ACTA ARITHMETICA
Online First version

Minimal additive complements for $\{m, h\}$-sets with $h \equiv 1 \pmod m$

by

Wanyan Guo and Min Tang

Abstract. Let C and W be two sets of integers. If $C + W = \mathbb{Z}$, then we say that C is an additive complement to W. If no proper subset of C is an additive complement to W, then we say that C is a minimal additive complement to W. Let m and h be two positive integers with $m \geq 2$, $h \equiv 1 \pmod m$. We show that there exists an infinite, not eventually periodic set $W = \{w_i\}_{i=1}^{\infty}$ such that $w_{i+1} - w_i \in \{m, h\}$ for all i and W admits a minimal additive complement.

1. Introduction. Let $\mathbb{Z}$ be the set of all integers and $\mathbb{N}$ be the set of all nonnegative integers. For $C, W \subseteq \mathbb{Z}$ and integer a, we define

$$C + W = \{c + w : c \in C,\ w \in W\},$$
$$a + W = \{a + w : w \in W\}.$$

For integers a and b, $[a, b]$ is defined as the set of all integers between a and b. For a positive integer k and set A, we define $A \pmod k = \{i \in [0, k-1] : i \equiv a \pmod k, a \in A\}$. If there exists a positive integer T such that $x + T \in X$ for all sufficiently large integers $x \in X$, then we call X *eventually periodic* with period T. The set C is called an *additive complement* to W if $C + W = \mathbb{Z}$ and if no proper subset of C is an additive complement to W, then the set C is called a *minimal additive complement* to W.

This concept of additive complement is already present in the works of Newman [16] and Weinstein [17]. In 2011, Nathanson [15] initiated the study of minimal complements in infinite abelian groups. He proved the following theorem.

Theorem A (see [15, Theorem 8]). *Let W be a nonempty, finite set of integers. In $\mathbb{Z}$, every complement to W contains a minimal complement to W.*

2020 *Mathematics Subject Classification*: Primary 11B13.
Key words and phrases: additive complement, minimal additive complement, eventually periodic set.
Received 20 November 2025; revised 9 June 2026.
Published online 21 July 2026.

DOI: 10.4064/aa251120-28-2

Nathanson [15] also posed the following problem.

PROBLEM 1.1 (see [15, Problem 11]). *Let W be an infinite set of integers. Does there exist a minimal complement to W? Does there exist a complement to W that does not contain a minimal complement?*

In 2012, Chen and Yang [10] showed that if $W \subseteq \mathbb{Z}$ is unbounded both above and below, then W has a minimal complement.

THEOREM B (see [10, Theorem 1]). *Let W be a set of integers with $\inf W = -\infty$ and $\sup W = +\infty$. Then there exists a minimal complement to W.*

Observing that the case of $\inf W$ or $\sup W$ finite can be reduced to $W = \{1 = w_1 < w_2 < \cdots\}$, Chen and Yang [10] showed that removing an extremely sparse set from $\mathbb{N}$ can still yield a set that does not have a minimal additive complement.

THEOREM C (see [10, Theorem 2]). *Let $W = \{1 = w_1 < w_2 < \cdots\}$ be a set of integers and $\overline{W} = (\mathbb{Z} \cap (0, +\infty)) \setminus W = \{\overline{w_1} < \overline{w_2} < \cdots\}$.*

(1) *If $\limsup_{i\to+\infty}(w_{i+1} - w_i) = +\infty$, then there exists a minimal complement to W.*
(2) *If $\lim_{i\to+\infty}(\overline{w_{i+1}} - \overline{w_i}) = +\infty$, then there does not exist a minimal complement to W.*

In 2019, Kiss, Sándor and Yang [13] investigated sets W with

$$\limsup_{i\to+\infty} (w_{i+1} - w_i) < +\infty$$

which have a minimal complement. They studied the existence of minimal complements for "eventually periodic" subsets of $\mathbb{Z}$. A central problem in this topic is to classify infinite sets in groups that admit minimal complements. This topic has attracted widespread attention; for related problems see [1–8, 14, 18]. In 2019, Kiss, Sándor and Yang ingeniously constructed an infinite, non-eventually-periodic set $W \subseteq \mathbb{N}$ with $w_{i+1} - w_i \in \{1, 2\}$ which has a minimal additive complement.

THEOREM D (see [13, Theorem 4]). *There exists an infinite, non-eventually-periodic set $W \subseteq \mathbb{N}$ such that $w_{i+1} - w_i \in \{1, 2\}$ for all i and there exists a minimal complement to W.*

Recently, Chen and Tang [9] generalized Theorem D.

THEOREM E (see [9, Theorem 1.3]). *Let c and d be distinct positive integers with $c \mid d$. Then there exists an infinite, non-eventually-periodic set $W = \{w_1, w_2, \ldots\}$ such that $w_{i+1} - w_i \in \{c, d\}$ for all positive integers i and there exists a minimal complement to W.*

Inspired by the work of Kiss, Sándor and Yang (see [13, Theorem 4]) and Nathanson (see [15, Theorem 8]), we attempt to study whether there exists an infinite, non-eventually-periodic set $W=\{w_i\}_{i=1}^{\infty}$ such that $w_{i+1}-w_i\in\{2,3\}$ for all i for which there exists a minimal complement to W. In [11], our idea is to first use Kiss, Sándor and Yang's method to construct a set $V=\{v_i\}_{i=1}^{\infty}$ such that $v_{i+1}-v_i\in\{1,2\}$ for all i and there exists a minimal complement C to V, and then choose a set $W\subset V$ that satisfies $w_{i+1}-w_i\in\{2,3\}$ for all i. We find that if $|\mathbb{Z}\setminus(C+W)|<+\infty$, then we can use Nathanson's method to construct a minimal complement to W based on C.

Theorem F (see [11]). *Let $\{b_i\}_{i=1}^{\infty}$, $\{V_i\}_{i=1}^{\infty}$ and $C=\{c_i\}_{i=1}^{\infty}$ be such that*

(i) $b_1=-1$, $V_1=\{1,\dots,12\}$, $c_1=-6$*;*
(ii) *b_i is the largest negative integer $\notin V_{i-1}+\{c_1,\dots,c_{i-1}\}$ for $i\ge 2$;*
(iii) *$c_i<b_i+2c_{i-1}-3$ for all $i\ge 2$;*
(iv) *for $i\ge 2$,*

$$V_i=V_{i-1}\cup\Big([-2c_{i-1},-2c_i-1]\setminus\bigcup_{j=1}^{i-1}\{-c_i+b_j\}\Big).$$

There exists a non-eventually-periodic set $W=\{w_i\}_{i=1}^{\infty}$ satisfying $b_i-c_i\in W$ and $w_{i+1}-w_i\in\{2,3\}$ for all i. In particular, if $|\mathbb{Z}\setminus(C+W)|<+\infty$, then there exists a minimal complement to W.

The construction of Theorem F does not necessarily yield a pair (C,W) satisfying $|\mathbb{Z}\setminus(C+W)|<+\infty$. Let $m\ge 2$ be a positive integer. In this paper, to overcome this limitation, we directly construct an infinite, non-eventually-periodic set $W=\{w_i\}_{i=1}^{\infty}$ with $w_{i+1}-w_i\in\{m,m+1\}$ which has a minimal complement.

Theorem 1.2. *Let m and h be positive integers with $m\ge 2$ and $h\equiv 1\ (\mathrm{mod}\ m)$. There exists an infinite, non-eventually-periodic set $W=\{w_i\}_{i=1}^{\infty}\subseteq\mathbb{N}$ such that $w_{i+1}-w_i\in\{m,h\}$ for all i and there exists a minimal complement to W.*

We pose the following problem.

Problem 1.3. *Let m and h be positive integers with $m\ge 2$ and $h\equiv 1\ (\mathrm{mod}\ m)$. Does there exist an infinite, non-eventually-periodic set $W=\{w_i\}_{i=1}^{\infty}\subseteq\mathbb{N}$ such that $w_{i+1}-w_i\in\{m,h\}$ for all i and there does not exist any minimal complement to W?*

In light of Theorem 1.2, it is natural to ask: Let $m\ge 2$ and $1\le r\le m-1$ be positive integers with $(r,m)=1$. Does there exist an infinite, not eventually periodic set $W\subseteq\mathbb{N}$ such that $w_{i+1}-w_i\in\{m,m+r\}$ for all i?

Inspired by the construction framework presented in this paper, He and Tang [12] introduced two new parameters to control the construction of W and to find a uniqueness witness for each element in C. They dealt with the case where $m \geq 3$ and $2 \leq r \leq m-1$. The following result was formally published prior to the present paper.

THEOREM G (see [12, Theorem 1.1]). *Let $m \geq 3$ and $2 \leq r \leq m-1$ be positive integers with $(r, m) = 1$. There exists an infinite, non-eventually-periodic set $W \subseteq \mathbb{N}$ such that $w_{i+1} - w_i \in \{m, m+r\}$ for all i and there exists a minimal complement to W.*

2. Proof of Theorem 1.2. By Theorem E, it is sufficient to consider $h > 1$. Write $h = km+1$ with $k \geq 1$. We can construct $\{d_i\}_{i=1}^{\infty}, \{W_i\}_{i=1}^{\infty}$ and $\{c_i^{(1)}, c_i^{(2)}, \ldots, c_i^{(m)}\}_{i=1}^{\infty}$ as follows:

(i) $d_1 = -1$, $c_1^{(j)} = -km^3 - j$, $j = 1, \ldots, m$,
$$W_1 = \{mi : i = 0, \ldots, 2km^2 + 2\};$$
(ii) d_i is the largest negative integer $\notin W_{i-1} + \{c_1^{(1)}, \ldots, c_1^{(m)}, \ldots, c_{i-1}^{(1)}, \ldots, c_{i-1}^{(m)}\}$ for $i \geq 2$;
(iii) for all $i \geq 2$, choose $c_i^{(j)}$ ($j = 1, \ldots, m$) such that
$$c_i^{(1)} \leq \min\{d_i + 2c_{i-1}^{(m)} - km^3 - 2km^2 - m + 1, \\ 3c_{i-1}^{(1)} - (km+1)(2m-1) - 2(m-1)\}$$
and $c_i^{(j)} = c_i^{(1)} - j + 1$;
(iv) for each $i \geq 2$, we shall consider the following steps to construct the elements of W which fall in each interval $(-2c_{i-1}^{(m)}, -2c_i^{(m)}]$.

STEP 1. Construct the elements of W which lie in $(-2c_{i-1}^{(m)}, d_i - c_i^{(1)}]$.

Let s_i be the least nonnegative residue of $d_i - c_i^{(1)} + 2c_{i-1}^{(m)}$ modulo m.

By (iii), we have $d_i - c_i^{(1)} + 2c_{i-1}^{(m)} - m(km+1) \geq km^3 + km^2 - 1$, thus
$$\left\lfloor \frac{\Delta_i}{m} \right\rfloor - ks_i \geq km^2, \tag{2.1}$$
where $\Delta_i = d_i - c_i^{(1)} + 2c_{i-1}^{(m)} - m(km+1)$.

If $s_i = 0$, then choose
$$V_{i-1}^{(1)} = \{-2c_{i-1}^{(m)} + (km+1)l : l = 1, \ldots, m\} \\ \cup \left\{-2c_{i-1}^{(m)} + m(km+1) + mj : j = 1, \ldots, \left\lfloor \frac{\Delta_i}{m} \right\rfloor \right\}. \tag{2.2}$$

If $1 \le s_i \le m-1$, then choose

$$V_{i-1}^{(1)} = \{-2c_{i-1}^{(m)} + (km+1)l : l = 1,\dots,m+s_i\} \cup \left\{-2c_{i-1}^{(m)} + m(km+1)(m+s_i) + mj : j = 1,\dots,\left\lfloor \frac{\Delta_i}{m} \right\rfloor - ks_i\right\}. \tag{2.3}$$

By the construction of $V_{i-1}^{(1)}$ in (2.2) and (2.3), we know that

$$\max V_{i-1}^{(1)} = d_i - c_i^{(1)}, \qquad i = 2,3,\dots. \tag{2.4}$$

Combininig this with (2.1)–(2.3), we know that for all $i \ge 2$,

$$d_i - c_i^{(1)} - ml \in V_{i-1}^{(1)}, \qquad l = 1,\dots,km^2. \tag{iv-a}$$

For $j = 1,\dots,m-1$, we have

$$d_i - (km+1)j - c_i^{(j+1)} = d_i - kmj - c_i^{(1)} \in V_{i-1}^{(1)},$$

thus, for any $1 \le s \le m$ and $s \ne j+1$, we have

$$d_i - (km+1)j - c_i^{(s)} \notin V_{i-1}^{(1)}. \tag{iv-b}$$

Noting that

$$W_1 + \{c_1^{(1)},\dots,c_1^{(m)}\} = [-km^3 - m, km^3 + 2m - 1],$$

we have $d_2 = -km^3 - m - 1$, thus $d_2 - d_1 = -km^3 - m$.

For $l = 0,\dots,km^2$, we have

$$\begin{aligned} d_i - ml &= (d_i - c_i^{(1)} - ml) + c_i^{(1)}, \\ d_i - (ml+1) &= (d_i - c_i^{(1)} - ml) + c_i^{(2)}, \\ &\cdots \\ d_i - (ml+m-1) &= (d_i - c_i^{(1)} - ml) + c_i^{(m)}. \end{aligned} \tag{2.5}$$

By (2.4), (2.5) and (iv-a), we have

$$\{d_i - ml,\dots,d_i - (ml+m-1)\} \subseteq V_{i-1}^{(1)} + \{c_i^{(1)},\dots,c_i^{(m)}\}.$$

By (ii) we know that $d_{i+1} < d_i - (km^3 + m - 1)$. That is, we have

$$d_{i+1} - d_i \le -km^3 - m.$$

STEP 2. Construct the elements which lie in $(d_i - c_i^{(1)}, d_1 - c_i^{(1)} - 1]$.

To construct all these elements, we first construct the elements of $V_{i-1,0}^{(2)}$ which lie in this interval. Let t_i be the least nonnegative residue of $d_{i-1} - d_i$ modulo m.

If $t_i = 0$, then choose

$$V^{(2)}_{i-1,0} = \{d_{i-1} - c^{(1)}_i - 1 - (km+1)l : l = 0, \dots, m-1\} \cup \left\{d_i - c^{(1)}_i + mj : j = 1, \dots, \left\lfloor \frac{d_{i-1} - d_i}{m} \right\rfloor - km + (k-1)\right\}. \tag{2.6}$$

If $1 \le t_i \le m-1$, then choose

$$\begin{aligned} V^{(2)}_{i-1,0} = {} & \{d_{i-1} - c^{(1)}_i - 1 - (km+1)l : l = 0, \dots, m-1\} \\ & \cup \{d_i - c^{(1)}_i + (km+1)l : l = 1, \dots, t_i\} \\ & \cup \Big\{d_i - c^{(1)}_i + (km+1)t_i + mj : \\ & \qquad j = 1, \dots, \left\lfloor \frac{d_{i-1} - d_i}{m} \right\rfloor - kt_i - km + (k-1)\Big\}. \end{aligned} \tag{2.7}$$

By the construction of $V^{(2)}_{i-1,0}$ in (2.6) and (2.7), we know that the distance between the first element of $V^{(2)}_{i-1,0}$ and $d_i - c^{(1)}_i$ is either m or $km+1$. Thus, for any $2 \le j \le m$,

$$d_i - c^{(j)}_i = d_i - c^{(1)}_i + j - 1 \notin V^{(2)}_{i-1,0}. \tag{iv-c}$$

Note that the distance of the adjacent elements in

$$V^{(2)}_{i-1,0} \cap [d_{i-1} - c^{(1)}_i - km^2 + (k-1)m, d_{i-1} - c^{(1)}_i - 1]$$

is $km+1$. Thus, for any $1 \le j \le m-1$,

$$d_{i-1} - c^{(s)}_i - (km+1)j = d_{i-1} - c^{(1)}_i - 1 - (km+1)j + s \notin V^{(2)}_{i-1,0}, \tag{iv-d}$$

where $s = 1, \dots, m$.

Noting that

$$[d_{i-1} - c^{(1)}_i, d_1 - c^{(1)}_i - 1] = \bigcup_{u=1}^{i-2} [d_{i-u} - c^{(1)}_i, d_{i-u-1} - c^{(1)}_i - 1],$$

if $i - 1 \ge 2$, then we will proceed to choose all the elements of $V^{(2)}_{i-1,u}$ which fall in $[d_{i-u} - c^{(1)}_i, d_{i-u-1} - c^{(1)}_i - 1]$ for all $u = 1, \dots, i-2$.

Let $p_{i,u}$ be the least nonnegative residue of $d_{i-u-1} - d_{i-u}$ modulo m.

If $p_{i,u} = 0$, then choose

$$\begin{aligned} V^{(2)}_{i-1,u} = {} & \{d_{i-u-1} - c^{(1)}_i - 1 - (km+1)l : l = 0, \dots, m-1\} \\ & \cup \left\{d_{i-u} - c^{(1)}_i + km + mj : j = 0, \dots, \left\lfloor \frac{d_{i-u-1} - d_{i-u}}{m} \right\rfloor - km - 1\right\}. \end{aligned} \tag{2.8}$$

If $1 \le p_{i,u} \le m-1$, then choose

$$V^{(2)}_{i-1,u} = \{d_{i-u-1} - c_i^{(1)} - 1 - (km+1)l : l = 0,\dots,m-1\} \cup \{d_{i-u} - c_i^{(1)} + km + (km+1)l : l = 0,\dots,p_{i,u}\} \cup \Big\{d_{i-u} - c_i^{(1)} + km + (km+1)p_{i,u} + mj : j = 1,\dots,\Big\lfloor \frac{d_{i-u-1}-d_{i-u}}{m} \Big\rfloor - kp_{i,u} - km - 1\Big\}. \tag{2.9}$$

By (2.8) and (2.9), we know that the first element in $V^{(2)}_{i-1,u}$ is $d_{i-u} - c_i^{(1)} + km$, thus

$$d_{i-u} - c_i^{(j)} \notin V^{(2)}_{i-1,u}, \quad j = 1,\dots,m. \tag{iv-e}$$

Note that the distance of consecutive elements in

$$V^{(2)}_{i-1,u} \cap \big[d_{i-u-1} - c_i^{(1)} - km^2 + (k-1)m, d_{i-u-1} - c_i^{(1)} - 1\big]$$

is $km+1$. Thus, fixing an integer $1 \le j \le m-1$, for all $s = 1,\dots,m$, we have

$$d_{i-u-1} - c_i^{(s)} - (km+1)j = d_{i-u-1} - c_i^{(1)} - 1 - (km+1)j + s \notin V^{(2)}_{i-1,u}. \tag{iv-f}$$

Write

$$V^{(2)}_{i-1} = \bigcup_{u=0}^{i-2} V^{(2)}_{i-1,u}.$$

Then

$$\max V^{(2)}_{i-1} = d_1 - c_i^{(1)} - 1, \quad i = 2,3,\dots.$$

Step 3. Construct the elements of $V^{(3)}_{i-1}$ which lie in $\big(d_1 - c_i^{(1)} - 1, -2c_i^{(m)}\big]$. Let q_i be the least nonnegative residue of $-2c_i^{(m)} - d_1 + c_i^{(1)}$ modulo m. If $q_i = 0$, then choose

$$V^{(3)}_{i-1} = \Big\{d_1 - c_i^{(1)} + km + mj : j = 0,\dots,\Big\lfloor \frac{-2c_i^{(m)} - d_1 + c_i^{(1)}}{m} \Big\rfloor - k\Big\}.$$

If $1 \le q_i \le m-1$, then choose

$$V^{(3)}_{i-1} = \{d_1 - c_i^{(1)} + km + (km+1)l : l = 0,\dots,q_i\} \cup \Big\{d_1 - c_i^{(1)} + km + (km+1)q_i + mj : j = 1,\dots,\Big\lfloor \frac{-2c_i^{(m)} - d_1 + c_i^{(1)}}{m} \Big\rfloor - kq_i - k\Big\}.$$

Then

$$\max V^{(3)}_{i-1} = -2c_i^{(m)}, \quad i = 2,3,\dots.$$

By the construction of $V_{i-1}^{(3)}$, the first element in $V_{i-1}^{(3)}$ is $d_1 - c_i^{(1)} + km$, and thus

$$\text{(iv-g)} \qquad d_1 - c_i^{(j)} \notin V_{i-1}^{(3)}, \qquad j = 1, \dots, m.$$

For $i = 2, 3, \dots$, let

$$W_i = W_{i-1} \cup V_{i-1}^{(1)} \cup V_{i-1}^{(2)} \cup V_{i-1}^{(3)}. \tag{2.10}$$

Write

$$W = \bigcup_{i=1}^{\infty} W_i, \qquad C = \bigcup_{i=1}^{\infty} \{c_i^{(1)}, \dots, c_i^{(m)}\}.$$

Since condition (iii) provides only an upper bound on $c_i^{(1)}$, we may choose $c_i^{(1)} \to -\infty$ as $i \to \infty$. Consequently, $-2c_i^{(m)} \to +\infty$. Thus, each newly constructed block of W lies strictly to the right of the previous blocks, and all sufficiently large elements of W are positive integers. Therefore, $W \subset \mathbb{N}$. Moreover, consecutive elements of W satisfy

$$w_{i+1} - w_i \in \{m, h\}, \qquad h = km + 1. \tag{2.11}$$

In fact, the initial set W_1 is an arithmetic progression with common difference m. So, (2.11) holds trivially. During the inductive construction of W, every new element added maintains the property that its difference from the previously added element (which is either within the same block or the last element of the previous block) is either m or $km + 1$. By induction, (2.11) holds for the entire infinite set W.

Since $d_1 = -1$ and $d_{i+1} - d_i \le -km^3 - m$ for all $i \ge 1$, it follows that $d_k \to -\infty$. By (ii) we have $(-\infty, km^3 + 2m - 1] \subseteq W + C$. For any integer $n \ge km^3 + 2m$, there exists an $i \ge 2$ such that $-c_{i-1}^{(m)} \le n < -c_i^{(m)}$. Hence

$$-c_i^{(1)} + d_1 < -c_{i-1}^{(m)} - c_i^{(1)} \le n - c_i^{(1)} < \cdots < n - c_i^{(m)} < -2c_i^{(m)}.$$

Since

$$-c_{i-1}^{(m)} - c_i^{(1)} - (d_1 - c_i^{(1)} + km) = -c_{i-1}^{(m)} - d_1 - km \ge km^3 + m + 1 - km,$$

we have

$$-c_{i-1}^{(m)} - c_i^{(1)} \ge d_1 - c_i^{(1)} + km^3 + km + 1 > d_1 - c_i^{(1)} + km^2 + m - 1.$$

Hence,

$$n - c_i^{(1)}, \dots, n - c_i^{(m)} \in \left[d_1 - c_i^{(1)} + km^2 + m - 1, -2c_i^{(m)}\right].$$

Write

$$I_i = [d_1 - c_i^{(1)} + km^2 + m - 1, -2c_i^{(m)}].$$

Within this interval, all elements of the set $V_{i-1}^{(3)}$ lie in the same fixed residue class modulo m. In fact, the elements of $V_{i-1}^{(3)}$ form arithmetic progressions with common difference m after the initial $km+1$ shifts. Hence, in the interval I_i the elements of $V_{i-1}^{(3)}$ appear periodically with step m.

Therefore, exactly one of $n-c_i^{(1)},\dots,n-c_i^{(m)}$ is in $V_{i-1}^{(3)}$. Hence,

$$n \in W_i + \{c_i^{(1)},\dots,c_i^{(m)}\}.$$

So, $W+C=\mathbb{Z}$.

Next, we prove that the complement C is minimal. For $i=2,3,\dots$, write

$$C_{i-1}=\{c_1^{(1)},c_1^{(2)},\dots,c_1^{(m)},\dots,c_{i-1}^{(1)},c_{i-1}^{(2)},\dots,c_{i-1}^{(m)}\}.$$

First, for $i=2,3,\dots$, we have the following fact:

$$d_i-(km+1)(s-1)\notin W_{i-1}+C_{i-1}. \tag{iv-h}$$

Indeed, by (i) we have

$$W_1+C_1=[-km^3-m,\ km^3+2m-1],\qquad d_2=-km^3-m-1.$$

Thus, for all $s=2,\dots,m$, we have $d_2-(km+1)(s-1)\notin W_1+C_1$.

For $i=2,3,\dots$, by (2.10) we have

$$\begin{aligned} W_i+C_i &= (W_{i-1}\cup V_{i-1})+(C_{i-1}\cup\{c_i^{(1)},\dots,c_i^{(m)}\}) \\ &= (W_{i-1}+C_{i-1})\cup(V_{i-1}+C_{i-1})\cup(W_{i-1}+\{c_i^{(1)},\dots,c_i^{(m)}\}) \\ &\quad \cup(V_{i-1}^{(1)}+\{c_i^{(1)},\dots,c_i^{(m)}\})\cup(V_{i-1}^{(2)}+\{c_i^{(1)},\dots,c_i^{(m)}\}) \\ &\quad \cup(V_{i-1}^{(3)}+\{c_i^{(1)},\dots,c_i^{(m)}\}), \end{aligned} \tag{2.12}$$

where $V_{i-1}=V_{i-1}^{(1)}\cup V_{i-1}^{(2)}\cup V_{i-1}^{(3)}$.

Combining this with the construction in Step 1, write

$$w^{(i-1)}=-2c_{i-1}^{(m)}+(km+1)(m+s_i).$$

If $s_i=0$, then

$$\begin{aligned} V_{i-1}^{(1)} &= \{w^{(i-1)}-j(km+1): j=0,\dots,m-1\} \\ &\quad \cup\{w^{(i-1)}+mj: j=1,\dots,\lfloor \varDelta_i/m\rfloor\}. \end{aligned}$$

Thus,

$$\begin{aligned} &V_{i-1}^{(1)}+\{c_i^{(1)},\dots,c_i^{(m)}\} \\ &= \bigcup_{\lambda=1}^{m-1}[w^{(i-1)}-\lambda(km+1)+c_i^{(m)},w^{(i-1)}-\lambda(km+1)+c_i^{(1)}]\cup[w^{(i-1)}+c_i^{(m)},d_i]. \end{aligned} \tag{2.13}$$

If $1\le s_i\le m-1$, then

$$\begin{aligned} V_{i-1}^{(1)} &= \{w^{(i-1)}-j(km+1): j=0,\dots,m+s_i-1\} \\ &\quad \cup\{w^{(i-1)}+mj: j=1,\dots,\lfloor \varDelta_i/m\rfloor-ks_i\}. \end{aligned}$$

Thus,

$$V_{i-1}^{(1)} + \{c_i^{(1)}, \dots, c_i^{(m)}\} \tag{2.14}$$
$$= \bigcup_{\lambda=1}^{m+s_i-1} [w^{(i-1)} - \lambda(km+1) + c_i^{(m)}, w^{(i-1)} - \lambda(km+1) + c_i^{(1)}] \cup [w^{(i-1)} + c_i^{(m)}, d_i].$$

Noting that $\min(V_{i-1} + C_{i-1}) > -2c_{i-1}^{(m)} + c_{i-1}^{(m)} > 0$ and by (iii) we have

$$\begin{aligned} w^{(i-1)} + c_i^{(m)} &= c_i^{(m)} - 2c_{i-1}^{(m)} + (km+1)(m+s_i) \\ &\le c_i^{(1)} - (m-1) - 2(c_{i-1}^{(1)} - (m-1)) + (km+1)(2m-1) \\ &\le c_{i-1}^{(m)} = \min(W_{i-1} + C_{i-1}). \end{aligned}$$

Combining this with (2.12), we obtain

$$d_{i+1} = w^{(i-1)} + c_i^{(m)} - 1.$$

Noting that

$$\begin{aligned} \max(W_{i-1} + \{c_i^{(1)}, \dots, c_i^{(m)}\}) &= -2c_{i-1}^{(m)} + c_i^{(1)} \\ &< -2c_{i-1}^{(m)} + km + 1 + c_i^{(m)} \\ &= \min(V_{i-1}^{(1)} + \{c_i^{(1)}, \dots, c_i^{(m)}\}), \end{aligned}$$

together with (2.13) and (2.14), for any $s = 2, \dots, m$ we have

$$d_{i+1} - (km+1)(s-1) \notin W_i + C_i.$$

Second, we shall show every $c_i^{(1)}$ is necessary. For any positive integer i, write $d_i = c + w$ with $c \in C$ and $w \in W$. We shall prove that $c = c_i^{(1)}$.

By (i), we have $d_1 - c_1^{(1)} = km^3 \in W_1$ and

$$d_1 - c_1^{(2)}, \dots, d_1 - c_1^{(m)} \notin W_1.$$

By (2.4), we have

$$d_i - c_i^{(1)} \in W \quad \text{for all } i \ge 2. \tag{2.15}$$

Moreover, by (iv-c),

$$d_i - c_i^{(2)}, \dots, d_i - c_i^{(m)} \notin W. \tag{2.16}$$

Fix an $i \ge 2$. By (iv-e) we know that for any $j \ge i+1$,

$$d_i - c_j^{(1)}, d_i - c_j^{(2)}, \dots, d_i - c_j^{(m)} \notin V_{j-1}^{(2)}.$$

By (iv-g), for all $j \ge 2$,

$$d_1 - c_j^{(1)}, d_1 - c_j^{(2)}, \dots, d_1 - c_j^{(m)} \notin V_{j-1}^{(3)}.$$

Hence, for any fixed $i \geq 1$, if $j \geq i+1$, then

$$d_i - c_j^{(1)}, d_i - c_j^{(2)}, \ldots, d_i - c_j^{(m)} \notin W. \tag{2.17}$$

That is, if $d_i = c + w$, then $c \neq c_j^{(1)}, \ldots, c_j^{(m)}$ for all positive integers $j > i$.

By (iii) and the construction of $V_{i-1}^{(1)}$ in Step 1, we have

$$\begin{aligned} \min(W \setminus W_{i-1}) &\geq -2c_{i-1}^{(m)} + km + 1 = -2(c_{i-1}^{(1)} - m + 1) + km + 1 \\ &= -2c_{i-1}^{(1)} + (k+2)m - 1, \end{aligned}$$

thus for all $j \leq i-1$, we have

$$d_i - c_j^{(1)} \leq d_i - c_{i-1}^{(1)} < \cdots < d_i - c_{i-1}^{(m)} < -2c_{i-1}^{(1)};$$

it follows that for all positive integers $j \leq i-1$,

$$d_i - c_j^{(1)}, \ldots, d_i - c_j^{(m)} \notin W \setminus W_{i-1}.$$

By (ii), we know that

$$d_i \notin W_{i-1} + \{c_1^{(1)}, \ldots, c_1^{(m)}, \ldots, c_{i-1}^{(1)}, \ldots, c_{i-1}^{(m)}\}.$$

Hence,

$$d_i \notin W + \{c_1^{(1)}, \ldots, c_1^{(m)}, \ldots, c_{i-1}^{(1)}, \ldots, c_{i-1}^{(m)}\}. \tag{2.18}$$

So, by (2.15)–(2.18), if $d_i = c + w$, then $c = c_i^{(1)}$.

Third, for all $s = 2, \ldots, m$, we shall show that every $c_i^{(s)}$ is necessary.

For any positive integer i, write $d_i - (km+1)(s-1) = c + w$ with $c \in C$ and $w \in W$. We shall prove that $c = c_i^{(s)}$.

By (i), we have

$$d_1 - (km+1)(s-1) - c_1^{(s)} = km^3 - k(s-1)m \in W_1$$

and $d_1 - (km+1)(s-1) - c_1^{(j)} \notin W_1$ for all $j \neq s$. By (iv-a) we have

$$d_i - (km+1)(s-1) - c_i^{(s)} = d_i - c_i^{(1)} - km(s-1) \in W \quad \text{for all } i \geq 2.$$

Moreover, by (iv-b), we find that for all $i \geq 2$,

$$d_i - (km+1)(s-1) - c_i^{(j)} \notin W \quad \text{for all } j \neq s.$$

For all $i \geq 1$, by (iv-d) we know that

$$d_i - c_{i+1}^{(j)} - (km+1)(s-1) \notin W \quad \text{for all } j = 1, \ldots, m.$$

For any $i \geq 1$, by (iv-f) for all $j \geq i+2$,

$$d_i - c_j^{(\lambda)} - (km+1)(s-1) \notin W, \quad \lambda = 1, \ldots, m.$$

Hence, for all positive integers $j > i$ and $\lambda = 1, \ldots, m$, we have

$$d_i - c_j^{(\lambda)} - (km+1)(s-1) \notin W.$$

That is, if

$$d_i - (km+1)(s-1) = c + w,$$

then

$$c \neq c_j^{(1)}, c_j^{(2)}, \dots, c_j^{(m)}$$

for all positive integers $j > i$.

For all $j \leq i-1$, we have

$$\begin{aligned} d_i - c_j^{(1)} - (km+1)(s-1) &\leq d_i - c_{i-1}^{(1)} - (km+1)(s-1) \\ &< d_i - c_{i-1}^{(2)} - (km+1)(s-1) \\ &< \cdots \\ &< d_i - c_{i-1}^{(m)} - (km+1)(s-1) \\ &< -2c_{i-1}^{(1)}. \end{aligned}$$

It follows that for all positive integers $j \leq i-1$ and $\lambda = 1, \dots, m$ we have

$$d_i - c_j^{(\lambda)} - (km+1)(s-1) \notin W \setminus W_{i-1}.$$

Moreover, by (iv-h) we have

$$d_i - (km+1)(s-1) \notin W_{i-1} + \{c_1^{(1)}, \dots, c_1^{(m)}, \dots, c_{i-1}^{(1)}, \dots, c_{i-1}^{(m)}\}.$$

Hence,

$$d_i - (km+1)(s-1) \notin W + \{c_1^{(1)}, \dots, c_1^{(m)}, \dots, c_{i-1}^{(1)}, \dots, c_{i-1}^{(m)}\}.$$

So, for any $s = 2, \dots, m$, if $d_i - (km+1)(s-1) = c + w$, then $c = c_i^{(s)}$.

Therefore, C is a minimal complement to W.

For $i = 1, 2, \dots,$ write $R_i = -2c_i^{(m)}$. Since condition (iii) grants complete freedom to choose $c_i^{(1)}$ arbitrarily negative, we can choose $c_i^{(1)}$ such that

$$\lim_{i\to\infty} R_i = +\infty, \qquad \lim_{i\to\infty} (R_{i+1} - R_i) = +\infty.$$

Suppose that W is eventually periodic with period T and starting point N. We can choose an index j such that $R_j + m(km+1) > N$ and $R_j + m(km+1) + T < R_{j+1}$. By the construction in Step 1, we can choose suitable $c_{j+1}^{(1)}$ to ensure $R_j + m(km+1) + T \notin W$ by appropriately controlling the value of s_{j+1}. In all, W is infinite and not eventually periodic.

This completes the proof of Theorem 1.2.

Acknowledgements. We are grateful to the anonymous referee of the first version of this paper for the valuable suggestions concerning the references. We also sincerely thank the reviewer of the revised version for insightful comments that helped improve the clarity and make the proofs more explicit. We thank Dr. Shiqiang Chen and Ms. Wenjing He for their careful examination of this paper. We are grateful for the dedicated guidance provided by the editors during the six rounds of revisions, which significantly improved the quality of our manuscript.

Funding. This work is supported by the National Natural Science Foundation of China (Grant No. 12371003).

References

[1] N. Alon, N. Kravitz and M. Larson, *Inverse problems for minimal complements and maximal supplements*, J. Number Theory 223 (2021), 307–324.
[2] A. Biswas and J. P. Saha, *Infinite co-minimal pairs in the integers and integral lattices*, in: Combinatorial and Additive Number Theory IV, Springer Proc. Math. Statist., 347, Springer, Cham, 2021, 41–71.
[3] A. Biswas and J. P. Saha, *Infinite co-minimal pairs involving lacunary sequences and generalisations to higher dimensions*, Ramanujan J. 57 (2022), 1445–1462.
[4] A. Biswas and J. P. Saha, *Minimal additive complements in finitely generated abelian groups*, Ramanujan J. 57 (2022), 215–238.
[5] A. Biswas and J. P. Saha, *On additive co-minimal pairs*, J. Number Theory 223 (2021), 350–370.
[6] A. Biswas and J. P. Saha, *On minimal complements in groups*, Ramanujan J. 55 (2021), 823–847.
[7] A. Biswas and J. P. Saha, *On non-minimal complements*, Adv. Appl. Math. 130 (2021), art. 102226, 25 pp.
[8] A. Burcroff and N. Luntzlara, *Sets arising as minimal additive complements in the integers*, J. Number Theory 247 (2023), 15–34.
[9] S. Q. Chen and M. Tang, *On minimal additive complements*, Ramanujan J. 69 (2026), art. 16, pp.
[10] Y. G. Chen and Q. H. Yang, *On a problem of Nathanson related to minimal additive complements*, SIAM J. Discrete Math. 26 (2012), 1532–1536.
[11] W. Y. Guo and M. Tang, *On minimal additive complements*, Bull. Austral. Math. Soc. 112 (2025), 410–417.
[12] W. J. He and M. Tang, *Minimal additive complements of not eventually periodic sets with bounded gaps*, Discrete Math. 349 (2026), art. 115121, 9 pp.
[13] S. Z. Kiss, C. Sándor and Q. H. Yang, *On minimal additive complements of integers*, J. Combin. Theory Ser. A 162 (2019), 344–353.
[14] A. Kwon, *A note on minimal additive complements of integers*, Discrete Math. 342 (2019), 1912–1918.
[15] M. B. Nathanson, *Problems in additive number theory, IV: nets in groups and shortest length g-adic representations*, Int. J. Number Theory 7 (2011), 1999–2017.
[16] D. J. Newman, *Complements of finite sets of integers*, Michigan Math. J. 14 (1967), 481–486.
[17] G. Weinstein, *Minimal complementary sets*, Trans. Amer. Math. Soc. 212 (1975), 131–137.
[18] F. Zhou, *On eventually periodic sets as minimal additive complements*, Electron. J. Combin. 30 (2023), art. 4.34, 40 pp.

Wanyan Guo, Min Tang (corresponding author)
School of Mathematics and Statistics
Anhui Normal University
Wuhu 241002, P. R. China
E-mail: gwy202411@163.com
tmzzz2000@163.com