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ON THE EXISTENCE OF SOLUTIONS FOR A NONLINEAR COUPLED SYSTEM IN ORLICZ–SOBOLEV SPACES

Abstract. This paper establishes the existence of a capacity solution for a system of two elliptic PDEs with perturbations. These equations are relevant to the stationary thermistor problem. Notably, we achieve this result without imposing the Δ_2-condition on the N-function.

1. Introduction. In this paper, we prove the existence of solutions for the coupled nonlinear elliptic system

$$\begin{cases} Au + \dfrac{m(|u|)}{|u|}u = \rho(u)|\nabla\varphi|^2 & \text{in } \Omega, \\ -\operatorname{div}(\rho(u)\nabla\varphi) + \varphi = g & \text{in } \Omega, \\ u = \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ satisfying the segment property, $Au = -\operatorname{div}\big(a(x,u)\frac{m(|\nabla u|)}{|\nabla u|}\nabla u\big)$ is a Leray–Lions operator defined on $W_0^1 L_M(\Omega)$, u and φ represent the temperature and the electric potential, g is a positive and bounded source term, and the function $\rho \in C(\mathbb{R}) \cap L^\infty(\Omega)$, which represents the electric conductivity, is supposed to be such that $\rho(s) > 0$ for all $s \in \mathbb{R}$.

System (1.1) is a mathematical model generalizing the so-called thermistor problems [2, 5, 6, 9, 14]. Here, we have $\rho(s) > 0$ for all $s \in \mathbb{R}$ and $\rho(s) \to 0$ as $s \to +\infty$. Consequently, the second equation in (1.1) is not uniformly elliptic so that no a priori estimate for $\nabla\varphi$ will be available. Therefore, the search for weak solutions to system (1.1) is not suitable in this context. In

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order to avoid this difficulty, we consider the function $\Phi = \rho(u)\nabla\varphi$ as a whole and then show that it belongs to $L^2(\Omega)^N$. A new formulation of the problem (1.1) will lead us to the introduction of the concept of capacity solution adapted to our setting. Furthermore, the fact that the function ρ is unbounded is not the only obstacle we will encounter when resolving problem (1.1). Indeed, we work in the framework of Orlicz–Sobolev space without assuming the Δ_2-condition on the N-functions.

Solvability of the thermistor problem. This has been considered by many authors. In particular, Xu [16, 17, 18] studied a modified version of the stationary and the evolution thermistor problem in classical Sobolev spaces with $p = 2$. González Montesinos and Ortegón Gallego [5] generalized the result obtained by Xu [16] to $p \geq 2$. Later on, Moussa et al. [11] studied the problem in Orlicz–Sobolev spaces. Recently, Ouyahya et al. [12] proved the existence of a capacity solution of a coupled system governed by an anisotropic nonlinear elliptic equation of the form $A(u) = \rho(u)|\nabla\varphi|^2$ in Ω and $\varphi = \varphi_0$ on $\partial\Omega$, where the operator A given by

$$A(u) = \sum_{i=1}^{d} \partial_i(a_i(x, u, \partial_i u))$$

maps the anisotropic Orlicz–Sobolev space $W^1L_{\boldsymbol{\mathcal{M}}}(\Omega)$ onto its dual, with $\boldsymbol{\mathcal{M}} = (M_1, \dots, M_d)$. Some other results on existence of a weak solution to a perturbed thermistor problem in Sobolev spaces with $p = 2$ are given in [3].

In this framework, by considering the nonuniformly elliptic character of the second equation of (1.1) we prove, under the assumptions given below, the existence of a capacity solution, in the sense of Definition 3.2, for system (1.1) in the classical Orlicz–Sobolev space. The proof is generally based on Schauder's fixed point theorem combined with a result on the existence and uniqueness of a weak solution to a certain elliptic problem.

Outline of the paper. In Section 2 we introduce some preliminaries and some technical results sequel. Section 3 is devoted to specifying the assumptions on data and presenting the concept of capacity solutions to system (1.1). Section 4 deals with solvability in two theorems: we prove the existence of a weak solution to system (1.1) via Schauder's fixed point theorem and we prove the existence of a capacity solution to (1.1).

2. Preliminaries. We begin by recalling some definitions, lemmas and properties of Orlicz spaces [1, 10, 13] and then we introduce Orlicz–Sobolev spaces.

2.1. N-functions. The basic concept in an Orlicz normed space is that of an N-function.

DEFINITION 2.1. A function $M\colon \mathbb{R} \to \mathbb{R}$ is called an *N-function* if it satisfies the following conditions:

(i) M is convex in $\mathbb{R}$: $M(\lambda s_1 + (1-\lambda)s_2) \le \lambda M(s_1) + (1-\lambda)M(s_2)$ for all $s_1, s_2 \in \mathbb{R}$ and for all $\lambda \in [0,1]$.
(ii) M is an even function: $M(s) = M(-s)$ for all $s \in \mathbb{R}$.
(iii) M is continuous.
(iv) $M(0) = 0$ and $M(s) > 0$ for all $s \in \mathbb{R}$.
(v) $M(s)/s \to 0$ as $s \to 0$ and $M(s)/s \to +\infty$ as $s \to +\infty$.

An N-function M is said to satisfy the Δ_2-*condition* if, for some $k > 0$,

$$M(2s) \le k\, M(s) \text{ for all } s \ge 0.$$

An equivalent definition of an N-function [10] is a function M that admits the representation

$$M(s) = \int_0^{|s|} m(\sigma)\, \mathrm{d}\sigma$$

for some nondecreasing and right-continuous function $m\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $m(s) > 0$ for all $s > 0$ and $m(s) \to +\infty$ as $s \to +\infty$. The N-function $\bar{M}$, *complementary* or *conjugate* to M, is defined by

$$\bar{M}(s) = \int_0^{|s|} \bar{m}(\sigma)\, \mathrm{d}\sigma,$$

where $\bar{m}\colon \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\bar{m}(t) = \sup\{s \mid m(s) \le t\}$.

We have *Young's inequality*

$$|ts| \le M(t) + \bar{M}(s) \quad \text{for all } t, s \in \mathbb{R}.$$

Furthermore, equality takes place if and only if $s = m(t)$ or $t = \bar{m}(s)$. Now, let Ω be an open set in $\mathbb{R}^N$, $N \in \mathbb{N}$. The *Orlicz class* $\mathcal{L}_M(\Omega)$ (resp. the *Orlicz space* $L_M(\Omega)$) is defined as the set of (resp. equivalence classes of) real-valued Lebesgue measurable functions u in Ω such that

$$\int_\Omega M(u(x))\, \mathrm{d}x < +\infty \quad \Big(\text{resp. } \int_\Omega M\Big(\frac{u(x)}{\lambda}\Big)\, \mathrm{d}x < +\infty \text{ for some } \lambda > 0\Big).$$

Notice that $L_M(\Omega)$ is a Banach space under the *Luxemburg norm* namely

$$\|u\|_M = \inf\Big\{\lambda > 0 \;\Big|\; \int_\Omega M(u(x)/\lambda)\, \mathrm{d}x \le 1\Big\},$$

and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$. Indeed, $L_M(\Omega)$ is the linear hull of $\mathcal{L}_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_M(\Omega)$. The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2-condition, for all s or for s large according to whether Ω has infinite measure or not. The dual of $E_M(\Omega)$ can be identified with $L_{\bar{M}}(\Omega)$ by means of the duality pairing $\int_\Omega u(x)v(x)\,\mathrm{d}x$, and the dual norm on $L_{\bar{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\bar{M}}$.

In $L_M(\Omega)$ we define the *Orlicz norm* $\|u\|_{(M)}$ by

$$\|u\|_{(M)} = \sup \int_\Omega u(x)v(x)\,\mathrm{d}x, \tag{2.1}$$

where the supremum is taken over all $v \in E_{\bar{M}(\Omega)}$ such that $\|v\|_{\bar{M}} \leq 1$. It turns out that the norms $\|\cdot\|_M$ and $\|\cdot\|_{(M)}$ are equivalent. In fact,

$$\|u\|_M \leq \|u\|_{(M)} \leq 2\|u\|_M \quad \text{for all } u \in L_M(\Omega). \tag{2.2}$$

Also, the *Hölder inequality* holds:

$$\int_\Omega |u(x)v(x)|\,\mathrm{d}x \leq \|u\|_M \|v\|_{(\bar{M})} \quad \text{for all } u \in L_M(\Omega) \text{ and } v \in L_{\bar{M}}(\Omega),$$

and by (2.2),

$$\int_\Omega |u(x)v(x)|\,\mathrm{d}x \leq 2\|u\|_M \|v\|_{\bar{M}} \quad \text{for all } u \in L_M(\Omega) \text{ and } v \in L_{\bar{M}}(\Omega).$$

In particular, if Ω has finite measure, Hölder's inequality yields the continuous inclusion $L_M(\Omega) \subset L^1(\Omega)$.

An important inequality in $L_M(\Omega)$ is the following:

$$\int_\Omega M(u(x))\,\mathrm{d}x \leq \|u\|_{(M)} \quad \text{for all } u \in L_M(\Omega) \text{ such that } \|u\|_{(M)} \leq 1, \tag{2.3}$$

from which we readily deduce

$$\int_\Omega M\left(\frac{u(x)}{\|u\|_{(M)}}\right)\mathrm{d}x \leq 1 \quad \text{for all } u \in L_M(\Omega) \setminus \{0\}. \tag{2.4}$$

Definition 2.2. We say that $(u_n) \subset L_M(\Omega)$ converges to $u \in L_M(\Omega)$ for *modular convergence* in $L_M(\Omega)$ if, for some $\lambda > 0$,

$$\int_\Omega M\left(\frac{u_n(x) - u(x)}{\lambda}\right)\mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

Modular convergence is weaker than convergence in the norm of $L_M(\Omega)$. However, it is enough for our purposes. The next result tells us that modular convergence in L_M implies convergence in the weak-$*$ topology $\sigma(L_M, L_{\bar{M}})$.

Lemma 2.3 ([4, 7]). *Let $(u_n) \subset L_M(\Omega)$, $u \in L_M(\Omega)$ and $v \in L_{\bar{M}}(\Omega)$ be such that $u_n \to u$ with respect to modular convergence. Then the following hold:*

(1) *$u_n v \to uv$ strongly in $L^1(\Omega)$. In particular, $\int_\Omega u_n v \to \int_\Omega uv$.*
(2) *If $(v_n) \subset L_{\bar{M}}(\Omega)$ is such that $v_n \to v$ with respect to modular convergence, then $u_n v_n \to uv$ strongly in $L^1(\Omega)$.*

2.2. Orlicz–Sobolev space. We now turn to Orlicz–Sobolev spaces. $W^1 L_M(\Omega)$ (resp. $W^1 E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). This is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|\nabla^\alpha u\|_M. \tag{2.5}$$

The spaces $W^1 L_M(\Omega)$ and $W^1 E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M(\Omega)$, we will use the weak topologies $\sigma(\prod L_M, \prod L_{\bar{M}})$ and $\sigma(\prod L_M, \prod E_{\bar{M}})$. The space $W_0^1 E_M(\Omega)$ is defined as the (norm) closure of $\mathcal{D}(\Omega)$ in $W^1 E_M(\Omega)$ and the space $W_0^1 L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\bar{M}})$-closure of $\mathcal{D}(\Omega)$ in $W^1 L_M(\Omega)$.

The following spaces of distributions will also be used:

$$W^{-1} L_{\bar{M}}(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega) \;\Big|\; f = \sum_{\alpha\leq 1} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_{\bar{M}}(\Omega) \Big\},$$

$$W^{-1} E_{\bar{M}}(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega) \;\Big|\; f = \sum_{\alpha\leq 1} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_{\bar{M}}(\Omega) \Big\}.$$

LEMMA 2.4 ([11]). *Let Ω be a bounded and open set in $\mathbb{R}^N$. Assume that $m(t) \geq t$ for all $t \geq 0$. Then, the following continuous embeddings hold:*

$$L_M(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow E_{\bar{M}} \hookrightarrow L_{\bar{M}}(\Omega).$$

In particular,

$$H^1(\Omega) \hookrightarrow W^1 L_{\bar{M}}(\Omega) \quad \text{and} \quad H^{-1}(\Omega) \hookrightarrow W^{-1} E_{\bar{M}}(\Omega) \hookrightarrow W^{-1} L_{\bar{M}}(\Omega).$$

REMARK 2.5. Assume that $m(t) \geq t$ for all $t \geq 0$. Then

$$\int_\Omega v^2 \,\mathrm{d}x \leq 2 \int_\Omega M(v) \,\mathrm{d}x \quad \text{for all } v \in \mathcal{L}_M(\Omega). \tag{2.6}$$

THEOREM 2.6 ([15]). *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set with locally Lipschitz boundary. Then the embedding $W^1 L_M(\Omega) \hookrightarrow E_M(\Omega)$ is compact. Furthermore, the compact embedding $W_0^1 L_M(\Omega) \hookrightarrow E_M(\Omega)$ holds without the locally Lipschitz boundary assumption.*

3. Essential assumptions and main result. We assume that $\Omega \subset \mathbb{R}^N$ is an open and bounded set satisfying the segment property. We now state the assumptions on the differential operator in divergence form

$$A\colon W_0^1 L_M(\Omega) \to W^{-1} L_{\bar{M}}(\Omega),$$
$$u \mapsto A(u) = \operatorname{div}\bigg(a(x,u) \frac{m(|\nabla u|)}{|\nabla u|} \nabla u \bigg).$$

(A_1) The function $a\colon \Omega \times \mathbb{R} \to \mathbb{R}$, $a = a(x,s)$, is a Carathéodory function (measurable in x for all s and continuous in s for almost every $x \in \Omega$)

such that

$$\beta_1 \leq a(x,s) \leq \beta_2, \tag{3.1}$$

where $0 < \beta_1 \leq \beta_2$ are two real numbers.

(A_2) $\rho \in C(\mathbb{R})$ and there exists $\bar{\rho} \in \mathbb{R}$ such that for all $s \in \mathbb{R}$,

$$0 < \rho(s) \leq \bar{\rho}. \tag{3.2}$$

Remark 3.1. Assume that (A_1) is true. It is straightforward to show that for a bounded set $B \subset \mathbb{R}$, for a.e. $x \in \Omega$, and all $s \in B$ and $\zeta, \zeta' \in \mathbb{R}^N$,

$$\left[a(x,s)\frac{m(|\zeta|)}{|\zeta|}\zeta - a(x,s)\frac{m(|\zeta'|)}{|\zeta'|}\zeta'\right](\zeta - \zeta') \to +\infty \tag{3.3}$$

as $|\zeta| \to +\infty$, uniformly in s.

Now we are ready to state the definition of a capacity solution to problem (1.1).

Definition 3.2. A triplet (u, φ, Φ) is called a *capacity solution* to problem (1.1) if the following conditions are fulfilled:

(C_1) $u \in W_0^1 L_M(\Omega)$, $\varphi \in L^\infty(\Omega)$ and $\Phi \in L^2(\Omega)^N$.

(C_2) (u, φ, Φ) satisfies the following system of differential equations in the sense of distributions:

$$\begin{cases} -\operatorname{div}\left(a(x,u)\dfrac{m(|\nabla u|)}{|\nabla u|}\nabla u\right) + \dfrac{m(|u|)}{|u|}u = \operatorname{div}(\varphi\Phi) - \varphi^2 + \varphi g & \text{in } \Omega, \\ \operatorname{div}\Phi + \varphi = g & \text{in } \Omega. \end{cases}$$

(C_3) For every $S \in C_0^1(\mathbb{R})$ (that is, $S \in C^1(\mathbb{R})$ and has compact support in $\mathbb{R}$), one has $S(u)\varphi \in H_0^1(\Omega)$, and

$$S(u)\Phi = \rho(u)[\nabla(S(u)\varphi) - \varphi\nabla S(u)].$$

Remark 3.3. Notice that the concept of capacity solution involves a third component, namely, $\Phi \in L^2(\Omega)^N$. The original problem only has two unknowns so that there must be a relationship between Φ and the couple (u, φ); this relationship has been expressed by (C_3). In fact, assume that $u \in L^\infty(\Omega)$ and take $S \in C_0^1(\mathbb{R})$ such that $S \equiv 1$ in the interval $[-\|u\|_\infty, \|u\|_\infty]$. Then we deduce that $\varphi \in H^1(\Omega)$, $\Phi = \rho(u)\nabla\varphi$ and $\rho(u)|\nabla\varphi|^2 = \operatorname{div}(\varphi\Phi) - \varphi^2 + \varphi g$. Consequently, bounded capacity solutions are weak solutions. In the general case where $u \notin L^\infty(\Omega)$ the expression given in (C_3) allows us to define the gradient of φ pointwise almost everywhere as

$$\Phi\chi_{\{|u|<K\}} = \rho(u)\nabla\varphi\chi_{\{|u|<K\}} \quad \text{for any } K > 0.$$

4. An existence result. In this section we establish the main result of this article.

THEOREM 4.1. *Let $g \geq 0$ be a function in $L^\infty(\Omega)$. Under the assumptions (A_1) and (A_2), system* (1.1) *admits a capacity solution.*

In order to prove Theorem 4.1, we will first prove the existence of a weak solution to problem (1.1) under the following assumption:

(A_3) $\rho \in C(\mathbb{R})$ and there exist ρ_1 and $\rho_2 \in \mathbb{R}$ such that for all $s \in \mathbb{R}$,

$$0 < \rho_1 \leq \rho(s) \leq \rho_2. \tag{4.1}$$

THEOREM 4.2. *Let $g \geq 0$ be a function in $L^\infty(\Omega)$. Assume (A_1) and (A_3). Then there exists a weak solution (u, φ) to problem* (1.1), *that is, $u \in W_0^1 L_M(\Omega)$, $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and*

$$\int_\Omega a(x,u)\frac{m(|\nabla u|)}{|\nabla u|}\nabla u\nabla v + \int_\Omega \frac{m(|u|)}{|u|}uv = \int_\Omega \rho(u)|\nabla\varphi|^2 v \quad \textit{for all } v \in W_0^1 L_M(\Omega), \tag{4.2}$$

$$\int_\Omega \rho(u)\nabla\varphi\nabla\psi + \int_\Omega \varphi\psi = \int_\Omega g\psi \qquad \textit{for all } \psi \in H_0^1(\Omega).$$

REMARK 4.3. The second integral on the left-hand side in (4.2) is well defined. Indeed, let $u \in W_0^1 L_M(\Omega)$. Then there exists $\lambda_0 > 0$ such that $\int_\Omega M(|u|/\lambda_0) < \infty$. From Young's inequality we have, for $t \geq 0$,

$$\bar{M}(m(t)) \leq tm(t),$$

hence

$$\frac{\lambda t m(t)}{\lambda} = \lambda \int_{t/\lambda}^{2t/\lambda} m(s)\,ds \leq \int_0^{2t/\lambda} m(s)\,ds \leq \lambda M(2t/\lambda) \leq 2\lambda_0 M(t/\lambda_0),$$

where $\lambda = 2\lambda_0$. Consequently, by using Young's inequality, we get

$$\begin{aligned}\int_\Omega \frac{m(|u|)}{|u|}uv\,\mathrm{d}x &\leq \int_\Omega m(|u|)|v|\,\mathrm{d}x \leq \lambda\Big(\int_\Omega \bar{M}(m(|u|))\,\mathrm{d}x + \int_\Omega M(|v|/\lambda)\,\mathrm{d}x\Big)\\ &\leq \lambda\Big(2\lambda_0 \int_\Omega M(|u|/\lambda_0) + \int_\Omega M(|v|/\lambda)\,\mathrm{d}x\Big)\\ &< \infty.\end{aligned}$$

REMARK 4.4. In the sequel, the constants may differ from one occurrence to another.

Proof of Theorem 4.2. We use Schauder's fixed point theorem to prove the existence of a weak solution to (4.2).

Let $\omega \in E_M(\Omega)$ and consider the elliptic problem

$$\begin{cases}\operatorname{div}(\rho(\omega)\nabla\varphi) + \varphi = g & \text{in } \Omega,\\ \varphi = 0 & \text{on } \partial\Omega.\end{cases} \tag{4.3}$$

Using $(\varphi - k)^+$ as a test function in (4.3), with $k = \|g\|_{L^\infty(\Omega)}$, we get

$$\begin{aligned}\int_\Omega \rho(\omega)\nabla\varphi\nabla(\varphi - k)^+ \,\mathrm{d}x + \int_\Omega \varphi(\varphi - k)^+ \,\mathrm{d}x &= \int_\Omega g(\varphi - k)^+ \,\mathrm{d}x,\\ &\le \|g\|_{L^\infty(\Omega)} \int_\Omega (\varphi - k)^+ \,\mathrm{d}x \le k \int_\Omega (\varphi - k)^+ \,\mathrm{d}x.\end{aligned}$$

Thus,

$$\int_\Omega \rho(\omega)\nabla\varphi\nabla(\varphi - k)^+ \,\mathrm{d}x + \int_\Omega (\varphi - k)(\varphi - k)^+ \,\mathrm{d}x \le 0.$$

From this inequality, we deduce that $(\varphi - k)^+ = 0$. This means that, recalling the definition of k,

$$\|\varphi\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\Omega)}. \tag{4.4}$$

Taking φ as a test function in (4.3) we obtain

$$\int_\Omega \rho(\omega)|\nabla\varphi|^2 \,\mathrm{d}x + \int_\Omega \varphi^2 \,\mathrm{d}x = \int_\Omega g\varphi \,\mathrm{d}x.$$

Then

$$\rho_1 \int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x \le \|g\|^2_{L^\infty(\Omega)} |\Omega|,$$

hence

$$\int_\Omega |\nabla\varphi|^2 \,\mathrm{d}x \le C(\rho_1, g) = C. \tag{4.5}$$

This means that $\rho(\omega)|\nabla\varphi|^2 \in L^1(\Omega)$.

Now we consider the elliptic problem

$$\begin{cases} -\operatorname{div}\left(a(x,\omega)\dfrac{m(|\nabla u|)}{|\nabla u|}\nabla u\right) + \dfrac{m(|u|)}{|u|}u = \rho(\omega)|\nabla\varphi|^2 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

LEMMA 4.5. *The second member of* $(4.6)_1$ *belongs to* $W^{-1}E_{\bar{M}}(\Omega)$.

Proof. Take $\psi = \phi\varphi$ as a test function in (4.3). Then

$$\int_\Omega \rho(\omega)|\nabla\varphi|^2\phi \,\mathrm{d}x + \int_\Omega \rho(u)\nabla\varphi\nabla\phi\varphi \,\mathrm{d}x + \int_\Omega \varphi^2\phi = \int_\Omega g\varphi\phi \,\mathrm{d}x.$$

Hence

$$\int_\Omega \rho(\omega)|\nabla\varphi|^2\phi \,\mathrm{d}x = -\int_\Omega \rho(u)\nabla\varphi\nabla\phi\varphi \,\mathrm{d}x - \int_\Omega \varphi^2\phi + \int_\Omega g\varphi\phi \,\mathrm{d}x.$$

Thus

$$\rho(u)|\nabla\varphi|^2 = \operatorname{div}(\rho(u)\varphi\nabla\varphi) - \varphi^2 + g\varphi.$$

Since $\rho(u)\varphi\nabla\varphi \in L^2(\Omega)^N$ and $-\varphi^2 + g\varphi \in L^2(\Omega)$, we get $\operatorname{div}(\rho(u)\varphi\nabla\varphi) - \varphi^2 + g\varphi \in H^{-1}(\Omega)$. Hence, using Lemma 2.4, we get the result. ∎

The new variational formulation of (4.6) is

$$\begin{cases} u\in W_0^1L_M(\Omega), \\ \int_\Omega a(x,\omega)\dfrac{m(|\nabla u|)}{|\nabla u|}\nabla u\nabla v\,\mathrm{d}x+\int_\Omega \dfrac{m(|u|)}{|u|}uv\,\mathrm{d}x \\ \qquad =-\int_\Omega \rho(\omega)\varphi\nabla\varphi\nabla v\,\mathrm{d}x-\int_\Omega \varphi^2 v+\int_\Omega g\varphi v\,\mathrm{d}x \qquad \text{for all } v\in W_0^1L_M(\Omega). \end{cases} \tag{4.7}$$

By Lemma 4.5, we have $\rho(\omega)|\nabla\varphi|^2=\operatorname{div}(\rho(u)\varphi\nabla\varphi)-\varphi^2+g\varphi\in W^{-1}E_{\bar{M}}(\Omega)$. The proof of the existence and uniqueness of solution to (4.7) is a straightforward application of the result in [8].

Now, we are going to prove the estimates

$$\int_\Omega M(|\nabla u|)\,\mathrm{d}x\le C, \tag{4.8}$$

$$\|a(\cdot,\omega)\frac{m(|\nabla u|)}{|\nabla u|}\nabla u\|_{\bar{M}}\le C, \tag{4.9}$$

$$\left\|\frac{m(|u|)}{|u|}u\right\|_{\bar{M}}\le C. \tag{4.10}$$

To do so, we first prove that $|\nabla u|\in\mathcal{L}_M(\Omega)$. Indeed, let $\lambda>0$ be such that $|\nabla u|/\lambda\in\mathcal{L}_M(\Omega)$. Since $\varphi\in H^1(\Omega)\subset W^1L_{\bar{M}}(\Omega)$, there exists $\mu>0$ such that $\frac{1}{\mu}\rho_2\|\varphi_0\|_{L^\infty(\Omega)}\nabla\varphi\in\mathcal{L}_{\bar{M}}(\Omega)$. By taking $v=u$ as a test function in (4.7), from (2.6), (4.1), (4.4), and Young's inequality we obtain

$$\begin{aligned}\frac{\alpha}{\lambda\mu}\int_\Omega M(\nabla u)\,\mathrm{d}x &\le \frac{1}{\lambda\mu}\int_\Omega a(x,\omega)\frac{m(|\nabla u|)}{|\nabla u|}\nabla u\nabla u\,\mathrm{d}x+\frac{1}{\lambda\mu}\int_\Omega\frac{m(|u|)}{|u|}uu\,\mathrm{d}x\\ &=-\frac{1}{\lambda\mu}\int_\Omega\rho(\omega)\varphi\nabla\varphi\nabla u\,\mathrm{d}x-\frac{1}{\lambda\mu}\int_\Omega\varphi^2u+\frac{1}{\lambda\mu}\int_\Omega g\varphi u\,\mathrm{d}x\\ &\le\frac{1}{\lambda\mu}\int_\Omega|\rho(\omega)\varphi\nabla\varphi\nabla u|\,\mathrm{d}x+\frac{2}{\lambda\mu}\|g\|^2_{L^\infty(\Omega)}\int_\Omega|u|\\ &\le\frac{1}{\lambda\mu}\int_\Omega|\rho(\omega)\varphi\nabla\varphi\nabla u|\,\mathrm{d}x+\frac{1}{\lambda\mu}C\int_\Omega|u|.\end{aligned}$$

Using the Young and Poincaré inequalities, we get

$$\begin{aligned}\frac{\alpha}{\lambda\mu}\int_\Omega M(\nabla u)\,\mathrm{d}x &\le\int_\Omega\bar{M}(\rho_2\|g\|_{L^\infty(\Omega)}\nabla\varphi/\mu)\,\mathrm{d}x+\int_\Omega M(\nabla u/\lambda)\,\mathrm{d}x+C\int_\Omega\frac{|\nabla u|}{\lambda}\\ &\le\int_\Omega\bar{M}(\rho_2\|g\|_{L^\infty(\Omega)}\nabla\varphi/\mu)\,\mathrm{d}x+\int_\Omega M(\nabla u/\lambda)\,\mathrm{d}x+\frac{C^2}{2}|\Omega|\\ &\quad+\frac{1}{2}\int_\Omega\left|\frac{\nabla u}{\lambda}\right|^2,\end{aligned}$$

where we denote by $|\Omega|$ the Lebesgue measure of Ω. In view of (2.6),

$$\frac{\alpha}{\lambda\mu}\int_\Omega M(\nabla u)\,\mathrm{d}x \le \int_\Omega \bar{M}(\rho_2\|g\|_{L^\infty(\Omega)}\nabla\varphi/\mu)\,\mathrm{d}x + \int_\Omega M(\nabla u/\lambda)\,\mathrm{d}x + \frac{C^2}{2}|\Omega| + \int_\Omega M(|\nabla u|/\lambda) < \infty.$$

This proves that $\nabla u \in \mathcal{L}_M(\Omega)$.

For estimate (4.8), using Young's inequality we obtain

$$\begin{aligned}\alpha\int_\Omega M(|\nabla u|)\,\mathrm{d}x &\le \int_\Omega a(x,\omega)\frac{m(|\nabla u|)}{|\nabla u|}\nabla u\nabla u\,\mathrm{d}x + \int_\Omega m(|u|)|u|\,\mathrm{d}x\\ &\le \int_\Omega |\rho(\omega)\varphi\nabla\varphi\nabla u|\,\mathrm{d}x + \int_\Omega \varphi^2|u| + \int_\Omega |g\varphi|\,|u|\,\mathrm{d}x\\ &\le \frac{1}{\alpha}\rho_2^2\|g\|^2_{L^\infty(\Omega)}\int_\Omega|\nabla\varphi|^2 + \frac{\alpha}{4}\int_\Omega|\nabla u|^2 + 2\|g\|^2_{L^\infty(\Omega)}\int_\Omega|u|\\ &\le C + \frac{\alpha}{4}\int_\Omega|\nabla u|^2 + \frac{(2\|g\|^2_{L^\infty(\Omega)})^2}{\alpha\epsilon}|\Omega| + \frac{\alpha\epsilon}{4}\int_\Omega|u|^2,\end{aligned}$$

and from the Poincaré's inequality and (2.6), we get

$$\begin{aligned}\alpha\int_\Omega M(|\nabla u|)\,\mathrm{d}x &\le C + \frac{\alpha}{4}\int_\Omega|\nabla u|^2 + \frac{\alpha\epsilon C(\Omega)}{4}\int_\Omega|\nabla u|^2\\ &\le C + \frac{\alpha}{2}\int_\Omega M(|\nabla u|) + \frac{\alpha\epsilon C(\Omega)}{2}\int_\Omega M(|\nabla u|).\end{aligned}$$

Take $\epsilon = 1/(2C(\Omega))$ in the last inequality to obtain (4.8).

Now, we show (4.9). Notice that from the last inequality, we get

$$\int_\Omega a(x,\omega)\frac{m(|\nabla u|)}{|\nabla u|}\nabla u\nabla u\,\mathrm{d}x \le \alpha C. \tag{4.11}$$

On the other hand, by monotonicity, for any $\phi \in W_0^1E_M(\Omega)$ such that $\|\nabla\phi\|_{(M)} = 1/(k+1)$ for some $k > 0$, we have

$$0 \le \int_\Omega\left(a(x,\omega)\frac{m(|\nabla u|)}{|\nabla u|}\nabla u - a(x,\omega)\frac{m(|\nabla\phi|)}{|\nabla\phi|}\nabla\phi\right)(\nabla u - \nabla\phi)\,\mathrm{d}x.$$

It follows by (4.11) and Young's inequality that

$$\begin{aligned}\int_\Omega a(x,\omega)\frac{m(|\nabla u|)}{|\nabla u|}\nabla u\nabla\phi\,\mathrm{d}x &\le \int_\Omega a(x,\omega)\frac{m(|\nabla u|)}{|\nabla u|}\nabla u\nabla u\,\mathrm{d}x\\ &\quad - \int_\Omega a(x,\omega)\frac{m(|\nabla\phi|)}{|\nabla\phi|}\nabla\phi(\nabla u - \nabla\phi)\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\le \alpha C + \beta_2 \int_\Omega m(|\nabla\phi|)|\nabla u|\,\mathrm{d}x + \beta_2 \int_\Omega m(|\nabla\phi|)|\nabla\phi|\,\mathrm{d}x \\
&\le \alpha C + 2\beta_2 \int_\Omega \left[\bar{M}\left(\frac{m(|\nabla\phi|)}{2}\right) + M(|\nabla u|)\right]\mathrm{d}x \\
&\quad + 2\beta_2 \int_\Omega \left[\bar{M}\left(\frac{m(|\nabla\phi|)}{2}\right) + M(|\nabla\phi|)\right]\mathrm{d}x \\
&\le \alpha C + 6\beta_2 \int_\Omega M(|\nabla\phi|) + 2\beta_2 \int_\Omega M(|\nabla u|) \le C
\end{aligned}
$$

for all $\phi \in W_0^1 E_M(\Omega)$ such that $\|\nabla\phi\|_{(M)} = 1/(k+1)$, and by considering the dual norm on $L_{\bar{M}}(\Omega)$, we get (4.9).

For (4.10), from (4.9), we deduce

$$
\begin{aligned}
\int_\Omega \frac{m(|u|)}{|u|} u\phi\,\mathrm{d}x \le \frac{1}{\lambda\mu}\int_\Omega \rho(\omega)\varphi\nabla\varphi\nabla\phi\,\mathrm{d}x \\
- \frac{1}{\lambda\mu}\int_\Omega \varphi^2\phi\,\mathrm{d}xt + \frac{1}{\lambda\mu}\int_\Omega g\varphi\phi\,\mathrm{d}x \\
- \int_\Omega a(x,\omega)\frac{m(|\nabla u|)}{|\nabla u|}\nabla u\nabla\phi\,\mathrm{d}x
\end{aligned}
\tag{4.12}
$$

for all $\phi \in W_0^1 E_M(Q)$ such that $\|\nabla\phi\|_{(M)} = 1/(k+1)$ for some $k > 0$. Hence, by considering the dual norm on $L_{\bar{M}}(Q)$, we get (4.10).

We define the operator $T : E_M(\Omega) \ni \omega \mapsto T(\omega) = u \in W_0^1 L_M(\Omega)$, with u being the unique solution to (4.7). We are going to show that T satisfies the conditions of Schauder's fixed point theorem. By Theorem 2.6 we have the compact embedding $W_0^1 L_M(\Omega) \hookrightarrow E_M(\Omega)$. Therefore, T maps $E_M(\Omega)$ into itself and, due to (4.8), T is a compact operator. Also, due to (4.8), for $R > 0$ large enough we have $T(B_R) \subset B_R$ where $B_R = \{v \in E_M(\Omega) \mid \|v\|_M \le R.\}$

Now, we prove that T is a continuous operator. Indeed, let $(\omega_n) \subset B_R$ be such that

$$
\omega_n \to \omega \quad \text{strongly in } E_M(\Omega),
\tag{4.13}
$$

and consider the functions corresponding to ω_n and ω, that is, $u_n = T(\omega_n)$, φ_n, $u = T(\omega)$, and φ. From (4.13), up to a subsequence, we have $\omega_n \to \omega$ a.e. in Ω. Then we put $H_n = \rho(\omega_n)\varphi_n\nabla\varphi_n$ and $H = \rho(\omega)\varphi\nabla\varphi$. We have to show that $H_n \to H$ strongly in $L^2(\Omega)^N$. It remains to show that $\varphi_n \to \varphi$ in $H^1(\Omega)$. To this end, we note that φ_n satisfies

$$
\operatorname{div}(\rho(\omega_n)\nabla\varphi_n) + \varphi_n = g, \qquad \varphi_n = 0 \text{ on } \partial\Omega,
\tag{4.14}
$$

while φ satisfies

$$
\operatorname{div}(\rho(\omega)\nabla\varphi) + \varphi = g, \qquad \varphi = 0 \text{ on } \partial\Omega.
\tag{4.15}
$$

Hence, using $\varphi_n - \varphi$ as a test function in both (4.14) and (4.15), we get

$$\int_\Omega \rho(\omega_n)\nabla\varphi_n\nabla(\varphi_n-\varphi)+\int_\Omega \varphi_n(\varphi_n-\varphi) = \int_\Omega \rho(\omega)\nabla\varphi\nabla(\varphi_n-\varphi)+\int_\Omega \varphi(\varphi_n-\varphi).$$

Inserting $-\rho(\omega_n)\nabla\varphi\nabla(\varphi_n-\varphi)$ in the last integral, we get

$$\int_\Omega \rho(\omega_n)|\nabla(\varphi_n-\varphi)|^2+\int_\Omega (\varphi_n-\varphi)^2 = \int_\Omega (\rho(\omega)-\rho(\omega_n))\nabla\varphi\nabla(\varphi_n-\varphi).$$

Then using (A_4) and Hölder's inequality yields

$$\int_\Omega |\nabla(\varphi_n-\varphi)|^2+\int_\Omega (\varphi_n-\varphi)^2 \le C\int_\Omega |\rho(\omega)-\rho(\omega_n)|^2|\nabla\varphi|^2. \tag{4.16}$$

Since ρ is continuous, $\rho(\omega_n)-\rho(\omega)\to 0$ a.e. in Ω. Furthermore,

$$|\rho(\omega)-\rho(\omega_n)|^2|\nabla\varphi|^2 \le C|\nabla\varphi|^2.$$

Due to (4.5), we can apply the Lebesgue convergence theorem to obtain

$$\int_\Omega |\rho(\omega)-\rho(\omega_n)|^2|\nabla\varphi|^2 \to 0 \quad \text{as } n\to\infty. \tag{4.17}$$

Thus, using (4.16) combined with (4.17),

$$\varphi_n \to \varphi \quad \text{in } H^1(\Omega). \tag{4.18}$$

Then it is easy to check that

$$H_n \to H \quad \text{strongly in } L^2(\Omega)^N. \tag{4.19}$$

Now, we will show that

$$u_n \to u \quad \text{strongly in } E_M(\Omega).$$

Using the same argument as above we find that (u_n) is bounded in $W_0^1L_M(\Omega)$ by some positive constant not depending on n. Consequently, in view of Theorem 2.6, there exist $v\in E_M(\Omega)$ and a subsequence of (u_n), not relabeled, such that

$$u_n \to v \quad \text{strongly in } E_M(\Omega) \text{ and a.e. in } \Omega. \tag{4.20}$$

We have to show that $v = u$. To do so, we will use the following result.

Lemma 4.6. *There exists a subsequence (not relabeled) such that*

$$\nabla u_n \to \nabla v \quad \text{a.e. in } \Omega, \tag{4.21}$$

This result is a special case of Step 2 below. For the proof of Lemma 4.6, one may repeat the arguments given in the proof of Step 2.

The variational formulation of the equation corresponding to u_n is

$$\int_\Omega a(x,\omega_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n \nabla\phi\,\mathrm{d}x + \int_\Omega \frac{m(|u_n|)}{|u_n|}.u_n.\phi\,\mathrm{d}x = -\int_\Omega H_n\nabla\phi\,\mathrm{d}x - \int_\Omega \varphi_n^2\phi + \int_\Omega g\varphi_n\phi\,\mathrm{d}x. \tag{4.22}$$

In view of (4.18) and (4.19), we get

$$\int_\Omega H_n\nabla\phi\,\mathrm{d}x - \int_\Omega \varphi_n^2\phi + \int_\Omega g\varphi_n\phi\,\mathrm{d}x \to \int_\Omega H\nabla\phi\,\mathrm{d}x - \int_\Omega \varphi^2\phi + \int_\Omega g\varphi\phi\,\mathrm{d}x. \tag{4.23}$$

On the other hand, since $\omega_n \to \omega$ a.e. in Ω and using Lemma 4.6, we get

$$a(x,\omega_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}.\nabla u_n \to a(x,\omega)\frac{m(|\nabla v|)}{|\nabla v|}\nabla v \text{ a.e. in } \Omega.$$

Furthermore, from (4.9), up to a subsequence we get

$$a(x,\omega_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n \rightharpoonup a(x,\omega)\frac{m(|\nabla v|)}{|\nabla v|}\nabla v \quad \text{weak-}* \text{ in } L_{\bar{M}}(\Omega)^N. \tag{4.24}$$

From (4.20) and (4.10), we also get

$$\frac{m(|u_n|)}{|u_n|}u_n \rightharpoonup \frac{m(|v|)}{|v|}v \quad \text{weakly in } L_{\bar{M}}(Q). \tag{4.25}$$

By using (4.24), we obtain

$$\int_\Omega a(x,\omega_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n \nabla\phi\,\mathrm{d}x + \int_\Omega \frac{m(|u_n|)}{|u_n|}u_n\phi\,\mathrm{d}x \to \int_\Omega a(x,\omega)\frac{m(|\nabla v|)}{|\nabla v|}\nabla v\nabla\phi\,\mathrm{d}x + \int_\Omega \frac{m(|v|)}{|v|}v\phi\,\mathrm{d}x. \tag{4.26}$$

In view of (4.22), (4.23) and (4.26), we get

$$\int_\Omega a(x,\omega)\frac{m(|\nabla v|)}{|\nabla v|}\nabla v\nabla\phi\,\mathrm{d}x + \int_\Omega \frac{m(|v|)}{|v|}v\phi\,\mathrm{d}x = \int_\Omega g\varphi\phi\,\mathrm{d}x - \int_\Omega H\nabla\phi\,\mathrm{d}x - \int_\Omega \varphi^2\phi.$$

Thus, $v = u$ and the operator T is continuous. Finally, according to the Schauder fixed point theorem, T has at least one fixed point, which defines a weak solution to our problem. This completes the proof of Theorem 4.2. ∎

Proof of Theorem 4.1. The proof is divided into several steps. We prove first the existence of an approximate solution, deriving a priori estimates, and we show the strong convergence in $L^1(\Omega)$, up to a subsequence, of both u_n and φ_n, where (u_n, φ_n) is a weak solution to the approximate problem of (1.1). Then we pass to the limit in the approximate solutions.

Step 1: *Approximate problems and a priori estimates.* We define the truncation function at height $K > 0$, T_K, by

$$T_K(s) = s \quad \text{if } |s| \le K, \qquad T_K(s) = K\frac{s}{|s|} \quad \text{if } |s| > K,$$

and we introduce the following regularization of the data for every $n \in \mathbb{N}^*$:

$$\rho_n(s) = \rho(s) + \frac{1}{n}. \tag{4.27}$$

The approximate problem is given as follows:

$$-\operatorname{div}\left(a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n\right) + \frac{m(|u_n|)}{|u_n|}u_n = \rho_n(u_n)|\nabla\varphi_n|^2 \quad \text{in } \Omega, \tag{4.28}$$

$$\operatorname{div}(\rho_n(u_n)\nabla\varphi_n) + \varphi_n = g \text{ in } \Omega, \tag{4.29}$$

$$u_n = 0, \quad \varphi_n = 0 \quad \text{on } \partial\Omega. \tag{4.30}$$

In view of (3.2), we have

$$n^{-1} \le \rho_n(s) \le \bar{\rho} + 1 = \rho_3 \quad \text{for all } s \in \mathbb{R}. \tag{4.31}$$

Thus, from Theorem 4.2, we get the existence of a weak solution (u_n, φ_n) to system (4.28)–(4.30).

We use the same argument as in (4.4) to get

$$\|\varphi_n\|_{L^\infty(\Omega)} \le \|g\|_{L^\infty(\Omega)}. \tag{4.32}$$

Therefore, there exists a function $\varphi \in L^\infty(\Omega)$ and a subsequence (not relabeled) such that

$$\varphi_n \to \varphi \quad \text{weak-}* \text{ in } L^\infty(\Omega). \tag{4.33}$$

Now, taking φ_n as a test function in (4.29), we have

$$\int_\Omega \rho_n(u_n)|\nabla\varphi_n|^2\,\mathrm{d}x + \int_\Omega \varphi_n\varphi_n\,\mathrm{d}x = \int_\Omega g\varphi_n\,\mathrm{d}x.$$

Thus,

$$\int_\Omega \rho_n(u_n)|\nabla\varphi_n|^2\,\mathrm{d}x \le \int_\Omega g\varphi_n\,\mathrm{d}x \le \|g\|^2_{L^\infty(\Omega)}|\Omega|.$$

Hence

$$\int_\Omega \rho_n(u_n)|\nabla\varphi_n|^2\,\mathrm{d}x \le C \quad \text{for all } n \ge 1. \tag{4.34}$$

Consequently, the sequence $(\rho_n(u_n)\nabla\varphi_n)$ is bounded in $L^2(\Omega)^N$. Thus, there exists a function $\Phi \in L^2(\Omega)^N$ and a subsequence (not relabeled) such that

$$\rho_n(u_n)\nabla\varphi_n \rightharpoonup \Phi \quad \text{weakly in } L^2(\Omega)^N. \tag{4.35}$$

This weak limit function $\Phi \in L^2(\Omega)^N$ is in fact the third component of the triplet appearing in Definition 3.2 of a capacity solution.

By taking u_n as a test function in (4.28), we get

$$\int_\Omega a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n \nabla u_n \,\mathrm{d}x + \int_\Omega \frac{m(|u_n|)}{|u_n|} u_n u_n \,\mathrm{d}x = -\int_\Omega \rho_n(u_n)\varphi_n \nabla\varphi_n \nabla u_n \,\mathrm{d}x - \int_\Omega \varphi_n^2\phi + \int_\Omega g\varphi_n\phi \,\mathrm{d}x. \tag{4.36}$$

Since $u_n \in W_0^1 L_M(\Omega)$ and $\varphi_n \in H^1(\Omega) \subset W^{-1}L_{\bar{M}}(\Omega)$, there exist $\lambda, \mu > 0$ such that $\nabla u_n/\lambda \in \mathcal{L}_M(\Omega)$ and $\rho_3\|g\|_{L^\infty(\Omega)}\nabla\varphi_n/\mu \in \mathcal{L}_{\bar{M}}(\Omega)$. In virtue of (2.6), (4.31), (4.32), and the Young and Poincaré inequalities, we obtain

$$\begin{aligned}
\frac{\alpha}{\lambda\mu}\int_\Omega M(\nabla u_n)\,\mathrm{d}x &\le \frac{1}{\lambda\mu}\int_\Omega a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n\nabla u_n\,\mathrm{d}x \\
&\quad + \frac{1}{\lambda\mu}\int_\Omega \frac{m(|u_n|)}{|u_n|}|u_n|^2\,\mathrm{d}x \\
&\le \frac{1}{\lambda\mu}\int_\Omega \rho_n(u_n)|\varphi_n|\,|\nabla\varphi_n|\,|\nabla u_n|\ \mathrm{d}x \\
&\quad - \frac{1}{\lambda\mu}\int_\Omega \varphi_n^2 u_n + \frac{1}{\lambda\mu}\int_\Omega g\varphi_n u_n \\
&\le \int_\Omega \bar{M}(\rho_3\|g\|_{L^\infty(\Omega)}\nabla\varphi_n/\mu)\,\mathrm{d}x + \int_\Omega M(|\nabla u_n|/\lambda))\,\mathrm{d}x \\
&\quad + C\int_\Omega M(|\nabla u_n|/\lambda) < \infty.
\end{aligned}$$

This implies that $|\nabla u_n| \in \mathcal{L}_M(\Omega)$. Moreover, by using the Young and Poincaré inequalities with (2.6) we get

$$\begin{aligned}
\alpha\int_\Omega M(\nabla u_n)\,\mathrm{d}x &\le \int_\Omega a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n\nabla u_n\,\mathrm{d}x + \int_\Omega \frac{m(|u_n|)}{|u_n|}|u_n|^2\,\mathrm{d}x \\
&\le \int_\Omega \rho_n(u_n)|\varphi_n|\,|\nabla\varphi_n|\,|\nabla u_n|\,\mathrm{d}x - \int_\Omega \varphi_n^2 u_n + \int_\Omega g\varphi_n u_n\,\mathrm{d}x \\
&\le \int_\Omega \sqrt{\rho_3}\|g\|_\infty\sqrt{\rho_n(u_n)}|\nabla\varphi_n|\,|\nabla u_n| + \int_\Omega \varphi_n^2|u_n| + \int_\Omega |g\varphi_n u_n| \\
&\le \frac{\rho_3\|g\|_{L^\infty(\Omega)}^2}{\alpha}\int_\Omega \rho_n(u_n)|\nabla\varphi_n(u_n)|^2 + \frac{\alpha}{2}\int_\Omega M(|\nabla u_n|) \\
&\quad + 2\|g\|_{L^\infty(\Omega)}^2\int_\Omega |u_n|
\end{aligned}$$

$$\leq C + \frac{\alpha}{2}\int_\Omega M(|\nabla u_n|)\,\mathrm{d}x + \frac{(2\|g\|^2_{L^\infty(\Omega)})^2}{\alpha\epsilon}|\Omega| + \frac{\alpha\epsilon}{4}\int_\Omega |u_n|^2\,\mathrm{d}x$$

$$\leq C + \frac{\alpha}{2}\int_\Omega M(|\nabla u_n|)\,\mathrm{d}x + \frac{\alpha\epsilon C(\Omega)}{4}\int_\Omega |\nabla u_n|^2\,\mathrm{d}x$$

$$\leq C + \frac{\alpha}{2}\int_\Omega M(|\nabla u_n|)\,\mathrm{d}x + \frac{\alpha\epsilon C(\Omega)}{2}\int_\Omega M(|\nabla u_n|)\,\mathrm{d}x.$$

Then, for $\epsilon = 1/(2C(\Omega))$, we get

$$\int_\Omega M(|\nabla u_n|)\,\mathrm{d}x \leq C. \tag{4.37}$$

We also get

$$\int_\Omega a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n\nabla u_n\,\mathrm{d}x \leq C. \tag{4.38}$$

Due to (4.37), the sequence (u_n) is bounded in $W_0^1L_M(\Omega)$. Hence, there exists a subsequence (not relabeled) and a function $u \in W_0^1L_M(\Omega)$ such that

$$u_n \to u \quad \text{in } W_0^1L_M(\Omega) \text{ for } \sigma\Big(\prod L_M, \prod E_{\bar{M}}\Big). \tag{4.39}$$

Since the embedding $W_0^1L_M(\Omega) \hookrightarrow E_M(\Omega)$ is compact, we deduce

$$u_n \to u \quad \text{strongly in } E_M(\Omega) \text{ and a.e. in } \Omega. \tag{4.40}$$

Now let $\phi \in W_0^1E_M(\Omega)$ be such that $\|\nabla\phi\|_{(M)} = 1/(k+1)$. From the monotonicity of the operator A and (4.38), we easily find that

$$\begin{aligned}\int_\Omega a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}.\nabla u_n\nabla\phi &\leq \int_\Omega a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}.\nabla u_n\nabla u_n \\ &\quad - \int_\Omega a(x,u_n)\frac{m(|\nabla\phi|)}{|\nabla\phi|}.\nabla\phi(\nabla u_n - \nabla\phi) \\ &\leq C + \beta_2\int_\Omega m(|\nabla\phi|)|\nabla u_n - \nabla\phi|\,\mathrm{d}x \leq C\end{aligned} \tag{4.41}$$

for all $n \geq 1$ and $\phi \in W_0^1E_M(\Omega)$ such that $\|\nabla\phi\|_{(M)} = 1/(k+1)$. Thus, by using the dual norm, it is easy to see that

$$\left(a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n\right) \quad \text{is bounded in } L_{\bar{M}}(\Omega)^N. \tag{4.42}$$

Then there exists $\delta \in L_{\bar{M}}(\Omega)^N$ and a subsequence (not relabeled) such that

$$a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n \to \delta \quad \text{weak-}* \text{ in } L_{\bar{M}}(\Omega)^N. \tag{4.43}$$

In view (4.41), we may show that

$$\frac{m(|u_n|)}{|u_n|}u_n \quad \text{is bounded in } L_{\bar{M}}(\Omega). \tag{4.44}$$

Thus, there exists $\delta' \in L_{\bar{M}}(\Omega)$ and a subsequence (not relabeled) such that

$$\frac{m(|u_n|)}{|u_n|}u_n \rightharpoonup \delta' \quad \text{weak-}* \text{ in } L_{\bar{M}}(\Omega). \tag{4.45}$$

According to (4.40), we get $\delta' = \frac{m(|u|)}{|u|}u$, thus

$$\frac{m(|u_n|)}{|u_n|}u_n \rightharpoonup \frac{m(|u|)}{|u|}u \quad \text{weak-}* \text{ in } L_{\bar{M}}(\Omega). \tag{4.46}$$

STEP 2: *Almost everywhere convergence of the gradients.* Let $(v_j) \subset \mathcal{D}(\Omega)$ be such that

- $v_j \to u$ in $W_0^1 L_M(\Omega)$ for modular convergence;
- $v_j \to u$ and $\nabla v_j \to \nabla u$ a.e. in Ω.

We shall prove the following proposition.

PROPOSITION 4.7. *Let (u_n, φ_n) be a solution of the approximate problem* (4.28)–(4.30). *Then there exists a subsequence* (*not relabeled*) *such that*

$$\nabla u_n \to \nabla u \qquad \text{a.e. in } \Omega, \tag{4.47}$$

as $n \to \infty$.

Proof. We denote by χ_s^j and χ_s, respectively, the characteristic functions of the sets

$$\Omega_s^j = \{x \in \Omega \mid |\nabla T_K(v_j)| \le s\}, \quad \Omega_s = \{x \in \Omega \mid |\nabla T_K(u)| \le s\}.$$

We will also denote by $\epsilon(n,j)$ and $\epsilon(n,j,s)$ any quantities such that

$$\lim_{j\to\infty}\lim_{n\to\infty}\epsilon(n,j) = 0, \quad \lim_{s\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\epsilon(n,j,s) = 0.$$

For any $\nu > 0$ and $j, n \ge 1$, we use the test function $\varphi_{n,j}^\nu = T_\nu(u_n - T_K(v_j))$ in (4.28) to obtain

$$\int_\Omega a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n \nabla T_\nu(u_n - T_K(v_j))\,\mathrm{d}x + \int_\Omega \frac{m(|u_n|)}{|u_n|}u_n \varphi_{n,j}^\nu\,\mathrm{d}x = \int_\Omega \rho_n(u_n)|\nabla\varphi_n|^2\varphi_{n,j}^\nu\,\mathrm{d}x. \tag{4.48}$$

From (4.32) and (4.34), we obtain

$$\int_\Omega a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n \nabla T_\nu(u_n - T_K(v_j))\,\mathrm{d}x + \int_\Omega \frac{m(|u_n|)}{|u_n|}u_n \varphi_{n,j}^\nu\,\mathrm{d}x \le C\nu. \tag{4.49}$$

Using (4.46), we have

$$\int_\Omega \frac{m(|u_n|)}{|u_n|} u_n T_\nu(u_n - T_K(v_j))\,\mathrm{d}x \to \int_\Omega \frac{m(|u|)}{|u|} u T_\nu(u - T_K(u))\,\mathrm{d}x = \epsilon(K). \tag{4.50}$$

On the other hand,

$$\begin{aligned}
&\int_\Omega a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n \nabla T_\nu(u_n - T_K(v_j))\,\mathrm{d}x \\
&= \int_{\{|u_n - T_K(v_j)|\le\nu\}} a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n \nabla(u_n - T_K(v_j))\,\mathrm{d}x \\
&= \int_{\{|u_n|>K\}\cap\{|u_n - T_K(v_j)|\le\nu\}} a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n \nabla(u_n - T_K(v_j))\,\mathrm{d}x \\
&\quad + \int_{\{|u_n|\le K\}\cap\{|u_n - T_K(v_j)|\le\nu\}} a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n \nabla(u_n - T_K(v_j))\,\mathrm{d}x \\
&= \int_{\{|T_K(u_n) - T_K(v_j)|\le\nu\}} a(x,T_K(u_n))\frac{m(|\nabla T_K(u_n)|)}{|\nabla T_K(u_n)|} \\
&\qquad\qquad \cdot \nabla T_K(u_n)(\nabla T_K(u_n) - \nabla T_K(v_j))\,\mathrm{d}x \\
&\quad + \int_{\{|u_n|>K\}\cap\{|u_n - T_K(v_j)|\le\nu\}} a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n \nabla u_n\,\mathrm{d}x \\
&\quad - \int_{\{|u_n|>K\}\cap\{|u_n - T_K(v_j)|\le\nu\}} a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n \nabla T_K(v_j)\,\mathrm{d}x.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\int_\Omega a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n \nabla T_\nu(u_n - T_K(v_j))\,\mathrm{d}x \\
&\ge \int_{\{|T_K(u_n) - T_K(v_j)|\le\nu\}} a(x,T_K(u_n))\frac{m(|\nabla T_K(u_n)|)}{|\nabla T_K(u_n)|} \\
&\qquad\qquad \cdot \nabla T_K(u_n)(\nabla T_K(u_n) - \nabla T_K(v_j))\,\mathrm{d}x \\
&\quad - \int_{\{|u_n|>K\}\cap\{|u_n - T_K(v_j)|\le\nu\}} a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n \nabla T_K(v_j)\,\mathrm{d}x.
\end{aligned} \tag{4.51}$$

Also, in the set $\{|u_n - T_K(v_j)| \le \nu\}$, we have

$$|u_n| \le |u_n - T_K(v_j)| + |T_K(v_j)| \le \nu + K,$$

and thus, we may write

$$\begin{aligned}(4.52)\qquad &\int_{\{|u_n|>K\}\cap\{|u_n-T_K(v_j)|\le\nu\}} a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n\nabla T_K(v_j)\,\mathrm{d}x\\ &= \int_{\{|u_n|>K\}\cap\{|u_n-T_K(v_j)|\le\nu\}} a(x,T_{\nu+K}(u_n))\frac{m(|\nabla T_{\nu+K}(u_n)|)}{|\nabla T_{\nu+K}(u_n)|}\\ &\qquad\cdot\nabla T_{\nu+K}(u_n)\nabla T_K(v_j)\,\mathrm{d}x.\end{aligned}$$

By using (4.51) and (4.52) we get

$$\begin{aligned}(4.53)\qquad &\int_\Omega a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n\nabla T_\nu(u_n-T_K(v_j))\,\mathrm{d}x\\ &\ge \int_{\{|T_K(u_n)-T_K(v_j)|\le\nu\}} a(x,T_K(u_n))\frac{m(|\nabla T_K(u_n)|)}{|\nabla T_K(u_n)|}\\ &\qquad\cdot\nabla T_K(u_n)(\nabla T_K(u_n)-\nabla T_K(v_j))\,\mathrm{d}x\\ &- \int_{\{|u_n|>K\}\cap\{|u_n-T_K(v_j)|\le\nu\}} a(x,T_{\nu+K}(u_n))\frac{m(|\nabla T_{\nu+K}(u_n)|)}{|\nabla T_{\nu+K}(u_n)|}\\ &\qquad\cdot\nabla T_{\nu+K}(u_n)\nabla T_K(v_j)\,\mathrm{d}x.\end{aligned}$$

We denote

$$\begin{aligned}F_1 = &\int_{\{|u_n|>K\}\cap\{|u_n-T_K(v_j)|\le\nu\}} a(x,T_{\nu+K}(u_n))\frac{m(|\nabla T_{\nu+K}(u_n)|)}{|\nabla T_{\nu+K}(u_n)|}\\ &\qquad\cdot\nabla T_{\nu+K}(u_n)\nabla T_K(v_j)\,\mathrm{d}x.\end{aligned}$$

Since $\big(a(x,T_{K+\nu}(u_n))\frac{m(|\nabla T_{\nu+K}(u_n)|)}{|\nabla T_{\nu+K}(u_n)|}\nabla T_{\nu+K}(u_n)\big)_n$ is bounded in $L_{\bar M}(\Omega)$, we get, for certain $\delta_{K+\nu}\in L_{\bar M}(\Omega)$, and a sequence u_n,

$$a(x,T_{K+\nu}(u_n))\frac{m(|\nabla T_{\nu+K}(u_n)|)}{|\nabla T_{\nu+K}(u_n)|}\nabla T_{\nu+K}(u_n)\to\delta_{K+\nu}\quad\text{weak-}*\text{ in } L_{\bar M}(\Omega).$$

Consequently,

$$\begin{aligned}&\int_{\{|u_n|>K\}\cap\{|u_n-T_K(v_j)|\le\nu\}} a(x,T_{\nu+K}(u_n))\frac{m(|\nabla T_{\nu+K}(u_n)|)}{|\nabla T_{\nu+K}(u_n)|}\\ &\qquad\cdot\nabla T_{\nu+K}(u_n)\nabla T_K(v_j)\,\mathrm{d}x\\ &\qquad\to\int_{\{|u|\ge K\}\cap\{|u-T_K(v_j)|\le\nu\}}\delta_{K+\nu}\nabla T_K(v_j)\,\mathrm{d}x\end{aligned}$$

as $n \to \infty$. From Lemma 2.3, we obtain, as $j \to \infty$,

$$\int_{\{|u|\geq K\}\cap\{|u-T_K(v_j)|\leq \nu\}} \delta_{K+\nu} \nabla T_K(v_j) \to \int_{\{|u|\geq K\}\cap\{|u-T_K(u)|\leq \nu\}} \delta_{K+\nu} \nabla T_K(u) = 0.$$

Since $\nabla T_K(u) = 0$ in the set $\{|u| \geq K\}$, this implies

$$F_1 = \epsilon(n, j). \tag{4.54}$$

Set $\Gamma = \Gamma(K, n, j, \nu) = \{|T_K(u_n) - T_K(v_j)| \leq \nu\}$. By using (4.49), (4.50) and (4.54) in (4.53) we get

$$\begin{aligned}\int_\Gamma a(x, T_K(u_n)) \frac{m(|\nabla T_K(u_n)|)}{|\nabla T_K(u_n)|} \nabla T_K(u_n)(\nabla T_K(u_n) - \nabla T_K(v_j))\,\mathrm{d}x \\ \leq C\nu + \epsilon(n, j) + \epsilon(K).\end{aligned} \tag{4.55}$$

On the other hand, notice that

$$\begin{aligned}&\int_\Gamma a(x, T_K(u_n)) \frac{m(|\nabla T_K(u_n)|)}{|\nabla T_K(u_n)|} \nabla T_K(u_n)(\nabla T_K(u_n) - \nabla T_K(v_j))\,\mathrm{d}x \\ &= \int_\Gamma a(x, T_K(u_n)) \frac{m(|\nabla T_K(u_n)|)}{|\nabla T_K(u_n)|} \nabla T_K(u_n)(\nabla T_K(u_n) - \nabla T_K(v_j)\chi_j^s) \\ &\quad + \int_\Gamma a(x, T_K(u_n)) \frac{m(|\nabla T_K(u_n)|)}{|\nabla T_K(u_n)|} \nabla T_K(u_n)(\nabla T_K(v_j)\chi_j^s - \nabla T_K(v_j)) \\ &= F_2 + F_3.\end{aligned} \tag{4.56}$$

The integral F_3 tends to 0 as, first n, then j, and then s go to infinity. Indeed, since

$$a(x, T_K(u_n)) \frac{m(|\nabla T_K(u_n)|)}{|\nabla T_K(u_n)|} \nabla T_K(u_n) \rightharpoonup \delta_K \quad \text{weak-}* \text{ in } L_{\bar{M}}(\Omega)^N,$$

as $n \to \infty$, and

$$\begin{aligned}&(\nabla T_K(v_j)\chi_j^s - \nabla T_K(v_j))\chi_{\{|T_K(u_n)-T_K(v_j)|\leq\nu\}} \\ &\qquad \to (\nabla T_K(v_j)\chi_j^s - \nabla T_K(v_j))\chi_{\{|T_K(u)-T_K(v_j)|\leq\nu\}}\end{aligned}$$

strongly in $E_M(\Omega)^N$ as $n \to \infty$, then

$$\lim_{n\to\infty} F_3 = \int_{\{|T_K(u)-T_K(v_j)|\leq\nu\}} \delta_K(\nabla T_K(v_j)\chi_j^s - \nabla T_K(v_j))\,\mathrm{d}x.$$

Finally letting j, then s, go to infinity, we deduce that

$$F_3 = \epsilon(n, j, s). \tag{4.57}$$

Therefore, by using (4.55)–(4.57), we have

$$F_2 \leq C\nu + \epsilon(n, j, s, K). \tag{4.58}$$

Consider

$$S_n = \Big[a(x,T_K(u_n))\frac{m(|\nabla T_K(u_n)|)}{|\nabla T_K(u_n)|}\nabla T_K(u_n) - a(x,T_K(u_n))\frac{m(|\nabla T_K(u)|)}{|\nabla T_K(u)|}\nabla T_K(u)\Big](\nabla T_K(u_n)-\nabla T_K(u)),$$

and let $0<\theta<1$ and denote $I_{n,r}=\int_{\Omega_r} S_n^\theta \,\mathrm{d}x$. We have

$$I_{n,r} = \int_{\Omega_r} S_n^\theta \chi_\Gamma + \int_{\Omega_r} S_n^\theta \chi_{\Gamma^c}. \tag{4.59}$$

By using Hölder's inequality we can bound the second term of the right-hand side by

$$\Big(\int_{\Omega_r} S_n \,\mathrm{d}x\Big)^\theta \cdot \Big(\int_{\Omega_r} \chi_{\Gamma^c}\,\mathrm{d}x\Big)^{1-\theta}.$$

Observe that

$$\begin{aligned}\int_{\Omega_r} S_n\,\mathrm{d}x = \Big[&\int_{\Omega_r} a(x,T_K(u_n))\frac{m(|\nabla T_K(u_n)|)}{|\nabla T_K(u_n)|}.\nabla T_K(u_n)\nabla T_K(u_n)\,\mathrm{d}x\\ &- \int_{\Omega_r} a(x,T_K(u_n))\frac{m(|\nabla T_K(u_n)|)}{|\nabla T_K(u_n)|}.\nabla T_K(u_n)\nabla T_K(u)\,\mathrm{d}x\\ &+ \int_{\Omega_r} a(x,T_K(u_n))\frac{m(|\nabla T_K(u)|)}{|\nabla T_K(u)|}.\nabla T_K(u)\nabla T_K(u)\,\mathrm{d}x\\ &- \int_{\Omega_r} a(x,T_K(u_n))\frac{m(|\nabla T_K(u)|)}{|\nabla T_K(u)|}.\nabla T_K(u)\nabla T_K(u_n)\,\mathrm{d}x\Big].\end{aligned}$$

The sequence

$$\Big(a(x,T_K(u_n))\frac{m(|\nabla T_K(u_n)|)}{|\nabla T_K(u_n)|}\nabla T_K(u_n)\Big)_n$$

is bounded in $L_{\bar M}(\Omega)^N$, $\big(\nabla T_K(u_n)\big)_n$ is bounded in $L_M(\Omega)^N$ and the sequence

$$\Big(a(x,T_K(u_n))\frac{m(|\nabla T_K(u)|)}{|\nabla T_K(u)|}\nabla T_K(u)\Big)_n$$

is bounded in $E_{\bar M}(\Omega_r)^N$. Then (S_n) is bounded in $L^1(\Omega_r)$. Thus there exists a constant $C_3>0$ such that

$$\int_{\Omega_r} S_n^\theta \chi_{\Gamma^c}\,\mathrm{d}x \le C_3 \,\mathrm{meas}(\Gamma^c)^{1-\theta}. \tag{4.60}$$

We use Hölder's inequality to deduce

$$\int_{\Omega_r} S_n^\theta \chi_\gamma \,\mathrm{d}x \le \Big(\int_{\Omega_r} 1 \,\mathrm{d}x\Big)^{1-\theta}\Big(\int_{\Gamma\cap\Omega_r} S_n \,\mathrm{d}x\Big)^{\theta} \le C_4\Big(\int_{\Gamma\cap\Omega_r} S_n \,\mathrm{d}x\Big)^{\theta}. \tag{4.61}$$

By the estimates (4.60) and (4.61), we get

$$I_{n,r} \le C_3 \operatorname{meas}(\Gamma^c)^{1-\theta} + C_4\Big(\int_{\Gamma\cap\Omega_r} S_n \,\mathrm{d}x\Big)^{\theta}. \tag{4.62}$$

Let $s \ge r$ and $r > 0$. We have

$$\begin{aligned}
&\int_{\Gamma\cap\Omega_r} S_n \,\mathrm{d}x \le \int_{\Gamma\cap\Omega_s} S_n \,\mathrm{d}x\\
&= \int_{\Gamma\cap\Omega_s} [a(x,T_K(u_n))\frac{m(|\nabla T_K(u_n)|)}{|\nabla T_K(u_n)|}\nabla T_K(u_n)\\
&\quad - a(x,T_K(u_n))\frac{m(|\nabla T_K(u)|)}{|\nabla T_K(u)|}\nabla T_K(u)\chi_s]\cdot[\nabla T_K(u_n) - \nabla T_K(u)\chi_s]\,\mathrm{d}x\\
&\le \int_{\Gamma}[a(x,T_K(u_n))\frac{m(|\nabla T_K(u_n)|)}{|\nabla T_K(u_n)|}\nabla T_K(u_n)\\
&\quad - a(x,T_K(u_n))\frac{m(|\nabla T_K(u)|)}{|\nabla T_K(u)|}\nabla T_K(u)\chi_s]\cdot[\nabla T_K(u_n) - \nabla T_K(u)\chi_s]\,\mathrm{d}x\\
&= \Big[\int_{\Gamma}[a(x,T_K(u_n))\frac{m(|\nabla T_K(u_n)|)}{|\nabla T_K(u_n)|}\nabla T_K(u_n)\\
&- a(x,T_K(u_n))\frac{m(|\nabla T_K(v_j)|)}{|\nabla T_K(v_j)|}\nabla T_K(v_j)\chi_j^s]\cdot[\nabla T_K(u_n) - \nabla T_K(v_j)\chi_j^s]\,\mathrm{d}x\\
&\quad + \int_{\Gamma} a(x,T_K(u_n))\frac{m(|\nabla T_K(u_n)|)}{|\nabla T_K(u_n)|}\nabla T_K(u_n)\cdot[\nabla T_K(v_j)\chi_j^s - \nabla T_K(u)\chi_s]\,\mathrm{d}x\\
&\quad + \int_{\Gamma}[a(x,T_K(u_n))\frac{m(|\nabla T_K(v_j)|)}{|\nabla T_K(v_j)|}\nabla T_K(v_j)\chi_j^s - a(x,T_K(u_n))\frac{m(|\nabla T_K(u)|)}{|\nabla T_K(u)|}\\
&\quad \cdot \nabla T_K(u)\chi_s]\cdot\nabla T_K(u_n)\,\mathrm{d}x\\
&\quad - \int_{\Gamma} a(x,T_K(u_n))\frac{m(|\nabla T_K(v_j)|)}{|\nabla T_K(v_j)|} nabla T_K(v_j)\chi_j^s\cdot\nabla T_K(v_j)\chi_j^s\,\mathrm{d}x\\
&\quad + \int_{\Gamma} a(x,T_K(u_n))\frac{m(|\nabla T_K(u)|)}{|\nabla T_K(u)|}\nabla T_K(u)\chi_s\cdot\nabla T_K(u)\chi_s\,\mathrm{d}x\Big]\\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

According to the estimate proved as above, we get $I_1 \to 0$, $I_2 \to 0$, $I_3 \to 0$, $I_4 \to 0$, and $I_5 \to 0$, as n, j and s tend to infinity.

Finally, we obtain

$$\limsup_{n\to\infty}\int\limits_{\Omega_r}\bigg(\bigg[a(x,T_K(u_n))\frac{m(|\nabla T_K(u_n)|)}{|\nabla T_K(u_n)|}.\nabla T_K(u_n) \\ - a(x,T_K(u_n))\frac{m(|\nabla T_K(u)|)}{|\nabla T_K(u)|}.\nabla T_K(u)\bigg]\cdot(\nabla T_K(u_n)-\nabla T_K(u))\bigg)^\theta\,\mathrm{d}x=0.$$

Consequently, there exist a subsequence (not relabeled) and a negligible subset $Z\subset\Omega$ such that for all $x\in\Omega\setminus Z$ one has

$$\bigg[a(x,T_K(u_n))\frac{m(|\nabla T_K(u_n)|)}{|\nabla T_K(u_n)|}\nabla T_K(u_n) \\ - a(x,T_K(u_n))\frac{m(|\nabla T_K(u)|)}{|\nabla T_K(u)|}\nabla T_K(u)\bigg](\nabla T_K(u_n)-\nabla T_K(u))\to 0 \text{ a.e. in } \Omega. \tag{4.63}$$

Fix $x\in\Omega\setminus Z$. Then, according to (3.3), the sequence $(\nabla T_K(u_n)(x))_n\subset\mathbb{R}^N$ is bounded. By extracting a convergent subsequence (not relabeled) we find that $\nabla T_K(u_n)(x)\to\zeta$ is for some $\zeta\in\mathbb{R}^N$.

Passing to the limit in (4.63) yields

$$\bigg(a(x,T_K(u)(x))\frac{m(|\zeta|)}{|\zeta|}\zeta \\ - a(x,T_K(u)(x))\frac{m(|\nabla T_K(u)|)}{|\nabla T_K(u)|}\nabla T_K(u)\bigg)\cdot(\zeta-\nabla T_K(u)(x))=0,$$

which, according to (3.1), it is only possible if $\zeta=\nabla T_K(u)(x)$. Therefore, for any $K>0$, we have deduced that, up to a subsequence, $\nabla T_K(u_n)\to\nabla T_K(u)$ a.e. in Ω. Since K is arbitrary, we finally obtain the desired result. ∎

REMARK 4.8. According to the almost everywhere convergence of $\nabla u_n\to\nabla u$, $u_n\to u$ and (A_1) we have $\delta=a(x,u)\frac{m(|\nabla u|)}{|\nabla u|}\nabla u$, that is,

$$a(x,u_n)\frac{m(|\nabla u_n|)}{|\nabla u_n|}\nabla u_n\to a(x,u)\frac{m(|\nabla u|)}{|\nabla u|}\nabla u \quad \text{weak-}*\text{ in } L_{\bar M}(\Omega)^N. \tag{4.64}$$

STEP 3: *L^1-convergence of (φ_n).* In this step, we prove that $\varphi_n\to\varphi$ strongly in $L^1(\Omega)$ up to a subsequence. This generalizes Lemma 4 of González Montesinos and Ortegón Gallego [6], which in turn is an adaptation of the original result due to Xu [16].

LEMMA 4.9 ([11]). *Let M be an N-function which admits the representation $M(t)=\int_0^{|t|}m(s)\,\mathrm{d}s$ and satisfies $s\le m(s)$ for all $s>0$. Let (u_n) be a bounded sequence in $W^1L_M(\Omega)$ such that $u_n\to u$ strongly in $E_M(\Omega)$. Then there exists a subsequence $(u_{n(k)})\subset(u_n)$ such that, for every $\epsilon>0$, there exists a constant value $\boldsymbol{M}=\boldsymbol{M}(\epsilon)$ and a function $\psi\in W^{1,1}(\Omega)$ satisfying* *the following properties:*

$$0 \le \psi \le 1, \tag{4.65}$$

$$\|\psi - 1\|_{L^1(\Omega)} + \|\nabla\psi\|_{L^1(\Omega)} \le \epsilon, \tag{4.66}$$

$$|u|, |u_{n(k)}| \le \boldsymbol{M} \quad \text{on } \{\psi > 0\} \text{ for all } k \ge 1. \tag{4.67}$$

LEMMA 4.10 ([6]). *Let* (u_n, φ_n) *be a weak solution to system* (4.28)–(4.30), $u \in E_M(\Omega)$ *and* $\varphi \in L^\infty(\Omega)$ *the limit functions appearing, respectively, in* (4.33) *and* (4.39). *Then, for any* $S \in C_0^1(\mathbb{R})$, *there exists a subsequence* (*not relabeled*) *such that*

$$S(u_n)\varphi_n \rightharpoonup S(u)\varphi \quad \text{weakly in } H^1(\Omega). \tag{4.68}$$

Moreover, if $0 \le S \le 1$, *then there exists a constant* $C > 0$, *independent of* S, *such that*

$$\limsup_{n\to\infty} \int_\Omega \rho_n(u_n)|\nabla[S(u_n)\varphi_n - S(u)\varphi]|^2 \le C\|S'\|_\infty(1 + \|S'\|_\infty). \tag{4.69}$$

LEMMA 4.11. *There exists a subsequence* $(\varphi_{n(k)}) \subset (\varphi_n)$ *such that*

$$\lim_{k\to\infty} \int_\Omega |\varphi_{n(k)} - \varphi| = 0. \tag{4.70}$$

Proof. The proof is almost identical to that of [11, Lemma 5.7]. ∎

STEP 4: *Passage to the limit.* Owing to (4.33), (4.35), (4.39), (4.43) and (4.44) the condition (C_1) of Definition 3.2 is fulfilled. The convergence (4.47) and Lemma 4.11 yield (C_2). In order to obtain (C_3), using Lemma 4.11 and Proposition 4.7 and with (4.68), it is enough to let $k \to \infty$ in

$$S(u_{n(k)})\rho_{n(k)}(u_{n(k)})\nabla\varphi_{n(k)} = \rho_{n(k)}(u_{n(k)})[\nabla(S(u_{n(k)})\varphi_{n(k)}) - \varphi_{n(k)}\nabla S(u_{n(k)})].$$

The proof of Theorem 4.1 is complete.

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