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Ramification groups of Galois extensions over local fields of positive characteristic with Galois group isomorphic to the group of unitriangular matrices

by

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Abstract. We study the ramification groups of finite Galois extensions L/K of a complete discrete valuation field K of equal characteristic $p > 0$ with perfect residue field and Galois group isomorphic to the group of unitriangular matrices $UT_n(\mathbb{F}_p)$ over $\mathbb{F}_p$. We show that the upper ramification breaks can be expressed as a linear function of the valuation of the entries of a matrix directly constructed from the coefficients of a defining equation of the extension. This allows us to compute the ramification groups without using any elements of $L - K$.

1. Introduction. Let K be a complete discrete valuation field of equal characteristic $p > 0$ with perfect residue field k. For any finite Galois extension of K, one can define the upper and lower ramification groups, or filtrations. When the extension is abelian, the jumps in the upper ramification filtration can be described in terms of the coefficients of a defining equation of the extension. For example, if L/K is an Artin–Schreier extension defined by $x^p + a = x$, where $a \in K$ satisfies $p \nmid v_K(a) < 0$, then the unique upper ramification break of L/K is given by $-v_K(a)$. Kanesaka and Sekiguchi [KS79] generalized this result to finite cyclic cases, using Witt vectors. Since every finite abelian group can be decomposed into a direct product of cyclic groups, this allows one to compute the upper ramification breaks of any finite abelian extension L/K in terms of the coefficients of a defining equation of the extension.

In the abelian case, the Hasse–Arf theorem asserts that the upper ramification breaks are integers. On the other hand, in the non-abelian setting, the situation is more complicated, and upper ramification breaks can be a

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fraction with a denominator of a divisor of the order of the Galois group. Specifically, [I25] calculates the upper ramification breaks for some finite Galois extensions of maximal nilpotency class and shows that in some cases, they belong to $\frac{1}{p}\mathbb{Z} - \mathbb{Z}$. Furthermore, [E23] shows that if the Galois group is non-abelian and of order p^3, then the extension can have non-integer upper ramification breaks, except for the case where the Galois group is isomorphic to the dihedral group D_8 of order 8.

In this paper, we study the ramification groups of totally wildly ramified finite Galois extensions whose Galois group is isomorphic to the group $UT_n(\mathbb{F}_p)$ of $n \times n$ upper triangular unipotent matrices, or unitriangular matrices, over $\mathbb{F}_p$. We have proved in our main result, Theorem 6.3, that when $3 \leq n \leq p+2$, the largest upper ramification break is given by a linear expression of the valuation of the entries of a matrix explicitly constructed from the coefficients of a defining equation of the extension. This result is an analogue of the previous findings by Kanesaka and Sekiguchi [KS79] in the abelian case: they showed that the largest upper ramification number of an Artin–Schreier–Witt extension of K defined by an equation $(x_0^p, \ldots, x_s^p) - (x_0, \ldots, x_s) = (a_0, \ldots, a_s)$ of Witt vectors is given by $\max_{0 \leq i \leq s}(-v_K(a_i)p^{s-i})$, and the equation (1.1) proved in Theorem 6.3 of this paper is a counterpart in the $UT_n(\mathbb{F}_p)$ case. Note that in the Kanesaka and Sekiguchi's result, the ramification number is derived only from the information of the elements a_i of K. Similarly, the left-hand side of the equation (1.1) consists of the valuation of the element $s_{R,i,j} \in K_R$, which can be calculated without depending on any elements of L not in K. Since every finite p-group can be embedded into $UT_n(\mathbb{F}_p)$ for some n, findings in these cases are expected to shed light on more general situations.

This paper consists of five sections:

In Section 2, we define a valuation on the subring $FT_n(M)$ of the ring of upper triangular matrices over a discrete valuation field M of positive characteristic, by extending the valuation on M. This valuation v_M satisfies $v_M(XY) \geq \min(v_M(X), v_M(Y))$ for any $X, Y \in FT_n(M)$, analogous to the formula $v_M(x+y) \geq \min(v_M(x), v_M(y))$ for any $x, y \in M$. This valuation of matrices is a partial analogue to the filtration on Witt vectors in [KS79].

In Section 3, we recall some properties of ramification numbers of separable extensions over a complete discrete valuation field K of equal characteristic p. We also define a totally wildly ramified separable extension K_R/K of degree q with a unique upper ramification break $\frac{R}{q-1}$, where R is a positive integer coprime to p and q is a power of p.

In Section 4, we introduce a totally wildly ramified Galois extension L/K with Galois group $G_{L/K}$ isomorphic to the group of unitriangular matrices $UT_n(\mathbb{F}_p)$ over $\mathbb{F}_p$. This extension has a defining equation of the form $X^{(p)}A$

$= X$ with $A \in UT_n(K)$. This equation is analogous to the defining equation $x^p + a = x$ for Artin–Schreier cases. We denote by Θ one of the solutions of $X^{(p)}A = X$, where $X^{(p)}$ denotes the matrix obtained by raising each entry of X to the pth power. Specifically, the entries of Θ generate L over K. We define an isomorphism $\iota_R : K \to K_R$ and extend it to separable closures. We also define the Galois extension L_R/K_R, where $L_R = \iota_R(L)$, and the matrices $\Theta_R = \iota_R(\Theta) \in UT_n(L_R)$ and $A_R = \iota_R(A) \in UT_n(K_R)$. In order to reduce the calculation of the ramification breaks to the Artin–Schreier case, we introduce another matrix D_R defined by A, A_R, Θ, and Θ_R, and approximate it by a matrix S_R constructed using only A and A_R.

In Section 5, we study the relationship between the valuations of the entries $s_{R,i,j}$ of S_R and the upper ramification breaks of the extension L/K. Specifically, our goal in this section is to prove that the formula

$$v_{K_R}(s_{R,i,j}) = -qr_{i,j} + R \tag{1.1}$$

holds for $1 \le i < j \le n$ if $v_{K_R}(s_{R,l,m}) < -q\mu_{l,m} + pR$ for all $i \le l < m \le j$, where $r_{i,j}$ is the largest upper ramification break of $K(\theta_{l,m}\ (i \le l < m \le j))/K$, and $\mu_{l,m}$ is a number calculated directly from the valuations of the entries of A. We also prove in Lemma 5.11 that if (1.1) holds for $R = R'$ for some positive integer R' coprime to p, then it holds for $R > R'$ as well. This lemma is crucial for completing the proof of the main theorem in Section 6.

In Section 6, we complete the proof of the main theorem, which states that the formula (1.1) holds for $j - i \le p + 1$, by showing that $v_{K_R}(s_{R,l,m}) < -q\mu_{l,m} + pR$ holds for all $i \le l < m \le j$ in this case. As a direct consequence, in Corollary 6.4, we also establish that $\mu_{i,j}$, which is easier to compute than $s_{R,i,j}$, can be used as an upper estimate for the upper ramification break, and provide a criterion for this estimate to be sharp.

Generalization of our result to any $j - i \in \mathbb{Z}_{\ge 1}$ or to Galois extensions with different Galois group structures merits further investigation, but we expect that the method of this paper may be adapted to such cases.

Our approach is inspired by that of [A95]. Abrashkin constructed a filtration on some Lie algebra that corresponds to the ramification filtration of a profinite Galois extension K_p/K. The upper ramification groups of K_p/K are described in terms of a specific set of generators of the Galois group $\mathrm{Gal}(K_p/K)$. This result allows one to deduce the ramification filtration of any Galois subextension of K_p/K, provided that the Galois group of the subextension can be expressed using the same set of generators. The field K_R and the matrices Θ, Θ_R, A, A_R, D_R, and S_R roughly correspond to K', $\mathcal{F}$, $\mathcal{F}'$, E, E', A_1, and A_1' in [A95].

The unique aspect of our approach is the use of the valuation v_M on the ring $FT_n(M)$, which allows us to significantly simplify the calculation of the valuations of the entries of the matrices S_R and D_R. Moreover, we show not only that the Artin–Schreier extension defined by $x^p + s_{R,i,j} = x$ over K_R has a ramification break connected to the upper ramification breaks of L/K, but also that we have an explicit formula (1.1) relating $r_{i,j}$ and $s_{R,i,j}$ in Sections 5 and 6. This formula essentially provides an algorithm to compute the largest ramification break using only the entries of the matrices A and A_R.

2. Valuation on upper triangular matrices. For any field F of characteristic $p > 0$, let $UT_n(F)$ (resp. $NT_n(F)$) denote the groups of $n \times n$ unipotent (resp. nilpotent) upper triangular matrices, or unitriangular (resp. niltriangular) matrices, over F, respectively. We denote the identity matrix and the zero matrix of size $n \times n$ by I and O, respectively. Let $FT_n(F) = \{iI + X \mid i \in \mathbb{F}_p,\ X \in NT_n(F)\}$, where I denotes the identity matrix. The set $FT_n(F)$ is closed under matrix multiplication and subtraction, and contains the identity matrix. Thus, $FT_n(F)$ is a subring of $M_n(F)$. This ring $FT_n(F)$ consists of upper triangular matrices whose diagonal entries are the same element in $\mathbb{F}_p$. Note that $FT_n(F)^\times = FT_n(F) - NT_n(F)$, where $FT_n(F)^\times$ denotes the group of invertible elements in $FT_n(F)$. Denote the algebraic (resp. separable) closure of F by $\overline{F}$ (resp. F_{sep}).

Note that if $X = (x_{i,j})_{i,j}, Y = (y_{i,j})_{i,j} \in FT_n(F)$ and $XY = (z_{i,j})_{i,j}$, then we have

$$z_{i,j} = \sum_{l=i}^{j} x_{i,l} y_{l,j} \tag{2.1}$$

for any $1 \le i < j \le n$.

Definition 2.1. Let $1 \le i < j \le n$. We define a *partition* of length s for (i,j) as a sequence $\lambda = \{\lambda_u\}_{u=0,\dots,s}$ of integers satisfying

$$i = \lambda_0 < \lambda_1 < \cdots < \lambda_{s-1} < \lambda_s = j. \tag{2.2}$$

We denote the length of a partition λ by $|\lambda|$. Let $\Lambda_{i,j}$ be the set of all partitions for (i,j).

Lemma 2.2. *Let $X = (x_{i,j})_{i,j} \in UT_n(M)$, and $X^{-1} = (x^*_{i,j})_{i,j}$. Then*

$$X^{-1} = \sum_{s=0}^{n-1} (-1)^s (X - I)^s, \tag{2.3}$$

and

$$x^*_{i,j} = \sum_{\lambda \in \Lambda_{i,j}} (-1)^{|\lambda|} \prod_{u=0}^{|\lambda|-1} x_{\lambda_u, \lambda_{u+1}}. \tag{2.4}$$

Proof. Since $X - I$ is a nilpotent upper triangular matrix of size $n \times n$, we have $(X - I)^n = O$. By Taylor expansion, we have (2.3). Hence, we have (2.4) by (2.1). ■

For any discrete valuation field M of equal characteristic $p > 0$, we denote the valuation, the ring of integers, and the maximal ideal of M by v_M, $\mathcal{O}_M$, and $\mathfrak{m}_M$, respectively. We also define $\mathcal{O}_{\overline{M}}$ and $\mathfrak{m}_{\overline{M}}$ for $\overline{M}$ in the same manner.

LEMMA 2.3. *Let $\nu_{i,j}$ $(1 \leq i < j \leq n)$ be rational numbers satisfying $\nu_{i,j} \geq \nu_{i,l} + \nu_{l,j}$ for all $1 \leq i < l < j \leq n$. Define subsets $H_{\leq\nu}$ and $H_{<\nu}$ of $UT_n(M)$ as follows:*

$$H_{\leq\nu} = \{X = (x_{i,j})_{i,j} \in UT_n(M) \mid \forall 1 \leq i < j \leq n,\ -v_M(x_{i,j}) \leq \nu_{i,j}\}, \tag{2.5}$$

$$H_{<\nu} = \{X = (x_{i,j})_{i,j} \in UT_n(M) \mid \forall 1 \leq i < j \leq n,\ -v_M(x_{i,j}) < \nu_{i,j}\}. \tag{2.6}$$

Then:

(i) *The subset $H_{\leq\nu}$ is a subgroup of $UT_n(M)$.*
(ii) *The subset $H_{<\nu}$ is a subgroup of $UT_n(M)$.*
(iii) *The subset $H_{<\nu}$ is a normal subgroup of $H_{\leq\nu}$.*
(iv) *Let $X = (x_{i,j})_{i,j} \in H_{\leq\nu}$, $Y = (y_{i,j})_{i,j} \in H_{<\nu}$, $XY = (z_{i,j})_{i,j}$, and $YX = (z'_{i,j})_{i,j}$. Then*

$$z_{i,j} \equiv x_{i,j} \equiv z'_{i,j} \pmod{\mathfrak{m}_M^{-\nu_{i,j}+1}} \tag{2.7}$$

for all $1 \leq i < j \leq n$.

Proof. (i) Let $X = (x_{i,j})_{i,j}, Y = (y_{i,j})_{i,j} \in H_{\leq\nu}$, $X^{-1} = (x^*_{i,j})_{i,j}$ and $XY = (z_{i,j})_{i,j}$. By (2.1), we have

$$v_M(z_{i,j}) \geq \min_{i \leq l < j}(v_M(x_{i,l}) + v_M(y_{l,j})) \tag{2.8}$$

for all $1 \leq i < j \leq n$. Since $X, Y \in H_{\leq\nu}$, we have $v_M(x_{i,l}) \geq -\nu_{i,l}$ for all $1 \leq i < l \leq j \leq n$ and $v_M(y_{l,j}) \geq -\nu_{l,j}$ for all $1 \leq i \leq l < j \leq n$. Thus, we obtain

$$v_M(x_{i,l}) + v_M(y_{l,j}) \geq -(\nu_{i,l} + \nu_{l,j}) \geq -\nu_{i,j} \tag{2.9}$$

for all $1 \leq i < l < j \leq n$ and

$$v_M(x_{i,j}) + v_M(y_{j,j}) \geq -\nu_{i,j}, \tag{2.10}$$

$$v_M(x_{i,i}) + v_M(y_{i,j}) \geq -\nu_{i,j}. \tag{2.11}$$

Therefore

$$v_M(z_{i,j}) \geq -\nu_{i,j}, \tag{2.12}$$

for all $1 \leq i < j \leq n$, and consequently $XY \in H_{\leq\nu}$.

Since

$$v_M\Big(\prod_{u=0}^{|\lambda|-1} x_{\lambda_u,\lambda_{u+1}}\Big) = \sum_{u=0}^{|\lambda|-1} v_M(x_{\lambda_u,\lambda_{u+1}}) \geq \sum_{u=0}^{|\lambda|-1} -\nu_{\lambda_u,\lambda_{u+1}} \geq -\nu_{i,j} \tag{2.13}$$

for all $1 \leq i < j \leq n$ and $\lambda \in \Lambda_{i,j}$, we deduce that $X^{-1} \in H_{\leq\nu}$ by Lemma 2.2. Therefore, $H_{\leq\nu}$ is a subgroup of $UT_n(M)$.

(ii) This follows from the same argument as (i).

(iii) Let $X = (x_{i,j})_{i,j} \in H_{\leq\nu}$, $Y = (y_{i,j})_{i,j} \in H_{<\nu}$ and $XYX^{-1} = (y'_{i,j})_{i,j}$. By (2.1),

$$y'_{i,j} = \sum_{i\leq l\leq m\leq j} x_{i,l}y_{l,m}x^*_{m,j} = \sum_{i\leq l<m\leq j} x_{i,l}y_{l,m}x^*_{m,j} + \sum_{l=i}^{j} x_{i,l}x^*_{l,j}. \tag{2.14}$$

Since $XX^{-1} = I$, we have

$$\sum_{l=i}^{j} x_{i,l}x^*_{l,j} = 0 \tag{2.15}$$

for all $1 \leq i < j \leq n$ by (2.1). Thus, we obtain

$$y'_{i,j} = \sum_{i\leq l<m\leq j} x_{i,l}y_{l,m}x^*_{m,j}. \tag{2.16}$$

Since

$$\begin{aligned} v_M(x_{i,l}y_{l,m}x^*_{m,j}) &= v_M(x_{i,l}) + v_M(y_{l,m}) + v_M(x^*_{m,j}) \\ &> -\nu_{i,l} - \nu_{l,m} - \nu_{m,j} \geq -\nu_{i,j} \end{aligned} \tag{2.17}$$

for all $1 \leq i \leq l < m \leq j \leq n$, we deduce that $v_M(y'_{i,j}) > -\nu_{i,j}$ for all $1 \leq i < j \leq n$. Therefore, we have $XYX^{-1} \in H_{<\nu}$, and consequently $H_{<\nu}$ is a normal subgroup of $H_{\leq\nu}$.

(iv) Since $YX = X(X^{-1}YX)$ and $H_{<\nu}$ is a normal subgroup of $H_{\leq\nu}$, it suffices to show the assertion for XY. Since $Y \in H_{<\nu}$, the first inequality in (2.9) for all $1 \leq i < l < j \leq n$ and the inequality in (2.11) are strict inequalities. Thus, we obtain the assertion for XY by (2.1). ■

We will now generalize the valuation $v_M : M \to \mathbb{Z}\cup\{+\infty\}$ to $FT_n(M) \to \mathbb{Q} \cup \{+\infty\}$.

Definition 2.4. Define maps $v_M,\ \tilde{v}_M : FT_n(M) \to \mathbb{Q} \cup \{+\infty\}$ by

$$v_M(X) = \min_{1\leq i<j\leq n} \left(\frac{v_M(x_{i,j})}{j-i}\right), \tag{2.18}$$

$$\tilde{v}_M(X) = \min_{\substack{1\leq i<j\leq n \\ (i,j)\neq(1,n)}} \left(\frac{v_M(x_{i,j})}{j-i}\right), \tag{2.19}$$

for any $X = (x_{i,j})_{i,j} \in FT_n(M)$.

From this definition, for any $X = (x_{i,j})_{i,j} \in FT_n(M)$, we can assert that

$$v_M(X) = \sup \{r \in \mathbb{Q} \mid \forall i,j \in \{1,\dots,n\},\ v_M(x_{i,j}) \geq (j-i)r\}, \tag{2.20}$$

$$\tilde{v}_M(X) = \sup \left\{ r \in \mathbb{Q} \;\middle|\; \begin{array}{l} \forall i,j \in \{1,\dots,n\}, \\ (i,j) \neq (1,n) \Rightarrow v_M(x_{i,j}) \geq (j-i)r \end{array} \right\}, \tag{2.21}$$

since $v_M(0) = +\infty$ and $v_M(x_{i,i}) \geq 0$ for $1 \leq i \leq n$.

For any $X = (x_{i,j})_{i,j} \in FT_n(M)$ and $m \in \mathbb{Z}_{\geq 0}$, we denote by $X^{(p^m)}$ the matrix obtained by raising each entry of X to the p^mth power.

LEMMA 2.5. *Let $X = (x_{i,j})_{i,j}$ and $Y = (y_{i,j})_{i,j}$ be arbitrary elements of $FT_n(M)$.*

(i) *$v_M(X) = +\infty$ if and only if X is diagonal.*
(ii) *$\tilde{v}_M(X) \geq v_M(X)$.*
(iii) *For any $i, j \in \mathbb{F}_p$, we have $v_M(iX + jY) \geq \min(v_M(X), v_M(Y))$ and $\tilde{v}_M(iX + jY) \geq \min(\tilde{v}_M(X), \tilde{v}_M(Y))$. In particular, adding integer multiples of I does not affect v_M and $\tilde{v}_M$.*
(iv) *$v_M(X^{(p)}) = pv_M(X)$ and $\tilde{v}_M(X^{(p)}) = p\tilde{v}_M(X)$.*
(v) *$v_M(XY) \geq \min(v_M(X), v_M(Y))$ and $\tilde{v}_M(XY) \geq \min(\tilde{v}_M(X), \tilde{v}_M(Y))$.*
(vi) *If X is nilpotent (i.e. not invertible), then we have $v_M(XY) \geq \min(v_M(X), \tilde{v}_M(Y))$ and $v_M(YX) \geq \min(v_M(X), \tilde{v}_M(Y))$.*
(vii) *If both X and Y are nilpotent, then $v_M(XY) \geq \min(\tilde{v}_M(X), \tilde{v}_M(Y))$.*
(viii) *If X is invertible and $v_M(X) > v_M(Y)$, then $v_M(XY) = v_M(YX) = v_M(Y)$.*
(ix) *If X is invertible and $\tilde{v}_M(X) > \tilde{v}_M(Y)$, then $\tilde{v}_M(XY) = \tilde{v}_M(YX) = \tilde{v}_M(Y)$.*
(x) *Suppose X is invertible. Then $v_M(X^{-1}) = v_M(X)$ and $\tilde{v}_M(X^{-1}) = \tilde{v}_M(X)$.*
(xi) *Take an integer c and suppose X is nilpotent. Fix $\gamma \in M$ satisfying $v_M(\gamma) = c$. Then*

$$v_M(\gamma X) \geq \min\left(\frac{c}{n-1}, c\right) + v_M(X), \tag{2.22}$$

$$\tilde{v}_M(\gamma X) \geq \min\left(\frac{c}{n-2}, c\right) + \tilde{v}_M(X). \tag{2.23}$$

Proof. Let $XY = (z_{i,j})_{i,j}$.

Assertions (i)–(iv) immediately follow from the properties of the valuation of M and the definitions of v_M and $\tilde{v}_M$. We will prove the remaining assertions.

(v) By (2.1), we have

$$\begin{aligned} v_M(z_{i,j}) &\geq \min_{i \leq l \leq j} (v_M(x_{i,l}) + v_M(y_{l,j})) \\ &\geq \min_{i \leq l \leq j} ((l-i)v_M(X) + (j-l)v_M(Y)) \\ &= (j-i)\min(v_M(X), v_M(Y)). \end{aligned} \tag{2.24}$$

Thus we get

$$v_M(XY) \geq \min(v_M(X), v_M(Y))$$

and

$$\tilde{v}_M(XY) \geq \min(\tilde{v}_M(X), \tilde{v}_M(Y)).$$

(vi) Since X is nilpotent, we have $x_{i,i} = 0$ for all $1 \leq i \leq n$. Therefore,

$$\begin{aligned} v_M(z_{i,j}) &\geq \min_{i+1\leq l\leq j}(v_M(x_{i,l}) + v_M(y_{l,j})) \\ &\geq \min_{i+1\leq l\leq j}((l-i)v_M(X) + (j-l)\tilde{v}_M(Y)) \\ &= v_M(X) + (j-i-1)\min(v_M(X), \tilde{v}_M(Y)) \\ &\geq (j-i)\min(v_M(X), \tilde{v}_M(Y)). \end{aligned} \tag{2.25}$$

We thus obtain $v_M(XY) \geq \min(v_M(X), \tilde{v}_M(Y))$. The assertion for YX follows from a similar argument.

(vii) Since X and Y are nilpotent, we have $x_{i,i} = y_{j,j} = 0$ for all $1 \leq i < j \leq n$. Thus,

$$\begin{aligned} v_M(z_{i,j}) &\geq \min_{i+1\leq l\leq j-1}(v_M(x_{i,l}) + v_M(y_{l,j})) \\ &\geq \min_{i+1\leq l\leq j-1}((l-i)\tilde{v}_M(X) + (j-l)\tilde{v}_M(Y)) \\ &= v_M(X) + v_M(Y) + (j-i-2)\min(\tilde{v}_M(X), \tilde{v}_M(Y)) \\ &\geq (j-i)\min(\tilde{v}_M(X), \tilde{v}_M(Y)) \end{aligned} \tag{2.26}$$

for all $1 \leq i < j \leq n$ with $j - i > 1$. We have $z_{i,i+1} = 0$, so we also have $v_M(z_{i,j}) \geq (j-i)\min(\tilde{v}_M(X), \tilde{v}_M(Y))$ for $1 \leq i < j \leq n$ with $j - i = 1$. Thus $v_M(XY) \geq \min(\tilde{v}_M(X), \tilde{v}_M(Y))$.

(viii) Let (i, j) be one of the indices where $v_M(y_{i,j}) = (j-i)v_M(Y)$. Then for all $i + 1 \leq l \leq j$, we get

$$\begin{aligned} v_M(x_{i,l}y_{l,j}) &= v_M(x_{i,l}) + v_M(y_{l,j}) \\ &\geq (l-i)v_M(X) + (j-l)v_M(Y) \\ &> (j-i)v_M(Y) = v_M(y_{i,j}) \\ &= v_M(x_{i,i}y_{i,j}). \end{aligned} \tag{2.27}$$

The last equality is due to $x_{i,i} \neq 0$, deduced from the assumption that X is invertible. Therefore, by (2.1), we obtain

$$v_M(z_{i,j}) = v_M(x_{i,i}y_{i,j}) = (j-i)v_M(Y). \tag{2.28}$$

This gives $v_M(XY) \leq v_M(Y)$ by the definition of v_M. Combining this inequality with (v), we get $v_M(XY) = v_M(Y)$. The assertion for YX follows from a similar argument.

(ix) This can be proved in the same way as (viii).

(x) Since X is invertible, $x_{1,1} \neq 0$ and $I - x_{1,1}^{-1}X$ is nilpotent. Then we have $(I - x_{1,1}^{-1}X)^n = O$, because the size of the matrix is $n \times n$. Hence, by Taylor expansion, it follows that

$$X^{-1} = x_{1,1}^{-1}\sum_{l=0}^{n-1}(I - x_{1,1}^{-1}X)^l. \tag{2.29}$$

Applying (iii) and (v), we conclude that $v_M(X^{-1}) \geq v_M(X)$. Applying this inequality to X^{-1} gives $v_M(X) \geq v_M(X^{-1})$, and hence $v_M(X) = v_M(X^{-1})$. A similar argument shows that $\tilde{v}_M(X^{-1}) = \tilde{v}_M(X)$.

(xi) By the definition of v_M, we have

$$\begin{aligned} v_M(\gamma X) &= \min_{1\leq l<m\leq n}\left(\frac{v_M(\gamma x_{l,m})}{m-l}\right) \\ &= \min_{1\leq l<m\leq n}\left(\frac{c}{m-l} + \frac{v_M(x_{l,m})}{m-l}\right) \\ &\geq \min_{1\leq l<m\leq n}\left(\frac{c}{m-l}\right) + \min_{1\leq l<m\leq n}\left(\frac{v_M(x_{l,m})}{m-l}\right) \\ &= \min\left(\frac{c}{n-1}, c\right) + v_M(X). \end{aligned} \tag{2.30}$$

The assertion for $\tilde{v}_M$ can be proved in the same manner. ∎

3. On ramification numbers. Let K be a complete discrete valuation field with algebraically closed residue field k of characteristic $p > 0$.

For any finite separable extension L/K, we denote by $G_{L/K}$ the set of the embeddings of L into K_{sep} over K. Note that this set coincides with the Galois group when L/K is Galois. We define the upper (resp. lower) ramification numbering $\{R^i_{L/K}\}_{i\in\mathbb{Q}_{\geq -1}}$ (resp. $\{R_{L/K,i}\}_{i\in\mathbb{Q}_{\geq -1}}$) on the data of equivalence relations over $G_{L/K}$, the function $\phi_{L/K}$ translating the lower ramification numbering to the upper ramification numbering, and the inverse $\psi_{L/K}$ of $\phi_{L/K}$ as in [D84, Appendice]. Note that the equivalence relation $R^i_{L/K}$ (resp. $R_{L/K,i}$) gets finer as i increases. When a rational number $i \geq -1$ satisfies $R^i_{L/K} \neq R^{i'}_{L/K}$ (resp. $R_{L/K,i} \neq R_{L/K,i'}$) for all $i' > i$, we call i an *upper* (resp. *lower*) *ramification break* of L/K. We denote the set of the upper (resp. lower) ramification breaks by $\mathcal{U}_{L/K}$ (resp. $\mathcal{L}_{L/K}$). Specifically,

$$\mathcal{U}_{L/K} = \{i \in \mathbb{Q} \mid \forall i' > i, R^i_{L/K} \neq R^{i'}_{L/K}\}, \tag{3.1}$$

$$\mathcal{L}_{L/K} = \{i \in \mathbb{Q} \mid \forall i' > i, R_{L/K,i} \neq R_{L/K,i'}\}. \tag{3.2}$$

Denote by $e_{L/K}$ the ramification index of L/K. We extend $v_K : K \to \mathbb{Z} \cup \{+\infty\}$ to $L \to \mathbb{Q} \cup \{+\infty\}$ by defining $v_K(x) = \frac{v_L(x)}{e_{L/K}}$ for any $x \in L$. We similarly extend $v_K, \tilde{v}_K : FT_n(K) \to \mathbb{Q} \cup \{+\infty\}$ to $FT_n(L) \to \mathbb{Q} \cup \{+\infty\}$ by defining $v_K(X) = \frac{v_L(X)}{e_{L/K}}$ and $\tilde{v}_K(X) = \frac{\tilde{v}_L(X)}{e_{L/K}}$ for any $X \in FT_n(L)$.

When L/K is Galois, we denote the ith upper (resp. lower) ramification group of $G_{L/K} = \mathrm{Gal}(L/K)$ by $G^i_{L/K}$ (resp. $G_{L/K,i}$).

Let us now cite from [D84] the properties of ramification numbers which we will use in this paper.

Lemma 3.1 ([D84, Appendice]). *For any separable extensions M/K and L/K of complete discrete valuation fields, we have the following assertions.*

(i) *Suppose $M \supset L$ and fix an embedding τ_L of L into $K_{\mathrm{sep}} = L_{\mathrm{sep}}$. Then for any $\tau_1, \tau_2 : M \to K_{\mathrm{sep}} \in G_{M/K}$ such that $\tau_1|_L = \tau_2|_L = \tau_L$ and $i \in \mathbb{Q}_{\geq -1}$, we have*

$$\tau_1 \equiv \tau_2 \ (\mathrm{mod}\ R_{M/L,i}) \iff \tau_1 \equiv \tau_2 \ (\mathrm{mod}\ R_{M/K,i}). \tag{3.3}$$

Consequently, $\mathcal{L}_{M/K} \supset \mathcal{L}_{M/L}$.

(ii) *Suppose $M \supset L$. Then for any $\tau_1, \tau_2 : M \to K_{\mathrm{sep}} \in G_{M/K}$ and $i \in \mathbb{Q}_{\geq -1}$, we have*

$$\begin{aligned} &\tau_1|_L \equiv \tau_2|_L \ (\mathrm{mod}\ R^i_{L/K}) \\ &\iff \exists \tau_1', \tau_2' \in G_{M/K} \text{ such that } \begin{cases} \tau_1'|_L = \tau_1|_L, \\ \tau_2'|_L = \tau_2|_L, \\ \tau_1' \equiv \tau_2' \ (\mathrm{mod}\ R^i_{M/K}). \end{cases} \end{aligned} \tag{3.4}$$

Consequently, $\mathcal{U}_{M/K} \supset \mathcal{U}_{L/K}$.

(iii) *Suppose $M \supset L$. Then for any $\tau_1, \tau_2 : M \to K_{\mathrm{sep}} \in G_{M/K}$ and $i \in \mathbb{Q}_{\geq -1}$, we have*

$$\begin{aligned} &\tau_1|_L \equiv \tau_2|_L \ (\mathrm{mod}\ R_{L/K,\phi_{M/L}(i)}) \\ &\iff \exists \tau_1', \tau_2' \in G_{M/K} \text{ such that } \begin{cases} \tau_1'|_L = \tau_1|_L, \\ \tau_2'|_L = \tau_2|_L, \\ \tau_1' \equiv \tau_2' \ (\mathrm{mod}\ R_{M/K,i}). \end{cases} \end{aligned} \tag{3.5}$$

Consequently, $\phi_{M/L}(\mathcal{L}_{M/K}) \supset \mathcal{L}_{L/K}$.

(iv) *Suppose $M \supset L$. Then*

$$\phi_{M/K} = \phi_{L/K} \circ \phi_{M/L}, \tag{3.6}$$

$$\psi_{M/K} = \psi_{M/L} \circ \psi_{L/K}. \tag{3.7}$$

(v) *We have*

$$\mathcal{L}_{L/K} = \{l \in \mathbb{Q}_{>-1} \mid \phi_{L/K}(x) \text{ is non-differentiable at } x = l\}, \tag{3.8}$$

$$\mathcal{U}_{L/K} = \{r \in \mathbb{Q}_{>-1} \mid \psi_{L/K}(x) \text{ is non-differentiable at } x = r\} \tag{3.9}$$

(vi) *We have*

$$\max \mathcal{U}_{LM/K} = \max(\mathcal{U}_{L/K} \cup \mathcal{U}_{M/K}). \tag{3.10}$$

Proof. (i)–(iv) See [D84, Appendice, Section A.4].

(v) This follows from [D84, Appendice, Proposition A.4.2], (iv), and the definitions of $\mathcal{L}_{L/K}$ and $\mathcal{U}_{L/K}$.

(vi) This follows from (ii). ∎

Lemma 3.2. *Let $M/L/K$ be a tower of finite totally ramified separable extensions and let K'/K be a finite totally ramified separable extension of degree e. Suppose that the following conditions hold.*

(a) $\mathcal{U}_{K'/K}$ *is a singleton* $\{r'\}$.
(b) $\mathcal{L}_{M/L}$ *is a singleton* $\{\psi_{M/K}(r)\}$
(c) $\max(\mathcal{U}_{L/K}) < r' < r$.

Then

$$\max \mathcal{U}_{LK'/K'} \leq r', \tag{3.11}$$

$$\max \mathcal{U}_{MK'/K'} = er - (e-1)r'. \tag{3.12}$$

Proof. Note that

$$\psi_{K'/K}(x) = \begin{cases} x & (-1 \leq x < r'), \\ ex - (e-1)r' & (x \geq r'). \end{cases} \tag{3.13}$$

By Lemma 3.1(v), condition (b) and the definition that $\psi_{M/L}$ is the inverse of $\phi_{M/L}$, we see that $\psi_{M/L}(x)$ has a unique non-differentiable point at $x = \phi_{M/L}(\psi_{M/K}(r))$. By Lemma 3.1(vi), we have $\phi_{M/L} \circ \psi_{M/K} = \psi_{L/K}$, and hence $\psi_{M/L}(x)$ has a unique non-differentiable point at $x = \psi_{L/K}(r)$. This implies

$$\mathcal{U}_{M/K} = \{r\} \cup \mathcal{U}_{L/K} \tag{3.14}$$

by Lemma 3.1(v). Therefore, by Lemma 3.1(vi), we have

$$\max \mathcal{U}_{MK'/K} = \max(\mathcal{U}_{M/K} \cup \mathcal{U}_{K'/K}) = \max(\{r, r'\} \cup \mathcal{U}_{L/K}) = r. \tag{3.15}$$

By condition (c) and Lemma 3.1(iv), we get

$$\psi_{MK'/K}(x) = \begin{cases} \psi_{MK'/K'}(x) & (-1 \leq x < r'), \\ \psi_{MK'/K'}(ex - (e-1)r') & (x \geq r'). \end{cases} \tag{3.16}$$

Therefore by Lemma 3.1(v) and condition (c), we get (3.12). By the same argument, we obtain (3.11). ∎

Suppose that K is of equal characteristic $p > 0$ and fix $t \in K$ such that $v_K(t) = -1$ for the rest of this paper. We will now introduce a separable extension K_R/K of K. Using this extension, we will show in Lemma 3.6 that the calculation of the ramification numbers of finite non-abelian Galois extensions of K can be reduced to the case of Artin–Schreier extensions.

Definition 3.3. Let R be a positive integer coprime to p. Let $K_R = K(T)$ be the extension of K defined by

$$T^q + T^{q-1} = t^R, \tag{3.17}$$

where q is a power of p.

Equation (3.17) yields

$$\left(\frac{t^{\frac{R}{q-1}}}{T}\right)^q - \frac{t^{\frac{R}{q-1}}}{T} = t^{\frac{R}{q-1}} \tag{3.18}$$

in $\overline{K}$. Therefore, $K_R(t^{\frac{1}{q-1}})$ is an Artin–Schreier extension of $K(t^{\frac{1}{q-1}})$, and consequently a separable extension of K.

We have $v_{K(t^{\frac{1}{q-1}})}(t^{\frac{R}{q-1}}) = -R$. Since $-R$ is coprime to p and negative, by (3.18), we have

$$[K_R : K] = [K_R(t^{\frac{1}{q-1}}) : K(t^{\frac{1}{q-1}})] = q.$$

LEMMA 3.4. *We have* $\mathcal{U}_{K_R/K} = \left\{\frac{R}{q-1}\right\}$.

Proof. It follows from (3.18) and the fact that $K(t^{\frac{1}{q-1}})/K$ is totally tamely ramified. ■

Let $t_R \in \overline{K}$ be the (qR)-th root of $t^R(1+T^{-1})^{-1}$ congruent to $t^{1/q}$ modulo $t^{1/q}\mathfrak{m}_{\overline{K}}$, where $\mathfrak{m}_{\overline{K}}$ denotes the maximal ideal of $\overline{K}$. Then we have $t_R^R = T$, since

$$t^R(1+T^{-1})^{-1} = T^q \tag{3.19}$$

by the definition of T.

LEMMA 3.5. *We have* $K_R = K(t_R)$ *and*

$$t^l = t_R^{ql}\left(1 + \sum_{i=1}^{\infty}\binom{l/R}{i} t_R^{-iR}\right) \tag{3.20}$$

for $l \in \mathbb{Z}$.

Proof. The inclusion $K_R = K(T) \subset K(t_R)$ follows from $t_R^R = T$. We will show the reverse inclusion.

By Hensel's lemma and the fact that the residue field of K_R is algebraically closed, it follows that the Rth roots of $1+T^{-1}$ and their inverses belong to K_R. Let z be the Rth root of $1+T^{-1}$ congruent to 1 modulo $\mathfrak{m}_{K_R}$. Thus, by the definition of t_R, we have

$$t_R^q = tz^{-1} \in K_R. \tag{3.21}$$

Take $c_1, c_2 \in \mathbb{Z}$ such that $qc_1 + Rc_2 = 1$. Then

$$t_R = (t_R^q)^{c_1}(t_R^R)^{c_2} = (tz^{-1})^{c_1}T^{c_2} \in K_R. \tag{3.22}$$

By (3.21), we have

$$t = t_R^q z. \tag{3.23}$$

By the definition of z and binomial expansion, we get (3.20). ■

Let $\iota_R : K \to K_R$ be the ring isomorphism defined by $t \mapsto t_R$ and $u \mapsto u^{1/q}$ for any $u \in k$. Since K_R/K is a separable extension, we can choose a common separable closure for both fields so that $K_{\mathrm{sep}} = K_{R,\mathrm{sep}}$. We then take an extension $\iota_R : K_{\mathrm{sep}} \to K_{R,\mathrm{sep}} = K_{\mathrm{sep}}$ of ι_R to this separable closure.

LEMMA 3.6. *Let $L/L'/K$ be a tower of finite separable totally ramified extensions and let $L_R = \iota_R(L)$ and $L'_R = \iota_R(L')$. Let K'_R/K_R be a separable extension. Suppose that the following conditions hold.*

(a) $LL_R = L'L_RK'_R$.
(b) $\mathcal{L}_{L/L'}$ *is a singleton* $\{\psi_{L/K}(r)\}$.
(c) $q\max(\mathcal{U}_{L'/K}) < R < (q-1)r$.

Then

$$\max\mathcal{U}_{K'_R/K_R} = qr - R. \tag{3.24}$$

Proof. By Lemma 3.1(vi), we have

$$\max\mathcal{U}_{LL_R/K_R} = \max(\mathcal{U}_{LK_R/K_R}\cup\mathcal{U}_{L_R/K_R}) = \max(\mathcal{U}_{LK_R/K_R}\cup\mathcal{U}_{L/K}). \tag{3.25}$$

By Lemma 3.2, we get

$$\max\mathcal{U}_{LK_R/K_R} = qr - R \tag{3.26}$$

By Lemma 3.1(iii) and conditions (b) and (c), we obtain

$$\max\mathcal{U}_{L/K} = r. \tag{3.27}$$

Therefore, we deduce that

$$\max\mathcal{U}_{LL_R/K_R} = \max(r, qr - R) = qr - R. \tag{3.28}$$

On the other hand, we have

$$\begin{aligned} \max\mathcal{U}_{LL_R/K_R} &= \max(\mathcal{U}_{L'K_R/K_R} \cup \mathcal{U}_{L_R/K_R} \cup \mathcal{U}_{K'_R/K_R}) \\ &= \max(\mathcal{U}_{L'K_R/K_R} \cup \mathcal{U}_{L/K} \cup \mathcal{U}_{K'_R/K_R}) \\ &= \max(\mathcal{U}_{L'K_R/K_R} \cup \mathcal{U}_{K'_R/K_R} \cup \{r\}) \end{aligned} \tag{3.29}$$

by condition (a) and (3.27). By Lemma 3.2, we have

$$\max\mathcal{U}_{L'K_R/K_R} \le \frac{R}{q-1} < r. \tag{3.30}$$

Therefore, we get

$$\max\mathcal{U}_{LL_R/K_R} = \max(\mathcal{U}_{K'_R/K_R} \cup \{r\}). \tag{3.31}$$

Comparing (3.28) and (3.31), we get the assertion. ∎

4. Galois extensions with Galois group isomorphic to unitriangular matrix group over $\mathbb{F}_p$. Let K be a complete discrete valuation field of equal characteristic $p > 0$ with algebraically closed residue field k and L be a totally wildly ramified finite Galois extension of K such that

$\mathrm{Gal}(L/K) \simeq UT_n(\mathbb{F}_p)$ for some $n \geq 2$. We identify the residue field of L with k. Define K_R as in Definition 3.3 for any positive integer R coprime to p. Define $\mathcal{L}_{M_1/M_2}$ and $\mathcal{U}_{M_1/M_2}$ for any finite separable extension M_1/M_2 of complete discrete valuation fields as in Section 3.

Definition 4.1. Every $x \in K$ can be uniquely written in the form $x = \sum_{i=v_K(x)}^{\infty} x_i t^{-i}$ with $x_i \in k$ for all i. Define $\mathrm{coef}_{K,i} : K \to k$ by

$$\mathrm{coef}_{K,i}(x) = x_i \tag{4.1}$$

for all $i \in \mathbb{Z}$.

Lemma 4.2. *There exists a matrix $A = (a_{i,j})_{1\leq i,j\leq n} \in UT_n(K)$ satisfying the following conditions.*

(a) *For all $1 \leq i < j \leq n$ and $l \in p\mathbb{Z} \cup \mathbb{Z}_{\geq 0}$,*

$$\mathrm{coef}_{K,l}(a_{i,j}) = 0. \tag{4.2}$$

(b) *The equation $X^{(p)}A = X$ defines L/K. In other words, if we take a solution $\Theta = (\theta_{i,j})_{1\leq i,j\leq n} \in UT_n(K_{\mathrm{sep}})$ of this equation, then $L = K(\theta_{i,j}\ (1 \leq i < j \leq n))$.*

Proof. By the same argument as in [I25, Lemma 1.9(a)], we see that there exists a matrix A satisfying condition (b). Since $X^{(p)}B^{(p)}AB^{-1} = X$ for any matrix $B \in UT_n(K)$ defines the same extension as L/K, we may replace A with $B^{(p)}AB^{-1}$.

Let $E_{i,j}$ be the matrix unit with a 1 at the (i,j)th entry for $1 \leq i < j \leq n$.

Let $1 \leq i < j \leq n$ and $x \in K$. Denote the matrix $(I+xE_{i,j})^{(p)}A(I+xE_{i,j})^{-1}$ by $A' = (a'_{l,m})_{l,m}$. Then

$$a'_{l,m} = \begin{cases} a_{l,m} + x^p - x & ((l,m) = (i,j)), \\ a_{l,m} + x^p a_{j,m} & (l = i, m > j), \\ a_{l,m} - x a_{l,i} & (l < i, m = j), \\ a_{l,m} & (\text{otherwise}). \end{cases} \tag{4.3}$$

This gives

$$(l,m) \neq (i,j) \wedge m - l \leq j - i \implies a'_{l,m} = a_{l,m}. \tag{4.4}$$

Note that for any $y \in K$, there exists $x \in K$ such that $\mathrm{coef}_{K,l}(y + x^p - x) = 0$ holds for all $l \in p\mathbb{Z} \cup \mathbb{Z}_{\geq 0}$, since the residue field k is algebraically closed. Hence, we can make A satisfy condition (a) by repeatedly replacing A by $(I + xE_{i,j})^{(p)}A(I + xE_{i,j})^{-1}$ for some appropriate $x \in K$ for each $1 \leq i < j \leq n$ in ascending order of indices (i,j). ∎

Fix $A = (a_{i,j})_{i,j}$ satisfying the conditions of Lemma 4.2 for the rest of this paper. Denote the (i,j)th entry of A^{-1} by $a^*_{i,j}$ for any $1 \leq i,j \leq n$. Denote $K(\theta_{l,m}\ (i \leq l < m \leq j))$ by $L_{i,j}$ for any $1 \leq i < j \leq n$.

Lemma 4.3. *For any* $1 \le i < j \le n$, *we have* $\mathcal{L}_{L_{i,j}/L_{i,j-1}L_{i+1,j}} = \{\max \mathcal{L}_{L_{i,j}/K}\}$ *and* $\mathcal{U}_{L_{i,j}/K} - \mathcal{U}_{L_{i,j-1}L_{i+1,j}/K} = \{\max \mathcal{U}_{L_{i,j}/K}\}$.

Proof. Let $l = j - i$. We prove this lemma by induction on l.

Suppose $l = 1$, then the assertion clearly holds, because $L_{i,j}/K$ is an Artin–Schreier extension and $\#\mathcal{U}_{L_{i,j}/K} = 1$.

Suppose $l \ge 2$. Let $r = \max \mathcal{L}_{L_{i,j}/K}$ and denote $G = G_{L_{i,j}/K}$. Note that we have $G = G_1$, since $L_{i,j}/K$ is totally wildly ramified. By [S62, IV, §2, Corollaire 1 of Proposition 9], we get

$$[G_r, G] \subset G_{r+1} = \{1\}. \tag{4.5}$$

Thus we have $G_r \subset Z(G)$. Note that $G \simeq UT_{j-i+1}(\mathbb{F}_p)$ and $G_{L_{i,j}/L_{i,j-1}L_{i+1,j}} = Z(G)$. Hence, $Z(G)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Thus it follows that $G_r = G_{L_{i,j}/L_{i,j-1}L_{i+1,j}}$. This yields

$$\mathcal{L}_{L_{i,j}/L_{i,j-1}L_{i+1,j}} = \{\max \mathcal{L}_{L_{i,j}/K}\}$$

and

$$\mathcal{U}_{L_{i,j}/K} - \mathcal{U}_{L_{i,j-1}L_{i+1,j}/K} = \{\max \mathcal{U}_{L_{i,j}/K}\}. \qquad \blacksquare$$

Let $r_{i,j} = \max \mathcal{U}_{L_{i,j}/K}$ for any $1 \le i < j \le n$. For convenience, we set $r_{i,i} = -1$ for any $1 \le i \le n$.

Corollary 4.4. *We have* $r_{i,j} > r_{l,m}$ *for any* $1 \le i \le l < m \le j \le n$ *satisfying* $(i, j) \ne (l, m)$.

Proof. It follows immediately from Lemma 4.3 and the definition of $r_{i,j}$. $\blacksquare$

Let $m_A = -v_K(A)$. Fix an integer $N > \log_p(n(p^{n(n-1)/2} + 1)m_A + p) + n(n-1)/2$ and let $q = p^N$ for the rest of this paper. Let R be a positive integer coprime to p. Define $K_R = K(T)$ to be the extension of K defined by $t^R = T^q + T^{q-1}$ as in Definition 3.3.

Lemma 4.5. *For any* $1 \le i < j \le n$ *such that* $j - i > 1$, *we have*

$$(q-1)r_{i,j} - q\max(r_{i,j-1}, r_{i+1,j}) > nm_A + p. \tag{4.6}$$

Proof. By the definition of upper ramification groups, we have

$$\mathcal{U}_{L_{i,j}/K} \in |\mathrm{Gal}(L_{i,j}/K)|^{-1}\mathbb{Z} = p^{-\frac{(j-i)(j-i+1)}{2}}\mathbb{Z}. \tag{4.7}$$

Thus $r_{i,j} > \max(r_{i,j-1}, r_{i+1,j})$ implies

$$r_{i,j} - \max(r_{i,j-1}, r_{i+1,j}) \ge p^{-\frac{(j-i)(j-i+1)}{2}}.$$

Since $q > p^{\frac{n(n-1)}{2}}(n(p^{\frac{n(n-1)}{2}} + 1)m_A + p)$ by definition, we get

$$q(r_{i,j} - \max(r_{i,j-1}, r_{i+1,j})) > n(p^{\frac{n(n-1)}{2}} + 1)m_A + p. \tag{4.8}$$

By Lemmas 3.1(ii) and 4.3, we have $r_{i,j} \le r_{1,n}$. By definition, we have $\psi_{L_{i,j}/K}(x) \ge x$ for any $x \ge -1$. Thus we have $r_{1,n} \le \max \mathcal{L}_{L_{1,n}/K}$. By

Lemma 4.3, we have

$$\mathcal{L}_{L_{1,n}/L_{1,n-1}L_{2,n}} = \{\max \mathcal{L}_{L_{1,n}/K}\}.$$

Note that $L_{1,n}/L_{1,n-1}L_{2,n}$ is an extension defined by an Artin–Schreier polynomial whose roots include $\theta_{1,n}$. Therefore,

$$\max \mathcal{L}_{L_{1,n}/K} \leq -v_{L_{1,n}}(\theta_{1,n}).$$

Hence, by Lemma 2.5, we obtain

$$\begin{aligned}(4.9)\qquad r_{i,j} &\leq -v_{L_{1,n}}(\theta_{1,n}) \leq -(n-1)v_{L_{1,n}}(\Theta)\\ &= -\frac{n-1}{p}v_{L_{1,n}}(A) = -\frac{n-1}{p}e_{L_{1,n}/K}v_K(A)\\ &= (n-1)p^{\frac{n(n-1)}{2}-1}m_A < np^{\frac{n(n-1)}{2}}m_A.\end{aligned}$$

Therefore we have

$$\begin{aligned}(4.10)\qquad (q-1)r_{i,j} - q\max(r_{i,j-1}, r_{i+1,j}) &\\ &= q(r_{i,j} - \max(r_{i,j-1}, r_{i+1,j})) - r_{i,j}\\ &> n(p^{\frac{n(n-1)}{2}}+1)m_A + p - np^{\frac{n(n-1)}{2}}m_A\\ &= nm_A + p. \ \blacksquare\end{aligned}$$

Definition 4.6. Take a common separable closure $K_{\mathrm{sep}} = K_{R,\mathrm{sep}}$ of K and K_R, and let $\iota_R : K_{\mathrm{sep}} \to K_{R,\mathrm{sep}} = K_{\mathrm{sep}}$ be an extension of the ring isomorphism $\iota_R : K \to K_R$ defined by $t \mapsto t_R$ and $u \mapsto u^{1/q}$ for any $u \in k$ as in Section 3. Define P, $\eta_R : K_{\mathrm{sep}} \to K_{R,\mathrm{sep}} = K_{\mathrm{sep}}$ by $P(x) = x^p - x$ and $\eta_R(x) = x - \iota_R(x)^q$ for any $x \in K_{\mathrm{sep}}$.

For any $X \in M_n(K_{\mathrm{sep}})$, we denote by $P(X)$, $\iota_R(X)$ and $\eta_R(X)$ the matrices obtained by applying P, ι_R and η_R respectively to each entry of X. Denote $K_R(\iota_R(\theta_{l,m})\ (i \leq l < m \leq j))$ by $L_{R,i,j}$.

Definition 4.7. We define matrices $\Theta_R = (\theta_{R,i,j})_{i,j}$, $A_R = (a_{R,i,j})_{i,j}$, $W_{R,e} = (w_{R,e,i,j})_{i,j}$, $\Delta_R = (\delta_{R,i,j})_{i,j}$, $D_R = (d_{R,i,j})_{i,j}$ as follows:

$$(4.11)\qquad \Theta_R = \iota_R(\Theta) \in UT_n(L_R),$$

$$(4.12)\qquad A_R = \iota_R(A) \in UT_n(K_R),$$

$$(4.13)\qquad W_{R,0} = I,$$

$$(4.14)\qquad W_{R,e} = A_R^{(p^{e-1})}W_{R,e-1} \in UT_n(K_R) \quad (1 \leq e \leq N),$$

$$(4.15)\qquad \Delta_R = \Theta_R - \Theta W_{R,N} \in NT_n(LL_R),$$

$$(4.16)\qquad D_R = P(\Delta_R) \in NT_n(LL_R).$$

Denote the (i,j)th entry of A_R^{-1} by $a^*_{R,i,j}$ for any $1 \leq i,j \leq n$.

The matrices Δ_R and D_R are nilpotent and the others are unipotent. Note that the matrices with subscript R depend on R.

Note that we have

$$\Theta_R^{(p)} A_R = \Theta_R, \tag{4.17}$$

$$\Theta_R^{(p^i)} = \Theta_R^{(p^{i+j})} W_{R,j}{}^{(p^i)} \qquad (i \geq 0,\ 0 \leq j \leq N), \tag{4.18}$$

$$W_{R,e} = W_{R,e-1}^{(p)} A_R \qquad (1 \leq e \leq N), \tag{4.19}$$

$$\Delta_R = -\eta_R(\Theta) W_{R,N}. \tag{4.20}$$

REMARK 4.8. We have

$$\begin{aligned} L_{i,j}K_R(\delta_{R,l,m}\ (i \leq l < m \leq j)) &= L_{R,i,j}(\delta_{R,l,m}\ (i \leq l < m \leq j)) \\ &= L_{i,j}L_{R,i,j} \end{aligned} \tag{4.21}$$

by the definition of Δ_R, and the fact that $\theta_{i,j} \in L_{i,j}$, $\theta_{R,i,j} \in L_{R,i,j}$ and $W_{R,N} \in UT_n(K_R)$. Thus the matrix Δ_R connects the extension L/K with the extension L_R/K_R. Note that $D_R = P(\Delta_R)$ and that D_R is not expressed in terms of the coefficients of the defining equation of L/K, since the definition of D_R includes Θ and Θ_R.

We will now show that if we have an extension K'_R/K_R satisfying the conditions of Lemma 3.6, we can extract the information of the ramification breaks of L/K by investigating K'_R/K_R. We will see that we can construct this extension by approximating D_R by a matrix in $NT_n(K_R)$.

LEMMA 4.9. *Let $1 \leq i < j \leq n$ and R be an integer coprime to p such that $q\max(r_{i,j-1}, r_{i+1,j}) < R < (q-1)r_{i,j}$. Assume that we have $x \in K_R$ satisfying $d_{R,i,j} - x \in \mathcal{O}_{L_{i,j}L_{R,i,j}}$. Then there exist $y, z \in K_R$ such that $x = y + P(z)$ and $v_{K_R}(y) = -qr_{i,j} + R$. Consequently, $v_{K_R}(x) \leq -qr_{i,j} + R < 0$ and the equality holds if $p \nmid v_{K_R}(x)$.*

Proof. We prove this lemma by applying Lemma 3.6 with $L = L_{i,j}$, $L' = L_{i,j-1}L_{i+1,j}$, and $K'_R = K_R(\xi)$, where $\xi \in K_{R,\mathrm{sep}} = K_{\mathrm{sep}}$ satisfies $P(\xi) = x$. The conditions (b) and (c) are satisfied by assumption. Therefore it suffices to show

$$L_{i,j}L_{R,i,j} = L_{i,j-1}L_{i+1,j}L_{R,i,j}K_R(\xi). \tag{4.22}$$

First, we will show that $d_{R,i,j} \in L_{i,j-1}L_{i+1,j}L_{R,i,j}$. Since Θ satisfies $\Theta = \Theta^{(p)}A$, we have

$$\theta_{i,j} = \theta_{i,j}^p + a_{i,j} + \sum_{l=i+1}^{j-1} \theta_{i,l}^p a_{l,j} \equiv \theta_{i,j}^p \pmod{L_{i,j-1}L_{i+1,j}}. \tag{4.23}$$

Hence, we have $P(\theta_{i,j}) \in L_{i,j-1}L_{i+1,j}$. By (4.15), we have

$$\begin{aligned} \delta_{R,i,j} &= \theta_{R,i,j} - \theta_{i,j} - w_{R,N,i,j} - \sum_{l=i+1}^{j-1} \theta_{i,l} w_{R,N,l,j} \\ &\equiv -\theta_{i,j} \pmod{L_{i,j-1}L_{i+1,j}L_{R,i,j}}. \end{aligned} \tag{4.24}$$

Therefore, we get $d_{R,i,j} = P(\delta_{R,i,j}) \in L_{i,j-1}L_{i+1,j}L_{R,i,j}$. Furthermore, we obtain $d_{R,i,j} - x \in \mathcal{O}_{L_{i,j-1}L_{i+1,j}L_{R,i,j}}$.

Since $L_{i,j-1}L_{i+1,j}L_{R,i,j}/K$ is a separable extension, the residue field of $L_{i,j-1}L_{i+1,j}L_{R,i,j}$ coincides with the residue field k of K. Hence, there exist $x_0 \in k$ and $x_1 \in \mathfrak{m}_{L_{i,j-1}L_{i+1,j}L_{R,i,j}}$ such that $d_{R,i,j} - x = x_0 + x_1$.

Since k is algebraically closed by assumption, there exists $\xi_0 \in k$ such that $P(\xi_0) = x_0$. Let

$$\xi_1 = -\sum_{l=0}^{\infty} x_1^{p^l}. \tag{4.25}$$

Then we have $\xi_1 \in \mathfrak{m}_{L_{i,j-1}L_{i+1,j}L_{R,i,j}}$ and $P(\xi_1) = x_1$. Thus,

$$P(\xi_0 + \xi_1) = d_{R,i,j} - x \quad \text{and} \quad \xi_0 + \xi_1 \in \mathcal{O}_{L_{i,j-1}L_{i+1,j}L_{R,i,j}}.$$

Hence, we obtain

$$\delta_{R,i,j} \equiv \xi \pmod{\mathcal{O}_{L_{i,j-1}L_{i+1,j}L_{R,i,j}}}.$$

Combining with (4.21), we get

$$\begin{aligned} L_{i,j}L_{R,i,j} &= L_{R,i,j}(\delta_{R,l,m}\ (i \le l < m \le j)) \\ &= L_{R,i,j-1}(\delta_{R,l,m}\ (i \le l < m \le j-1)) \\ &\quad \cdot L_{R,i+1,j}(\delta_{R,l,m}\ (i+1 \le l < m \le j)) \cdot L_{R,i,j}(\delta_{R,i,j}) \\ &= L_{i,j-1}L_{R,i,j-1} \cdot L_{i+1,j}L_{R,i+1,j} \cdot L_{R,i,j}(\delta_{R,i,j}) \\ &= L_{i,j-1}L_{i+1,j}L_{R,i,j}(\delta_{R,i,j}) = L_{i,j-1}L_{i+1,j}L_{R,i,j}(\xi). \end{aligned} \tag{4.26}$$

Since K_R is a subfield of $L_{R,i,j}$, we get (4.22), and hence the lemma is proved. ■

Definition 4.10. We now define matrices $S_R = (s_{R,l,m})_{l,m}$, $\varSigma_R = (\sigma_{R,l,m})_{l,m}$ as follows:

$$\begin{aligned} S_R &= (W_{R,N}^{(p)})^{-1}\eta_R(A)W_{R,N-1}^{(p)} \\ &= (W_{R,N}^{(p)})^{-1}AW_{R,N-1}^{(p)} - I \in NT_n(K_R), \end{aligned} \tag{4.27}$$

$$P(\varSigma_R) = S_R,\ \varSigma_R \in NT_n(K_{R,\mathrm{sep}}). \tag{4.28}$$

We will prove that S_R and $\varSigma_R$ serve as the approximations of D_R and $\varDelta_R$ respectively, for some appropriate R. Then the calculation of the ramification breaks of L/K is reduced to Artin–Schreier theory by Lemma 4.9.

There exist multiple choices for matrix $\varSigma_R$ that satisfy (4.28), but they are equivalent modulo $NT_n(\mathbb{F}_p)$ and we may arbitrarily choose one of them to be $\varSigma_R$.

Denote the (i,j)th entry of $W_{R,e}^{-1}$ by $w^*_{R,e,i,j}$.

By Lemma 2.5(v), (vi), (viii), (ix), and (x), we get

$$v_{K_R}(W_{R,0}) = +\infty,\ v_{K_R}(W_{R,e}) = -p^{e-1}m_A \qquad (1 \le e \le N), \tag{4.29}$$

$$\tilde{v}_{K_R}(W_{R,0}) = +\infty,\ \tilde{v}_{K_R}(W_{R,e}) = p^{e-1}\tilde{v}_{K_R}(A_R) \qquad (1 \le e \le N), \tag{4.30}$$

$$v_{K_R}(\Theta) = p^{-1}v_{K_R}(A) = -qp^{-1}m_A, \tag{4.31}$$

$$\tilde{v}_{K_R}(\Theta) = p^{-1}\tilde{v}_{K_R}(A) = qp^{-1}\tilde{v}_{K_R}(A_R), \tag{4.32}$$

$$v_{K_R}(\Theta_R) = -p^{-1}m_A, \tag{4.33}$$

$$\tilde{v}_{K_R}(\Theta_R) = p^{-1}\tilde{v}_{K_R}(A_R). \tag{4.34}$$

Note that for any $X \in FT_n(L)$, we have $v_{K_R}(X) = qv_{K_R}(\iota_R(X))$.

By the definition of S_R, we have $W_{R,N}^{(p)}S_R = \eta_R(A)W_{R,N-1}^{(p)}$. Therefore, we get

$$s_{R,i,j} + \sum_{l=i+1}^{j-1} w_{R,N,i,l}^p s_{R,l,j} = \eta_R(a_{i,j}) + \sum_{l=i+1}^{j-1} \eta_R(a_{i,l})w_{R,N-1,l,j}^p \tag{4.35}$$

for any $1 \le i < j \le n$, since S_R and $\eta_R(A)$ are nilpotent.

Lemma 4.11. *We have*

$$D_R - S_R = \Delta_R^{(p)}(I-(S_R+I)A_R) + (\Theta_R - I)^{(p)}S_RA_R + S_R(A_R - I). \tag{4.36}$$

Proof. We will first show

$$\Delta_R = (\Delta_R^{(p)}(S_R+I) - \Theta_R^{(p)}S_R)A_R. \tag{4.37}$$

By the definition of Δ_R, we have

$$\Delta_R^{(p)}(S_R+I) - \Theta_R^{(p)}S_R = \Theta_R^{(p)} - \Theta^{(p)}W_{R,N}^{(p)}(S_R+I). \tag{4.38}$$

By (4.27) and the definition of Θ, we have

$$\Theta^{(p)}W_{R,N}^{(p)}(S_R+I) = \Theta^{(p)}AW_{R,N-1}^{(p)} = \Theta W_{R,N-1}^{(p)}. \tag{4.39}$$

Hence we have

$$\Delta_R^{(p)}(S_R+I) - \Theta_R^{(p)}S_R = \Theta_R^{(p)} - \Theta W_{R,N-1}^{(p)}. \tag{4.40}$$

By (4.17) and (4.19) and the definition of Δ_R, we have

$$(\Delta_R^{(p)}(S_R+I) - \Theta_R^{(p)}S_R)A_R = \Theta_R - \Theta W_{R,N} = \Delta_R, \tag{4.41}$$

which is precisely (4.37).

By (4.37), we have

$$D_R = \Delta_R^{(p)} - \Delta_R = \Delta_R^{(p)}(I-(S_R+I)A_R) + \Theta_R^{(p)}S_RA_R. \tag{4.42}$$

Since

$$\Theta_R^{(p)}S_RA_R - S_R = (\Theta_R - I)^{(p)}S_RA_R + S_R(A_R - I), \tag{4.43}$$

we obtain (4.36). ∎

Proposition 4.12. *Let* $R_0 = q\max(r_{1,n-1}, r_{2,n}) + (n-1)m_A$ *and* $R > R_0$ *be an integer coprime to* p. *Let* $c = R - q\max(r_{1,n-1}, r_{2,n})$. *Suppose that the following conditions hold.*

(a) $\tilde{v}_{K_R}(t_R^c S_R) > -m_A$
(b) $\tilde{v}_{K_R}(t_R^c D_R) > -m_A$

Then we have

$$v_{K_R}(d_{R,1,n} - s_{R,1,n}) > c - (n-1)m_A > 0 \text{ and } v_{K_R}(D_R - S_R) > 0.$$

Proof. We will prove this proposition by evaluating the valuations of the three terms of the right-hand side of (4.36).

Applying Lemma 2.5(xi) to conditions (a) and (b), we get

$$\tilde{v}_{K_R}(S_R) \geq \left(\frac{c}{n-2} - m_A\right) > 0, \tag{4.44}$$

$$\tilde{v}_{K_R}(D_R) \geq \left(\frac{c}{n-2} - m_A\right) > 0. \tag{4.45}$$

Hence, we obtain

$$v_{K_R}(d_{R,i,j}) = v_{K_R}(\delta_{R,i,j}) > 0 \tag{4.46}$$

for all $1 \leq i < j \leq n$ except for $(i,j) = (1,n)$. Therefore, we have

$$\tilde{v}_{K_R}(D_R) = \tilde{v}_{K_R}(\varDelta_R), \tag{4.47}$$

$$\tilde{v}_{K_R}(t_R^c D_R) = \tilde{v}_{K_R}(t_R^c \varDelta_R) > -m_A \tag{4.48}$$

by condition (b).

It follows by (4.44) and Lemma 2.5(iii) and (v) that

$$\tilde{v}_{K_R}((S_R + I)A_R) \geq \min(\tilde{v}_{K_R}(S_R), \tilde{v}_{K_R}(A_R)) = \tilde{v}_{K_R}(A_R) \geq -m_A. \tag{4.49}$$

Therefore, by Lemma 2.5(xi), we obtain

$$\begin{aligned} &v_{K_R}(t_R^c \varDelta_R^{(p)}(I - (S_R + I)A_R)) \\ &\qquad = v_{K_R}(t_R^{-(p-1)c}(t_R^c \varDelta_R)^{(p)}(I - (S_R + I)A_R)) \\ &\qquad \geq \frac{(p-1)c}{n-1} + v_{K_R}((t_R^c \varDelta_R)^{(p)}(I - (S_R + I)A_R)). \end{aligned} \tag{4.50}$$

By Lemma 2.5(v), we deduce that

$$\begin{aligned} &v_{K_R}((t_R^c \varDelta_R)^{(p)}(I - (S_R + I)A_R)) \\ &\qquad \geq \min(v_{K_R}((t_R^c \varDelta_R)^{(p)}), v_{K_R}(I - (S_R + I)A_R)). \end{aligned} \tag{4.51}$$

Applying Lemma 2.5(iii) and (iv), we get

$$v_{K_R}((t_R^c \varDelta_R)^{(p)}) = p v_{K_R}(t_R^c \varDelta_R), \tag{4.52}$$

$$v_{K_R}(I - (S_R + I)A_R) = v_{K_R}((S_R + I)A_R) \tag{4.53}$$

By (4.48) and (4.49), it follows that

$$\min\left(v_{K_R}((t_R^c\Delta_R)^{(p)}), v_{K_R}(I-(S_R+I)A_R)\right) \geq \min(-pm_A, -m_A) = -pm_A. \tag{4.54}$$

Combining (4.50), (4.51), and (4.54), we get

$$v_{K_R}(t_R^c\Delta_R^{(p)}(I-(S_R+I)A_R)) \geq \frac{(p-1)c}{n-1} - pm_A. \tag{4.55}$$

Since $c > (n-1)m_A$ by definition, we have

$$\frac{(p-1)c}{n-1} - pm_A > ((p-1)-p)m_A = -m_A. \tag{4.56}$$

Hence we see that

$$v_{K_R}(t_R^c\Delta_R^{(p)}(I-(S_R+I)A_R)) > -m_A. \tag{4.57}$$

By Lemma 2.5(vi) and (v), we have

$$\begin{aligned} v_{K_R}(t_R^c(\Theta_R - I)^{(p)}S_RA_R) &\geq \min(v_{K_R}((\Theta_R-I)^{(p)}), \tilde{v}_{K_R}(t_R^cS_RA_R)) \\ &\geq \min(v_{K_R}((\Theta_R-I)^{(p)}), \tilde{v}_{K_R}(t_R^cS_R), \tilde{v}_{K_R}(A_R)). \end{aligned} \tag{4.58}$$

By condition (a), and Lemma 2.5(ii)–(iv), we get

$$\begin{aligned} \min(v_{K_R}((\Theta_R-I)^{(p)}), \tilde{v}_{K_R}(t_R^cS_R), \tilde{v}_{K_R}(A_R)) &\geq \min(pv_{K_R}(\Theta_R), -m_A, v_{K_R}(A_R)) \\ &\geq \min(pv_{K_R}(\Theta_R), -m_A). \end{aligned} \tag{4.59}$$

By (4.33), we obtain

$$\min(pv_{K_R}(\Theta_R), -m_A) = -m_A. \tag{4.60}$$

Hence, we get

$$v_{K_R}(t_R^c(\Theta_R-I)^{(p)}S_RA_R) \geq -m_A. \tag{4.61}$$

By condition (a), Lemma 2.5(iii), and (vii), we get

$$\begin{aligned} v_{K_R}(t_R^cS_R(I-A_R)) &\geq \min(\tilde{v}_{K_R}(t_R^cS_R), \tilde{v}_{K_R}(I-A_R{}^{-1})) \\ &= v_{K_R}(A_R) = -m_A. \end{aligned} \tag{4.62}$$

Therefore, by (4.36), (4.57), (4.61), (4.62), and Lemma 2.5(iii), we get

$$v_{K_R}(t_R^c(D_R-S_R)) \geq -m_A. \tag{4.63}$$

By Lemma 2.5(xi), we get

$$v_{K_R}(D_R-S_R) \geq \frac{c}{n-1} - m_A > 0. \tag{4.64}$$

Consequently, we have

$$v_{K_R}(d_{R,1,n} - s_{R,1,n}) \geq c-(n-1)m_A > 0. \ \blacksquare \tag{4.65}$$

Define $m_{A,i,j}$ and $R_{i,j}$ by

$$m_{A,i,j} = -\min_{i\le l<m\le j}\left(\frac{v_K(a_{l,m})}{m-l}\right), \tag{4.66}$$

$$R_{i,j} = q\max(r_{i,j-1}, r_{i+1,j}) + (j-i)m_{A,i,j} \tag{4.67}$$

for $1 \le i < j \le n$. Note that $-m_{A,i,j}$ is the valuation of the $(j-i+1) \times (j-i+1)$ submatrix of A consisting of the ith to jth columns and the ith to jth rows.

Lemma 4.13. *Let $1 \le i < j \le n$. Let $R > R_{i,j}$ be an integer coprime to p. Suppose we have $v_{K_R}(s_{R,i,j}) = v_{K_R}(d_{R,l,m}) = -qr_{l,m} + R$ for $i \le l < m \le j$ except for $(l,m) = (i,j)$. Then $v_{K_R}(d_{R,i,j} - s_{R,i,j}) > R - R_{i,j} > -qr_{i,j} + R$.*

Proof. Let $c = R - q\max(r_{i,j-1}, r_{i+1,j})$. Note that by Corollary 4.4, we have $R_{i,j} > R_{l,m}$ for $i \le l < m \le j$ such that $j-i > m-l$. By the assumption that $v_{K_R}(s_{R,l,m}) = v_{K_R}(d_{R,l,m}) = -qr_{l,m} + R$ for all $i \le l < m \le j$ except for $(l,m) = (i,j)$ and Corollary 4.4, we get

$$\begin{aligned} v_{K_R}(t_R^c s_{R,l,m}) = v_{K_R}(t_R^c d_{R,l,m}) &= -qr_{l,m} + R - c \\ &\ge 0 > -(m-l)m_{A,i,j} \end{aligned} \tag{4.68}$$

for all $i \le l < m \le j$ except for $(l,m) = (i,j)$. We apply Proposition 4.12 to the $(j-i+1) \times (j-i+1)$ submatrix of A consisting of ith to jth columns and ith to jth rows, setting R in the proposition to be our current R. Note that R_0 in the proposition equals $R_{i,j}$ in this setting, and the conditions of the proposition follow from (4.68). We get

$$v_{K_R}(d_{R,i,j} - s_{R,i,j}) > c - (j-i)m_{A,i,j} = R - R_{i,j}. \tag{4.69}$$

It follows from Lemma 4.5 that $qr_{i,j} > q\max(r_{i,j-1}, r_{i+1,j}) + nm_A + p$. Since we have $m_A \ge m_{A,i,j}$ by definition, we get

$$\begin{aligned} R_{i,j} &= q\max(r_{i,j-1}, r_{i+1,j}) + (j-i)m_{A,i,j} \\ &< q\max(r_{i,j-1}, r_{i+1,j}) + nm_A + p < qr_{i,j}, \end{aligned} \tag{4.70}$$

and we obtain $R - R_{i,j} > -qr_{i,j} + R$. ∎

5. Calculation of the valuations of the matrices and the ramification breaks. Let K be a complete discrete valuation field of equal characteristic $p > 0$ with algebraically closed residue field k and L be a totally wildly ramified finite Galois extension of K such that $\mathrm{Gal}(L/K) \simeq UT_n(\mathbb{F}_p)$ for some $n \ge 2$. Fix an integer N and $q = p^N$ as in Section 4, and define K_R as in Definition 3.3 for any positive integer R coprime to p. Define ι_R as in Definition 4.6. Fix A satisfying the conditions of Lemma 4.2 and define A_R, $W_{R,i}$, D_R, S_R as in Definition 4.7 and Definition 4.10. Define $L_{i,j}$, $r_{i,j}$, and $R_{i,j}$ as in Section 4. Define $\mathrm{coef}_{K,l} : K \to k$ as in Definition 4.1 for any $l \in \mathbb{Z}$.

In this section, we will relate the ramification breaks with the valuations of the entries of S_R. By the definition of S_R, we can express $s_{R,i,j}$ for $1 \leq i < j \leq n$ in terms of the coefficients of the defining equation of L/K.

Recall the definition of the partitions in Definition 2.1.

DEFINITION 5.1. Define $m_{i,j}$, m_λ, $\mu_{i,j}$ for $1 \leq i < j \leq n$ and $\lambda \in \Lambda_{i,j}$ as follows:

$$m_{i,j} = -v_K(a_{i,j}), \tag{5.1}$$

$$m_\lambda = \sum_{u=0}^{|\lambda|-1} m_{\lambda_u,\lambda_{u+1}}, \tag{5.2}$$

$$\mu_{i,j} = \max_{\lambda\in\Lambda_{i,j}} m_\lambda. \tag{5.3}$$

Note that we have

$$v_K(a^*_{i,j}) \geq -\mu_{i,j} \tag{5.4}$$

by Lemma 2.3(i) for $\nu_{i,j} = \mu_{i,j}$ for all $1 \leq i < j \leq n$.

By definition, we have

$$\mu_{i,j} \geq \mu_{i,l} + \mu_{l,j} \tag{5.5}$$

for any $1 \leq i < l < j \leq n$ and

$$\mu_{i,j} \geq m_{i,j} \tag{5.6}$$

for any $1 \leq i < j \leq n$. By the definition of A, we have $m_{i,i} = 0$ for all $1 \leq i \leq n$, and $p \nmid m_{i,j} > 0$ or $m_{i,j} = -\infty$ for all $1 \leq i < j \leq n$.

LEMMA 5.2. *We have $p \nmid m_{i,i+1} = r_{i,i+1} > 0$ for all $1 \leq i \leq n-1$.*

Proof. Since $\mathrm{Gal}(L/K) \simeq UT_n(\mathbb{F}_p)$, we have $[L : K] = p^{n(n-1)/2}$. On the other hand, $L_{i,j}/L_{i,j-1}L_{i+1,j}$ is an Artin–Schreier extension for all $1 \leq i < j \leq n$. Comparing the degrees of the extensions, we see that $L_{i,j}/L_{i,j-1}L_{i+1,j}$ is of degree p for all $1 \leq i < j \leq n$. Therefore, $K(\theta_{i,i+1})/K$ is not trivial, and consequently, $a_{i,i+1} = -P(\theta_{i,i+1}) \neq 0$. By the definition of A, we have $p \nmid m_{i,i+1} = -v_K(a_{i,i+1}) > 0$. Hence, we obtain $r_{i,i+1} = m_{i,i+1}$. ∎

LEMMA 5.3. *Let $1 \leq e \leq N$ and $1 \leq i < j \leq n$. We have*

$$w_{R,e,i,j} \equiv a_{R,i,j}^{p^{e-1}} \pmod{\mathfrak{m}_{K_R}^{-p^{e-1}\mu_{i,j}+1}}, \tag{5.7}$$

$$v_{K_R}(w_{R,e,i,j}) \geq -p^{e-1}\mu_{i,j}. \tag{5.8}$$

Equality in (5.8) *holds if and only if $m_{i,j} = \mu_{i,j}$. We also have*

$$w^*_{R,e,i,j} \equiv a_{R,i,j}^{*p^{e-1}} \pmod{\mathfrak{m}_{K_R}^{-p^{e-1}\mu_{i,j}+1}}, \tag{5.9}$$

$$v_{K_R}(w^*_{R,e,i,j}) \geq -p^{e-1}\mu_{i,j}. \tag{5.10}$$

Equality in (5.10) *holds if and only if equality in* (5.4) *holds for the pair (i,j).*

Proof. This lemma follows from the definition of $W_{R,e}$ and Lemma 2.3(i) and (iv) for $\nu_{i,j} = p^{e-1}\mu_{i,j}$ for all $1 \leq i < j \leq n$. ∎

For all $1 \leq i < j \leq n$, we have

$$\eta_R(a_{i,j}) = \sum_{l=-m_{i,j}}^{\infty} \mathrm{coef}_{K,l}(a_{i,j}) t_R^{-ql} \sum_{m=1}^{\infty} \binom{-l/R}{m} t_R^{-mR} \tag{5.11}$$

by (3.20). Thus,

$$v_{K_R}(\eta_R(a_{i,j})) = -qm_{i,j} + R. \tag{5.12}$$

Definition 5.4. Define $a_{i,j}^{[m]}$ and $s_{R,i,j}^{[m]}$ for $1 \leq i < j \leq n$ and $m \in \mathbb{Z}_{\geq 0}$ by

$$a_{i,j}^{[m]} = \sum_{l=-m_{i,j}}^{\infty} \mathrm{coef}_{K,l}(a_{i,j}) \binom{-l/R}{m} t_R^{-ql-mR}, \tag{5.13}$$

$$s_{R,i,j}^{[m]} = \sum_{i \leq i' < j' \leq j} w_{R,N,i,i'}^{*p} a_{i',j'}^{[m]} w_{R,N-1,j',j}^{p}. \tag{5.14}$$

Define $a'_{i,j}$ for $1 \leq i < j \leq n$ by

$$a'_{i,j} = \sum_{l=-m_{i,j}}^{\infty} l \, \mathrm{coef}_{K,l}(a_{i,j}) t^{-l}. \tag{5.15}$$

We define $\mathrm{coef}_{K_R,i}$ for K_R in the same manner as in Definition 4.1.

Lemma 5.5. *Take any $1 \leq i < j \leq n$ and $m \in \mathbb{Z}_{\geq 0}$. Then $a_{i,j}^{[m]}$, $s_{R,i,j}^{[m]}$, and $a'_{i,j}$ satisfy the following properties:*

(i)

$$a_{i,j} = \sum_{m=0}^{\infty} a_{i,j}^{[m]}. \tag{5.16}$$

(ii)

$$a_{i,j}^{[0]} = a_{R,i,j}^{q}. \tag{5.17}$$

(iii)

$$\eta_R(a_{i,j}) = \sum_{m=1}^{\infty} a_{i,j}^{[m]}. \tag{5.18}$$

(iv)

$$s_{R,i,j} = \sum_{m=1}^{\infty} s_{R,i,j}^{[m]}. \tag{5.19}$$

(v)

$$a_{i,j}^{[1]} = -\frac{t_R^{-R}}{R}\iota_R(a'_{i,j})^q \tag{5.20}$$

(vi)

$$\mathrm{coef}_{K_R,l}(a_{i,j}^{[m]}) \neq 0 \implies l \equiv mR \pmod{q}. \tag{5.21}$$

(vii)

$$v_{K_R}(a_{i,j}^{[m]}) = -qm_{i,j} + mR. \tag{5.22}$$

(viii)

$$v_K(a'_{i,j}) = -m_{i,j}. \tag{5.23}$$

(ix)

$$v_{K_R}\Big(\sum_{l=i}^{j-1} a_{i,l}^* a_{l,j}^{[m]}\Big) \geq -q\mu_{i,j} + mR. \tag{5.24}$$

(x)

$$v_K\Big(\sum_{l=i}^{j-1} a_{i,l}^* a'_{l,j}\Big) \geq -\mu_{i,j}. \tag{5.25}$$

(xi)

$$v_{K_R}(s_{R,i,j}^{[m]}) \geq -q\mu_{i,j} + mR. \tag{5.26}$$

Equality in (5.26) *holds if and only if equality in* (5.24) *holds.*

(xii)

$$v_{K_R}(s_{R,i,j}^{[m]}) \equiv mR \pmod{p}. \tag{5.27}$$

(xiii)

$$v_{K_R}(\eta_R(a_{i,j})) = -qm_{i,j} + R. \tag{5.28}$$

(xiv)

$$v_{K_R}(s_{R,i,j}) \geq -q\mu_{i,j} + R. \tag{5.29}$$

Equality in (5.29) *holds if and only if equality in* (5.25) *holds.*

(xv)

$$v_{K_R}(s_{R,i,i+1}) = -qm_{i,i+1} + R. \tag{5.30}$$

Proof. (i)–(viii) follow from (3.20), (5.11), and the definitions of $a_{i,j}^{[m]}$, $s_{R,i,j}^{[m]}$, and $a'_{i,j}$.

(ix) This follows from (vii), (5.4), (5.5) and (5.6).

(x) This follows from (viii), (5.4), (5.5) and (5.6).

(xi) Combining Lemma 5.3 and (vii) with (5.14), we get the inequality.

Since

$$v_{K_R}(w^{*p}_{R,N,i,i'}a^{[m]}_{i',j'}w^{p}_{R,N-1,j',j}) \geq -q\left(\mu_{i,i'} + m_{i',j'} + \frac{\mu_{j',j}}{p}\right) + mR \tag{5.31}$$
$$> -q\mu_{i,j} + mR$$

for all $i \leq i' < j' < j$, we obtain

$$s^{[m]}_{R,i,j} \equiv \sum_{i'=i}^{j-1} w^{*p}_{R,N,i,i'}a^{[m]}_{i',j} \pmod{\mathfrak{m}_{K_R}^{-q\mu_{i,j}+mR+1}}. \tag{5.32}$$

By (5.9) and (vii), we get

$$w^{*p}_{R,N,i,i'}a^{[m]}_{i',j} \equiv a^{*q}_{R,i,i'}a^{[m]}_{i',j} \pmod{\mathfrak{m}_{K_R}^{-q\mu_{i,j}+mR+1}}, \tag{5.33}$$

for all $i \leq i' < j' < j$, since $\mu_{i,i'} + m_{i',j} \leq \mu_{i,j}$. We have

$$\begin{aligned} \operatorname{coef}_{K_R,-qm_{i,i'}}(a^{*q}_{R,i,i'}) &= \operatorname{coef}_{K_R,-qm_{i,i'}}(\iota_R(a^{*}_{i,i'})^q) \\ &= \operatorname{coef}_{K_R,-qm_{i,i'}}(a^{*}_{i,i'}) \end{aligned} \tag{5.34}$$

by the definition of ι_R and t_R. Hence, we get the assertion.

(xii) By (vi), we obtain

$$\operatorname{coef}_{K_R,l}(x^p a^{[m]}_{i,j}) \neq 0 \implies l \equiv mR \pmod{p} \tag{5.35}$$

for any $x \in K_R$. Thus, by (5.14), we have

$$\operatorname{coef}_{K_R,l}(s^{[m]}_{R,i,j}) \neq 0 \implies l \equiv mR \pmod{p}. \tag{5.36}$$

This gives the assertion.

(xiii) By (iii) and (vii), we get the assertion.

(xiv) By (iv) and (xi), we get the inequality. Furthermore, we deduce that equality holds if and only if

$$v_{K_R}(s_{R,i,j}) = v_{K_R}(s^{[1]}_{R,i,j}) = -q\mu_{i,j} + R. \tag{5.37}$$

The first equality of (5.37) holds if the second equality holds, since $s^{[1]}_{R,i,j}$ is the only term in the right-hand side of (5.19) whose valuation in K can be equal to $-q\mu_{i,j} + R$. Thus, by (xi), (5.37) holds if and only if the equality in (5.24) for $m = 1$ holds. By (v), it follows that

$$\sum_{l=i}^{j-1} a^{*}_{i,l}a^{[1]}_{l,j} = -\frac{t_R^{-R}}{R}\Big(\sum_{l=i}^{j-1} a^{*}_{i,l}a'_{l,j} - \sum_{l=i}^{j-1} a^{*}_{i,l}\eta_R(a'_{l,j})\Big). \tag{5.38}$$

By the same argument as (5.12), we get $v_{K_R}(\eta_R(a'_{l,j})) = -qm_{i,j} + R$. Therefore, we have

$$v_{K_R}\left(\frac{t_R^{-R}}{R}\sum_{l=i}^{j-1} a^{*}_{i,l}\eta_R(a'_{l,j})\right) \geq -q\mu_{i,j} + 2R. \tag{5.39}$$

This implies that the equality in (5.24) for $m = 1$ holds if and only if

$$v_{K_R}\Big(-\frac{t_R^{-R}}{R}\sum_{l=i}^{j-1} a^*_{i,l}a'_{l,j}\Big) = -q\mu_{i,j} + R. \tag{5.40}$$

Equation (5.40) is equivalent to the equality in (5.25).

(xv) By (4.35), we get $s_{R,i,i+1} = \eta_R(a_{i,i+1})$. We get the assertion by (iii) and (vii). ∎

REMARK 5.6. Let $d : K \to \Omega^1_{K/k}$ be the canonical derivation of the Kähler differential. Then

$$da_{i,j} = \sum_{l=-m_{i,j}}^{\infty} -l\operatorname{coef}_{K,l}(a_{i,j})t^{-l-1}dt = -a'_{i,j}\frac{dt}{t}. \tag{5.41}$$

Thus the element $a'_{i,j} \in K$ corresponds to $da_{i,j}$ in [I25]. We will see that the results of this paper are consistent with the results of [I25] in Example 6.6.

LEMMA 5.7. *Let $1 \le i < j \le n$ and suppose $j - i > 1$ and the inequality*

$$\mu_{i,j} = m_{i,j} > \mu_{i,l} + \mu_{l,j} \tag{5.42}$$

holds for all $i < l < j$. Then

$$v_{K_R}(s_{R,i,j}) = v_{K_R}(\eta_R(a_{i,j})) = -qm_{i,j} + R. \tag{5.43}$$

Proof. By Lemma 5.3 and Lemma 5.5(xiii) and (xiv), we get

$$v_{K_R}\Big(\sum_{l=i+1}^{j-1}(\eta_R(a_{i,l})w^p_{R,N-1,l,j} - w^p_{R,N,i,l}s_{R,l,j})\Big) \ge -q\max_{i<l<j}(\mu_{i,l} + \mu_{l,j}) + R \tag{5.44}$$

Combining (4.35), (5.42), (5.44), and Lemma 5.5(xiii), we obtain (5.43). ∎

LEMMA 5.8. *Let $1 \le i < j \le n$, and suppose $\lambda \in \Lambda_{i,j}$ is one of the longest partitions satisfying*

$$\mu_{i,j} = m_\lambda. \tag{5.45}$$

Then

$$\mu_{i,j} = -\sum_{u=0}^{|\lambda|-1}\frac{v_{K_R}(s_{R,\lambda_u,\lambda_{u+1}}) - R}{q} \tag{5.46}$$

Proof. We have

$$\mu_{\lambda_u,\lambda_{u+1}} = m_{\lambda_u,\lambda_{u+1}} > \mu_{\lambda_u,m} + \mu_{m,\lambda_{u+1}} \tag{5.47}$$

for all $0 \le u \le |\lambda| - 1$ and $\lambda_u < m < \lambda_{u+1}$. Indeed, if $\mu_{\lambda_u,\lambda_{u+1}} = \mu_{\lambda_u,m} + \mu_{m,\lambda_{u+1}}$ for some $\lambda_u < m < \lambda_{u+1}$, then there exists a partition $\lambda' \in \Lambda_{\lambda_u,\lambda_{u+1}}$ satisfying

$$m_{\lambda_u,\lambda_{u+1}} = \mu_{\lambda_u,\lambda_{u+1}} = \mu_{\lambda_u,m} + \mu_{m,\lambda_{u+1}} = m_{\lambda'}. \tag{5.48}$$

Then we can make a partition satisfying (5.45) longer than λ by substituting the part $\lambda_u < \lambda_{u+1}$ of λ by λ'. This contradicts the maximality of the length of λ, and therefore we get the assertion by Lemma 5.7 and (5.47). ∎

Lemma 5.8 shows that there exists at least one partition satisfying (5.46).

LEMMA 5.9. *Let $1 \le i < j \le n$ and $R > R_{i,j}$ be an integer coprime to p. Let λ be the longest partition in $\Lambda_{i,j}$, i.e. λ satisfies $|\lambda| = j-i$ and $\lambda_u = i+u$ for $0 \le u \le j-i$. Suppose that λ is the only partition for (i,j) satisfying* (5.46) *in Lemma* 5.8. *Then*

$$\mu_{l,m} = m_{l,m} = \sum_{l'=l}^{m-1} m_{l',l'+1} \tag{5.49}$$

for all $i \le l < m \le j$.

Proof. By Lemma 5.5(xv), we get

$$\mu_{i,j} = \sum_{l=i}^{j-1} m_{l,l+1}. \tag{5.50}$$

For any $i \le l < m \le j$, we have

$$\begin{aligned}\mu_{i,j} &\ge \mu_{i,l} + \mu_{l,m} + \mu_{m,j} \\ &\ge \sum_{l'=i}^{l-1} m_{l',l'+1} + \sum_{l'=l}^{m-1} m_{l',l'+1} + \sum_{l'=m}^{j-1} m_{l',l'+1} = \sum_{l'=i}^{j-1} m_{l',l'+1}\end{aligned} \tag{5.51}$$

by definition. By (5.50), we see that the inequalities in (5.51) are equalities. Thus, by Lemma 5.5(xv), we have

$$-q\mu_{l,m} = -q\sum_{l'=l}^{m-1} m_{l',l'+1} = \sum_{l'=l}^{m-1}(v_{K_R}(s_{R,l',l'+1}) - R) \tag{5.52}$$

for all $i \le l < m \le j$. By Lemma 5.5(xiv) and (5.52), we get

$$v_{K_R}(s_{R,l,m}) - R \ge \sum_{l'=l}^{m-1}(v_{K_R}(s_{R,l',l'+1}) - R) \tag{5.53}$$

for all $i \le l < m \le j$. If the equality holds in (5.53) for some $i \le l < m \le j$ with $m - l > 1$, then

$$\begin{aligned}-q\mu_{i,j} = &\sum_{l'=i}^{l-1}(v_{K_R}(s_{R,l',l'+1}) - R) + (v_{K_R}(s_{R,l,m}) - R) \\ &+ \sum_{l'=m}^{j-1}(v_{K_R}(s_{R,l',l'+1}) - R).\end{aligned} \tag{5.54}$$

This implies that there exists a partition shorter than λ satisfying (5.46), which contradicts the assumption. Therefore the equality does not hold in (5.53) for all $i \le l < m \le j$ with $m - l > 1$. Hence, combining with (5.52), we deduce that equality in

$$v_{K_R}(s_{R,l,m}) \ge -q\mu_{l,m} + R \tag{5.55}$$

holds if and only if $m - l = 1$.

We will now show that $m_{l,m} = \mu_{l,m}$ for $i \le l < m \le j$ by induction on $m - l$.

⟨1⟩ Case $m - l = 1$. This is immediate from the definition of $\mu_{l,m}$.

⟨2⟩ Case $m-l > 1$. We will evaluate the valuation of each term in (4.35). By the induction hypothesis, we have $m_{l,l'} = \mu_{l,l'}$ and $m_{l',m} = \mu_{l',m}$ for all $l < l' < m$. By Lemma 5.3, we deduce that

$$v_{K_R}(w^p_{R,N,l,l'}) = -q\mu_{l,l'}, \tag{5.56}$$

$$v_{K_R}(w^p_{R,N-1,l',m}) = -qp^{-1}\mu_{l',m}. \tag{5.57}$$

Thus

$$v_{K_R}(\eta_R(a_{l,l'})w^p_{R,N-1,l',m}) = -q\left(m_{l,l'} + \frac{\mu_{l,m}}{p}\right) + R > -q\mu_{l,m} + R \tag{5.58}$$

for all $i \le l < l' < m \le j$, and

$$v_{K_R}(w^p_{R,N,l,l'}s_{R,l',m}) \ge -q(\mu_{l,l'} + \mu_{l',m}) + R = -q\mu_{l,m} + R \tag{5.59}$$

for all $i \le l < l' < m \le j$ by (5.55) and (5.56). Since equality in (5.55) holds if and only if $l = m-1$, equality in (5.59) also holds if and only if $l' = m-1$.

We now evaluate the valuation of each term of (4.35) for $(i,j) = (l,m)$ by (5.55), (5.58), and (5.59). Taking into account that equality in (5.55) does not hold for $l < m-1$ and that equality in (5.59) does not hold for $l' < m-1$, we deduce that $v_{K_R}(w^p_{R,N,l,m-1}s_{R,m-1,m} - \eta_R(a_{l,m})) > -q\mu_{l,m} + R$. Hence, we must have

$$v_{K_R}(w^p_{R,N,l,m-1}s_{R,m-1,m}) = v_{K_R}(\eta_R(a_{l,m})), \tag{5.60}$$

since equality holds in (5.59) if $l' = m - 1$. This gives $-q\mu_{l,m} + R = -qm_{l,m} + R$ by Lemma 5.5(xiii), and consequently, $\mu_{l,m} = m_{l,m}$. Combining with (5.52), we obtain the assertion. ∎

Lemma 5.10. *If $v_{K_R}(s_{R,i,j}) < -q\mu_{i,j} + pR$, then $p \nmid v_{K_R}(s_{R,i,j})$.*

Proof. Since

$$v_{K_R}(s^{[m]}_{R,i,j}) \ge -q\mu_{i,j} + mR \ge -q\mu_{i,j} + pR > v_{K_R}(s_{R,i,j}) \tag{5.61}$$

for $m \geq p$, we have

$$v_{K_R}(s_{R,i,j}) \geq \min_{1\leq m<p}(v_{K_R}(s^{[m]}_{R,i,j})). \tag{5.62}$$

By (5.27), $v_{K_R}(s^{[m]}_{R,i,j})$ $(1 \leq m < p)$ are different from each other. Therefore the equality holds in (5.62), and we get $p \nmid v_{K_R}(s_{R,i,j})$. ∎

Lemma 5.11. *Let R' be a positive integer coprime to p and suppose we have $v_{K_{R'}}(s_{R',i,j}) = -qr_{i,j}+R' < -q\mu_{i,j}+pR'$. Then $v_{K_R}(s_{R,i,j}) = -qr_{i,j}+R$ for any integer $R > R'$ congruent to R' modulo p.*

Proof. Combining Lemma 5.10 with $v_{K_{R'}}(s_{R',i,j}) \equiv R' \pmod p$, we get $v_{K_{R'}}(s^{[1]}_{R',i,j}) = -qr_{i,j} + R'$ and $v_{K_{R'}}(s^{[m]}_{R',i,j}) > -qr_{i,j} + R'$ for all $m \geq 2$.

Since $R \equiv R' \pmod p$, we have $\binom{-l/R}{m} \equiv \binom{-l/R'}{m} \pmod p$ for any integer l and $1 \leq m < p$. Therefore, by (5.14), we have $\mathrm{coef}_{K_{R'},pl+mR'}(s^{[m]}_{R',i,j}) = \mathrm{coef}_{K_R,pl+mR}(s^{[m]}_{R,i,j})$ for any integer l and $1 \leq m < p$. This implies $v_{K_R}(s^{[m]}_{R,i,j}) = v_{K_{R'}}(s^{[m]}_{R',i,j}) + m(R-R')$ for $1 \leq m < p$. Therefore we have

$$\begin{aligned} v_{K_R}(s^{[m]}_{R,i,j}) &= v_{K_{R'}}(s^{[m]}_{R',i,j}) + m(R-R') \\ &> -qr_{i,j} + R' + m(R-R') > -qr_{i,j} + R \end{aligned} \tag{5.63}$$

for $2 \leq m < p$ and

$$v_{K_R}(s^{[m]}_{R,i,j}) \geq -q\mu_{i,j} + pR > -qr_{i,j} + R \tag{5.64}$$

for $m \geq p$. Thus we have $v_{K_R}(s_{R,i,j}) = v_{K_R}(s^{[1]}_{R,i,j}) = -qr_{i,j} + R$. ∎

Proposition 5.12. *Let $1 \leq i < j \leq n$. Fix an integer $R > R_{i,j}$ coprime to p, where $R_{i,j}$ is defined as in (4.67). Suppose we have $v_{K_R}(s_{R,l,m}) < -q\mu_{l,m} + pR$ for $i \leq l < m \leq j$. Then we have $v_{K_R}(s_{R,i,j}) = v_{K_R}(d_{R,i,j}) = -qr_{i,j} + R$.*

Proof. We will prove the claim by induction on $j - i$.

⟨1⟩ Case $j-i = 1$. We have $s_{R,i,i+1} = \eta_R(a_{i,i+1})$. By (5.12) and Lemma 5.2, we have

$$v_{K_R}(s_{R,i,i+1}) = -qm_{i,i+1} + R = -qr_{i,i+1} + R.$$

By (4.20) and the definition of D_R, we have

$$d_{R,i,i+1} = -P(\eta_R(\theta_{i,i+1})) = \eta_R(a_{i,i+1}).$$

Hence

$$v_{K_R}(d_{R,i,i+1}) = v_{K_R}(s_{R,i,i+1}). \tag{5.65}$$

⟨2⟩ Case $j - i > 1$. Note that by Corollary 4.4, we have $R_{i,j} > R_{l,m}$ for $i \leq l < m \leq j$ such that $j-i > m-l$. Therefore by the induction hypothesis, we get $v_{K_R}(s_{R,l,m}) = v_{K_R}(d_{R,l,m}) = -qr_{l,m} + R$ for all $i \leq l < m \leq j$ except for $(l,m) = (i,j)$. Hence, by Lemma 4.13, we get $v_{K_R}(d_{R,i,j} - s_{R,i,j}) > R - R_{i,j} > -qr_{i,j} + R$. Thus, it remains to show $v_{K_R}(s_{R,i,j}) = -qr_{i,j} + R$.

$\langle$2-1$\rangle$ Case $R_{i,j} < R < (q-1)r_{i,j}$. Since $v_{K_R}(d_{R,i,j} - s_{R,i,j}) > R - R_{i,j} > 0$, by Lemma 4.9 and Lemma 5.10, we have $v_{K_R}(s_{R,i,j}) = -qr_{i,j} + R$.

$\langle$2-2$\rangle$ Case $R \geq (q-1)r_{i,j}$. By Lemma 4.5, we have $R_{i,j} < (q-1)r_{i,j} - p$. We can take an integer $R' \leq R$ such that $R' \equiv R \pmod{p}$ and $R_{i,j} \leq R' < (q-1)r_{i,j}$.

Therefore by Case $\langle$2-1$\rangle$and Lemma 5.11, we get

$$v_{K_R}(s_{R,i,j}) = -qr_{i,j} + R. \blacksquare$$

Proposition 5.13. *Let $1 \leq i < j \leq n$ and $R > R_{i,j}$ be an integer coprime to p. Suppose we have $v_{K_R}(s_{R,l,m}) = v_{K_R}(d_{R,l,m}) = -qr_{l,m} + R$ for $i \leq l < m \leq j$ except for $(l,m) = (i,j)$ and $v_{K_R}(s_{R,i,j}) \leq -qr_{i,j} + R$. Let λ be a partition for (i,j) satisfying* (5.46) *in Lemma* 5.8. *Then we have $v_{K_R}(s_{R,i,j}) \leq -q\mu_{i,j} + |\lambda|R$ and the equality holds if and only if $|\lambda| = 1$.*

Proof. By (5.46), we have

$$-q\mu_{i,j} + |\lambda|R = \sum_{u=0}^{|\lambda|-1} v_{K_R}(s_{R,\lambda_u,\lambda_{u+1}}). \tag{5.66}$$

$\langle$1$\rangle$ Case $|\lambda| = 1$. In this case, (5.66) becomes $v_{K_R}(s_{R,i,j}) = -q\mu_{i,j} + R$.

$\langle$2$\rangle$ Case $|\lambda| > 1$. By assumption and (5.66), we get

$$-q\mu_{i,j} + |\lambda|R = \sum_{u=0}^{|\lambda|-1} (-qr_{\lambda_u,\lambda_{u+1}} + R). \tag{5.67}$$

By Corollary 4.4, we have $R > R_{i,j} > qr_{\lambda_u,\lambda_{u+1}}$ and $r_{i,j} > r_{\lambda_u,\lambda_{u+1}}$ for all $0 \leq u \leq |\lambda| - 1$. Therefore,

$$-qr_{\lambda_u,\lambda_{u+1}} + R > -qr_{\lambda_u,\lambda_{u+1}} + R_{i,j} > 0, \tag{5.68}$$

$$-qr_{\lambda_u,\lambda_{u+1}} + R > -qr_{i,j} + R. \tag{5.69}$$

Therefore,

$$\begin{aligned} -q\mu_{i,j} + |\lambda|R &> (|\lambda| - 1) \cdot 0 + 1 \cdot (-qr_{i,j} + R) \\ &= -qr_{i,j} + R \geq v_{K_R}(s_{R,i,j}). \blacksquare \end{aligned} \tag{5.70}$$

6. Case $j - i \leq p+1$. Let K be a complete discrete valuation field of equal characteristic $p > 0$ with algebraically closed residue field k and L be a totally wildly ramified finite Galois extension of K such that $\mathrm{Gal}(L/K) \simeq UT_n(\mathbb{F}_p)$

for some $n \geq 2$. Fix an integer $N > \log_p(n(p^{\frac{n(n-1)}{2}} + 1)m_A + p) + \frac{n(n-1)}{2}$ and $q = p^N$, and define K_R as in Definition 3.3 for any positive integer R coprime to p. Fix A satisfying the conditions of 4.2 and define A_R, $W_{R,i}$, D_R, S_R as in Definitions 4.7 and 4.10. Define $r_{i,j}$ as in Section 4, $m_{i,j}$ and $\mu_{i,j}$ as in Definition 5.1, and $a_{i,j}^{[m]}$ and $s_{R,i,j}^{[m]}$ as in Definition 5.4 for $1 \leq i < j \leq n$ and $m \in \mathbb{Z}$. Define $\mathrm{coef}_{K,l} : K \to k$ as in Definition 4.1 for any $l \in \mathbb{Z}$.

We will show in this section that the premise of Proposition 5.12 holds for $1 \leq i < j \leq n$ with $j - i \leq p + 1$.

Lemma 6.1. *Let $m > m' \geq 2$ be integers. Let $x_{i,j}$ $(1 \leq i \leq j \leq m)$ be integers satisfying $x_{i,j} + x_{j,l} = x_{i,l}$ for all $1 \leq i \leq j \leq l \leq m$. Then there exists $1 \leq i < j \leq m$ such that $x_{i,j} \equiv 0 \pmod{m'}$.*

Proof. By the pigeonhole principle, there exists $1 \leq i < j \leq m$ such that $x_{1,i} \equiv x_{1,j} \pmod{m'}$. Then we have $x_{i,j} = x_{1,j} - x_{1,i} \equiv 0 \pmod{m'}$. ∎

Lemma 6.2. *Let $1 \leq i < j \leq n$ and $R > R_{i,j}$ be an integer coprime to p. Suppose that $j - i \leq p + 1$ and $v_{K_R}(s_{R,l,m}) = v_{K_R}(d_{R,l,m}) = -qr_{l,m} + R < -q\mu_{l,m} + pR$ for $i \leq l < m \leq j$ except for $(l,m) = (i,j)$. Then $v_{K_R}(s_{R,i,j}) \leq -q\mu_{i,j} + \min(p, j-i)R$, and the equality holds if and only if $j - i = 1$.*

Proof. Let $\lambda \in \Lambda_{i,j}$ be one of the shortest partitions satisfying (5.46) in Lemma 5.8.

⟨1⟩ Case $|\lambda| \leq p$ and $R_{i,j} < R < (q-1)r_{i,j}$. By Lemma 4.13, we get

$$v_{K_R}(d_{R,i,j} - s_{R,i,j}) > R - R_{i,j} > 0. \tag{6.1}$$

Hence, by Lemma 4.9, we get $v_{K_R}(s_{R,i,j}) \leq -qr_{i,j} + R$. Then the assertion follows from Proposition 5.13.

⟨2⟩ Case $|\lambda| \leq p$ and $R \geq (q-1)r_{i,j}$. By Lemma 4.5, we have $R_{i,j} < (q-1)r_{i,j} - p$. We can take an integer $R' \leq R$ such that $R' \equiv R \pmod{p}$ and $R_{i,j} \leq R' < (q-1)r_{i,j}$. By Proposition 5.12 and ⟨1⟩, we get $v_{K_{R'}}(s_{R',i,j}) = -qr_{i,j} + R' < -q\mu_{i,j} + pR'$. By Lemma 5.11, we have $v_{K_R}(s_{R,i,j}) = -qr_{i,j} + R$. Hence, the assertion follows from Proposition 5.13.

⟨3⟩ Case $|\lambda| = p + 1$. We will prove by contradiction that this case does not happen.

By assumption, we have $|\lambda| = p + 1 = j - i$. By Lemma 5.9, we have

$$m_{l,m} = \sum_{l'=l}^{m-1} m_{l',l'+1} \tag{6.2}$$

for all $i \leq l < m \leq j$.

Then by Lemma 6.1, there exists $i \leq l < m \leq j$ such that $p \mid m_{l,m}$. This contradicts the definition of $m_{l,m}$. ∎

THEOREM 6.3. *Fix A satisfying the conditions of Lemma 4.2 and let $\Theta = (\theta_{i,j})_{i,j}$ be a solution of the equation $X^{(p)}A = X$. Fix an integer N and $q = p^N$ as in Section 4, and define K_R as in Definition 3.3. Define S_R as in Definition 4.10 and $r_{i,j}$ as the largest upper ramification jump of $L_{i,j}/K$, where $L_{i,j} = K(\theta_{l,m}\ (i \le l < m \le j))$ for $1 \le i < j \le n$, as in Section 4.*

Let $1 \le i < j \le n$ and $j - i \le p+1$. Then we have $v_{K_R}(s_{R,i,j}) = v_{K_R}(s^{[1]}_{R,i,j}) = -qr_{i,j} + R$ for any integer $R > R_{i,j}$ coprime to p.

Proof. By using Lemma 6.2 and Proposition 5.12 inductively, we get $v_{K_R}(s_{R,i,j}) = -qr_{i,j} + R < -q\mu_{i,j} + pR$. By the same argument as in the proof of Lemma 5.10, we get $v_{K_R}(s_{R,i,j}) = v_{K_R}(s^{[1]}_{R,i,j})$. ∎

This implies that we can calculate $r_{i,j}$ for $j - i \le p+1$ by calculating $s_{R,i,j} \in K_R$ for sufficiently large R.

COROLLARY 6.4. *Let $1 \le i < j \le n$ and $j - i \le p+1$. Then we have $r_{i,j} \le \mu_{i,j}$, and equality holds if and only if equality in (5.25) holds.*

Proof. This follows from Theorem 6.3 and Lemma 5.5(xiv). ∎

REMARK 6.5. Note that we have $\mathcal{U}_{L/K} \supset \{r_{i,j}\}_{1\le i<j\le n}$ and $\max\mathcal{U}_{L/K} = r_{1,n}$, but we do not always have $\mathcal{U}_{L/K} = \{r_{i,j}\}_{1\le i<j\le n}$. For example, let $p > 2, n = 3$ and suppose that $L_{1,2}/K$ is defined by $P(x) = t^2$ and $L_{2,3}/K$ is defined by $P(x) = t^2 + t$. Then we have $r_{1,2} = r_{2,3} = 2 < r_{1,3}$. However, $L_{1,2}L_{2,3}$ includes the roots of $P(x) - t$, so $1 \in \mathcal{U}_{L_{1,2}L_{2,3}/K} \subset \mathcal{U}_{L/K}$.

In general, $\mathcal{U}_{L/K} = \{r_{i,j}\}_{1\le i<j\le n}$ if $r_{i,j}$ for $1 \le i < j \le n$ do not coincide with each other at all, since $|\mathcal{U}_{L/K}| \le n(n-1)/2$.

On the other hand, we have $|\{r_{i,j}\}| \ge n-1$, since $r_{1,2} < r_{1,3} < \cdots < r_{1,n}$ by Corollary 4.4.

EXAMPLE 6.6. Consider the case $n = 3$. Suppose R is a sufficiently large positive integer coprime to p.

By definition, we have

$$s_{R,1,3} = a^{[1]}_{1,3} + a^{[1]}_{1,2} w^p_{R,N-1,2,3} - w^p_{R,N,1,2} a^{[1]}_{2,3}. \tag{6.3}$$

By the definition of W_R, we have

$$w_{R,N,1,2} = \sum_{e=0}^{N-1} a^{p^e}_{R,1,2}, \tag{6.4}$$

$$w_{R,N-1,2,3} = \sum_{e=0}^{N-2} a^{p^e}_{R,2,3}. \tag{6.5}$$

Therefore, we get

$$s^{[1]}_{R,1,3} = a^{[1]}_{1,3} - a^q_{R,1,2} a^{[1]}_{2,3} + \sum_{e=1}^{N-1} (a^{[1]}_{1,2} a^{p^e}_{R,2,3} - a^{p^e}_{R,1,2} a^{[1]}_{2,3}). \tag{6.6}$$

In addition, we have

$$v_{K_R}(a_{1,3}^{[1]}) = -qm_{1,3} + R, \tag{6.7}$$

$$v_{K_R}(a_{R,1,2}^{q}a_{2,3}^{[1]}) = -q(m_{1,2} + m_{2,3}) + R, \tag{6.8}$$

$$v_{K_R}(a_{1,2}^{[1]}a_{R,2,3}^{p^e}) = -q\left(m_{1,2} + \frac{m_{2,3}}{p^{N-e}}\right) + R \geq -q(m_{1,2} + m_{2,3}) + R \quad (1 \leq e \leq N-1), \tag{6.9}$$

$$v_{K_R}(a_{R,1,2}^{p^e}a_{2,3}^{[1]}) = -q\left(\frac{m_{1,2}}{p^{N-e}} + m_{2,3}\right) + R \geq -q(m_{1,2} + m_{2,3}) + R \quad (1 \leq e \leq N-1). \tag{6.10}$$

Since A satisfies condition (a) of Lemma 4.2, by Lemma 5.5(vi), we have

$$v_{K_R}(a_{1,2}^{[1]}a_{R,2,3}^{p^e} - a_{R,1,2}^{p^e}a_{2,3}^{[1]}) \in \left(\frac{q}{p^e}\mathbb{Z} + R\right) - \left(\frac{q}{p^{e-1}}\mathbb{Z} + R\right) \tag{6.11}$$

for $1 \leq e \leq N-1$, and consequently

$$v_{K_R}\Big(\sum_{e=1}^{N-1}(a_{1,2}^{[1]}a_{R,2,3}^{p^e} - a_{R,1,2}^{p^e}a_{2,3}^{[1]})\Big) \notin q\mathbb{Z} + R. \tag{6.12}$$

By Lemma 5.5(vi), we have

$$\operatorname{coef}_{K_R,l}(a_{1,3}^{[1]} - a_{R,1,2}^{q}a_{2,3}^{[1]}) \neq 0 \Rightarrow l \in q\mathbb{Z} + R, \tag{6.13}$$

and consequently

$$v_{K_R}(a_{1,3}^{[1]} - a_{R,1,2}^{q}a_{2,3}^{[1]}) \in q\mathbb{Z} + R. \tag{6.14}$$

Therefore, we have

$$\begin{aligned} v_{K_R}(s_{R,1,3}^{[1]}) & \\ &= \min\Big(v_{K_R}(a_{1,3}^{[1]} - a_{R,1,2}^{q}a_{2,3}^{[1]}), v_{K_R}\Big(\sum_{e=1}^{N-1}(a_{1,2}^{[1]}a_{R,2,3}^{p^e} - a_{R,1,2}^{p^e}a_{2,3}^{[1]})\Big)\Big) \end{aligned} \tag{6.15}$$

In [I25], it is shown that we may take A such that at least one of the following conditions holds.

(a) $m_{1,2} \neq m_{2,3}$,
(b) $m_{1,2} = m_{2,3}$ and

$$\frac{\operatorname{coef}_{K,-m_{1,2}}(a_{1,2})}{\operatorname{coef}_{K,-m_{2,3}}(a_{2,3})} \notin \mathbb{F}_p. \tag{6.16}$$

If one of these conditions holds, we have

$$(6.17)\qquad v_{K_R}(a_{1,2}^{[1]}a_{R,2,3}^{p^{N-1}} - a_{R,1,2}^{p^{N-1}}a_{2,3}^{[1]}) = -q\max\left(m_{1,2} + \frac{m_{2,3}}{p}, \frac{m_{1,2}}{p} + m_{2,3}\right) + R.$$

Since

$$(6.18)\qquad \begin{aligned} v_{K_R}(a_{1,2}^{[1]}a_{R,2,3}^{p^{e}} - a_{R,1,2}^{p^{e}}a_{2,3}^{[1]}) &\geq -q\max\left(m_{1,2} + \frac{m_{2,3}}{p^{N-e}}, \frac{m_{1,2}}{p^{N-e}} + m_{2,3}\right) + R \\ &> -q\max\left(m_{1,2} + \frac{m_{2,3}}{p}, \frac{m_{1,2}}{p} + m_{2,3}\right) + R \end{aligned}$$

for $1 \leq e \leq N-2$, we get

$$(6.19)\qquad v_{K_R}\left(\sum_{e=1}^{N-1}(a_{1,2}^{[1]}a_{R,2,3}^{p^{e}} - a_{R,1,2}^{p^{e}}a_{2,3}^{[1]})\right) = -q\max\left(m_{1,2} + \frac{m_{2,3}}{p}, \frac{m_{1,2}}{p} + m_{2,3}\right) + R.$$

Therefore, by (6.15) and Theorem 6.3, we get

$$(6.20)\quad r_{1,3} = \max\left(-\frac{v_{K_R}(a_{1,3}^{[1]} - a_{R,1,2}^{q}a_{2,3}^{[1]}) - R}{q}, m_{1,2}+\frac{m_{2,3}}{p}, \frac{m_{1,2}}{p}+m_{2,3}\right).$$

This result corresponds to the result given in [I25, Example 3.8.1].

Example 6.7. Consider the case $n = 4$. Suppose R is a sufficiently large positive integer coprime to p.

Let

$$(6.21)\qquad a_0' = a_{1,4}^{[1]} - a_{R,1,2}^{q}a_{2,4}^{[1]} - (a_{R,1,3}^{q} - a_{R,1,2}^{q}a_{R,2,3}^{q})a_{3,4}^{[1]},$$

$$(6.22)\qquad a_0'' = 0$$

$$(6.23)\qquad \begin{aligned} \mu_0' &= \max(m_{1,4}, m_{1,2} + m_{2,4}, m_{1,3} + m_{3,4}, m_{1,2} + m_{2,3} + m_{3,4}) \\ &= \mu_{1,4}, \end{aligned}$$

and

$$(6.24)\qquad \begin{aligned} a_e' = {}& a_{1,3}^{[1]}a_{R,3,4}^{\frac{q}{p^e}} + a_{1,2}^{[1]}a_{R,2,4}^{\frac{q}{p^e}} - a_{R,1,3}^{\frac{q}{p^e}}a_{3,4}^{[1]} \\ &+ a_{R,1,2}^{\frac{q}{p^e}}(a_{R,2,3}^{q}a_{3,4}^{[1]} - a_{2,4}^{[1]}) - a_{R,1,2}^{q}a_{2,3}^{[1]}a_{R,3,4}^{\frac{q}{p^e}}, \end{aligned}$$

$$(6.25)\qquad a_e'' = a_{R,1,2}^{\frac{q}{p^e}}\sum_{e'=1}^{e}a_{R,2,3}^{\frac{q}{p^{e'}}}a_{3,4}^{[1]} - \sum_{e'=1}^{e}a_{R,1,2}^{\frac{q}{p^{e'}}}a_{2,3}^{[1]}a_{R,3,4}^{\frac{q}{p^e}} - a_{R,1,2}^{\frac{q}{p^e}}a_{2,3}^{[1]}\sum_{e'=1}^{e-1}a_{R,3,4}^{\frac{q}{p^{e'}}},$$

$$\mu'_e = \max\left(m_{1,2} + \frac{m_{2,4}}{p^e}, \frac{m_{1,2}}{p^e} + m_{2,4}, m_{1,3} + \frac{m_{3,4}}{p^e}, \frac{m_{1,3}}{p^e} + m_{3,4}, \frac{m_{1,2}}{p^e} + m_{2,3} + m_{3,4}, m_{1,2} + m_{2,3} + \frac{m_{3,4}}{p^e}\right) \tag{6.26}$$

for $1 \leq e \leq N-1$. Note that

$$v_{K_R}(a'_e) \geq -q\mu'_e + R, \tag{6.27}$$

$$v_{K_R}(a''_e) > -q\mu'_e + R \tag{6.28}$$

for $0 \leq e \leq N-1$, and

$$\mu'_e > \mu'_{e'} \tag{6.29}$$

for $0 \leq e < e' \leq N-1$. We also have

$$\mathrm{coef}_{K_R,l}(a'_e) \neq 0 \implies l \in (p^{N-e}\mathbb{Z} + R) - (p^{N-e+1}\mathbb{Z} + R), \tag{6.30}$$

$$\mathrm{coef}_{K_R,l}(a''_e) \neq 0 \implies l \in p^{N-e}\mathbb{Z} + R \tag{6.31}$$

for $0 \leq e \leq N-1$, by Lemma 5.5(vi).

By Lemma 2.2 and (5.14), we get

$$\begin{aligned} s^{[1]}_{R,1,4} = a^{[1]}_{1,4} &+ a^{[1]}_{1,3} w^p_{R,N-1,3,4} + a^{[1]}_{1,2} w^p_{R,N-1,2,4} \\ &- (w^p_{R,N,1,3} - w^p_{R,N,1,2} w^p_{R,N,2,3}) a^{[1]}_{3,4} \\ &- w^p_{R,N,1,2}(a^{[1]}_{2,4} + a^{[1]}_{2,3} w^p_{R,N-1,3,4}). \end{aligned} \tag{6.32}$$

By the definition of W_R and (2.1), we get

$$w^p_{R,e,i,i+1} = \sum_{u=N-e}^{N-1} a^{\frac{q}{p^u}}_{R,i,i+1} \qquad (i = 1, 2, 3,\ 1 \leq e \leq N), \tag{6.33}$$

$$\begin{aligned} w^p_{R,e,i,i+2} = &\sum_{u=N-e}^{N-1} a^{\frac{q}{p^u}}_{R,i,i+2} \\ &+ \sum_{u=N-e}^{N-1} \sum_{u'=u+1}^{N-1} a^{\frac{q}{p^u}}_{R,i,i+1} a^{\frac{q}{p^{u'}}}_{R,i+1,i+2} \qquad (i = 1, 2,\ 1 \leq e \leq N), \end{aligned} \tag{6.34}$$

and consequently

$$\begin{aligned} &w^p_{R,N,i,i+2} - w^p_{R,N,i,i+1} w^p_{R,N,i+1,i+2} \\ &\qquad = \sum_{u=0}^{N-1} a^{\frac{q}{p^u}}_{R,i,i+2} - \sum_{u=0}^{N-1} \sum_{u'=0}^{u-1} a^{\frac{q}{p^u}}_{R,i,i+1} a^{\frac{q}{p^{u'}}}_{R,i+1,i+2} \end{aligned} \tag{6.35}$$

for $i = 1, 2$. By substituting (6.33) and (6.35) into (6.32), we get

$$
\begin{aligned}
(6.36)\qquad s^{[1]}_{R,1,4} &= a^{[1]}_{1,4} + a^{[1]}_{1,3} \sum_{e=1}^{N-1} a^{\frac{q}{p^e}}_{R,3,4} + a^{[1]}_{1,2} \sum_{e=1}^{N-1} a^{\frac{q}{p^e}}_{R,2,4} \\
&\quad - \sum_{e=0}^{N-1} a^{\frac{q}{p^e}}_{R,1,3} a^{[1]}_{3,4} + \sum_{e=0}^{N-1} a^{\frac{q}{p^e}}_{R,1,2} \sum_{e'=0}^{e} a^{\frac{q}{p^{e'}}}_{R,2,3} a^{[1]}_{3,4} \\
&\quad - \sum_{e=0}^{N-1} a^{\frac{q}{p^e}}_{R,1,2} a^{[1]}_{2,4} - \sum_{e'=0}^{N-1} a^{\frac{q}{p^{e'}}}_{R,1,2} a^{[1]}_{2,3} \sum_{e=1}^{N-1} a^{\frac{q}{p^e}}_{R,3,4} \\
&= a^{[1]}_{1,4} - a^{q}_{R,1,2} a^{[1]}_{2,4} - (a^{q}_{R,1,3} - a^{q}_{R,1,2} a^{q}_{R,2,3}) a^{[1]}_{3,4} \\
&\quad + \sum_{e=1}^{N-1} (a^{[1]}_{1,3} a^{\frac{q}{p^e}}_{R,3,4} + a^{[1]}_{1,2} a^{\frac{q}{p^e}}_{R,2,4} - a^{\frac{q}{p^e}}_{R,1,3} a^{[1]}_{3,4} - a^{\frac{q}{p^e}}_{R,1,2} a^{[1]}_{2,4}) \\
&\quad + \sum_{e=1}^{N-1} a^{\frac{q}{p^e}}_{R,1,2} \sum_{e'=0}^{e} a^{\frac{q}{p^{e'}}}_{R,2,3} a^{[1]}_{3,4} - \sum_{e'=0}^{N-1} \sum_{e=1}^{N-1} a^{\frac{q}{p^{e'}}}_{R,1,2} a^{[1]}_{2,3} a^{\frac{q}{p^e}}_{R,3,4}.
\end{aligned}
$$

By separating the terms and changing the order of the summations, we can rewrite the last two terms as follows:

$$
\begin{aligned}
(6.37)\qquad & \sum_{e=1}^{N-1} a^{\frac{q}{p^e}}_{R,1,2} \sum_{e'=0}^{e} a^{\frac{q}{p^{e'}}}_{R,2,3} a^{[1]}_{3,4} - \sum_{e'=0}^{N-1} \sum_{e=1}^{N-1} a^{\frac{q}{p^{e'}}}_{R,1,2} a^{[1]}_{2,3} a^{\frac{q}{p^e}}_{R,3,4} \\
&= \sum_{e=1}^{N-1} a^{\frac{q}{p^e}}_{R,1,2} a^{q}_{R,2,3} a^{[1]}_{3,4} + \sum_{e=1}^{N-1} a''_e.
\end{aligned}
$$

Therefore, we get

$$
(6.38)\qquad s^{[1]}_{R,1,4} = \sum_{e=0}^{N-1} a'_e + \sum_{e=0}^{N-1} a''_e.
$$

By (6.27)–(6.29), we obtain

$$
(6.39)\qquad s^{[1]}_{R,1,4} \equiv \sum_{e=0}^{e'} a'_e + \sum_{e=0}^{e'-1} a''_e \pmod{\mathfrak{m}_{K_R}^{-q\mu'_{e'}+R+1}}
$$

for $0 \le e' \le N-1$.

Suppose equality holds in (6.27) for some $e \in \mathbb{Z}_{\ge 1}$ and let e_0 be one of such e. By (6.30) and (6.31), we have

$$
(6.40)\qquad v_{K_R}\Big(\sum_{e=0}^{e_0-1} a'_e + \sum_{e=0}^{e_0-1} a''_e\Big) \in p^{N-e_0+1}\mathbb{Z} + R,
$$

$$
(6.41)\qquad v_{K_R}(a'_{e_0}) = -q\mu'_{e_0} + R \in (p^{N-e_0}\mathbb{Z} + R) - (p^{N-e_0+1}\mathbb{Z} + R).
$$

Therefore, combining these with (6.39), we deduce that

$$v_{K_R}(s_{R,1,4}^{[1]}) = \min\Big(v_{K_R}\Big(\sum_{e=0}^{e_0-1} a'_e + \sum_{e=0}^{e_0-1} a''_e\Big), -q\mu'_{e_0} + R\Big). \tag{6.42}$$

By Theorem 6.3, we deduce that

$$r_{1,4} = \max\Big(-\frac{v_{K_R}(\sum_{e=0}^{e_0-1} a'_e + \sum_{e=0}^{e_0-1} a''_e) - R}{q}, \mu'_{e_0}\Big). \tag{6.43}$$

Note that the result (6.20) in Example 6.6 has a similar form to (6.43). To clarify the similarity, we make a similar argument in the case $n = 3$. Then the condition that either (a) or (b) holds in Example 6.6 corresponds to the condition that the equality in (6.27) holds for $e = 1$. This implies that we have $e_0 = 1$ in the case $n = 3$. Then we can deduce the equation (6.20) in the same way as (6.43).

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