\gamma^{-1}$ for $\psi_\gamma$ and $\varphi_\gamma$ given by

$$\varphi_\gamma(\xi) := -\frac{p_\gamma}{i}\xi \quad \text{and} \quad \psi_\gamma(\xi) := \frac{(1 - w_\gamma i)\xi + w_\gamma}{w_\gamma \xi + (1 + w_\gamma i)}.$$

Using (3.5) and applying the same argument as in [JKS13, proofs of Lemmas 2.4 and 2.5], we obtain the following:

Lemma 3.5. *There exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ we have*

$$\Big| n^2 \mu_1(\{\rho = n\}) - \sum_{\gamma \in P_0} w_\gamma^{-2} \tilde{\Phi}_\gamma \tilde{\Phi}'_\gamma \Big| \le \frac{C}{n}.$$

*In particular, $\kappa = \sum_{\gamma \in P_0} w_\gamma^{-2} \tilde{\Phi}_\gamma \tilde{\Phi}'_\gamma$.*

**4. A multifractal analysis for the cusp winding process.** Since for all $\alpha \in [1, \infty]$ the set $B(\alpha)$ is a countable union of bi-Lipschitz images of $B(\alpha) \cap \mathcal{D}$, we have $b(\alpha) = \dim_H(B(\alpha) \cap \mathcal{D})$ for all $\alpha \in [1, \infty]$. Since $\rho(\xi) = X_2(\xi)$ for each $\xi \in \mathcal{D}$, we therefore have, for each $\alpha \in [1, \infty]$,

$$b(\alpha) = \dim_H \Big\{ \xi \in \Lambda_c(G) \cap \mathcal{D} : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \rho \circ T_{\mathcal{D}}^i(\xi) = \alpha \Big\}. \tag{4.1}$$

The following theorem can be proved by the same methods as in [A25].

Theorem 4.1. *For each $\alpha \in (1, \infty)$ there exists a unique $q(\alpha) \in (0, \infty)$ such that*

$$p(b(\alpha), q(\alpha)) = -q(\alpha)\alpha \quad \text{and} \quad \int \rho \circ \pi \, d\widetilde{\mu}_{b(\alpha), q(\alpha)} = \alpha.$$

*Moreover, the functions $b$ and $q$ are real-analytic on $(1, +\infty)$, $b$ is strictly increasing, and* $\lim_{\alpha \to \infty} b(\alpha) = 1$.

The following lemma is proved as in [AJ25, Lemma 4.2].

Lemma 4.2. *We have* $\lim_{\alpha \to \infty} q(\alpha)\alpha = 0$.

Using Proposition 2.1 and the proof of [AJ25, Lemma 4.3], we obtain the following:

Lemma 4.3. *We have*

$$\lim_{\alpha \to \infty} \frac{1 - b(\alpha)}{\lambda(\mu_1)^{-1} \int_\alpha^\infty q(t)\, dt} = 1.$$

**4.1. Asymptotic analysis of a Dirichlet series.** We define the functions

$$\mathrm{Li}(u, q) = \sum_{\ell=1}^{\infty} \frac{e^{-\ell q}}{\ell^u} \quad \text{and} \quad \mathrm{Li}_1(u, q) = \sum_{\ell=1}^{\infty} \frac{e^{-\ell q}}{\ell^u} \log \ell, \qquad q > 0,\ 0 \le u < 1.$$

For all $0 < u < 1$ the following asymptotic expansions are well known. Let $\Gamma$ be the Gamma function and let $\zeta$ be the Riemann zeta function. We have, as $q \to 0$,

$$(4.2)\qquad \begin{aligned} \mathrm{Li}(u,q) &= \Gamma(1-u)q^{u-1} + \zeta(u) + O(q), \\ \mathrm{Li}_1(u,q) &= -\zeta'(u) + \Gamma'(1-u)q^{u-1} - \Gamma(1-u)q^{u-1}\log q + O(q), \end{aligned}$$

where the constant in $O(q)$ is independent of $u$. If moreover $q^{u-1} \to 1$, then

$$(4.3)\qquad \mathrm{Li}(u,q) \sim -\log q \quad \text{and} \quad \mathrm{Li}_1(u,q) \sim \tfrac{1}{2}(\log q)^2 \quad \text{as } q \to 0.$$

The proof of (4.2) makes use of the Mellin transform and the meromorphic extension of $\zeta$ (see [FS09, Theorem VI.7]). The asymptotic in (4.3) is an immediate consequence of (4.2) and the Laurent expansions of $\zeta$ at 1 and $\Gamma$ at 0.

Note that for $u = 1$ we have, as $q \to 0$,

$$\mathrm{Li}(1,q) = -\log(1-e^{-q}) \sim -\log q.$$

To investigate the asymptotics of the cusp winding spectrum, we derive an asymptotic expansion of the following Dirichlet series.

Let

$$\begin{aligned} a_\ell &= \kappa^{-1}\widetilde{\mu}_{1,0}(\{\rho\circ\pi = \ell\})\ell^2, \\ D_u(q) := D(u,q) &:= \sum_{\ell=1}^{\infty} a_\ell \frac{e^{-\ell q}}{\ell^u}, \qquad q>0,\ 0 \le u < 1. \end{aligned}$$

The asymptotic expansion will be derived from the associated Dirichlet series

$$L(z) := \sum_{\ell=1}^{\infty} \frac{a_\ell}{\ell^z}, \qquad \mathrm{Re}(z) > 1.$$

By the uniqueness of the conformal measure on $\Sigma_B$ and the invariant probability measure absolutely continuous to the conformal measure (see Theorems 3.1 and 3.2), we obtain

$$(4.4)\qquad \widetilde{\mu}_{1,0} = \mu_1 \circ \pi.$$

Thus, for all $l \in \mathbb{N}$ we have

$$(4.5)\qquad |\widetilde{\mu}_{1,0}(\{\rho\circ\pi = l\})l^2 - \kappa| \le C/l,$$

where $C$ denotes the constant obtained in Lemma 3.5.

Proposition 4.4. *The Dirichlet series $L$ admits a meromorphic extension to the right half-plane $\{\mathrm{Re}(z) > 0\}$ whose only pole is at $z = 1$. Moreover,*

$$L(z) = \zeta(z) - \gamma - 1 + O(z-1) \qquad \textit{as } z \to 1.$$

*Proof.* Let $u \in (1,\infty)$. By Abel's summation formula, for all $N \ge 2$ we have

$$\sum_{\ell=1}^{N} \frac{a_\ell}{\ell^u} = \frac{\sum_{k=1}^{N} a_k}{N^u} + u\int_1^N \frac{\sum_{k=1}^{[x]} a_k}{x^{u+1}}\,dx, \qquad \sum_{\ell=1}^{N} \frac{1}{\ell^u} = \frac{N}{N^u} + u\int_1^N \frac{[x]}{x^{u+1}}\,dx.$$

Noting that $u > 1$ and using (4.5), we obtain

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} a_k}{N^u} = 0 \quad \text{and thus} \quad L(u) = \zeta(u) + u \int_1^\infty \frac{\sum_{k=1}^{[x]} a_k - [x]}{x^{u+1}}\, dx.$$

By (4.5) there exists a constant $C \geq 1$ such that for all $x \in [1, \infty)$ we have

$$\Big|\sum_{k=1}^{[x]} a_k - [x]\Big| \leq C \sum_{k=1}^{[x]} \frac{1}{k}.$$

This implies that for all $\epsilon > 0$ and $z \in \mathbb{C}$ with $|z| > \epsilon$, the integral $\int_1^\infty \sum_{k=1}^{[x]} (a_k - [x])/x^{z+1}\, dx$ converges uniformly, and therefore defines a holomorphic function on $\{|z| > \epsilon\}$. By an elementary calculation and another application of Abel's summation formula, we obtain

$$\begin{aligned}\int_1^\infty \frac{\sum_{l=1}^{[x]} a_l - [x]}{x^2}\, dx &= \sum_{l=1}^{\infty} (l^{-1} a_l - l^{-1}) \\ &= \sum_{l=1}^{\infty} \Big(\sum_{k \geq l} a_k k^{-2} - l^{-1}\Big) - 1 = -\gamma - 1. \quad \blacksquare\end{aligned}$$

Proposition 4.4 allows us to partially apply the argument from the proof of [FS09, Theorem VI.7]. We obtain the following.

THEOREM 4.5. *Let $0 < \epsilon < 1$. For $1 - \epsilon/2 \leq u < 1$ we have, as $q \to 0$,*

$$D_u(q) = \Gamma(1-u) q^{u-1} + L(u) + O(q^{1-\epsilon}),$$

*where the $O$-constant is independent of $u$. In particular,*

$$D(u,q) = -\log q - \gamma - 1 + O((1-u)(\log q)^2 q^{u-1}) + O(q^{1-\epsilon}) \quad \text{as } q \to 0,$$

*where both $O$-constants are independent of $u$.*

*Proof.* Let $u > 0$. By Lemma 3.5 and [FGD95, Lemma 2], the Mellin transform $D_u^*$ of $D_u$ is given by $D_u^*(z) = L(u+z)\Gamma(z)$ for $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 1-u$. By Mellin's inversion theorem, for all $q \in (0, \infty)$ and $K > 1-u$ we obtain

$$D_u(q) = \frac{1}{2\pi i} \int_{K-i\infty}^{K+i\infty} L(u+z)\Gamma(z) q^{-z}\, dz.$$

Let $0 < \epsilon < 1$ and let $u$ satisfying $1 - \epsilon/2 \leq u$ be arbitrary. By Lemma 3.5 and the complex version of Stirling's formula, we obtain

$$\lim_{M\to\infty} \Big| \int_{K+iM}^{-(1-\epsilon)+iM} L(u+z)\Gamma(z) q^{-z}\, dz \Big| = 0,$$

$$\lim_{M\to-\infty} \Big| \int_{-(1-\epsilon)+iM}^{K+iM} L(u+z)\Gamma(z) q^{-z}\, dz \Big| = 0.$$

Hence, by the residue theorem, for all $q \in (0, \infty)$ we have

$$D_u(q) = L(u) + \Gamma(1-u)q^{u-1} + \eta_D(u, q), \tag{4.6}$$

where $\eta_D(u, q) := \int_{-(1-\epsilon)-i\infty}^{-(1-\epsilon)+i\infty} L(u+z)\Gamma(z)q^{-z}\, dz$. Since $2/\epsilon \le u-(1-\epsilon) \le \epsilon$, we have

$$\eta_D(u, q) = O(q^{1-\epsilon}), \tag{4.7}$$

where the implied constants in (4.7) do not depend on $u$. On the other hand, for all $q \in (0, 1]$ we have

$$\begin{aligned} (u) &= \frac{1}{u-1} + \gamma_0 - \gamma_1(u-1) + O((u-1)^2) && \text{as } u \to 1, \\ \Gamma(u) &= \frac{1}{u} - \gamma_0 + \frac{1}{12}(6\gamma_0^2 + \pi^2)u + O(u^2) && \text{as } u \to 0, \\ q^{-u} &= 1 - (\log q)u + O((\log q)^2 u^2 q^{-u}) && \text{as } u \to 0, \end{aligned}$$

where the $O$-constants do not depend on $q$, and $\gamma_1$ denotes the Stieltjes constant. Hence, by Proposition 4.4, (4.6) and (4.7), we are done. ∎

**4.2. Proof of the main theorem.** The next lemma is proved in the same way as [AJ25, Lemma 4.4].

LEMMA 4.6. *For each $0 < \epsilon < 1/2$ there exists a constant $C_1 > 0$ such that for all $(b, q) \in [1/2 + \epsilon, 1] \times (0, 1]$ and all $l \ge 1$,*

$$\begin{aligned} &|\widetilde{\mu}_{1,0}(\{\rho \circ \pi = l\}) - e^{p(b,q)+ql}\widetilde{\mu}_{b,q}(\{\rho \circ \pi = l\})| \\ &\qquad \le C_1 l^{-2b}(\|\widetilde{\Phi}_{1,0} - \widetilde{\Phi}_{b,q}\|_\beta + |1-b|\log l + \|\widetilde{m}_{1,0} - \widetilde{m}_{b,q}\|_\beta). \end{aligned}$$

LEMMA 4.7. *We have*

$$\begin{aligned} &|e^{p(b(\alpha),q(\alpha))}\alpha - \kappa D(2b(\alpha)-1, q(\alpha))| \\ &\quad = O\big(\mathrm{Li}(2b(\alpha)-1, q(\alpha))\big)(\|\widetilde{\Phi}_{1,0} - \widetilde{\Phi}_{b(\alpha),q(\alpha)}\|_\beta + \|\widetilde{m}_{1,0} - \widetilde{m}_{b(\alpha),q(\alpha)}\|_\beta) \\ &\qquad + O\big(\mathrm{Li}_1(2b(\alpha)-1, q(\alpha))(1-b(\alpha))\big) \qquad \textit{as } \alpha \to \infty. \end{aligned}$$

*Proof.* Let $0 < \epsilon < 1/2$. By Theorem 4.1 and Lemma 4.2, there exists $M > 1$ such that for all $\alpha \in (M, \infty)$ we have $(b(\alpha), q(\alpha)) \in [1/2+\epsilon, 1] \times (0, 1]$. By Theorem 4.1, for all $\alpha \in (M, \infty)$ we have

$$\begin{aligned} &\left| e^{p(b(\alpha),q(\alpha))}\alpha - \sum_{l=1}^{\infty} \frac{\widetilde{\mu}_{1,0}(\{\rho \circ \pi = l\}) l^2 e^{-q(\alpha)l}}{l^{2b(\alpha)-1}} \right| \\ &\quad \le \sum_{l=1}^{\infty} l |e^{p(b(\alpha),q(\alpha))}\widetilde{\mu}_{b(\alpha),q(\alpha)}(\{\rho \circ \pi = l\}) - e^{-q(\alpha)l}\widetilde{\mu}_{1,0}(\{\rho \circ \pi = l\})| \\ &\qquad + \sum_{l=1}^{\infty} \frac{\widetilde{\mu}_{1,0}(\{\rho \circ \pi = l\}) l^2 e^{-q(\alpha)l}}{l^{2b(\alpha)-1}} |l^{2(b(\alpha)-1)} - 1|. \end{aligned}$$

By the mean value theorem, there exists a constant $\tilde{C}_2 > 0$ such that for all $l \in \mathbb{N}$ and $\alpha \in (M, \infty)$ we have $|l^{2(b(\alpha)-1)} - 1| \le \tilde{C}_2(\log l)|1 - b(\alpha)|$. Hence, by (4.5) and Lemma 4.6, we are done. ∎

LEMMA 4.8. *The function* $(1 - b(\alpha))\alpha$ *is bounded.*

*Proof.* For $\alpha \in (1, \infty)$ we put $M_\alpha := \bigcup_{i=1}^{[\alpha]} E_i$ and $K_\alpha := M_\alpha^{\mathbb{N}_0} \cap \Sigma_B$. By using essentially the same argument as in the proof of [CV06, Lemma 17], we can show that $(1 - \dim_H(\pi(K_\alpha)))\alpha$ is bounded. Therefore, it is sufficient to show that for all $\alpha \in (1, \infty)$ we have $\dim_H(\pi(K_\alpha)) \le b(\alpha)$. Let $\alpha \in (1, \infty)$. For all $\beta \in [1, [\alpha]]$ we define $B_\alpha(\beta) := \pi(K_\alpha) \cap B(\beta)$ and $b_\alpha(\beta) = \dim_H(B_\alpha(\beta))$. By [BS01], there exists $\beta \in (1, [\alpha])$ such that $b_\alpha(\beta) = \dim_H(\pi(K_\alpha))$. Since $b_\alpha(\beta) \le b(\beta) \le b(\alpha)$ by Theorem 4.1, we obtain $\dim_H(\pi(K_\alpha)) \le b(\alpha)$. ∎

By Theorem 4.1, Lemma 4.2, (3.3), (3.1) and Lemma 3.5 we have, as $\alpha \to \infty$,

$$\alpha \asymp \mathrm{Li}(2b(\alpha) - 1, q(\alpha)) \asymp D(2b(\alpha) - 1, q(\alpha)), \tag{4.8}$$

where $f \asymp g$ means that $f/g$ is bounded away from zero and from infinity.

For $\alpha \in (1, \infty)$ we define

$$u(\alpha) := 2b(\alpha) - 1.$$

Note that for all $\alpha \in (1, \infty)$ we have $u(\alpha) < 1$ and $\lim_{\alpha\to\infty} u(\alpha) = 1$.

LEMMA 4.9. *We have*

$$\lim_{\alpha\to\infty} q(\alpha)^{u(\alpha)-1} = 1.$$

*Proof.* By (4.2) and Lemma 4.8,

$$\limsup_{\alpha\to\infty} q(\alpha)^{u(\alpha)-1} \text{ is bounded.}$$

Thus, we obtain

$$q(\alpha)^{u(\alpha)-1} - 1 = O\big((1 - u(\alpha)) \log q(\alpha)\big) \quad \text{as } \alpha \to \infty. \tag{4.9}$$

By (4.2) and the Laurent expansions of $\zeta$ at 1 and $\Gamma$ at 0 we have, as $\alpha \to \infty$,

$$(1 - u(\alpha))\alpha \asymp q(\alpha)^{u(\alpha)-1} - 1. \tag{4.10}$$

Therefore, by (4.9),

$$(1 - u(\alpha))\alpha = O\big((1 - u(\alpha)) \log q(\alpha)\big) \quad \text{as } \alpha \to \infty,$$

which yields

$$\alpha = O(\log q(\alpha)) \quad \text{as } \alpha \to \infty.$$

Combining this with Lemma 4.3, we obtain

$$\lim_{\alpha\to\infty} (1 - u(\alpha))\alpha = 0 \tag{4.11}$$

Therefore, the lemma follows from (4.10). ∎

*Proof of Theorem 1.1.* Note that, by Ruelle's formula (see [MU03, Proposition 2.6.13]), Lemma 4.2, (3.3) and (3.1) we have, as $\alpha \to \infty$,

$$\frac{\partial}{\partial b} p(b(\alpha), q(\alpha)) \asymp \mathrm{Li}_1(2b(\alpha), q(\alpha)).$$

Hence, by the mean value theorem,

$$|p(1, q(\alpha)) - p(b(\alpha), q(\alpha))| = O(1 - b(\alpha)) \quad \text{as } \alpha \to \infty. \tag{4.12}$$

By (4.3), (4.8) and Lemma 4.9, as $\alpha \to \infty$,

$$-\log q(\alpha) \asymp \alpha. \tag{4.13}$$

Combining Lemma 4.3 with (4.13) we have, as $\alpha \to \infty$,

$$-\log(1 - b(\alpha)) \asymp \alpha. \tag{4.14}$$

Since $p(b(\alpha), q(\alpha)) = -\alpha q(\alpha)$ by Theorem 4.1, it follows from (4.12), (4.13) and (4.14) that as $\alpha \to \infty$,

$$-\log |p(1, q(\alpha))| \gg \alpha,$$

where $f \gg g$ means that $f/g$ is bounded away from zero. By Lemma 4.7 and Proposition 3.4, as $\alpha \to \infty$,

$$-\log |e^{p(b(\alpha),q(\alpha))}\alpha - \kappa D(u(\alpha), q(\alpha))| \gg \alpha.$$

Since $e^x = 1 + O(x)$ as $x \to 0$, we have $\alpha - e^{p(b(\alpha),q(\alpha))}\alpha = O(q(\alpha)\alpha^2)$ as $\alpha \to \infty$. Hence, as $\alpha \to \infty$,

$$-\log |\alpha - \kappa D(u(\alpha), q(\alpha))| \gg \alpha.$$

By Theorem 4.5 this implies, as $\alpha \to \infty$,

$$-\log |\alpha - \kappa(-\log q(\alpha) - \gamma - 1)| \gg \alpha.$$

By Lemma 4.3 we obtain

$$\lim_{\alpha \to \infty} (1 - b(\alpha)) e^{\kappa^{-1}\alpha} = \lambda(\mu_1)^{-1} \kappa e^{-\gamma - 1}.$$

The proof of Theorem 1.1 is complete. ∎

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