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A sharp Harnack bound for a nonlocal heat equation

by

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Summary. A sharp double-sided Harnack bound is derived for positive solutions of a fractional-order heat equation.

1. Introduction. Consider the classical linear heat equation, $\partial_t w - \partial_{xx} w = 0$, posed in an infinite strip $S_T = \mathbb{R} \times (0,T)$ for some $T \in (0,+\infty]$. If we restrict our attention to positive solutions, then the following important lower bound can be deduced:

$$\partial_t \ln w - |\partial_x \ln w|^2 \geq -\frac{1}{2t} \quad \text{in } S_T. \tag{1.1}$$

This result is an instance of a family of estimates derived by Aronson and Bénilan [1] to tackle the problem of regularity of solutions to the porous medium equation. A highly consequential generalisation to the setting of Riemannian manifolds, due to Li and Yau [9], has led to (1.1) being commonly associated with their names in the literature.

Note that inequality (1.1) makes no reference to the initial moment and applies to *all positive smooth solutions* regardless of their origin. In particular, no further assumptions on the asymptotic behaviour of solutions are required. On the contrary, it is this remarkable generality of the Aronson–Bénilan–Li–Yau (ABLY) bound that imposes restrictions on the admissible spatial growth of solutions and the nature of the initial object. It should be stressed that (1.1) is *sharp* – a rare and desirable property in the analysis of partial differential equations. Indeed, the bound is satisfied identically when

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evaluated on the fundamental solution,

$$g(x-x_*,t) := \frac{1}{\sqrt{4\pi t}}\,\mathrm{e}^{-\frac{|x-x_*|^2}{4t}},$$

irrespective of the location x_* of the source at the initial instant.

Estimate (1.1) is also called a *differential Harnack bound* for the linear heat flow. In line with this terminology, (1.1) may be appropriately integrated (see [7, 9]), to reveal the well-known classical parabolic Harnack estimate, due independently to Hadamard and Pini [6, 12],

$$\frac{w(x_2,t_2)}{w(x_1,t_1)} \geq \sqrt{\frac{t_1}{t_2}}\,\mathrm{e}^{-\frac{|x_2-x_1|^2}{4(t_2-t_1)}}, \tag{1.2}$$

where x_1 and x_2 are arbitrary and $0 < t_1 < t_2 < T$.

The differential bound (1.1) is *sharper* than the integrated one (1.2), as the latter reduces to identity only for $u(x,t) = g(x-x_*,t)$, with the initial mass concentrated at a particular point, related to (x_i,t_i) through

$$x_* = \frac{x_1t_2 - x_2t_1}{t_2-t_1}.$$

Observe moreover that for the bound to be useful, there needs to be a nonzero gap between t_1 and t_2, i.e., we cannot compare the solution at two different points in space at the same instant of time. This is a characteristic feature of parabolic Harnack bounds [11].

The following note is inspired by a natural problem, put forward in [5], of finding an appropriate extension for the ABLY inequality to the context of nonlocal diffusion and in particular to the canonical model of the fractional heat equation

$$\partial_t u + (-\Delta)^{\alpha/2}u = 0 \quad \text{ on } S_T$$

with $0 < \alpha < 2$. There are numerous definitions of the fractional power of the Laplace operator (see e.g. [8]), none of which will be employed in any of the considerations below. However, to fix ideas, we may opt for the standard potential-theoretic definition:

$$(-\Delta)^{\alpha/2}f(x) := C(n,\alpha)\,\mathrm{P.V.}\int_{\mathbb{R}} \frac{f(x)-f(y)}{|x-y|^{n+\alpha}}\,\mathrm{d}y,$$

where

$$C(n,\alpha) := \frac{2^\alpha \Gamma\left(\frac{n+\alpha}{2}\right)}{\pi^{n/2}|\Gamma(-\alpha/2)|}.$$

For the purpose of this note we will restrict attention to one space dimension and $\alpha = 1$, i.e., the *half-Laplacian*, where

$$\partial_t u + (-\Delta)^{1/2}u = 0 \quad \text{ on } S_T, \tag{1.3}$$

with

$$(-\Delta)^{1/2} f(x) = -\frac{1}{\pi} \, \mathrm{P.V.} \int\limits_{\mathbb{R}} \frac{f(x)-f(y)}{|x-y|^2} \, \mathrm{d}y.$$

Before we begin, let us define the notion of a strong solution. Following [2], we will say that $u(x,t)$ is a *strong solution* of the fractional heat equation (1.3) in the strip S_T if

- $\partial_t u \in C(\mathbb{R} \times (0,T))$,
- $u \in C(\mathbb{R} \times [0,T))$,
- the equation (1.3) is satisfied pointwise for every $(x,t) \in S_T$.

The same definition, with obvious modifications, applies to the classical heat equation.

One promising line of inquiry is to find a fractional counterpart of the ABLY inequality. Note that (1.1) is equivalent to

$$\partial_{xx} \ln w \geq -\frac{1}{2t}$$

in S_T. This form of the inequality has been generalised to the fractional setting in [15] where the following elegant inequality has been proven.

THEOREM 1.1 ([15, Thm. 3.2 + Prop. 3.3]). *Let $u : S_T \mapsto (0,\infty)$ be a strong solution to the fractional heat equation* (1.3)*. Then the Li–Yau-type inequality*

$$-(-\Delta)^{1/2} \ln u \geq -\frac{1}{2t}$$

holds in S_T.

This inequality may then be employed to derive a Harnack bound.

THEOREM 1.2 ([15, Thm. 5.2]). *Let $0 < t_1 < t_2 < \infty$ and $x_1, x_2 \in \mathbb{R}$. If u is a positive strong solution of* (1.3) *on $\mathbb{R} \times [0,\infty)$, then*

$$\frac{u(x_2,t_2)}{u(x_1,t_1)} \geq \sqrt{\frac{t_1}{t_2}} \, \mathrm{e}^{-C[1+\frac{|x_2-x_1|^2}{(t_2-t_1)^2}]} \tag{1.4}$$

for some positive constant C.

For a more general statement and further results consult [15]. In that work, the authors explain how their bound differs from the Hadamard–Pini estimate. In particular, (1.4) does not result in equality when applied to the fractional heat kernel with an appropriate choice of (x_i,t_i). Moreover, due to the polynomial behaviour of the heat kernel, it is expected that the sharp Harnack bound would display a polynomial rather than exponential decay. Lastly, it would be desirable not to assume continuity of the solution at the initial time, thus allowing for generalised initial conditions.

Another result in this direction is provided in [4]. There the authors consider a weaker class of solutions, the very weak solutions. If we assume that the solution is also strong, then we obtain the following result in our setting.

Theorem 1.3 ([4, Thm. 8.2]). *Let u be a positive strong solution of* (1.3) *for $t > 0$. Suppose moreover that the initial condition is dominated by the fractional heat kernel away from the origin, in the following sense:*

$$0 \le u_0(x) \le \frac{1}{1+|x|^2}$$

for $|x| \ge R_0 \ge 0$. Then, for all $x_1, x_2 \in \mathbb{R}$ and $t_1, t_2 > 0$, we have

$$u(x_2,t_2) \le C\frac{t_2}{t_1}\left[1+\frac{\sqrt{|t_2-t_1|}+\big||x_1|^2-|x_2|^2\big|}{\sqrt{t_2}+|x_2|^2}\right]u(x_1,t_1) \tag{1.5}$$

for some constant C depending on R_0.

The bound (1.5) does reflect the decay rate of the fractional heat kernel. Since t_1 and t_2 need not be ordered, this bound, unlike the Hadamard–Pini estimate, is double-sided. This interesting feature is a result of nonlocality of the fractional flow. Here, however, a constraint is placed on the initial condition. Since one of the uses of Harnack bounds is to obtain constraints on the initial data [7], it is desirable to have bounds, derivation of which avoids introducing restrictions on the initial datum.

Our contribution in this note concerns obtaining an *unconditional* bound, that is, apart from the solution being classical and positive, we do not impose any further restrictions on the spatial growth of solutions and of the initial data.

In order to demonstrate that no additional requirement is needed, we will refer to the fractional counterpart of the classical uniqueness theorem of Widder [2, 16]. Our approach applies to both the classical heat equation and its fractional counterpart.

Let us first re-prove (1.2) through the convolution formula for the heat equation. To begin, suppose that w is a nonnegative strong solution on $\mathbb{R} \times [0,T)$ with initial condition w_0. By Widder's representation and uniqueness theorems [16], the solution is unique and expressed by the integral

$$w(x,t) = \int_{\mathbb{R}} g(x-y,t)w_0(y)\,\mathrm{d}y$$

with the kernel

$$g(x,t) = \frac{1}{\sqrt{4\pi t}}\,\mathrm{e}^{-\frac{|x|^2}{4t}}.$$

Since this kernel is strictly positive, nothing prevents us from performing an elementary estimate

$$\begin{aligned} w(x_2,t_2) &= \int_{\mathbb{R}} \frac{g(x_2-y,t_2)}{g(x_1-y,t_1)} g(x_1-y,t_1) w_0(y)\,\mathrm{d}y \\ &\geq \left[\inf_{y\in\mathbb{R}} \frac{g(x_2-y,t_2)}{g(x_1-y,t_1)}\right] \int_{\mathbb{R}} g(x_1-y,t_1) w_0(y)\,\mathrm{d}y \\ &= w(x_1,t_1) \inf_{y\in\mathbb{R}} \frac{g(x_2-y,t_2)}{g(x_1-y,t_1)}. \end{aligned}$$

It is easy to see that whenever $0<\sigma_1<\sigma_2<+\infty$, we have

$$\frac{|x_1-y|^2}{\sigma_1} - \frac{|x_2-y|^2}{\sigma_2} \geq -\frac{|x_2-x_1|^2}{\sigma_2-\sigma_1},$$

which in turn, when applied to the heat kernel, yields

$$\mathrm{e}^{-\frac{|x_2-y|^2}{4\sigma_2}} / \mathrm{e}^{-\frac{|x_1-y|^2}{4\sigma_1}} \geq \mathrm{e}^{-\frac{|x_2-x_1|^2}{4(\sigma_2-\sigma_1)}},$$

i.e.,

$$\sqrt{\frac{\sigma_2}{\sigma_1}}\, \frac{g(x_2-y,\sigma_2)}{g(x_1-y,\sigma_1)} \geq \sqrt{4\pi(\sigma_2-\sigma_1)}\, g(x_2-x_1,\sigma_2-\sigma_1). \tag{1.6}$$

This inequality is sharp and yields equality for $y=x_*$ with

$$x_* := \frac{x_1\sigma_2 - x_2\sigma_1}{\sigma_2-\sigma_1}.$$

Hence, with $t_i=\sigma_i$, we find

$$w(x_2,t_2) \geq w(x_1,t_1)\sqrt{\frac{t_1}{t_2}}\sqrt{4\pi(t_2-t_1)}\, g(x_2-x_1,t_2-t_1),$$

which is (1.2).

Thus, we arrive at the desired classical estimate, the only issue being that the initial data (and so the solution class covered) is restricted, as compared to the technique resting on the ABLY inequality.

We will now lift this restriction. Suppose w is a positive smooth solution in $\mathbb{R}\times(0,T)$. Choose $0<\tau<t_1$ and consider w^τ, the restriction of w to $\mathbb{R}\times[\tau,T)$. Clearly, w^τ is a strong solution on its domain and by the Widder representation and uniqueness theorems we have

$$w^\tau(x,t) = \int_{\mathbb{R}^n} g(x-y,t-\tau) w(y,\tau)\,\mathrm{d}y$$

with $w^\tau = w$ in $\mathbb{R}\times(\tau,T)$. We can now perform the same estimate as before but with $\sigma_i = t_i-\tau$ in place of t_i. In effect, we find

$$\frac{w(x_2,t_2)}{w(x_1,t_1)} = \frac{w^\tau(x_2,t_2)}{w^\tau(x_1,t_1)} \geq \sqrt{\frac{t_1-\tau}{t_2-\tau}}\sqrt{4\pi(t_2-t_1)}\, g(x_2-x_1,t_2-t_1).$$

We are free to apply this procedure with any choice of $\tau \in (0, t_1)$ and so

$$\frac{w(x_2,t_2)}{w(x_1,t_1)} \geq \sqrt{4\pi(t_2-t_1)}\, g(x_2-x_1, t_2-t_1) \sup_{0<\tau<t_1} \sqrt{\frac{t_1-\tau}{t_2-\tau}}$$
$$= \sqrt{\frac{t_1}{t_2}} \sqrt{4\pi(t_2-t_1)}\, g(x_2-x_1, t_2-t_1).$$

Thus, provided the solution we work with is smooth and positive in $\mathbb{R}\times(0,T)$, we do not need to impose additional constraints on the initial condition.

We see that it is the Widder representation and uniqueness theorems that ensure that our straightforward estimation covers the proper class of solutions, without unnecessary additional restrictions. To apply a similar reasoning to the fractional heat flow we need an appropriate generalisation of these theorems. Indeed, such a result is available [2] and below we cite a version tailored to our needs.

Theorem 1.4 ([2, Thms. 1.4 and 2.1]). *Let v be a nonnegative strong solution of the problem*

$$\partial_t v + (-\Delta)^{1/2} v = 0 \qquad \text{in } \mathbb{R}\times[\tau, T)$$

with $v(\cdot,\tau) = \nu \in C(\mathbb{R})$. Then v is unique and admits the representation

$$v(x,t) = \int_{\mathbb{R}} k(x-y, t-\tau)\nu(y)\,\mathrm{d}y$$

with

$$k(x,t) := \frac{1}{\pi t}\frac{1}{1+|x|^2/t^2}.$$

In particular, since the kernel is smooth, the solution needs to be smooth in $\mathbb{R}\times(\tau,T)$. Actually, in this one-dimensional context and for the square root of the Laplace operator, the above representation was known to Widder even before his celebrated representation theorem for the classical heat flow [10].

First, we will derive a simple counterpart of (1.2) that relies on a lemma inspired by the appealing inequality (1.6).

Lemma 1.5. *Let $0 < \sigma_1 < \sigma_2 < +\infty$ and $x_1, x_2 \in \mathbb{R}$. Then*

$$\frac{\sigma_2}{\sigma_1}\frac{k(x_2-y,\sigma_2)}{k(x_1-y,\sigma_1)} \geq \pi(\sigma_2-\sigma_1)k(x_2-x_1,\sigma_2-\sigma_1).$$

Proof. Set $\lambda = \frac{|x_2-x_1|^2}{(\sigma_2-\sigma_1)^2}$. We need to show that

$$\frac{1+|x_1-y|^2/\sigma_1^2}{1+|x_2-y|^2/\sigma_2^2} \geq \frac{1}{1+\frac{|x_2-x_1|^2}{(\sigma_2-\sigma_1)^2}},$$

which is equivalent to

$$\sigma_2^2|x_1-y|^2(1+\lambda) - \sigma_1^2|x_2-y|^2 + \lambda\sigma_1^2\sigma_2^2 \geq 0.$$

This in turn may be rephrased as a quadratic inequality in y:

$$Ay^2 + By + C \geq 0$$

with

$$\begin{cases} A = (1+\lambda)\sigma_2^2 - \sigma_1^2, \\ B = 2\sigma_1^2 x_2 - 2(1+\lambda)\sigma_2^2 x_1, \\ C = (1+\lambda)\sigma_2^2 x_1^2 - \sigma_1^2 x_2^2 + \lambda\sigma_1^2\sigma_2^2. \end{cases}$$

Since $\lambda \geq 0$ and $\sigma_2 > \sigma_1$, we have $A > 0$. Hence, it suffices to check that the discriminant is not positive. After a brief calculation, this amounts to the requirement that

$$4\sigma_1^2\sigma_2^2(1+\lambda)|x_2 - x_1|^2 - 4\lambda\sigma_1^2\sigma_2^2[(1+\lambda)\sigma_2^2 - \sigma_1^2] \leq 0.$$

Now, $\sigma_i \neq 0$ and by definition $|x_2 - x_1|^2 = \lambda(\sigma_2 - \sigma_1)^2$. Hence, the above simplifies further to

$$\lambda[(1+\lambda)(2\sigma_1\sigma_2 - \sigma_1^2) - \sigma_1^2] \geq 0,$$

which is true, since $\lambda \geq 0$ and $\sigma_2 > \sigma_1$. ∎

THEOREM 1.6. *Let u be a positive smooth solution of the fractional heat equation* (1.3) *in $\mathbb{R} \times (0,T)$. Then, given $0 < t_1 < t_2 < T$ and $x_1, x_2 \in \mathbb{R}$, we have*

$$\frac{u(x_2,t_2)}{u(x_1,t_1)} \geq \frac{t_1}{t_2} \frac{1}{1 + \frac{|x_2-x_1|^2}{(t_2-t_1)^2}}.$$

Proof. Choose $0 < \tau < t_1$ and restrict u to $\mathbb{R} \times [\tau, T)$, where it becomes a strong solution. By the Widder-type representation and uniqueness theorem [2], we may write

$$u(x,t) = \int_{\mathbb{R}} k(x-y, t-\tau) u(y,\tau)\,\mathrm{d}y.$$

Further, due to Lemma 1.5 with $\sigma_i = t_i - \tau$, we have

$$\begin{aligned} u(x_2,t_2) &= \int_{\mathbb{R}} \frac{k(x_2-y, t_2-\tau)}{k(x_1-y, t_1-\tau)} k(x_1-y, t_1-\tau) u(y,\tau)\,\mathrm{d}y \\ &\geq \frac{t_1-\tau}{t_2-\tau} \frac{1}{1 + \frac{|x_2-x_1|^2}{(t_2-t_1)^2}} u(x_1,t_1). \end{aligned}$$

The above inequality is valid for all $\tau \in (0, t_1)$ and so we may optimise by taking τ arbitrarily small, arriving at

$$u(x_2,t_2) \geq \frac{t_1}{t_2} \frac{1}{1 + \frac{|x_2-x_1|^2}{(t_2-t_1)^2}} u(x_1,t_1),$$

as required. ∎

The above theorem enjoys some of the expected properties of a Harnack bound for (1.3) in that it is related to the fractional heat kernel and its decay properties. However, it provides only a one-sided estimate, despite the expectation of a double-sided bound, and it fails to be sharp even for the fractional heat kernel, regardless of the location of the source.

Next, we move on to our main result, i.e., an optimal fractional counterpart of the Hadamard–Pini bound.

THEOREM 1.7. *Let u be a positive classical solution of* (1.3) *on* $\mathbb{R}\times(0,T)$. *Given* $0 < t_1, t_2 < T$ *and* $x_1, x_2 \in \mathbb{R}$, *we have*

$$M_* \leq \frac{u(x_2,t_2)}{u(x_1,t_1)} \leq M^* \tag{1.7}$$

with

$$M_* = \frac{t_1^2 + t_2^2 + |x_1 - x_2|^2 - \sqrt{\kappa_0}}{2t_1t_2},$$
$$M^* = \frac{t_1^2 + t_2^2 + |x_1 - x_2|^2 + \sqrt{\kappa_0}}{2t_1t_2},$$

where

$$\kappa_0 = (|x_2 - x_1|^2 + |t_2 - t_1|^2)(|x_2 - x_1|^2 + |t_2 + t_1|^2).$$

Moreover,

$$M_* = \frac{k(x_2 - x_*, t_2)}{k(x_1 - x_*, t_1)} \quad \text{and} \quad M^* = \frac{k(x_2 - x^*, t_2)}{k(x_1 - x^*, t_1)}$$

where

$$x_* = \frac{x_1^2 - x_2^2 - t_2^2 + t_1^2 + \sqrt{\kappa_0}}{2(x_2 - x_1)},$$
$$x^* = \frac{x_1^2 - x_2^2 - t_2^2 + t_1^2 - \sqrt{\kappa_0}}{2(x_2 - x_1)}$$

when $x_1 \neq x_2$, *and* $x_* = x^* = x_1 = x_2$ *otherwise.*

Before laying out the proof, it is worth stressing that, just as in the estimate of [4], our result provides bounds irrespective of the order of t_1 and t_2.

The formulae for M_* and M^* are presented in a possibly simple form that emphasises the spatial $|x_2 - x_1|$ and temporal $|t_2 - t_1|$ distance between the points (x_1, t_1) and (x_2, t_2). In the Hadamard–Pini bound (1.2), the counterpart of M_* depends on the points (x_i, t_i) through the distances $|x_2 - x_1|$ and $|t_2 - t_1|$ alone. In the fractional case however our estimate is also sensitive to the life-span of the solution prior to the instants t_1 and t_2.

Proof of Theorem 1.7. Fix (x_1, t_1), (x_2, t_2) and $0 < \tau < \min\{t_1, t_2\}$. As before, when considered on $\mathbb{R} \times [\tau, T)$, the solution u is strong and due to the representation formula on $C(\mathbb{R}\times[\tau,T))$ we may write

$$\begin{aligned} u(x_2,t_2) &= \int_{\mathbb{R}} k(x_2-y,t_2-\tau)u(y,\tau)\,\mathrm{d}y \\ &= \int_{\mathbb{R}} \frac{k(x_2-y,t_2-\tau)}{k(x_1-y,t_1-\tau)} k(x_1-y,t_1-\tau)u(y,\tau)\,\mathrm{d}y. \end{aligned}$$

Denote

$$m(y) := \frac{k(x_2-y,t_2-\tau)}{k(x_1-y,t_1-\tau)}.$$

Since u is positive we have

$$\inf_y m(y) \le \frac{u(x_2,t_2)}{u(x_1,t_1)} \le \sup_y m(y). \tag{1.8}$$

In the calculations below it will be expedient to introduce new coordinates:

$$\omega(y) := \frac{1}{2}\left(\frac{x_2-y}{t_2-\tau}+\frac{x_1-y}{t_1-\tau}\right) \quad \text{and} \quad \rho(y) := \frac{1}{2}\left(\frac{x_2-y}{t_2-\tau}-\frac{x_1-y}{t_1-\tau}\right).$$

In particular,

$$\omega-\rho = \frac{x_1-y}{t_1-\tau} \quad \text{and} \quad \omega+\rho = \frac{x_2-y}{t_2-\tau}.$$

We have

$$m(y) = \frac{t_1-\tau}{t_2-\tau}\,\frac{k\left(\frac{x_2-y}{t_2-\tau},1\right)}{k\left(\frac{x_1-y}{t_1-\tau},1\right)} = \frac{t_1-\tau}{t_2-\tau}\,\frac{1+|\omega-\rho|^2}{1+|\omega+\rho|^2}. \tag{1.9}$$

This function is smooth and approaches $\frac{t_2-\tau}{t_1-\tau}$ as $|y|\to+\infty$. Hence, there exist global bounds and they may be computed explicitly. To this end it will be more convenient to work with the function

$$H(y) := \ln\left[\frac{k(x_2-y,t_2-\tau)}{k(x_1-y,t_1-\tau)}\right].$$

We find that

$$H' = 2\frac{\omega-\rho}{1+|\omega-\rho|^2}(\omega'-\rho') - 2\frac{\omega+\rho}{1+|\omega+\rho|^2}(\omega'+\rho')$$

or equivalently

$$\begin{aligned} &\frac{(1+|\omega+\rho|^2)(1+|\omega-\rho|^2)}{2}H' \\ &\qquad = (\omega-\rho)(\omega'-\rho')(1+|\omega+\rho|^2) \\ &\qquad\quad - (\omega+\rho)(\omega'+\rho')(1+|\omega-\rho|^2) \\ &\qquad = -2(\rho\omega)' - 2(\omega-\rho)(\omega+\rho)(\rho'\omega-\rho\omega'). \end{aligned} \tag{1.10}$$

Now

$$4\rho\omega = \frac{|x_2-y|^2}{(t_2-\tau)^2} - \frac{|x_1-y|^2}{(t_1-\tau)^2}, \qquad -2(\rho\omega)' = \frac{x_2-y}{(t_2-\tau)^2} - \frac{x_1-y}{(t_1-\tau)^2}$$

and

$$-2(\rho'\omega - \rho\omega') = -\frac{x_2 - x_1}{(t_2-\tau)(t_1-\tau)}.$$

Hence

$$-2(\rho\omega)' - 2(\omega-\rho)(\omega+\rho)(\rho'\omega - \rho\omega') = \frac{x_2 - y}{(t_2-\tau)^2} - \frac{x_1 - y}{(t_1-\tau)^2} - \frac{(x_2-y)(x_1-y)(x_2-x_1)}{(t_2-\tau)^2(t_1-\tau)^2}.$$

Thus, if we multiply (1.10) through $(t_2-\tau)^2(t_1-\tau)^2$ we get

$$\frac{(|x_2-y|^2 + (t_2-\tau)^2)(|x_1-y|^2+(t_1-\tau)^2)}{2} H' = Q(y)$$

with

$$Q(y) := (x_2-y)(t_1-\tau)^2 - (x_1-y)(t_2-\tau)^2 - (x_2-y)(x_1-y)(x_2-x_1).$$

Thus, the sign of H' coincides with the sign of Q. Now Q is simply a quadratic function in y,

$$Q(y) = Ay^2 + By + C$$

with

$$\begin{aligned} &A = x_1 - x_2, \qquad B = x_2^2 - x_1^2 + (t_2-\tau)^2 - (t_1-\tau)^2,\\ &C = (t_1-\tau)^2 x_2 - (t_2-\tau)^2 x_1 - x_1x_2(x_2-x_1). \end{aligned}$$

In order to obtain explicit bounds on $m(y)$ we will divide the argument into four cases which we will later unite into one expression.

Case 1: *$x_1 = x_2$ and $t_1 < t_2$.* We have $A = 0$ and $B > 0$. The global minimum of $m(y)$ is attained at $y = x_1 = x_2$, while the supremum is given by the asymptotic value of $m(y)$ as $|y| \to +\infty$:

$$\min_y m(y) = \frac{t_1-\tau}{t_2-\tau} \quad \text{and} \quad \sup_y m(y) = \frac{t_2-\tau}{t_1-\tau}.$$

Case 1′: *$x_1 = x_2$ and $t_1 > t_2$.* Now $A = 0$ but $B < 0$ and a symmetric argument yields

$$\min_y m(y) = \frac{t_2-\tau}{t_1-\tau} \quad \text{and} \quad \sup_y m(y) = \frac{t_1-\tau}{t_2-\tau}.$$

Case 2: *$x_1 < x_2$.* When $x_1 \neq x_2$, the discriminant of Q,

$$\kappa = (|x_2-x_1|^2 + |(t_2-\tau)-(t_1-\tau)|^2)(|x_2-x_1|^2 + |(t_2-\tau)+(t_1-\tau)|^2),$$

is strictly positive. Let

$$y_\pm := \frac{x_1^2 - x_2^2 - (t_2-\tau)^2 + (t_1-\tau)^2 \pm \sqrt{\kappa}}{2(x_1-x_2)}$$

be the roots of Q. Since $A<0$ we have $0<y_+<y_-<+\infty$. Also,

$$\min_y m(y) = m(y_+) \quad \text{and} \quad \max_y m(y) = m(y_-).$$

When (1.10) is evaluated at $y_\pm$ we find

$$(\omega-\rho)(\omega'-\rho')(1+|\omega+\rho|^2) = (\omega+\rho)(\omega'+\rho')(1+|\omega-\rho|^2).$$

We also note that at $y_\pm$ we have $(\omega+\rho)(\omega'+\rho')\neq 0$. This follows since otherwise we would have

$$(\omega+\rho)(\omega'+\rho') = -\frac{x_2-y_\pm}{(t_2-\tau)^2} = 0 \quad \text{and} \quad (\omega-\rho)(\omega'-\rho') = -\frac{x_1-y_\pm}{(t_1-\tau)^2} = 0,$$

which would in turn force $y_-=y_+=x_1=x_2$, contrary to assumption. We can therefore write

$$\frac{1+|\omega-\rho|^2}{1+|\omega+\rho|^2} = \frac{(\omega-\rho)(\omega'-\rho')}{(\omega+\rho)(\omega'+\rho')} = \left(\frac{t_2-\tau}{t_1-\tau}\right)^2 \frac{x_1-y_\pm}{x_2-y_\pm}.$$

Putting this together with (1.9) we obtain

$$m(y_\pm) = \frac{t_1-\tau}{t_2-\tau}\,\frac{1+|\omega-\rho|^2}{1+|\omega+\rho|^2} = \frac{t_2-\tau}{t_1-\tau}\,\frac{x_1-y_\pm}{x_2-y_\pm}.$$

Next,

$$\min_y m(y) = m(y_+) = \frac{t_2-\tau}{t_1-\tau}\,\frac{(t_2-\tau)^2-(t_1-\tau)^2+|x_1-x_2|^2-\sqrt{\kappa}}{(t_2-\tau)^2-(t_1-\tau)^2-|x_1-x_2|^2-\sqrt{\kappa}} \tag{1.11}$$

and

$$\max_y m(y) = m(y_-) = \frac{t_2-\tau}{t_1-\tau}\,\frac{(t_2-\tau)^2-(t_1-\tau)^2+|x_1-x_2|^2+\sqrt{\kappa}}{(t_2-\tau)^2-(t_1-\tau)^2-|x_1-x_2|^2+\sqrt{\kappa}}. \tag{1.12}$$

CASE $2'$: $x_1>x_2$. With $A>0$ a symmetrical reasoning yields $0<y_-<y_+<+\infty$ with

$$\min_y m(y) = m(y_+) \quad \text{and} \quad \max_y m(y) = m(y_-).$$

All of the above calculations are valid for any $0<\tau<\min\{t_1,t_2\}$ and we will now argue that optimal bounds are obtained in the limit of $\tau\to 0$.

In Case 1 we may define

$$M_* := \sup_\tau \frac{t_1-\tau}{t_2-\tau} = \frac{t_1}{t_2} \quad \text{and} \quad M^* := \inf_\tau \frac{t_2-\tau}{t_1-\tau} = \frac{t_2}{t_1},$$

while in Case $1'$,

$$M_* := \sup_\tau \frac{t_2-\tau}{t_1-\tau} = \frac{t_2}{t_1} \quad \text{and} \quad M^* := \inf_\tau \frac{t_1-\tau}{t_2-\tau} = \frac{t_1}{t_2}.$$

For Cases 2 and $2'$ we note that when τ is considered a variable then the roots $y_\pm$ as well as $m(y_\pm)$ become functions of τ. Define

$$m_*(\tau) := m(y_+(\tau),\tau) \quad \text{and} \quad m^*(\tau) := m(y_-(\tau),\tau),$$

where $m(y_+(\tau),\tau)$ and $m(y_-(\tau),\tau)$ are given by (1.11) and (1.12) respectively.

Due to the definition of $m_*(\tau)$, $m^*(\tau)$, the representation Theorem 1.4 and the bound (1.8) we have

$$m_*(\tau)v(x_1,t_1) \le v(x_2,t_2) \quad \text{and} \quad v(x_2,t_2) \le m^*(\tau)v(x_1,t_1)$$

for any nonnegative strong solution v. In particular, for the initial data $\nu(x) = k(x-\xi,\varepsilon)$ for some arbitrary source $\xi \in \mathbb{R}$ we obtain the unique positive strong solution $v(x,t) = k(x-\xi, t-\tau+\varepsilon)$ to which the bound also applies:

$$m_*(\tau) \le \frac{k(x_2-\xi, t_2-\tau+\varepsilon)}{k(x_1-\xi, t_1-\tau+\varepsilon)} \le m^*(\tau).$$

When we let $\varepsilon \to 0$ we see that $m_*(\tau)$ and $m^*(\tau)$ are respectively the largest lower and the least upper bound on the whole set of solutions admissible by Theorem 1.4. By the uniqueness of solutions these bounds can only get tighter when $\tau \to 0$. The formulae for $m_*(\tau)$ and $m^*(\tau)$ are valid when $\tau = 0$ so that the optimal bounds are as follows:

$$M_* := \sup_\tau m_*(\tau) = m_*(0) = \frac{t_2}{t_1}\,\frac{t_2^2 - t_1^2 + |x_1-x_2|^2 - \sqrt{\kappa_0}}{t_2^2 - t_1^2 - |x_1-x_2|^2 - \sqrt{\kappa_0}},$$
$$M^* := \inf_\tau m^*(\tau) = m^*(0) = \frac{t_2}{t_1}\,\frac{t_2^2 - t_1^2 + |x_1-x_2|^2 + \sqrt{\kappa_0}}{t_2^2 - t_1^2 - |x_1-x_2|^2 + \sqrt{\kappa_0}}$$

with

$$\kappa_0 = (|x_2-x_1|^2 + |t_2-t_1|^2)(|x_2-x_1|^2 + |t_2+t_1|^2).$$

These bounds may be further restated as

$$M_* = \frac{t_1^2 + t_2^2 + |x_1-x_2|^2 - \sqrt{\kappa_0}}{2t_1t_2}, \tag{1.13}$$

$$M^* = \frac{t_1^2 + t_2^2 + |x_2-x_1|^2 + \sqrt{\kappa_0}}{2t_1t_2}. \tag{1.14}$$

It may seem at first sight that M_* and M^* do not depend on whether $t_1 < t_2$ or vice versa and it may therefore not be obvious that they reduce to Case 1 and Case $1'$ when we set $x_1 = x_2$. Observe, however, that the required dependence on the order of t_1 and t_2 is provided by the discriminant κ_0:

$$\sqrt{\kappa_0} = |t_2-t_1|\,|t_2+t_1| = \begin{cases} t_2^2 - t_1^2 & \text{when } t_1 < t_2, \\ t_1^2 - t_2^2 & \text{when } t_1 > t_2. \end{cases}$$

Thus we may take formulae (1.13) and (1.14) to be valid in all cases and combine the bounds into one statement independent of the mutual relation of (x_1,t_1) and (x_2,t_2):

$$\frac{t_1^2+t_2^2+|x_1-x_2|^2-\sqrt{\kappa_0}}{2t_1t_2} \le \frac{u(x_2,t_2)}{u(x_1,t_1)} \le \frac{t_1^2+t_2^2+|x_1-x_2|^2+\sqrt{\kappa_0}}{2t_1t_2}$$

for all $(x_1,t_1),(x_2,t_2) \in \mathbb{R}\times(0,T)$.

Finally, let us denote $x_* := y_+(0)$ and $x^* := y_-(0)$. Then

$$m_* = \frac{k(x_2-x_*,t_2)}{k(x_1-x_*,t_1)} \quad \text{and} \quad m^* = \frac{k(x_2-x^*,t_2)}{k(x_1-x^*,t_1)}.$$

Note that $x_* \le x^*$ when $x_1 \le x_2$, and $x_* > x^*$ otherwise. ∎

2. Discussion. One striking difference between the Hadamard–Pini bound (1.2) and its fractional counterpart (1.7) is the double-sidedness of the latter. When perceived as a model of diffusion that – morally speaking – should share some broad characteristics with the standard heat flow, it may appear surprising that the fractional flow admits an upper bound. Upon closer inspection we see, however, that replacing the Laplace operator with its fractional power radically constrains the space of initial data, or – if we choose not to refer to the initial condition – the admissible growth of solutions in the spatial direction. This limitation is a necessary prerequisite for using $(-\Delta)^{1/2}$ in the first place. No such *a priori* constraint is found in the local case. The relatively narrow domain of the fractional operator means that some of the intended applications of the Harnack bound become irrelevant. For example, the Aronson–Bénilan–Li–Yau inequality may be used to characterise those positive smooth solutions of the heat equation that exist on a strip $\mathbb{R}\times(0,T)$, leading in effect to Tikhonov-type conditions [7]. The necessity of such constraints stems from the fact that positive smooth solutions of the heat equation may blow up in finite time, in case there is so much heat "tucked away at space-infinity" that the averaging process commanded by the diffusion operator cannot redistribute it efficiently enough and there comes a time when the solution becomes unbounded everywhere. Following this line of thought, we may use the Hadamard–Pini bound to characterise those initial conditions that give rise to global-in-time solutions. Considerations of this type are unnecessary for the fractional heat equation (1.3), since *all positive smooth solutions are global* [2, 14]. This circumstance is not limited to the particular instance of the fractional heat flow considered here but applies more broadly to other exponents and nonlocal operators.

The classical heat equation may be seen as an endpoint case of the family of fractional heat equations parametrised by the order of the fractional Laplacian operator. It seems reasonable to expect that the associated Harnack bounds and differential Harnack bounds would also form parametrised families of inequalities. The Hadamard–Pini bound is one-sided, while its fractional counterpart is double-sided. This adds another layer of difficulty to the problem of finding a proper extension of the powerful Li–Yau-type argument to the fractional case. In the case of the standard heat flow, the transition from the differential Harnack bound to the Hadamard–Pini estimate involves integration along a straight line segment (a geodesic segment in the setting of Riemannian manifolds) connecting the space-time points being considered. Adjusting this argument to accommodate two paths – one optimal for the lower bound and one optimal for the upper bound – is not obvious, especially given the fundamental nature of the geodesic path used in the original argument.

In this note we addressed the diffusion process driven by the "half-Laplace" operator and we took advantage of the explicit form of the heat kernel. The aim was to obtain, by direct computation, a sharp Harnack bound, free of the artifacts brought about by standard estimation techniques. Although there is no such representation of fractional heat kernels for other powers of the Laplace operator, precise asymptotic bounds have been available for a long time [3, 13]. The exact form of the Harnack bound (1.7) sheds light on the way in which the relative position of the two space-time points manifests itself in the estimate. A generalisation of this result to other powers of the operator and arbitrary spatial dimension will be addressed in a separate paper.

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