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CONSTRUCTIONS OF ROTA–BAXTER OPERATORS BY L-R SMASH PRODUCTS

BY

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Abstract. Let A and H be two cocommutative Hopf algebras such that A is an H-bimodule Hopf algebra. Suppose that $R : A \to A$ is a linear map and B is a Rota–Baxter operator of H. We characterize the Rota–Baxter operators on the L-R smash product $A \natural H$ and give necessary and sufficient conditions for $\overline{B}(a \natural h) = B(h_1) \triangleright R(a) \triangleleft B(h_2) \natural B(h_3)$ to be a Rota–Baxter operator of $A \natural H$ for $a \in A$ and $h \in H$. Then we consider the dual case, and construct a Rota–Baxter co-operator on the L-R smash coproduct $C \ltimes H$, where C and H are commutative Hopf algebras and C is an H-bicomodule Hopf algebra.

Introduction. Rota–Baxter operators for associative algebras were first introduced by G. Baxter [4] as a tool for studying integral operators in the theory of probability and mathematical statistics. An algebra endowed with a Rota–Baxter operator is called a Rota–Baxter algebra. These algebras relate to many topics in mathematical physics. For example, Rota–Baxter operators on associative algebras satisfy the famous (operator form of) classical Yang–Baxter equation on the Lie algebra which is the commutator of the associative algebra [1–3]. Rota–Baxter algebras appeared in connection with the work of Connes and Kreimer on renormalization theory in perturbative quantum field theory [11, 13]. They are also related to Loday's dendriform algebras [16, 17].

The notion of Rota–Baxter operators on Lie algebras was introduced independently by A. A. Belavin and V. G. Drinfeld [5] and M. A. Semenov-Tyan-Shanskii [10] in their research of the solutions of the classical Yang–Baxter equation. Let $(L, [\,,])$ be a Lie algebra. A *Rota–Baxter operator* of weight 1 on L is a linear map $B : L \to L$ such that for all $x, y \in L$,

$$[B(x), B(y)] = B([x, B(y)] + [B(x), y] + [x, y]).$$

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It turns out that on quadratic Lie algebras skew-symmetric solutions of the classical Yang–Baxter equation are in one-to-one correspondence with skew-symmetric Rota–Baxter operators.

Recently the notion of Rota–Baxter operators on groups was introduced in [15]. Let G be a group, a map $B : G \to G$ is called a Rota–Baxter operator of G if for all $g, h \in G$,

$$B(g)B(h) = B(gB(g)hB(g)^{-1}).$$

A group G with a Rota–Baxter operator B is called a *Rota–Baxter group*. It was pointed out in [15] that if (G, B) is a Rota–Baxter Lie group, then the tangent map of B at the identity is a Rota–Baxter operator of weight 1 on the Lie algebra of G. Later in [14] the notion of Rota–Baxter group was naturally generalized to the notion of *Rota–Baxter Hopf algebra*, which is a cocommutative Hopf algebra H with a coalgebra map $B : H \to H$ satisfying for all $g, h \in H$,

$$B(g)B(h) = B(g_1 B(g_2)hS(B(g_3))).$$

Smash products and crossed-type products are among the most fundamental ways to build larger algebras from a Hopf algebra action. Given a Hopf algebra H acting on an algebra A, the (left) smash product $A \# H$ embraces the algebra structure of A together with the action of H in a single associative algebra. Such constructions play an indispensable role in quantum algebra, Hopf–Galois theory and noncommutative geometry [19]. Recently Zhu et al. [23] showed how to construct Rota–Baxter operators on cocommutative smash product Hopf algebras, which is inspired by the well known fact that any pointed cocommutative Hopf algebra H over an algebraically closed field F is isomorphic to the smash product $U(L) \# F[G]$, where G is a group and $U(L)$ is the universal enveloping algebra of a Lie algebra L.

The purpose of this paper is to extend these constructions from smash products to the more flexible framework of L-R smash (co)products emerging from deformation quantization [6–9]. The L-R smash product simultaneously involves a left and a right action (dually, left and right coactions), and it encompasses several important crossed constructions as special cases. In particular, it provides a unifying setting that includes diagonal crossed products [18] and, in suitable finite-dimensional commutative situations, recovers twisted smash products [20]. Moreover under explicit compatibility conditions the L-R smash product carries a Hopf algebra structure [21]. Therefore developing Rota–Baxter operators on L-R smash (co)products yields Rota–Baxter structures on a broad family of Hopf algebras that arise from double or crossed constructions and deformation-type considerations.

Beyond the conceptual unification, there are concrete reasons why Rota–Baxter operators on L-R smash (co)products are interesting. First, crossed-type Hopf algebras such as diagonal crossed products are closely tied to double constructions; consequently, Rota–Baxter operators defined on L-R smash products can be transferred to algebras related to quantum doubles and similar factorization structures. Second, Rota–Baxter operators provide a framework for generating richer algebraic structures (splittings, new products, and Yang–Baxter type operators) on the underlying algebras and on associated commutator Lie algebras. Thus explicit Rota–Baxter constructions on L-R smash products provide new, computable examples in which these links can be explored. Third, from a practical viewpoint, L-R smash products often appear when one needs to combine left and right symmetries; having Rota–Baxter operators compatible with both sides can be useful when constructing or comparing operator-theoretic data across related crossed products.

Explicitly, let A and H be two cocommutative Hopf algebras such that $(A, \triangleleft, \triangleright)$ is an H-bimodule Hopf algebra. Assume that $R : A \to A$ is a linear map and B is a Rota–Baxter operator on H. Define a linear map $\overline{B} : A \natural H \to A \natural H$ by

$$\overline{B}(a \natural h) = B(h_1) \triangleright R(a) \triangleleft B(h_2) \natural B(h_3)$$

for all $a \in A$ and $h \in H$. We will give necessary and sufficient conditions for $\overline{B}$ to be a Rota–Baxter operator on $A \natural H$. Finally, assume that C and H are commutative Hopf algebras and C is an H-bicomodule Hopf algebra. We will examine the dual case and characterize the Rota–Baxter co-operator on the L-R smash coproduct Hopf algebras $C \ltimes H$.

The paper is organized as follows. In Section 1 we recall some basic definitions and results on L-R smash product and smash coproduct. In Section 2 we investigate how to construct Rota–Baxter operators on L-R smash product Hopf algebras and give necessary and sufficient conditions for existence of Rota–Baxter operators on L-R smash product. Moreover, to motivate our efforts we give some examples. In Section 3, we will study the dual case and give necessary and sufficient conditions for existence of a Rota–Baxter co-operator on L-R smash coproduct Hopf algebras.

1. Preliminaries. Throughout this paper, k is a fixed field, and all vector spaces and tensor product are over k. For a coalgebra C, we will use Heyneman–Sweedler's notation $\Delta(c) = c_1 \otimes c_2$ for any $c \in C$ (summation omitted).

Let H be a bialgebra and A a bialgebra. Then A is called an *H-bimodule bialgebra* if A is an H-bimodule with the left action $\triangleright : H \otimes A \to A$ and the right action $\triangleleft : A \otimes H \to A$ such that for all $h \in H$ and $a, b \in A$,

$$\left.\begin{aligned} &h \triangleright (ab) = (h_1 \triangleright a)(h_2 \triangleright b),\ h \triangleright 1_A = \varepsilon(h)1_A,\\ &(ab) \triangleleft h = (a \triangleleft h_1)(b \triangleleft h_2),\ 1_A \triangleleft h = \varepsilon(h)1_A, \end{aligned}\right\}\text{bimodule algebra} \tag{1.1}$$

$$\left.\begin{aligned} &(h \triangleright a)_1 \otimes (h \triangleright a)_2 = h_1 \triangleright a_1 \otimes h_2 \triangleright a_2,\\ &(a \triangleleft h)_1 \otimes (a \triangleleft h)_2 = a_1 \triangleleft h_1 \otimes a_2 \triangleleft h_2,\\ &\varepsilon(h \triangleright a) = \varepsilon(a \triangleleft h) = \varepsilon(a)\varepsilon(h). \end{aligned}\right\}\text{bimodule coalgebra} \tag{1.2}$$

Let (H, S_H) be a Hopf algebra and A an H-bimodule algebra. Then we have the L-R smash product $A \natural H$ (which is equal to $A \otimes H$ as a vector space) with the following multiplication:

$$(a \natural h)(b \natural g) = (a \triangleleft g_2)(h_1 \triangleright b) \natural h_2 g_1, \tag{1.3}$$

and the unit $1_A \natural 1_H$ for all $a, b \in A$ and $g, h \in H$. Furthermore, if (A, S_A) is a Hopf algebra and A is an H-bimodule bialgebra, then $A \natural H$ becomes a Hopf algebra with the comultiplication, counit and antipode given by

$$\Delta(a \natural h) = a_1 \natural h_1 \otimes a_2 \natural h_2, \tag{1.4}$$

$$\varepsilon(a \natural h) = \varepsilon_A(a)\varepsilon_H(h), \tag{1.5}$$

$$S(a \natural h) = S_H(h_3) \triangleright S_A(a) \triangleleft S_H(h_2) \natural S_H(h_1), \tag{1.6}$$

under the following conditions:

$$h_1 \triangleright a \otimes h_2 = h_2 \triangleright a \otimes h_1, \tag{1.7}$$

$$a \triangleleft h_1 \otimes h_2 = a \triangleleft h_2 \otimes h_1. \tag{1.8}$$

Note that when H is cocommutative, the identities (1.7) and (1.8) naturally hold.

We call A an *H-bimodule Hopf algebra* if S_A is H-bilinear.

A bialgebra C is called an *H-bicomodule bialgebra* if C is an H-bicomodule with the left coaction $\rho^l : C \to H \otimes C,\ c \mapsto c_{(-1)} \otimes c_{(0)}$, and the right coaction $\rho^r : C \to C \otimes H,\ c \mapsto c_{[0]} \otimes c_{[1]}$, such that for all $c \in C$,

(1.9)
$$\left.\begin{aligned} &(ab)_{(-1)} \otimes (ab)_{(0)} = a_{(-1)}b_{(-1)} \otimes a_{(0)}b_{(0)},\\ &(ab)_{[0]} \otimes (ab)_{[1]} = a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]},\\ &1_{A(-1)} \otimes 1_{A(0)} = 1_H \otimes 1_A,\ 1_{A[0]} \otimes 1_{A[1]} = 1_A \otimes 1_H, \end{aligned}\right\}\text{bicomodule algebra}$$

(1.10)
$$\left.\begin{aligned} &c_{(-1)} \otimes c_{(0)1} \otimes c_{(0)2} = c_{1(-1)}c_{2(-1)} \otimes c_{1(0)} \otimes c_{2(0)},\\ &c_{[0]1} \otimes c_{[0]2} \otimes c_{[1]} = c_{1[0]} \otimes c_{2[0]} \otimes c_{1[1]}c_{2[1]},\\ &\varepsilon(c_{(0)})c_{(-1)} = \varepsilon(c)1_H,\ \varepsilon(c_{[0]})c_{[1]} = \varepsilon(c)1_H. \end{aligned}\right\}\text{bicomodule coalgebra}$$

Assume that a Hopf algebra (C, S_C) is an H-bicomodule bialgebra and denote $C \otimes H$ by $C \ltimes H$ and the elements $c \otimes h$ by $c \ltimes h$. Furthermore, assume that

$$c_{(-1)}h \otimes c_{(0)} = hc_{(-1)} \otimes c_{(0)}, \tag{1.11}$$

$$c_{[0]} \otimes c_{[1]}h = c_{[0]} \otimes hc_{[1]}, \tag{1.12}$$

for all $c \in C$ and $h \in H$. Then $C \ltimes H$ becomes a Hopf algebra (called the *L-R smash coproduct*) with the following structures:

$$(c \ltimes g)(d \ltimes h) = cd \ltimes gh, \quad 1_{C\ltimes H} = 1_C \ltimes 1_H, \tag{1.13}$$

$$\Delta(c \ltimes h) = (c_{1[0]} \ltimes c_{2(-1)}h_1) \otimes (c_{2(0)} \ltimes h_2c_{1[1]}), \tag{1.14}$$

$$\varepsilon(c \ltimes h) = \varepsilon(c)\varepsilon(h), \tag{1.15}$$

$$S(c \ltimes h) = S_C(c_{(0)[0]}) \ltimes S_H(c_{(-1)}c_{(0)[1]}h). \tag{1.16}$$

Note that when H is commutative, the identities (1.11) and (1.12) naturally hold. We call C an *H-bicomodule Hopf algebra* if S_C is H-bicolinear.

2. Rota–Baxter operators on L-R smash products. In this section, we will construct an operator $\overline{B}$ on the L-R smash product $A \natural H$ and give necessary and sufficient conditions for $\overline{B}$ to be a Rota–Baxter operator.

Definition 2.1 ([14]). Let (H, S) be a cocommutative Hopf algebra. A coalgebra map $B : H \to H$ is called a *Rota–Baxter operator* on H if for all $x, y \in H$,

$$B(x)B(y) = B(x_1B(x_2)yS(B(x_3))). \tag{2.1}$$

By [14, Corollary 1] we know that if H is a cocommutative Hopf algebra with antipode S, then S is a Rota–Baxter operator on H.

Since a Rota–Baxter operator B is a coalgebra map, we have $\Delta(B(1)) = B(1) \otimes B(1)$, and $B(1)S(B(1)) = \varepsilon(B(1)) = 1, B(1)^{-1} = S(B(1))$. Hence

$$B(1)B(1) = B(1B(1)1S(B(1))) = B(1),$$

which implies $B(1) = 1$. For details one can refer to [14].

Theorem 2.2. *Let A and H be two cocommutative Hopf algebras such that $(A, \triangleleft, \triangleright)$ is an H-bimodule Hopf algebra. Assume that $R : A \to A$ is a linear map and B is a Rota–Baxter operator on H. Define a linear map $\overline{B} : A \natural H \to A \natural H$ by*

$$\overline{B}(a \natural h) = B(h_1) \triangleright R(a) \triangleleft B(h_2) \natural B(h_3) \tag{2.2}$$

for all $a \in A$ and $h \in H$. Then $\overline{B}$ is a Rota–Baxter operator on $A \natural H$ if and only if R is a Rota–Baxter operator on A and satisfies the following conditions:

$$R[(a_1R(a_2) \triangleleft h_1)b(h_2 \triangleright S_A(R(a_3)))] = (S_H(B(h)) \triangleright R(a))R(b), \tag{2.3}$$

$$R(g_1B(g_2) \triangleright b \triangleleft S_H(B(g_3))) = R(b) \triangleleft S_H(B(g)), \tag{2.4}$$

for all $a, b \in A$ and $g, h \in H$.

Proof. If $\overline{B}$ is a Rota–Baxter operator on $A \natural H$, it must satisfy the identity (2.1), that is, for all $a, b \in A$ and $g, h \in H$,

$$\overline{B}(a \natural g)\overline{B}(b \natural h) = \overline{B}[(a_1 \natural g_1)\overline{B}(a_2 \natural g_2)(b \natural h)S(\overline{B}(a_3 \natural g_3))]. \tag{2.5}$$

For the left side of (2.5), we compute

$$\begin{aligned}
&\overline{B}(a \natural g)\overline{B}(b \natural h)\\
&\quad = [B(g_1) \triangleright R(a) \triangleleft B(g_2) \natural B(g_3)][B(h_1) \triangleright R(b) \triangleleft B(h_2) \natural B(h_3)]\\
&\quad = (B(g_1) \triangleright R(a) \triangleleft B(g_2)B(h_1))(B(g_3)B(h_2) \triangleright R(b) \triangleleft B(h_3)) \natural B(g_4)B(h_4),
\end{aligned}$$

and for the right side of (2.5), we compute

$$\begin{aligned}
&\overline{B}[(a_1 \natural g_1)\overline{B}(a_2 \natural g_2)(b \natural h)S(\overline{B}(a_3 \natural g_3))]\\
&\overset{(2.2)}{=} \overline{B}[(a_1 \natural g_1)(B(g_2) \triangleright R(a_2) \triangleleft B(g_3) \natural B(g_4))(b \natural h)S(B(g_5) \triangleright R(a_3) \triangleleft B(g_6) \natural B(g_7))]\\
&\overset{(1.6)}{=} \overline{B}[(a_1 \natural g_1)(B(g_2) \triangleright R(a_2) \triangleleft B(g_3) \natural B(g_4))(b \natural h)\\
&\qquad (S_H(B(g_7))B(g_5) \triangleright S_A(R(a_3)) \triangleleft B(g_6)S_H(B(g_8))) \natural S_H(B(g_9))]\\
&\quad = \overline{B}[(a_1 \natural g_1)(B(g_2) \triangleright R(a_2) \triangleleft B(g_3) \natural B(g_4))(b \natural h)(S_A(R(a_3)) \natural S_H(B(g_5)))]\\
&\overset{(1.3)}{=} \overline{B}[((a_1 \triangleleft B(g_6))(g_1B(g_2) \triangleright R(a_2) \triangleleft B(g_3)) \natural g_4B(g_5)))(b \natural h)(S_A(R(a_3)) \natural S_H(B(g_7)))]\\
&\overset{(1.3)}{=} \overline{B}[((a_1 \triangleleft B(g_8)h_1)(g_1B(g_2) \triangleright R(a_2) \triangleleft B(g_3)h_2)(g_4B(g_5) \triangleright b) \natural g_6B(g_7)h_3)\\
&\qquad (S_A(R(a_3)) \natural S_H(B(g_9)))]\\
&\overset{(1.3)}{=} \overline{B}[((a_1 \triangleleft B(g_{10})h_1S_H(B(g_{11})))(g_1B(g_2) \triangleright R(a_2) \triangleleft B(g_3)h_2S_H(B(g_{12})))\\
&\qquad (g_4B(g_5) \triangleright b \triangleleft S_H(B(g_{13})))(g_6B(g_7)h_3 \triangleright S_A(R(a_3))) \natural g_8B(g_9)h_4S_H(B(g_{14}))]\\
&\overset{(2.2)}{=} B(g_8B(g_9)h_4S_H(B(g_{18}))) \triangleright R[(a_1 \triangleleft B(g_{14})h_1S_H(B(g_{15})))\\
&\qquad (g_1B(g_2) \triangleright R(a_2) \triangleleft B(g_3)h_2S_H(B(g_{16})))\\
&\qquad (g_4B(g_5) \triangleright b \triangleleft S_H(B(g_{17})))(g_6B(g_7)h_3 \triangleright S_A(R(a_3)))] \triangleleft B(g_{10}B(g_{11})h_5S_H(B(g_{19})))\\
&\qquad \natural B(g_{12}B(g_{13})h_6S_H(B(g_{20})))\\
&\quad = B(g_8B(g_9)h_4S_H(B(g_{10}))) \triangleright R[(a_1 \triangleleft B(g_{17})h_1S_H(B(g_{18})))\\
&\qquad (g_1B(g_2) \triangleright R(a_2) \triangleleft B(g_3)h_2S_H(B(g_{19})))\\
&\qquad (g_4B(g_5) \triangleright b \triangleleft S_H(B(g_{20})))(g_6B(g_7)h_3 \triangleright S_A(R(a_3)))] \triangleleft B(g_{11}B(g_{12})h_5S_H(B(g_{13})))\\
&\qquad \natural B(g_{14}B(g_{15})h_6S_H(B(g_{16})))\\
&\overset{(2.1)}{=} B(g_8)B(h_4) \triangleright R[(a_1 \triangleleft B(g_{11})h_1S_H(B(g_{12})))(g_1B(g_2) \triangleright R(a_2) \triangleleft B(g_3)h_2S_H(B(g_{13})))\\
&\qquad (g_4B(g_5) \triangleright b \triangleleft S_H(B(g_{14})))(g_6B(g_7)h_3 \triangleright S_A(R(a_3)))] \triangleleft B(g_9)B(h_5)) \natural B(g_{10})B(h_6)\\
&\quad = B(g_1)B(h_1) \triangleright R[(a_1 \triangleleft B(g_2)h_2S_H(B(g_3)))(g_4B(g_5) \triangleright R(a_2) \triangleleft B(g_6)h_3S_H(B(g_7)))\\
&\qquad (g_8B(g_9) \triangleright b \triangleleft S_H(B(g_{10})))(g_{11}B(g_{12})h_4 \triangleright S_A(R(a_3)))] \triangleleft B(g_{13})B(h_5)) \natural B(g_{14})B(h_6),
\end{aligned}$$

where the second identity uses the bilinearity of S_A, and the third, the eighth and the tenth identities use the cocommutativity of H. Therefore the identity (2.5) holds if and only if

$$(2.6)\qquad (B(g_1)\triangleright R(a)\triangleleft B(g_2)B(h_1))(B(g_3)B(h_2)\triangleright R(b)\triangleleft B(h_3))\natural B(g_4)B(h_4)$$
$$= B(g_1)B(h_1)\triangleright R[(a_1\triangleleft B(g_2)h_2S_H(B(g_3)))(g_4B(g_5)\triangleright R(a_2)\triangleleft B(g_6)h_3S_H(B(g_7)))$$
$$(g_8B(g_9)\triangleright b\triangleleft S_H(B(g_{10})))(g_{11}B(g_{12})h_4\triangleright S_A(R(a_3)))]\triangleleft B(g_{13})B(h_5))\natural B(g_{14})B(h_6)$$

for all $a, b \in A$ and $g, h \in H$.

($\Rightarrow$): Assume that $\overline{B}$ is a Rota–Baxter operator on $A\natural H$. Since $\overline{B}$ is a coalgebra map, we have

$$\Delta(\overline{B}(a\natural 1)) = \overline{B}(a_1\natural 1)\otimes\overline{B}(a_2\natural 1).$$

Hence $\Delta(R(a)) = R(a_1)\otimes R(a_2)$, which means that R is a coalgebra map. Applying $\mathrm{id}\otimes\varepsilon$ to (2.6) we have

$$B(g_1)B(h_1)\triangleright R[(a_1\triangleleft B(g_2)h_2S_H(B(g_3)))(g_4B(g_5)\triangleright R(a_2)\triangleleft B(g_6)h_3S_H(B(g_7)))$$
$$(g_8B(g_9)\triangleright b\triangleleft S_H(B(g_{10})))(g_{11}B(g_{12})h_4\triangleright S_A(R(a_3)))]\triangleleft B(g_{13})B(h_5)$$
$$= (B(g_1)\triangleright R(a)\triangleleft B(g_2)B(h_1))(B(g_3)B(h_2)\triangleright R(b)\triangleleft B(h_3)).$$

Then

$$B(h_1)\triangleright R[(a_1\triangleleft B(g_1)h_2S_H(B(g_2)))(g_3B(g_4)\triangleright R(a_2)\triangleleft B(g_5)h_3S_H(B(g_6)))$$
$$(g_7B(g_8)\triangleright b\triangleleft S_H(B(g_9)))(g_{10}B(g_{11})h_4\triangleright S_A(R(a_3)))]\triangleleft B(g_{12})$$
$$= (R(a)\triangleleft B(g))(B(h)\triangleright R(b)).$$

This is equivalent to

$$(2.7)\qquad R[(a_1\triangleleft B(g_1)h_1S_H(B(g_2)))$$
$$(g_3B(g_4)\triangleright(((R(a_2)\triangleleft B(g_5)h_2S_H(B(g_6)))(b\triangleleft S_H(B(g_7)))(h_3\triangleright S_A(R(a_3))))))]$$
$$= (S_H(B(h))\triangleright R(a))(R(b)\triangleleft S_H(B(g))).$$

Now taking $g = 1_H$ we obtain

$$R[(a_1R(a_2)\triangleleft h_1)b(h_2\triangleright S_A(R(a_3)))] = (S_H(B(h))\triangleright R(a))R(b).$$

Furthermore, setting $h = 1_H$, we have

$$R(a_1R(a_2)bS_A(R(a_3))) = R(a)R(b).$$

Again in the identity (2.7), let $a = 1_A$ and $h = 1_H$ to get

$$R(g_1B(g_2)\triangleright b\triangleleft S_H(B(g_3))) = R(b)\triangleleft S_H(B(g)).$$

($\Leftarrow$): First of all, for $g\in H$ and $a\in A$, we have

$$(2.8)\qquad R(g_1B(g_2)\triangleright a)\triangleleft B(g_3)$$
$$= R(g_1B(g_2)\triangleright(a\triangleleft B(g_3))\triangleleft S_H(B(g_4)))\triangleleft B(g_5)$$
$$\overset{(2.4)}{=} R(a\triangleleft B(g_1))\triangleleft S_H(B(g_2))B(g_3)$$
$$= R(a\triangleleft B(g)).$$

Hence

$$R(a)\lhd B(g)=R(g_1B(g_2)\rhd(S_H(g_3B(g_4))\rhd a))\lhd B(g_5)$$
$$=R(g_1B(g_2)\rhd(S_H(g_4B(g_5))\rhd a))\lhd B(g_3)$$
$$\overset{(2.8)}{=}R(S_H(g_1B(g_2))\rhd a\lhd B(g_3)). \tag{2.9}$$

For all $a,b\in A$ and $g,h\in H$, we only need to prove the identity (2.7):

$$\begin{aligned}
&R[(a_1\lhd B(g_1)h_1S_H(B(g_2)))\\
&(g_3B(g_4)\rhd(((R(a_2)\lhd B(g_5)h_2S_H(B(g_6)))(b\lhd S_H(B(g_7)))(h_3\rhd S_A(R(a_3))))))]\\
&= R[g_1B(g_2)\rhd((S_H(g_3B(g_4))\rhd a_1\lhd B(g_5)h_1S_H(B(g_6)))\\
&\quad (R(a_2)\lhd B(g_7)h_2S_H(B(g_8)))(b\lhd S_H(B(g_9)))(h_3\rhd S_A(R(a_3))))]\\
&= R[g_1B(g_2)\rhd((S_H(g_4B(g_5))\rhd a_1\lhd B(g_6)h_1)\\
&\quad (R(a_2)\lhd B(g_7)h_2)b(h_3\rhd S_A(R(a_3))\lhd B(g_8)))\lhd S_H(B(g_3))]\\
&\overset{(2.4)}{=} R[(S_H(g_1B(g_2))\rhd a_1\lhd B(g_3)h_1)\cdot\\
&\quad \cdot(\underline{R(a_2)\lhd B(g_4)}h_2)b(h_3\rhd S_A(R(a_3))\lhd B(g_5)))]\lhd S_H(B(g_6))\\
&\overset{(2.9)}{=} R[((S_H(g_1B(g_2))\rhd a_1\lhd B(g_3))\lhd h_1)(R(S_H(g_4B(g_5))\rhd a_2\lhd B(g_6))\lhd h_2)b\\
&\quad (h_3\rhd S_A(\underline{R(a_3)\lhd B(g_7)})))]\lhd S_H(B(g_8))\\
&\overset{(2.9)}{=} R[((S_H(g_1B(g_2))\rhd a_1\lhd B(g_3))R(S_H(g_4B(g_5))\rhd a_2\lhd B(g_6)))\lhd h_1)b\\
&\quad (h_2\rhd S_A(R(S_H(g_7B(g_8))\rhd a_3\lhd B(g_9))))]\lhd S_H(B(g_{10}))\\
&\overset{(2.3)}{=} [(S_H(B(h))\rhd R(S_H(g_1B(g_2))\rhd a)\lhd B(g_3))R(b)]\lhd S_H(B(g_4))\\
&\overset{(2.9)}{=} [(S_H(B(h))\rhd R(a)\lhd B(g_1))R(b)]\lhd S_H(B(g_2))\\
&= (S_H(B(h))\rhd R(a))(R(b)\lhd S_H(B(g_3)). \ \blacksquare
\end{aligned}$$

Remark 2.3. In the above theorem, when the right action is trivial, that is, $a\lhd h=\varepsilon(h)a$ for all $a\in A$ and $h\in H$, we recover [23, Theorem 3.3].

Corollary 2.4. *Let A and H be two cocommutative Hopf algebras such that $(A,\lhd,\rhd)$ is an H-bimodule Hopf algebra. Assume that $R:A\to A$ is a linear map. Define a linear map $\overline{B}:A\natural H\to A\natural H$ by*

$$\overline{B}(a\natural h)=S_H(h_1)\rhd R(a)\lhd S_H(h_2)\natural S_H(h_3) \tag{2.10}$$

for all $a\in A$ and $h\in H$. Then $\overline{B}$ is a Rota–Baxter operator on $A\natural H$ if for all $a,b\in A$ and $h\in H$, the following equations hold:

$$R[(a_1R(a_2)\lhd h_1)b(h_2\rhd S_A(R(a_3)))]=(h\rhd R(a))R(b), \tag{2.11}$$

$$R(a\lhd h)=R(a)\lhd h. \tag{2.12}$$

Proof. Since for a cocommutative Hopf algebra the square of the antipode coincides with the identity, the result is straightforward. $\blacksquare$

COROLLARY 2.5. *Let A and H be two cocommutative Hopf algebras such that $(A, \triangleleft, \triangleright)$ is an H-bimodule Hopf algebra. Assume that B is a Rota–Baxter operator on H. Define a linear map $\overline{B} : A \natural H \to A \natural H$ by*

$$\overline{B}(a \natural h) = B(h_1) \triangleright S_A(a) \triangleleft B(h_2) \natural B(h_3) \tag{2.13}$$

for all $a \in A$ and $h \in H$. Then $\overline{B}$ is a Rota–Baxter operator on $A \natural H$ if and only if for all $a \in A$ and $h \in H$, the following equations hold:

$$h \triangleright S_A(a) = S_H(B(h)) \triangleright S_A(a), \tag{2.14}$$

$$h_1 B(h_2) \triangleright a = \varepsilon(h)a. \tag{2.15}$$

Proof. The proof is straightforward and left to the reader. ∎

Let H be a cocommutative Hopf algebra. Since H has a trivial R-matrix, i.e. $R = 1 \otimes 1$, the category ${}_H\mathbb{M}_H$ of H-bimodules is a symmetric category. If $(L, [\,,])$ is a Lie algebra with left and right H-actions such that

$$h \triangleright [x, y] \triangleleft h' = [h_1 \triangleright x \triangleleft h'_1, h_2 \triangleright y \triangleleft h'_2]$$

for all $x, y \in L$ and $h, h' \in H$, then $(L, [\,,])$ is a Lie algebra in the category ${}_H\mathbb{M}_H$. Then by [12, Theorem 2.6], $U(L)$ is a Hopf algebra in ${}_H\mathbb{M}_H$. So we have the L-R smash product $U(L) \natural H$. Consider the Lie algebra $\mathrm{sl}_2(k)$ with the basis x, y, z and the multiplication

$$[z, x] = 2x, \quad [z, y] = -2y, \quad [x, y] = z.$$

In [14, Example 1] Goncharov gave a Rota–Baxter operator R on $A = U(\mathrm{sl}_2(k))$ as follows:

$$R(x^i z^j y^k) = \begin{cases} 0, & i > 0, \\ (-1)^{j+k} \dfrac{y^k z^j}{2^j}, & i = 0. \end{cases} \tag{2.16}$$

Obviously $R(x) = 0$, $R(y) = -y$, $R(z) = -z/2$. We aim to construct Rota–Baxter operators $\overline{B}$ on $A \natural H$ of the form (2.10).

EXAMPLE 2.6. Let $C_2 = \{1, g \mid g^2 = 1\}$ and H the group algebra $k[C_2]$. It is clear that the antipode S_H of H is the identity map. Now we define the left and right actions of H on $\mathrm{sl}_2(k)$ as follows:

$$g \triangleright x = -x, \quad g \triangleright y = -y, \quad g \triangleright z = z,$$
$$x \triangleleft g = -x, \quad y \triangleleft g = -y, \quad z \triangleleft g = z.$$

Then by [12, Theorem 2.6], A is an H-bimodule Hopf algebra.

Just as in [23, Example 3.8], the morphism R given in (2.16) is H-bilinear, hence condition (2.12) is satisfied. Now we need to verify (2.11). We only present the detailed computations in the case of a being one of the generators

x, y and z in (2.11). Let $a' = x^i z^j y^k \in A$. For x, we compute

$$\begin{aligned} &R[(x_1 R(x_2) \triangleleft g)a'(g \triangleright S_A(R(x_3)))] \\ &\qquad = R[(x \triangleleft g)a' + (R(x) \triangleleft g)a' + a'(g \triangleright S_A(R(x)))] \\ &\qquad = R(-xa') = 0 = (g \triangleright R(x))R(a'). \end{aligned}$$

For y, we compute

$$\begin{aligned} &R[(y_1 R(y_2) \triangleleft g)a'(g \triangleright S_A(R(y_3)))] \\ &\qquad = R[(y \triangleleft g)a' + (R(y) \triangleleft g)a' + a'(g \triangleright S_A(R(y)))] \\ &\qquad = R(-ya' + ya' - a'y) = R(-a'y) = -R(x^i z^j y^{k+1}), \end{aligned}$$

and

$$(g \triangleright R(y))R(a') = yR(x^i z^j y^k).$$

If $i > 0$, then $R(x^i z^j y^{k+1}) = yR(x^i z^j y^k) = 0$. If $i = 0$, then

$$-R(z^j y^{k+1}) = (-1)^{j+k} \frac{y^{k+1} z^j}{2^j} = yR(z^j y^k).$$

Hence we have

$$R[(y_1 R(y_2) \triangleleft g)a'(g \triangleright S_A(R(y_3)))] = (g \triangleright R(y))R(a').$$

For z, we compute

$$\begin{aligned} &R[(z_1 R(z_2) \triangleleft g)a'(g \triangleright S_A(R(z_3)))] \\ &\quad = R[(zR(1) \triangleleft g)a'(g \triangleright S_A(R(1))) + (R(z) \triangleleft g)a'(g \triangleright S_A(R(1))) \\ &\qquad + (1 \triangleleft g)a'(g \triangleright S_A(R(z)))] \\ &\quad = R[zR(1)a' S_A(R(1)) + R(z)a' S_A(R(1)) + a' S_A(R(g \triangleright z))] \\ &\quad = R[z_1 R(z_2)a' S_A(R(z_3))] = R(z)R(a') = -\frac{z}{2}R(a') = (g \triangleright R(z))R(a'). \end{aligned}$$

Therefore we obtain a Rota–Baxter operator $\overline{B}$ on $A \natural H$ as follows:

$$\overline{B}(x^i z^j y^k \natural g^l) = \begin{cases} 0, & i > 0, \\ (-1)^{j+k} \dfrac{y^k z^j}{2^j} \natural g^l, & i = 0. \end{cases}$$

Example 2.7. Let $C_3 = \{1, g \mid g^3 = 1\}$ and ω be the primitive 3th root of unity. Define the left and right actions of $H = k[C_3]$ on $\mathrm{sl}_2(k)$ as follows:

$$\begin{aligned} g \triangleright x = \omega x, \quad g \triangleright y = \omega^2 y, \quad g \triangleright z = z, \\ x \triangleleft g = \omega x, \quad y \triangleleft g = \omega^2 y, \quad z \triangleleft g = z. \end{aligned}$$

One can verify that $\mathrm{sl}_2(k)$ is an H-bimodule and thus $A = U(\mathrm{sl}_2(k))$ is an H-bimodule Hopf algebra. It is easy to see that the morphism R given in (2.16) is H-bilinear, and we only need to check condition (2.11). Following the method used in Example 2.6, we set $a' = x^i z^j y^k \in A$. For x, we compute

$$\begin{aligned} &R[(x_1 R(x_2) \triangleleft g)a'(g \triangleright S_A(R(x_3)))] \\ &\qquad = R[(x \triangleleft g)a' + (R(x) \triangleleft g)a' + a'(g \triangleright S_A(R(x)))] \\ &\qquad = R(\omega x a') = 0 = (g \triangleright R(x))R(a'). \end{aligned}$$

For y, we compute

$$\begin{aligned} &R[(y_1 R(y_2) \triangleleft g)a'(g \triangleright S_A(R(y_3)))] \\ &\qquad = R[(y \triangleleft g)a' + (R(y) \triangleleft g)a' + a'(g \triangleright S_A(R(y)))] \\ &\qquad = R((y \triangleleft g)a' - (y \triangleleft g)a' + a'(g \triangleright y)) \\ &\qquad = \omega^2 R(a'y) = \omega^2 R(x^i z^j y^{k+1}), \end{aligned}$$

and

$$(g \triangleright R(y))R(a') = -\omega^2 y R(x^i z^j y^k).$$

If $i > 0$, then $R(x^i z^j y^{k+1}) = yR(x^i z^j y^k) = 0$. If $i = 0$, then

$$\omega^2 R(z^j y^{k+1}) = \omega^2 (-1)^{j+k+1} \frac{y^{k+1} z^j}{2^j} = -\omega^2 y R(z^j y^k).$$

Hence we have

$$R[(y_1 R(y_2) \triangleleft g)a'(g \triangleright S_A(R(y_3)))] = (g \triangleright R(y))R(a').$$

For z, we also obtain

$$R[(z_1 R(z_2) \triangleleft g)a'(g \triangleright S_A(R(z_3)))] = (g \triangleright R(z))R(a').$$

Therefore we have a Rota–Baxter operator $\overline{B}$ on $A \natural H$ as follows: for all $i, j, k \in \mathbb{N}$ and $l \in \{0, 1, 2\}$,

$$\overline{B}(x^i z^j y^k \natural g^l) = \begin{cases} 0, & i > 0, \\ (-1)^{j+k} \omega^{(3-l)k} \dfrac{y^k z^j}{2^j} \natural g^{3-l}, & i = 0. \end{cases}$$

EXAMPLE 2.8. Let $C_2 = \{1, g \mid g^2 = 1\}$ and $C_3 = \{1, h \mid h^3 = 1\}$ be the cyclic group of order 2 and 3, respectively. Define the left and right actions of $k[C_2]$ on $k[C_3]$ as follows:

$$\begin{aligned} g \triangleright h &= h, \qquad & g \triangleright h^2 &= h^2, \\ h \triangleleft g &= h^2, \qquad & h^2 \triangleleft g &= h. \end{aligned}$$

It is straightforward to verify that $k[C_3]$ is a $k[C_2]$-bimodule Hopf algebra; hence we have the L-R smash product $k[C_3] \natural k[C_2]$. And by a routine verification one finds that $k[C_3] \natural k[C_2] \cong k[S_3]$ as Hopf algebras, where S_3 is the symmetric group on three symbols.

(1) Let $R : k[C_3] \to k[C_3]$ be the trivial Rota–Baxter operator, i.e., $R(h^i)=1$ for $i = 0, 1, 2$. Then (2.11) is satisfied, and

$$\overline{B}(h^i \natural g) = S_H(g) \triangleright R(h^i) \triangleleft S_H(g) \natural S_H(g) = 1 \natural g, \qquad i = 0, 1, 2.$$

(2) Define $R : k[C_3] \to k[C_3]$ by

$$R(h) = h^2, \qquad R(h^2) = h.$$

Then for any $b \in k[C_3]$, let $a = h$ and the left side of (2.11) reduces to

$$R((hR(h) \triangleleft g)b(g \triangleright S_{k[C_3]}(R(h)))) = R(b(g \triangleright h)) = R(bh),$$

and the right side reduces to

$$(g \triangleright R(h))R(b) = (g \triangleright h^2)R(b) = h^2R(b).$$

Setting $b = 1, h, h^2$ respectively, we can verify that (2.11) holds. One can check that (2.11) also holds for $a = h^2$. Thus

$$\overline{B}(h \natural g) = g \triangleright R(h) \triangleleft g \natural g = g \triangleright h^2 \triangleleft g \natural g = h \natural g,$$
$$\overline{B}(h^2 \natural g) = g \triangleright R(h) \triangleleft g \natural g = g \triangleright h \triangleleft g \natural g = h^2 \natural g,$$

which implies that $\overline{B} = \mathrm{id}_{k[C_3]\natural k[C_2]}$.

3. Rota–Baxter co-operator on L-R smash coproduct. In this section we will consider the dual case and characterize a class of Rota–Baxter co-operators on an L-R smash coproduct.

Definition 3.1 ([22]). Let (H, S) be a commutative Hopf algebra. An algebra map $B : H \to H$ is called a *Rota–Baxter co-operator* on H if for all $x \in H$,

$$B(x_1) \otimes B(x_2) = B(x)_1B(B(x)_2S(B(x)_4)) \otimes B(x)_3. \tag{3.1}$$

Proposition 3.2. *Let H be a commutative Hopf algebra with a Rota–Baxter co-operator B. Then $\varepsilon \circ B = \varepsilon$.*

Proof. For all $x \in H$, applying $\varepsilon \otimes \varepsilon$ to both sides of (3.1), we have

$$\varepsilon(B(x_1))\varepsilon(B(x_2)) = \varepsilon(B(B(x)_1S(B(x)_2))) = \varepsilon(B(x)),$$

that is, $(\varepsilon \circ B) * (\varepsilon \circ B) = \varepsilon \circ B$. Since $B(1) = 1$, we get

$$\varepsilon(B(x_1))\varepsilon(B(S(x_2))) = \varepsilon(B(x_1S(x_2))) = \varepsilon(x),$$

that is, $(\varepsilon \circ B) * (\varepsilon \circ B \circ S) = \varepsilon$. Therefore $\varepsilon \circ B = \varepsilon$. ∎

Theorem 3.3. *Let C and H be two commutative Hopf algebras such that (C, ρ^l, ρ^r) is an H-bicomodule Hopf algebra. Assume that $R : C \to C$ is a linear map and B is a Rota–Baxter co-operator on H. Define a linear map $\widetilde{B} : C \ltimes H \to C \ltimes H$ by*

$$\widetilde{B}(c \ltimes h) = R(c_{(0)[0]}) \ltimes B(c_{(-1)}c_{(0)[1]}h) \tag{3.2}$$

for all $c \in C$ and $h \in H$. Then $\widetilde{B}$ is a Rota–Baxter co-operator on $C \ltimes H$ if and only if R is a Rota–Baxter co-operator on C and satisfies the following conditions:

$$\begin{aligned}(3.3)\quad & R(c)_{1[0]1}R(R(c)_{1[0]2}S_C(R(c)_{3(0)})) \otimes R(c)_2 \otimes R(c)_{1[1]}R(c)_{3(-1)} \\ &= R(c_{1(0)}) \otimes R(c_2) \otimes B(S_H(c_{1(-1)})),\end{aligned}$$

$$\begin{aligned}(3.4)\quad & R(c)_{[0](-1)1}B(R(c)_{[0](-1)2}S_H(R(c)_{[1]})) \otimes R(c)_{[0](0)} \\ &= B(S_H(c_{[1]})) \otimes R(c_{[0]}).\end{aligned}$$

Proof. For all $c \in C$ and $h \in H$, we first compute

$$\begin{aligned}
&(c \ltimes h)_1 \otimes (c \ltimes h)_2 \otimes (c \ltimes h)_3 \otimes (c \ltimes h)_4 \\
\overset{(1.14)}{=}\ & (c_{1[0]} \ltimes c_{2(-1)}h_1)_1 \\
& \otimes (c_{1[0]} \ltimes c_{2(-1)}h_1)_2 \otimes (c_{2(0)} \ltimes h_2c_{1[1]})_1 \otimes (c_{2(0)} \ltimes h_2c_{1[1]})_2 \\
\overset{(1.14)}{=}\ & \underline{c_{1[0]1[0]}} \ltimes \underline{c_{1[0]2(-1)}}c_{2(-1)1}h_1 \otimes \underline{c_{1[0]2(0)}} \ltimes c_{2(-1)2}h_2\underline{c_{1[0]1[1]}} \\
& \otimes c_{2(0)1[0]} \ltimes c_{2(0)2(-1)}h_3\underline{c_{1[1]1}} \otimes c_{2(0)2(0)} \ltimes h_4\underline{c_{1[1]2}}c_{2(0)1[1]} \\
\overset{(1.10)}{=}\ & c_{1[0][0]} \ltimes c_{2[0](-1)}c_{3(-1)1}h_1 \otimes c_{2[0](0)} \ltimes c_{3(-1)2}h_2c_{1[0][1]} \\
& \otimes c_{3(0)1[0]} \ltimes c_{3(0)2(-1)}h_3c_{1[1]1}c_{2[1]1} \otimes c_{3(0)2(0)} \ltimes h_4c_{1[1]2}c_{2[1]2}c_{3(0)1[1]} \\
=\ & c_{1[0]} \ltimes c_{2[0](-1)}c_{3(-1)1}c_{4(-1)1}h_1 \otimes c_{2[0](0)} \ltimes c_{3(-1)2}c_{4(-1)2}h_2c_{1[1]1} \\
& \otimes c_{3(0)[0]} \ltimes c_{4(0)(-1)}h_3c_{1[1]2}c_{2[1]1} \otimes c_{4(0)(0)} \ltimes h_4c_{1[1]3}c_{2[1]2}c_{3(0)[1]} \\
=\ & c_{1[0]} \ltimes c_{2[0](-1)}c_{3(-1)1}c_{4(-1)1}h_1 \otimes c_{2[0](0)} \ltimes c_{3(-1)2}c_{4(-1)2}h_2c_{1[1]1} \\
& \otimes c_{3(0)[0]} \ltimes c_{4(-1)3}h_3c_{1[1]2}c_{2[1]1} \otimes c_{4(0)} \ltimes h_4c_{1[1]3}c_{2[1]2}c_{3(0)[1]},
\end{aligned}$$

where the last two identities use the coassociativity of a comodule. Hence we obtain

$$\begin{aligned}(3.5)\quad & (c \ltimes h)_1 \otimes (c \ltimes h)_2 \otimes (c \ltimes h)_3 \otimes (c \ltimes h)_4 \\
&= c_{1[0]} \ltimes c_{2[0](-1)}c_{3(-1)1}c_{4(-1)1}h_1 \otimes c_{2[0](0)} \ltimes c_{3(-1)2}c_{4(-1)2}h_2c_{1[1]1} \\
&\quad \otimes c_{3(0)[0]} \ltimes c_{4(-1)3}h_3c_{1[1]2}c_{2[1]1} \otimes c_{4(0)} \ltimes h_4c_{1[1]3}c_{2[1]2}c_{3(0)[1]}.\end{aligned}$$

For all $c \in C$ and $h \in H$, we compute

$$\begin{aligned}
&(c \ltimes h)_1\widetilde{B}((c \ltimes h)_2S((c \ltimes h)_4)) \otimes (c \ltimes h)_3 \\
\overset{(3.5)}{=}\ & (c_{1[0]} \ltimes c_{2[0](-1)}c_{3(-1)1}c_{4(-1)1}h_1)\widetilde{B}((c_{2[0](0)} \ltimes c_{3(-1)2}c_{4(-1)2}h_2c_{1[1]1})\cdot \\
& \cdot S(c_{4(0)} \ltimes h_4c_{1[1]3}c_{2[1]2}c_{3(0)[1]})) \otimes c_{3(0)[0]} \ltimes c_{4(-1)3}h_3c_{1[1]2}c_{2[1]1} \\
\overset{(1.16)}{=}\ & (c_{1[0]} \ltimes c_{2[0](-1)}c_{3(-1)1}c_{4(-1)1}h_1) \\
& \widetilde{B}(c_{2[0](0)}S_C(c_{4(0)[0]}) \ltimes c_{3(-1)2}c_{4(-1)2}h_2c_{1[1]1}S_H(c_{4(-1)4}c_{4(0)[1]}h_4c_{1[1]3}c_{2[1]2}c_{3(0)[1]})) \\
& \otimes c_{3(0)[0]} \ltimes c_{4(-1)3}h_3c_{1[1]2}c_{2[1]1}
\end{aligned}$$

$$\overset{(3.2)}{=} (c_{1[0]} \ltimes c_{2[0](-1)1}c_{3(-1)1}c_{4(-1)1}h_1)(R(c_{2[0](0)}S_C(c_{4(0)[0](0)}))$$
$$\ltimes B(c_{2[0](-1)2}c_{4(0)[0](-1)}c_{2[1]1}c_{4(0)[1]1}c_{3(-1)2}c_{4(-1)2}h_2c_{1[1]1}\cdot$$
$$\cdot S_H(c_{4(-1)4}c_{4(0)[1]2}h_4c_{1[1]3}c_{2[1]3}c_{3(0)[1]}))) \otimes c_{3(0)[0]} \ltimes c_{4(-1)3}h_3c_{1[1]2}c_{2[1]2}$$
$$\overset{(1.13)}{=} c_{1[0]}R(c_{2[0](0)}S_C(c_{4(0)}))$$
$$\ltimes c_{2[0](-1)1}c_{3(-1)1}c_{4(-1)1}h_1B(c_{2[0](-1)2}c_{3(-1)2}c_{4(-1)2}h_2c_{1[1]1}c_{2[1]1}S_H(h_4c_{1[1]3}c_{2[1]3}c_{3(0)[1]}))$$
$$\otimes c_{3(0)[0]} \ltimes c_{4(-1)3}h_3c_{1[1]2}c_{2[1]2}$$
$$\overset{(1.10)}{=} c_{1[0]1}R(c_{1[0]2(0)}S_C(c_{3(0)}))$$
$$\ltimes c_{1[0]2(-1)1}c_{2(-1)1}c_{3(-1)1}h_1B(c_{1[0]2(-1)2}c_{2(-1)2}c_{3(-1)2}h_2c_{1[1]1}S_H(h_4c_{1[1]3}c_{2(0)[1]}))$$
$$\otimes c_{2(0)1[0]} \ltimes c_{3(-1)3}h_3c_{1[1]2}$$
$$\overset{(1.10)}{=} c_{1[0]1}R(c_{1[0]2(0)}S_C(c_{2(0)2(0)}))$$
$$\ltimes c_{1[0]2(-1)1}c_{2(-1)1}h_1B(c_{1[0]2(-1)2}c_{2(-1)2}h_2c_{1[1]1}S_H(h_4c_{1[1]3}c_{2(0)1[1]}))$$
$$\otimes c_{2(0)1[0]} \ltimes c_{2(0)2(-1)}h_3c_{1[1]2}$$
$$\overset{h=B(x)}{=} c_{1[0]1}R(c_{1[0]2(0)}S_C(c_{2(0)2(0)}))$$
$$\ltimes c_{1[0]2(-1)1}c_{2(-1)1}B(x_1)B(c_{1[0]2(-1)2}c_{2(-1)2}c_{1[1]1}S_H(c_{1[1]3}c_{2(0)1[1]}))$$
$$\otimes c_{2(0)1[0]} \ltimes c_{2(0)2(-1)}B(x_2)c_{1[1]2}.$$

If $\widetilde{B}$ is a Rota–Baxter co-operator, it must satisfy (3.1), that is, for all $c \in C$ and $h \in H$,

$$\widetilde{B}((c \ltimes h)_1) \otimes \widetilde{B}((c \ltimes h)_2) = (c \ltimes h)_1\widetilde{B}((c \ltimes h)_2S((c \ltimes h)_4)) \otimes (c \ltimes h)_3. \tag{3.6}$$

For the right side of (3.6), we compute

$$\widetilde{B}(c \ltimes h)_1\widetilde{B}(\widetilde{B}(c \ltimes h)_2S(\widetilde{B}(c \ltimes h)_4)) \otimes \widetilde{B}(c \ltimes h)_3$$
$$= (R(c_{(0)[0]}) \ltimes B(c_{(-1)}c_{(0)[1]}h))_1\cdot$$
$$\cdot \widetilde{B}((R(c_{(0)[0]}) \ltimes B(c_{(-1)}c_{(0)[1]}h))_2S((R(c_{(0)[0]}) \ltimes B(c_{(-1)}c_{(0)[1]}h))_4))$$
$$\otimes (R(c_{(0)[0]}) \ltimes B(c_{(-1)}c_{(0)[1]}h))_3$$
$$= R(c_{(0)[0]})_{1[0]1}R(R(c_{(0)[0]})_{1[0]2(0)}S_C(R(c_{(0)[0]})_{2(0)2(0)}))$$
$$\ltimes R(c_{(0)[0]})_{1[0]2(-1)1}R(c_{(0)[0]})_{2(-1)1}$$
$$\cdot B(c_{(-1)1}c_{(0)[1]1}h_1)B(R(c_{(0)[0]})_{1[0]2(-1)2}R(c_{(0)[0]})_{2(-1)2}R(c_{(0)[0]})_{1[1]1}$$
$$\cdot S_H(R(c_{(0)[0]})_{1[1]3}R(c_{(0)[0]})_{2(0)1[1]}))$$
$$\otimes R(c_{(0)[0]})_{2(0)1[0]} \ltimes R(c_{(0)[0]})_{2(0)2(-1)}B(c_{(-1)2}c_{(0)[1]2}h_2)R(c_{(0)[0]})_{1[1]2}.$$

For the left side of (3.6), we obtain

$$\widetilde{B}((c \ltimes h)_1) \otimes \widetilde{B}((c \ltimes h)_2)$$
$$= \widetilde{B}(c_{1[0]} \ltimes c_{2(-1)}h_1) \otimes \widetilde{B}(c_{2(0)} \ltimes h_2c_{1[1]})$$
$$= R(c_{1[0](0)[0]}) \ltimes B(c_{1[0](-1)}c_{1[0](0)[1]}c_{2(-1)}h_1) \otimes R(c_{2(0)(0)[0]}) \ltimes B(c_{2(0)(-1)}c_{2(0)(0)[1]}h_2c_{1[1]})$$
$$= R(c_{1[0][0](0)}) \ltimes B(c_{1[0][0](-1)}c_{1[0][1]}c_{2(-1)1}h_1) \otimes R(c_{2(0)[0]}) \ltimes B(c_{2(-1)2}c_{2(0)[1]}h_2c_{1[1]})$$
$$= R(c_{1[0](0)}) \ltimes B(c_{1[0](-1)}c_{1[1]1}c_{2(-1)1}h_1) \otimes R(c_{2(0)[0]}) \ltimes B(c_{2(-1)2}c_{2(0)[1]}h_2c_{1[1]2}).$$

Therefore (3.6) is equivalent to

$$\begin{aligned}
&R(c_{(0)[0]})_{1[0]1}R(R(c_{(0)[0]})_{1[0]2(0)}S_C(R(c_{(0)[0]})_{2(0)2(0)}))\ltimes R(c_{(0)[0]})_{1[0]2(-1)1}R(c_{(0)[0]})_{2(-1)1}\\
&\qquad\cdot B(c_{(-1)1}c_{(0)[1]1}h_1)B(R(c_{(0)[0]})_{1[0]2(-1)2}R(c_{(0)[0]})_{2(-1)2}R(c_{(0)[0]})_{1[1]1}\\
&\qquad\cdot S_H(R(c_{(0)[0]})_{1[1]3}R(c_{(0)[0]})_{2(0)1[1]}))\\
&\qquad\otimes R(c_{(0)[0]})_{2(0)1[0]}\ltimes R(c_{(0)[0]})_{2(0)2(-1)}B(c_{(-1)2}c_{(0)[1]2}h_2)R(c_{(0)[0]})_{1[1]2}\\
&\quad= R(c_{1[0](0)})\ltimes B(c_{1[0](-1)}c_{1[1]1}c_{2(-1)1}h_1)\otimes R(c_{2(0)[0]})\ltimes B(c_{2(-1)2}c_{2(0)[1]}h_2c_{1[1]2}).
\end{aligned}$$

By a long and tedious computation, the above identity is equivalent to

$$\begin{aligned}
(3.7)\qquad &R(c)_{1[0]}R(R(c)_{2(0)1[0]})R(S_C(R(c)_{2(0)3(0)}))\\
&\otimes B(R(c)_{1[1]1})B(S_H(R(c)_{1[1]3}))R(c)_{2(-1)1}B(R(c)_{2(-1)2})B(R(c)_{2(0)1[1]1})\\
&\quad B(S_H(R(c)_{2(0)1[1]3}))B(S_H(R(c)_{2(0)2[1]}))\\
&\otimes R(c)_{2(0)2[0]}\otimes R(c)_{1[1]2}R(c)_{2(0)1[1]2}R(c)_{2(0)3(-1)}\\
&= R(c_{1(0)})\otimes B(S_H(c_{2[1]}))\otimes R(c_{2[0]})\otimes B(S_H(c_{1(-1)})).
\end{aligned}$$

($\Rightarrow$): Assume that $\widetilde{B}$ is a Rota–Baxter co-operator on $C\ltimes H$. Then for all $c,d\in C$, we have

$$\begin{aligned}
\widetilde{B}((c\ltimes 1)(d\ltimes 1)) &= R((cd)_{(0)[0]})\ltimes B((cd)_{(-1)}(cd)_{(0)[1]})\\
&= R(c_{(0)[0]}d_{(0)[0]})\ltimes B(c_{(-1)}d_{(-1)}c_{(0)[1]}d_{(0)[1]}),
\end{aligned}$$

and

$$\widetilde{B}(c\ltimes 1)\widetilde{B}(d\ltimes 1) = R(c_{(0)[0]})R(d_{(0)[0]})\ltimes B(c_{(-1)}c_{(0)[1]})B(d_{(-1)}d_{(0)[1]}).$$

Since $\widetilde{B}$ is an algebra map,

$$\begin{aligned}
&R(c_{(0)[0]}d_{(0)[0]})\ltimes B(c_{(-1)}d_{(-1)}c_{(0)[1]}d_{(0)[1]})\\
&\qquad= R(c_{(0)[0]})R(d_{(0)[0]})\ltimes B(c_{(-1)}c_{(0)[1]})B(d_{(-1)}d_{(0)[1]}).
\end{aligned}$$

Applying $\mathrm{id}\otimes\varepsilon$ to the above identity we obtain $R(cd)=R(c)R(d)$, that is, R is an algebra morphism.

Then apply $\mathrm{id}\otimes\varepsilon\otimes\mathrm{id}\otimes\varepsilon$ to (3.7) and let $h=1$. We have

$$R(c)_1R(R(c)_2S_C(R(c)_4))\otimes R(c)_3 = R(c_1)\otimes R(c_2).$$

Hence R is a Rota–Baxter co-operator on C.

Applying $\mathrm{id}\otimes\varepsilon\otimes\mathrm{id}\otimes\mathrm{id}$ to (3.7) and setting $h=1$, we obtain

$$\begin{aligned}
&R(c)_{1[0]}R(R(c)_{2[0]})R(S_C(R(c)_{4(0)}))\otimes R(c)_3\otimes R(c)_{1[1]}R(c)_{2[1]}R(c)_{4(-1)}\\
&\qquad= R(c_{1(0)})\otimes R(c_2)\otimes B(S_H(R(c_{1(-1)}))),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&R(c)_{1[0]1}R(R(c)_{1[0]2}S_C(R(c)_{3(0)}))\otimes R(c)_2\otimes R(c)_{1[1]}R(c)_{3(-1)}\\
&\qquad= R(c_{1(0)})\otimes R(c_2)\otimes B(S_H(c_{1(-1)})).
\end{aligned}$$

Again applying $\varepsilon\otimes \mathrm{id}\otimes \mathrm{id}\otimes\varepsilon$ to (3.7) and letting $h=1$, we have

$$\begin{aligned}R(c)_{(-1)1}B(R(c)_{(-1)2})B(S_H(R(c)_{(0)[1]}))\otimes R(c)_{(0)[0]}\\ = B(S_H(R(c_{[1]})))\otimes R(c_{[0]}),\end{aligned}$$

which is equivalent to the identity (3.4).

($\Leftarrow$): Assume that R is a Rota–Baxter co-operator on C and the identities (3.3) and (3.4) are satisfied. We only need to verify (3.7). For all $c\in C$ and $h\in H$, we have

$$\begin{aligned}&R(c_{[0]})_{(-1)1}B(R(c_{[0]})_{(-1)2})B(c_{[1]})\otimes R(c_{[0]})_{(0)}\\ &\ = R(c_{[0]})_{[0][0](-1)1}B(R(c_{[0]})_{[0][1]})B(S_H(R(c_{[0]})_{[1]}))B(R(c_{[0]})_{[0][0](-1)2})B(c_{[1]})\\ &\quad\ \otimes R(c_{[0]})_{[0][0](0)}\\ &\ = R(c_{[0]})_{[0](-1)1}B(R(c_{[0]})_{[0](0)[1]})B(S_H(R(c_{[0]})_{[1]}))B(R(c_{[0]})_{[0](-1)2})B(c_{[1]})\\ &\quad\ \otimes R(c_{[0]})_{[0](0)[0]}\\ &\overset{(3.4)}{=} B(S_H(c_{[0][1]}))B(R(c_{[0][0]})_{[1]})B(c_{[1]})\otimes R(c_{[0][0]})_{[0]}\\ &\ = B(R(c)_{[1]})\otimes R(c)_{[0]}.\end{aligned}$$

and

$$\begin{aligned}R(c)_{(0)[0]}\otimes B(S_H(R(c)_{(-1)1}))S_H(R(c)_{(-1)2})B(R(c)_{(0)[1]})\\ = R(c)_{[0](0)}\otimes B(S_H(R(c)_{[0](-1)1}))S_H(R(c)_{[0](-1)2})B(R(c)_{[1]})\\ = R(c_{[0]})_{(0)}\otimes B(S_H(R(c_{[0]})_{(-1)3}))S_H(R(c_{[0]})_{(-1)4})R(c_{[0]})_{(-1)1}\\ \cdot B(R(c_{[0]})_{(-1)2})B(c_{[1]})\\ = R(c_{[0]})\otimes B(c_{[1]}).\end{aligned}\tag{3.8}$$

Then

$$\begin{aligned}&R(c)_{1[0]}R(R(c)_{2(0)1[0]})R(S_C(R(c)_{2(0)3(0)}))\\ &\quad\otimes B(R(c)_{1[1]1})B(S_H(R(c)_{1[1]3}))R(c)_{2(-1)1}B(R(c)_{2(-1)2})B(R(c)_{2(0)1[1]1})\\ &B(S_H(R(c)_{2(0)1[1]3}))B(S_H(R(c)_{2(0)2[1]}))\otimes R(c)_{2(0)2[0]}\otimes R(c)_{1[1]2}R(c)_{2(0)1[1]2}R(c)_{2(0)3(-1)}\\ &\overset{(S_H)}{=} R(c)_{1[0]}R(R(c)_{2(0)1[0]})R(S_C(R(c)_{2(0)3[0](0)}))\otimes R(c)_{2(-1)1}B(R(c)_{2(-1)2})\\ &\quad B(S_H(R(c)_{1[1]3}R(c)_{2(0)1[1]3}R(c)_{2(0)2[1]}R(c)_{2(0)3[1]2}))B(R(c)_{1[1]1})B(R(c)_{2(0)1[1]1})\\ &\quad\cdot B(R(c)_{2(0)3[1]1})\otimes R(c)_{2(0)2[0]}\otimes R(c)_{1[1]2}R(c)_{2(0)1[1]2}R(c)_{2(0)3[0](-1)}\\ &\overset{(1.10)}{=} R(c)_{1[0]}R(R(c)_{2(0)1[0]})R(S_C(R(c)_{2(0)3(0)[0]}))\otimes R(c)_{2(-1)1}B(R(c)_{2(-1)2})\\ &\quad B(S_H(R(c)_{1[1]3})R(c)_{2(0)1[1]3})R(c)_{2(0)2[1]})R(c)_{2(0)3(0)[1]2}))B(R(c)_{1[1]1})B(R(c)_{2(0)1[1]1})\\ &\quad\cdot B(R(c)_{2(0)3(0)[1]1})\otimes R(c)_{2(0)2[0]}\otimes R(c)_{1[1]2}R(c)_{2(0)1[1]2}R(c)_{2(0)3(-1)}\\ &\overset{(1.10)}{=} R(c)_{1[0]}R(R(c)_{2(0)1[0]})R(S_C(R(c)_{3(0)[0]}))\otimes R(c)_{2(-1)1}R(c)_{3(-1)1}\\ &B(R(c)_{2(-1)2}R(c)_{3(-1)2})B(S_H(R(c)_{1[1]3})R(c)_{2(0)1[1]3})R(c)_{2(0)2[1]})R(c)_{3(0)[1]2}))\\ &\cdot B(R(c)_{1[1]1})B(R(c)_{2(0)1[1]1})B(R(c)_{3(0)[1]1})\otimes R(c)_{2(0)2[0]}\otimes R(c)_{1[1]2}R(c)_{2(0)1[1]2}R(c)_{3(-1)3}\end{aligned}$$

$$
\begin{aligned}
&\overset{(1.10)}{=} R(c)_{1[0]}R(R(c)_{2(0)[0]})R(S_C(R(c)_{4(0)[0]}))\\
&\otimes R(c)_{2(-1)1}R(c)_{3(-1)1}R(c)_{4(-1)1}B(R(c)_{2(-1)2}R(c)_{3(-1)2}R(c)_{4(-1)2})\\
&\cdot B(S_H(R(c)_{1[1]3})R(c)_{2(0)[1]3})R(c)_{3(0)[1]})R(c)_{4(0)[1]2}))B(R(c)_{1[1]1})B(R(c)_{2(0)[1]1})\\
&\cdot B(R(c)_{4(0)[1]1})\otimes R(c)_{3(0)[0]}\otimes R(c)_{1[1]2}R(c)_{2(0)[1]2}R(c)_{4(-1)3}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(S_H)}{=} R(c)_{1[0](0)}R(R(c)_{2(0)[0][0]})R(S_C(R(c)_{4(0)[0](0)[0]}))\\
&\otimes\underline{R(c)_{1[0](-1)1}}R(c)_{2(-1)1}R(c)_{3(-1)1}R(c)_{4(-1)1}B(\underline{R(c)_{1[0](-1)2}}R(c)_{2(-1)2}R(c)_{3(-1)2}R(c)_{4(-1)2})\\
&\cdot B(S_H(R(c)_{1[1]3})R(c)_{2(0)[1]})R(c)_{3(0)[1]})R(c)_{4(0)[1]}))\underline{S_H(R(c)_{1[0](-1)4})}B(\underline{S_H(R(c)_{1[0](-1)3})})\\
&\cdot B(R(c)_{1[1]1})B(R(c)_{2(0)[0][1]1})B(R(c)_{4(0)[0](0)[1]})\\
&\otimes R(c)_{3(0)[0]}\otimes R(c)_{1[1]2}R(c)_{2(0)[0][1]2}R(c)_{4(0)[0](-1)}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(1.10)}{=} R(c)_{1(0)[0](0)}R(R(c)_{2(0)[0][0]})R(S_C(R(c)_{4(0)[0](0)[0]}))\\
&\otimes R(c)_{1(-1)1}R(c)_{2(-1)1}R(c)_{3(-1)1}R(c)_{4(-1)1}B(R(c)_{1(-1)2}R(c)_{2(-1)2}R(c)_{3(-1)2}R(c)_{4(-1)2})\\
&B(S_H(R(c)_{1(0)[1]3})R(c)_{2(0)[1]})R(c)_{3(0)[1]})R(c)_{4(0)[1]}))S_H(R(c)_{1(0)[0](-1)2})\\
&B(S_H(R(c)_{(0)1[0](-1)1}))B(R(c)_{1(0)[1]1})B(R(c)_{2(0)[0][1]1})B(R(c)_{4(0)[0](0)[1]})\\
&\otimes R(c)_{3(0)[0]}\otimes R(c)_{1(0)[1]2}R(c)_{2(0)[0][1]2}R(c)_{4(0)[0](-1)}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(1.10)}{=} R(c)_{(0)1[0](0)}R(R(c)_{(0)2[0][0]})R(S_C(R(c)_{(0)4[0](0)[0]}))\\
&\otimes R(c)_{(-1)1}B(R(c)_{(-1)2})B(S_H(R(c)_{(0)1[1]3})R(c)_{(0)2[1]})R(c)_{(0)3[1]})R(c)_{(0)4[1]}))\\
&S_H(R(c)_{(0)1[0](-1)2})B(S_H(R(c)_{(0)[0]1[0](-1)1}))\\
&B(R(c)_{(0)1[1]1})B(R(c)_{(0)2[0][1]1})B(R(c)_{(0)4[0](0)[1]})\\
&\otimes R(c)_{(0)3[0]}\otimes R(c)_{(0)1[1]2}R(c)_{(0)2[0][1]2}R(c)_{(0)4[0](-1)}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(1.10)}{=} R(c)_{(0)[0]1(0)[0]}R(R(c)_{(0)[0]2[0]})R(S_C(R(c)_{(0)[0]4(0)[0]}))\\
&\otimes R(c)_{(-1)1}B(R(c)_{(-1)2})B(S_H(R(c)_{(0)[1]}))S_H(R(c)_{(0)[0]1(-1)2})B(S_H(R(c)_{(0)[0]1(-1)1}))\\
&B(R(c)_{(0)[0]1(0)[1]1})B(R(c)_{(0)[0]2[1]1})B(R(c)_{(0)[0]4(0)[1]})\\
&\otimes R(c)_{(0)[0]3}\otimes R(c)_{(0)[0]1(0)[1]2}R(c)_{(0)[0]2[1]2}R(c)_{(0)[0]4(-1)}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(3.4)}{=} R(c_{[0]})_{1(0)[0]}R(\underline{R(c_{[0]})_{2[0][0]}})R(S_C(R(c_{[0]})_{4(0)[0]}))\\
&\otimes B(S_H(c_{[1]}))S_H(R(c_{[0]})_{1(-1)2})B(S_H(R(c_{[0]})_{1(-1)1}))\\
&B(R(c_{[0]})_{1(0)[1]1})\underline{B(R(c_{[0]})_{2[0][1]})}B(R(c_{[0]})_{4(0)[1]})\otimes R(c_{[0]})_3\\
&\otimes R(c_{[0]})_{1(0)[1]2}R(c_{[0]})_{2[1]}R(c_{[0]})_{4(-1)}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(3.8)}{=} R(c_{[0]})_{1(0)[0]}R(R(c_{[0]})_{2[0]})_{(0)[0]}\underline{R(S_C(R(c_{[0]})_{4(0)})_{[0]})}\\
&\otimes B(S_H(c_{[1]}))S_H(R(c_{[0]})_{1(-1)2})B(S_H(R(c_{[0]})_{1(-1)1}))B(R(c_{[0]})_{1(0)[1]1})\\
&B(S_H(R(R(c_{[0]})_{2[0]})_{(-1)1}))S_H(R(R(c_{[0]})_{2[0]})_{(-1)2})B(R(R(c_{[0]})_{2[0]})_{(0)[1]})\\
&\underline{B(S_C(R(c_{[0]})_{4(0)})_{[1]})}\otimes R(c_{[0]})_3\otimes R(c_{[0]})_{1(0)[1]2}R(c_{[0]})_{2[1]}R(c_{[0]})_{4(-1)}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(3.8)}{=} R(c_{[0]})_{1(0)[0]}R(R(c_{[0]})_{2[0]})_{(0)[0]}R(S_C(R(c_{[0]})_{4(0)}))_{(0)[0]}\\
&\otimes B(S_H(c_{[1]}))S_H(R(c_{[0]})_{1(-1)2})B(S_H(R(c_{[0]})_{1(-1)1}))B(R(c_{[0]})_{1(0)[1]1})\\
&B(S_H(R(R(c_{[0]})_{2[0]})_{(-1)1}))S_H(R(R(c_{[0]})_{2[0]})_{(-1)2})B(R(R(c_{[0]})_{2[0]})_{(0)[1]})\\
&B(S_H(R(S_C(R(c_{[0]})_{4(0)}))_{(-1)1}))S_H(R(S_C(R(c_{[0]})_{4(0)}))_{(-1)2})B(R(S_C(R(c_{[0]})_{4(0)}))_{(0)[1]})\\
&\otimes R(c_{[0]})_3\otimes R(c_{[0]})_{1(0)[1]2}R(c_{[0]})_{2[1]}R(c_{[0]})_{4(-1)}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(1.10)}{=} R(c_{[0]})_{1[0](0)[0]}R(R(c_{[0]})_{2[0]})_{(0)[0]}R(S_C(R(c_{[0]})_{4(0)}))_{(0)[0]}\\
&\otimes B(S_H(c_{[1]}))S_H(R(c_{[0]})_{1[0](-1)2})B(S_H(R(c_{[0]})_{1[0](-1)1}))B(R(c_{[0]})_{1[0](0)[1]})\\
&B(S_H(R(R(c_{[0]})_{2[0]})_{(-1)1}))S_H(R(R(c_{[0]})_{2[0]})_{(-1)2})B(R(R(c_{[0]})_{2[0]})_{(0)[1]})\\
&B(S_H(R(S_C(R(c_{[0]})_{4(0)}))_{(-1)1}))S_H(R(S_C(R(c_{[0]})_{4(0)}))_{(-1)2})B(R(S_C(R(c_{[0]})_{4(0)}))_{(0)[1]})\\
&\otimes R(c_{[0]})_3 \otimes R(c_{[0]})_{1[1]}R(c_{[0]})_{2[1]}R(c_{[0]})_{4(-1)}\\
&\overset{(1.10)}{=} R(c_{[0]})_{1[0]1(0)[0]}R(R(c_{[0]})_{1[0]2})_{(0)[0]}R(S_C(R(c_{[0]})_{3(0)}))_{(0)[0]}\\
&\otimes B(S_H(c_{[1]}))S_H(R(c_{[0]})_{1[0]1(-1)2})B(S_H(R(c_{[0]})_{1[0]1(-1)1}))B(R(c_{[0]})_{1[0]1(0)[1]})\\
&B(S_H(R(R(c_{[0]})_{1[0]2})_{(-1)1}))S_H(R(R(c_{[0]})_{1[0]2})_{(-1)2})B(R(R(c_{[0]})_{1[0]2})_{(0)[1]})\\
&B(S_H(R(S_C(R(c_{[0]})_{3(0)}))_{(-1)1}))S_H(R(S_C(R(c_{[0]})_{3(0)}))_{(-1)2})B(R(S_C(R(c_{[0]})_{3(0)}))_{(0)[1]})\\
&\otimes R(c_{[0]})_2 \otimes R(c_{[0]})_{1[1]}R(c_{[0]})_{3(-1)}\\
&= [R(c_{[0]})_{1[0]1}R(R(c_{[0]})_{1[0]2})R(S_C(R(c_{[0]})_{3(0)}))]_{(0)[0]}\\
&\otimes B(S_H(c_{[1]}))B(S_H[R(c_{[0]})_{1[0]1}R(R(c_{[0]})_{1[0]2}R(S_C(R(c_{[0]})_{3(0)})))]_{(-1)1})\\
&S_H[R(c_{[0]})_{1[0]1}R(R(c_{[0]})_{1[0]2})R(S_C(R(c_{[0]})_{3(0)}))]_{(-1)2}\\
&B[R(c_{[0]})_{1[0]1}R(R(c_{[0]})_{1[0]2}R(S_C(R(c_{[0]})_{3(0)})))]_{(0)[1]}\\
&\otimes R(c_{[0]})_2 \otimes R(c_{[0]})_{1[1]}R(c_{[0]})_{3(-1)}\\
&\overset{(3.3)}{=} R(c_{[0]1(0)})_{(0)[0]} \otimes B(S_H(c_{[1]}))B(S_H(R(c_{[0]1(0)})_{(-1)1}))S_H(R(c_{[0]1(0)})_{(-1)2})\\
&B(R(c_{[0]1(0)})_{(0)[1]}) \otimes R(c_{[0]2}) \otimes B(S_H(c_{[0]1(-1)}))\\
&\overset{(3.8)}{=} R(c_{[0]1(0)[0]}) \otimes B(S_H(c_{[1]}))B(c_{[0]1(0)[1]}) \otimes R(c_{[0]2}) \otimes B(S_H(c_{[0]1(-1)}))\\
&\overset{(1.10)}{=} R(c_{1[0](0)}) \otimes B(S_H(c_{1[1]2}c_{2[1]}))B(c_{1[1]1}) \otimes R(c_{2[0]}) \otimes B(S_H(c_{1[0](-1)}))\\
&= R(c_{1(0)}) \otimes B(S_H(c_{2[1]})) \otimes R(c_{2[0]}) \otimes B(S_H(c_{1(-1)})). \quad \blacksquare
\end{aligned}
$$

When the right H-coaction on C is trivial, that is, $c_{[0]} \otimes c_{[1]} = c \otimes 1_H$, we obtain the following results.

COROLLARY 3.4. *Let C and H be two commutative Hopf algebras such that (C, ρ^l) is a left H-comodule Hopf algebra. Assume that $R : C \to C$ is a linear map and B is a Rota–Baxter co-operator on H. Define a linear map $\widetilde{B} : C \ltimes H \to C \ltimes H$ by*

$$\widetilde{B}(c \ltimes h) = R(c_{(0)}) \ltimes B(c_{(-1)}h)$$

for all $c \in C$ and $h \in H$. Then $\widetilde{B}$ is a Rota–Baxter co-operator on $C \ltimes H$ if and only if R is a Rota–Baxter co-operator on C and satisfies the following conditions:

$$
\begin{aligned}
&R(c)_1R(R(c)_2S_C(R(c)_{3(0)})) \otimes R(c)_2 \otimes R(c)_{3(-1)}\\
&\qquad = R(c_{1(0)}) \otimes R(c_2) \otimes B(S_H(c_{1(-1)})), \qquad (3.9)
\end{aligned}
$$

$$R(c)_{(-1)1}B(R(c)_{(-1)2} \otimes R(c)_{(0)} = 1_H \otimes R(c). \qquad (3.10)$$

In Theorem 3.3, when the Rota–Baxter co-operators on C and H are replaced by S_C and S_H respectively, we get the following results directly.

Corollary 3.5. *Let C and H be two commutative Hopf algebras such that (C,ρ^l,ρ^r) is an H-bicomodule Hopf algebra. Assume that B is a Rota–Baxter co-operator on H. Define a linear map $\widetilde{B}: C \ltimes H \to C \ltimes H$ by*

$$\widetilde{B}(c \ltimes h) = S_C(c_{(0)[0]}) \ltimes B(c_{(-1)}c_{(0)[1]}h)$$

for all $c \in C$ and $h \in H$. Then $\widetilde{B}$ is a Rota–Baxter co-operator on $C \ltimes H$ if and only if R is a Rota–Baxter co-operator on C and satisfies the following conditions:

$$S_C(c_{1(0)}) \otimes c_{1(-1)} = S_C(c_{1(0)}) \otimes B(S_H(c_{1(-1)})), \tag{3.11}$$

$$c_{(-1)1}B(c_{(-1)2}) \otimes c_{(0)} = 1_H \otimes c. \tag{3.12}$$

Corollary 3.6. *Let C and H be two commutative Hopf algebras such that (C,ρ^l,ρ^r) is an H-bicomodule Hopf algebra. Assume that $R: C \to C$ is a linear map. Define a linear map $\widetilde{B}: C \ltimes H \to C \ltimes H$ by*

$$\widetilde{B}(c \ltimes h) = R(c_{(0)[0]}) \ltimes S_H(c_{(-1)}c_{(0)[1]}h)$$

for all $c \in C$ and $h \in H$. Then $\widetilde{B}$ is a Rota–Baxter co-operator on $C \ltimes H$ if and only if R is a Rota–Baxter co-operator on C and satisfies the following conditions

$$\begin{aligned} R(c)_{1[0]1}R(R(c)_{1[0]2}S_C(R(c)_{3(0)})) \otimes R(c)_2 \otimes R(c)_{1[1]}R(c)_{3(-1)} \\ = R(c_{1(0)}) \otimes R(c_2) \otimes c_{1(-1)}, \end{aligned} \tag{3.13}$$

$$R(c)_{[1]} \otimes R(c)_{[0]} = c_{[1]} \otimes R(c_{[0]}). \tag{3.14}$$

Example 3.7. Let C_2 and C_3 be the cyclic groups given in Example 2.8. Denote by k^{C_2} and k^{C_3} the linear duals of $k[C_2]$ and $k[C_3]$, respectively. Then k^{C_2} has a basis p_1, p_g dual to $1, g$, and k^{C_3} has a basis q_1, q_h, q_{h^2} dual to $1, h, h^2$. Define a k^{C_2}-bicomodule action on k^{C_3} by

$$\begin{aligned} &q_{1(-1)} \otimes q_{1(0)} = (p_1+p_g) \otimes q_1,\ q_{1[0]} \otimes q_{1[1]} = q_1 \otimes (p_1+p_g), \\ &q_{h(-1)} \otimes q_{h(0)} = (p_1+p_g) \otimes q_h,\ q_{h[0]} \otimes q_{h[1]} = q_h \otimes p_1 + q_{h^2} \otimes p_g, \\ &q_{h^2(-1)} \otimes q_{h^2(0)} = (p_1+p_g) \otimes q_{h^2},\ q_{h^2[0]} \otimes q_{h^2[1]} = q_{h^2} \otimes p_1 + q_h \otimes p_g. \end{aligned}$$

Then k^{C_3} is a k^{C_2}-bicomodule coalgebra.

(1) Define $R: k^{C_3} \to k^{C_3}$ by $R(x) = 1$ for all $x \in k^{C_3}$. Note that $p_1 + p_g$ is the unit of k^{C_2}. Then it is straightforward to verify that (3.13) and (3.14) are satisfied, and

$$\widetilde{B}(q_{h^i} \ltimes p_{g^j}) = 1 \ltimes p_{g^j}, \quad i = 0,1,2,\ j = 0,1.$$

is a Rota–Baxter co-operator on $k^{C_3} \ltimes k^{C_2}$.

(2) Define $R: k^{C_3} \to k^{C_3}$ by

$$R(q_1) = q_1, \quad R(q_h) = q_{h^2}, \quad R(q_{h^2}) = q_h.$$

Then $\widetilde{B}$ is the identity map on $k^{C_3} \ltimes k^{C_2}$, which is a Rota–Baxter cooperator since $k^{C_3} \ltimes k^{C_2}$ is cocommutative.

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