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Mykola Yaremenko

GABOR FRAMES AND τ-PSEUDODIFFERENTIAL OPERATORS ON LOCALLY COMPACT ABELIAN GROUPS

Abstract. This paper presents a comprehensive framework for time-frequency analysis on locally compact abelian groups. We introduce and study generalized versions of fundamental time-frequency objects, specifically the cross τ-ambiguity function and the cross τ-Wigner distribution. Building on these definitions, we define a class of τ-pseudodifferential operators via a duality approach, establishing that the modulation spaces $M^{p,r}_{\omega}(G)$ serve as their natural functional setting. A significant portion of our work is dedicated to the theory of Gabor frames on quasi-lattices. We prove that under mild conditions on the window function, the associated coefficient and reconstruction operators are bounded on modulation spaces, leading to unconditional convergence of the resulting frame expansions. This provides a powerful machinery for the phase-space analysis of pseudodifferential operators in a general group-theoretic context, unifying and extending classical results from Euclidean space to the setting of locally compact abelian groups.

1. Introduction. The rapid advancement of signal processing and quantum computing has generated significant interest in abstract time-frequency analysis on locally compact abelian groups [4, 15, 9]. While classical time-frequency analysis is deeply rooted in Euclidean space $\mathbb{R}^d$, many modern applications, particularly in quantum mechanics on lattices and signal processing on network structures, necessitate a more general framework. The theory of locally compact abelian (LCA) groups provides the natural setting

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for such generalizations, encompassing both continuous and discrete scenarios within a unified harmonic-analytic framework.

The origins of this field trace back to foundational work in quantum mechanics during the 1930s, which established time-frequency analysis as a branch of harmonic analysis. A pivotal development was Pontryagin's duality theory [13] for locally compact groups, which enabled a consistent generalization of the Fourier transform to this broad setting. In 1946, Gabor introduced a seminal approach to signal processing, proposing that "every function" admits a decomposition into a double series of time-frequency shifts. The modern framework of time-frequency analysis is deeply connected with the theory of pseudodifferential operators and Gabor frames on quasi-lattices [2, 3, 5]. The natural function spaces for this theory are the modulation spaces, which precisely characterize those functions representable via Gabor frames. For comprehensive background, we refer to [1, 14, 21].

This paper investigates Gabor systems on locally compact abelian groups and their applications to pseudodifferential operators. Our study is motivated by both theoretical considerations [2, 3, 5, 16, 17] and practical applications in mobile communications and quantum mechanics [7, 12, 19, 20].

The work of Marius Măntoiu and his collaborators has been particularly influential in the development of pseudodifferential theory on groups; we refer the reader to [11] for the general setting of type I groups and to [10] for spectral results on LCA groups.

The main contributions of the present work are threefold. First, we develop a comprehensive theory of generalized time-frequency distributions on LCA groups. Central to our approach are the cross τ-ambiguity function and the cross τ-Wigner distribution, defined respectively by

$$\operatorname{Am}_\tau(f,\psi)(h,\chi) = \int_G f(g\tau(h))\overline{\psi(g(\tau(h))^{-1})\chi(g)}\, d\mu(g),$$

$$W_\tau(f,\psi)(g,\chi) = \int_G f(g\tau(h))\overline{\psi(g(\tau(h))^{-1})\chi(h)}\, d\mu(h),$$

for $h, g \in G$ and $\chi \in \hat{G}$, where $\tau : G \to H$ is a topological isomorphism onto an open subgroup H of G.

Second, we introduce and analyze a general class of τ-pseudodifferential operators. For a symbol $a \in S'(G \times \hat{G})$, we define the operator $L_\tau(a) : S(G) \to S'(G)$ via the duality relation

$$\langle L_\tau(a)(f), \psi\rangle = \langle a, W_\tau(f,\psi)\rangle \quad \text{for all } \psi \in S(G).$$

We analyze these operators within modulation spaces $M^{p,r}_\omega(G)$, where $1 < p, r < \infty$, establishing these spaces as the natural functional setting. We prove boundedness results for symbols in appropriate modulation spaces and develop the symbolic calculus for these operators.

Third, we establish a comprehensive theory of Gabor frames on quasi-lattices in LCA groups. We prove that under mild conditions on the window function, the associated coefficient and reconstruction operators are bounded on modulation spaces. This leads to our main result on the unconditional convergence of Gabor frame expansions in these spaces, providing a powerful machinery for the phase-space analysis of pseudodifferential operators in a general group-theoretic context.

Our work builds upon recent theoretical advances. Cordero, Giacchi, Rodino, and Valenzano [2, 3] and Cornean, Helffer, and Purice [5] have obtained deep results in this area. Song et al. [18] introduced a simplified definition of the Weyl operator lift and described its properties. Zhang and Li [26] studied the graph fractional Fourier transform in Hilbert spaces, demonstrating its extension of traditional graph signal processing to continuous domains. Z. C. Zhang [28] investigated a generalization of the classical Wigner–Ville distribution, establishing properties including an anti-derivative property and a Moyal formula. Further applications can be found in [22, 23, 25, 27, 29].

The paper is organized as follows: Section 2 reviews the short-time Fourier transform and uncertainty principles on LCA groups. Section 3 develops the theory of Gabor frames on quasi-lattices, including our main results on boundedness of coefficient operators and unconditional convergence. Section 4 introduces the τ-Wigner distribution and τ-pseudodifferential operators, establishing their fundamental properties and boundedness in modulation spaces.

2. The short-time Fourier transform. Let G be a locally compact abelian Hausdorff group and $\hat{G}$ its dual group, consisting of all continuous group homomorphisms χ from G to the circle group $\mathbb{T}$. Let μ be a Haar measure on G. We define the Fourier transform $\mathcal{F}(f) = \hat{f} : \hat{G} \to \mathbb{C}$ of a function $f : G \to \mathbb{C}$ by

$$\mathcal{F}(f)(\chi) = \hat{f}(\chi) = \int_G \overline{\chi(g)} f(g)\, d\mu(g),$$

and similarly the inverse Fourier transform $\mathcal{F}^{-1}(\psi) : G \to \mathbb{C}$ of $\psi : \hat{G} \to \mathbb{C}$ is given by

$$\mathcal{F}^{-1}(\psi)(g) = \int_{\hat{G}} \chi(g)\psi(\chi)\, d\hat{\mu}(\chi).$$

REMARK 2.1. We always normalize the Haar measures so that the Fourier transform extends to a unitary operator from $L^2(G)$ onto $L^2(\hat{G})$ (Plancherel theorem). This normalization is used throughout the paper.

Let $\phi \neq 0$ be a fixed window function. The *short-time Fourier transform* (STFT) of a function f with respect to ϕ is defined by

$$V_\phi(f)(g,\chi) = \int\limits_G f(h)\overline{\phi(hg^{-1})\chi(h)}\, d\mu(h)$$

for all $(g,\chi) \in G \times \hat{G}$.

For a fixed $h \in G$, the *translation operator* τ_h is given by $\tau_h f(g) = f(gh^{-1})$ for all $g \in G$. For a fixed $\chi \in \hat{G}$, the *modulation operator* m_χ is given by $m_\chi f(g) = \chi(g)f(g)$ for all $g \in G$. Using this notation, we can rewrite the STFT as

$$V_\phi(f)(g,\chi) = \int\limits_G f(h)\overline{m_\chi \tau_g \phi(h)}\, d\mu(h) = \langle f, m_\chi \tau_g \phi\rangle$$

for all $g \in G$ and $\chi \in \hat{G}$.

EXAMPLE 2.2 (Gaussian window on $\mathbb{R}^d$). Take $G = \mathbb{R}^d$ and let $\phi(t) = e^{-\pi|t|^2}$ be the Gaussian. For the chirp signal $f(t) = e^{i\pi|t|^2}$, a direct computation gives

$$V_\phi(f)(g,\chi) = \int\limits_{\mathbb{R}^d} e^{i\pi|t|^2} e^{-\pi|t-g|^2} e^{-2\pi i \chi\cdot t}\, dt,$$

which is essentially a Gaussian in g and χ (up to a phase factor). This example illustrates how the STFT localizes the frequency content of a non-stationary signal.

EXAMPLE 2.3 (Discrete STFT on $\mathbb{Z}$). For $G = \mathbb{Z}$ (with counting measure) and a compactly supported window ϕ, say $\phi(k) = 1$ for $|k| \leq N$ and 0 otherwise, the STFT becomes

$$V_\phi(f)(n,\chi) = \sum_{k\in\mathbb{Z}} f(k)\overline{\phi(k-n)}e^{-2\pi i\chi k},$$

which is the usual discrete short-time Fourier transform used in digital signal processing. Here $\chi \in \widehat{\mathbb{Z}} \cong \mathbb{T}$, so $e^{-2\pi i\chi k}$ is considered with $\chi \in [0,1)$.

PROPOSITION 2.4. *Assume that $V_\phi(f)$ is well-defined. Then the following identity holds for all $g, z \in G$ and $\chi, \xi \in \hat{G}$:*

$$V_\phi(\tau_z m_\xi f)(g,\chi) = \overline{\chi(z)} V_\phi(f)(gz^{-1}, \chi\xi^{-1}).$$

Proof. For $g, z \in G$ and $\chi, \xi \in \hat{G}$, we have the operator identity

$$m_{\xi^{-1}}\tau_{z^{-1}} m_\chi \tau_g = \overline{\chi(z)} m_{\chi\xi^{-1}}\tau_{gz^{-1}}.$$

Therefore,

$$\begin{aligned} V_\phi(\tau_z m_\xi f)(g,\chi) &= \langle \tau_z m_\xi f, m_\chi \tau_g \phi\rangle \\ &= \langle f, m_{\xi^{-1}}\tau_{z^{-1}} m_\chi \tau_g \phi\rangle = \overline{\chi(z)} V_\phi(f)(gz^{-1}, \chi\xi^{-1}). \ \blacksquare \end{aligned}$$

We now recall the Donoho–Stark uncertainty principle [6].

DEFINITION 2.5. Let $\epsilon > 0$. A function $f \in L^2(G)$ is said to be *ϵ-concentrated* on a measurable set $B \subseteq G$ if

$$\int_{G\setminus B} |f(h)|^2 \, d\mu(h) \le \epsilon^2 \|f\|^2_{L^2(G)}.$$

THEOREM 2.6 (Analog Donoho–Stark theorem [24]). *Let G be a locally compact abelian Hausdorff group. Let $f \in L^2(G)$, $f \neq 0$, be ϵ_B-concentrated on $B \subseteq G$, and let its Fourier transform $\hat{f}$ be ϵ_Ω-concentrated on $\Omega \subseteq \hat{G}$. Then*

$$(1 - \epsilon_B - \epsilon_\Omega)^2 \le \mu(B)\hat{\mu}(\Omega).$$

In our recent work on Plancherel theory and uncertainty principles [24], we extended this result as follows. Assume K is a maximal compact subgroup of G endowed with a probability Haar measure μ_K. Suppose $f \in L^p(G/K)$ is ϵ_B-concentrated on $B = BK \subset G$, and there exists a function $f_\Omega \in L^p(G/K)$ such that $\operatorname{supp}(\mathcal{F}(f_\Omega)) \subset \Omega$ and $\|f - f_\Omega\|_{L^p} \le \epsilon_\Omega \|f\|_{L^p}$. Then we have the estimate

$$\left(\frac{1 - \epsilon_\Omega}{1 - \epsilon_B - \epsilon_\Omega}\right)^p \mu(B)\hat{\mu}(\Omega) \ge 1.$$

DEFINITION 2.7. A function f is called *ϵ_Ω-bandlimited to* a set $\Omega \subseteq \hat{G}$ *with tolerance* ϵ_Ω if there exists a function f_Ω such that $\operatorname{supp}(\mathcal{F}(f_\Omega)) \subset \Omega$ and $\|f - f_\Omega\|_{L^p} \le \epsilon_\Omega$.

Thus, we can reformulate [24, Theorem 6] as follows.

THEOREM 2.8 ([24, Theorem 6]). *Let $f \in L^p(G/K)$, $f \neq 0$, be ϵ_B-concentrated on $B = BK \subset G$ and ϵ_Ω-bandlimited to $\Omega \subseteq \hat{G}$ with tolerance ϵ_Ω. Then*

$$\left(\frac{1 - \epsilon_\Omega}{1 - \epsilon_B - \epsilon_\Omega}\right)^p \mu(B)\hat{\mu}(\Omega) \ge 1$$

for all $p \le 2$.

Similarly, the Lieb uncertainty principle [8] states that for all $p \in [2, \infty)$ and all $f, \phi \in L^2(G)$,

$$\int_{G\times\hat{G}} |V_\phi(f)(g,\chi)|^p \, d\hat{\mu}(\chi) \, d\mu(g) \le C(p) \|f\|^p_{L^2} \|\phi\|^p_{L^2}.$$

This implies that for all time-frequency representations, a function cannot be concentrated on sets of relatively small measure.

3. Gabor frames on locally compact abelian groups. By the structure theorem for locally compact abelian groups (see, e.g., [9]), every such group G is isomorphic to a direct product $\mathbb{R}^d \times \tilde{G}$, where $\tilde{G}$ is a locally compact group containing a compact open subgroup K. Typical examples include $\mathbb{R}^d \times \mathbb{T}^m \times \mathbb{Z}^n \times F$ with F a finite abelian group.

We recall that a *lattice* Λ in a locally compact Hausdorff group G is a discrete subgroup such that the quotient space G/Λ is compact. A *fundamental domain* for a lattice Λ is a relatively compact subset $U \subset G$ such that

$$\bigcup_{\alpha\in\Lambda} \alpha U = G \quad \text{and} \quad \alpha U \cap \beta U = \emptyset \quad \text{for all } \alpha \neq \beta \text{ in } \Lambda.$$

Let A and B be two invertible, real-valued $d \times d$ matrices. A *quasi-lattice* in $G \times \hat{G}$ can be defined as

$$\Lambda = (A\mathbb{Z}^d \times E_1) \times (B\mathbb{Z}^d \times E_2)$$

with a fundamental domain

$$U = (A[0,1)^d \times K) \times (B[0,1)^d \times K^{\perp}) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} [0,1)^{2d} \times (K \times K^{\perp}),$$

where the compact subgroup $K \subset \tilde{G}$ arises from the decomposition $G \cong \mathbb{R}^d \times \tilde{G}$, and $E_1 \subset \tilde{G}$, $E_2 \subset \tilde{G}$ are complete sets of coset representatives of $\tilde{G}/K$ and $\tilde{G}/K^{\perp}$ respectively.

We denote elements of $\Lambda \subset G \times \hat{G}$ by $\theta = (z, \xi) \in \Lambda$.

Definition 3.1. Let $\Lambda \subset G \times \hat{G}$ be a quasi-lattice and $\phi \in L^2(G)$. Then:

(1) The *Gabor system* generated by ϕ is $\{m_\xi \tau_z \phi : (z, \xi) \in \Lambda\}$.

(2) The *coefficient operator* $C_\phi : L^2(G) \to \ell^2(\Lambda)$ is defined by

$$C_\phi(f) = \{\langle f, m_\xi \tau_z \phi\rangle : (z, \xi) \in \Lambda\}.$$

(3) Its *adjoint* $C_\phi^* : \ell^2(\Lambda) \to L^2(G)$ is given by

$$C_\phi^*(\{c_{z,\xi}\}) = \sum_{(z,\xi)\in\Lambda} c_{z,\xi}\, m_\xi \tau_z \phi.$$

(4) The *Gabor frame operator* $\Gamma : L^2(G) \to L^2(G)$ is $\Gamma(f) = C_\phi^* C_\phi(f)$.

(5) The Gabor system is called a *Gabor frame* for $L^2(G)$ if there exist constants $0 < c_1 \le c_2 < \infty$ such that

$$c_1 \|f\|_{L^2}^2 \le \sum_{(z,\xi)\in\Lambda} |\langle f, m_\xi \tau_z \phi\rangle|^2 \le c_2 \|f\|_{L^2}^2 \quad \text{for all } f \in L^2(G).$$

(6) A Gabor frame is called *tight* if $c_1 = c_2$.

(7) A function $\psi \in L^2(G)$ is called a *dual window* for the frame if for all $f \in L^2(G)$,

$$\sum_{(z,\xi)\in\Lambda} \langle f, m_\xi \tau_z \phi\rangle m_\xi \tau_z \psi = \sum_{(z,\xi)\in\Lambda} \langle f, m_\xi \tau_z \psi\rangle m_\xi \tau_z \phi = f.$$

We now present an explicit example of a Gabor frame with a Gaussian window. Define the Gaussian function $\gamma : G \to \mathbb{C}$ by

$$\gamma(g_1, g_2) = \exp(-\pi g_1^2)\mathbf{1}_K(g_2) \quad \text{for } (g_1, g_2) \in \mathbb{R}^d \times \widetilde{G},$$

where K is a compact subgroup.

PROPOSITION 3.2. *Let $s \in (0,1)$ be fixed and let $\Lambda = s\mathbb{Z}^{2d} \times (E_1 \times E_2)$ be a quasi-lattice in $G \times \hat{G}$ as defined above. Then the Gabor system*

$$\{m_\xi \tau_z \gamma : (z, \xi) \in \Lambda\}$$

is a Gabor frame for $L^2(G)$.

Proof. By the structure theorem for LCA groups we may write $G \cong \mathbb{R}^d \times \widetilde{G}$, where $\widetilde{G}$ contains a compact open subgroup K. The window is taken as

$$\gamma(g_1, g_2) = e^{-\pi|g_1|^2}\,\mathbf{1}_K(g_2) = (\gamma_1 \otimes \gamma_2)(g_1, g_2), \qquad g_1 \in \mathbb{R}^d,\ g_2 \in \widetilde{G}.$$

Euclidean part. On $\mathbb{R}^d$ the Gaussian $\gamma_1(x) = e^{-\pi|x|^2}$ is a classical window. It is well known (see e.g. [8, Theorem 7.5.3]) that for every $s \in (0,1)$ the Gabor system

$$\{m_{\xi_1} \tau_{z_1} \gamma_1 : z_1, \xi_1 \in s\mathbb{Z}^d\}$$

is a Gabor frame for $L^2(\mathbb{R}^d)$. Let $A, B > 0$ be its frame bounds. Moreover, there exists a dual window $\widetilde{\gamma}_1 \in L^2(\mathbb{R}^d)$ (for instance the canonical dual) such that for every $f_1 \in L^2(\mathbb{R}^d)$ we have

$$f_1 = \sum_{z_1, \xi_1 \in s\mathbb{Z}^d} \langle f_1, m_{\xi_1} \tau_{z_1} \gamma_1 \rangle m_{\xi_1} \tau_{z_1} \widetilde{\gamma}_1$$

and also the analogous expansion with the roles of γ_1 and $\widetilde{\gamma}_1$ interchanged.

Compact part. On $\widetilde{G}$ we use the window $\gamma_2 = \mathbf{1}_K$, the indicator function of the compact open subgroup K. Normalize the Haar measure on $\widetilde{G}$ so that $\mu_{\widetilde{G}}(K) = 1$. Let $E_1 \subset \widetilde{G}$ be a complete set of representatives of the discrete quotient $\widetilde{G}/K$ and $E_2 \subset \widehat{\widetilde{G}}$ a complete set of representatives of $\widehat{\widetilde{G}}/K^\perp$ (where $K^\perp$ is the annihilator of K). Then the family

$$\{m_\chi \tau_g \gamma_2 : g \in E_1,\ \chi \in E_2\}$$

is an orthonormal basis of $L^2(\widetilde{G})$. Indeed:

- The translates $\tau_g \gamma_2$ for $g \in E_1$ have disjoint supports (the cosets gK) and form an orthonormal basis for the space they span because $\mu(K) = 1$.
- For a fixed g, the functions $\chi \mapsto \chi(\cdot)\mathbf{1}_{gK}(\cdot)$ with $\chi \in E_2$ are exactly the characters of the compact group K (shifted to gK); they constitute an orthonormal basis of $L^2(gK)$.

Consequently, the whole system is an orthonormal basis, hence a tight Gabor frame with frame bound 1, and γ_2 is self-dual.

Tensor product. The full Gabor system is

$$\{m_{(\xi_1,\chi)}\tau_{(z_1,g)}(\gamma_1\otimes\gamma_2) : (z_1,g)\in s\mathbb{Z}^d\times E_1,\ (\xi_1,\chi)\in s\mathbb{Z}^d\times E_2\}.$$

Because the Hilbert space $L^2(\mathbb{R}^d\times\widetilde{G})$ is the tensor product $L^2(\mathbb{R}^d)\otimes L^2(\widetilde{G})$ and the time-frequency shifts act independently on the two factors, the frame operator factorizes as the tensor product of the individual frame operators. Since the Euclidean part has frame bounds A, B and the compact part is an orthonormal basis (frame bounds $1, 1$), the whole system is a Gabor frame for $L^2(G)$ with the same bounds A, B. Moreover, a dual window is given by $\widetilde{\gamma}_1\otimes\gamma_2$; one verifies directly on the dense set of finite linear combinations of product functions $\psi_1\otimes\psi_2$ that

$$\sum_{(z,\xi)\in\Lambda}\langle\psi_1\otimes\psi_2, m_\xi\tau_z\gamma\rangle m_\xi\tau_z(\widetilde{\gamma}_1\otimes\gamma_2)=\psi_1\otimes\psi_2,$$

and similarly with the roles of γ and $\widetilde{\gamma}_1\otimes\gamma_2$ interchanged. By density the equalities extend to all of $L^2(G)$. ∎

We recall that the *modulation space* $M^{p,r}_\omega(G)$ consists of all tempered distributions f for which the (quasi-)norm

$$\|f\|_{M^{p,r}_\omega}=\Big(\int_{\hat{G}}\Big(\int_G|V_\rho(f)(g,\chi)|^p\omega(g,\chi)^p\,d\mu(g)\Big)^{r/p}d\hat{\mu}(\chi)\Big)^{1/r}$$

is finite. Here $0<p,r\le\infty$ (with the usual interpretation for the endpoints). For $p,r\ge 1$ this is a Banach space; for $0<p,r<1$ it is a quasi-Banach space. By convention, we denote $M^1_\omega(G)=M^{1,1}_\omega(G)$. The window ρ is a fixed non-zero Schwartz function; different choices yield equivalent norms.

In what follows we work with "moderate weights". A weight $\omega: G\times\hat{G}\to(0,\infty)$ is called *moderate* if there exists a submultiplicative weight v on $G\times\hat{G}$ and a constant $C>0$ such that

$$\omega(x+y)\le Cv(x)\omega(y)\quad\text{for all } x,y\in G\times\hat{G}$$

(the group operation written additively). For such a weight we extend it to a weight $\tilde{\omega}$ on the Heisenberg group $H_G=G\times\hat{G}\times\mathbb{T}$ by $\tilde{\omega}(g,\chi,\theta)=\omega(g,\chi)$. Then $\tilde{\omega}$ is left and right moderate on H_G.

The associated Heisenberg group H_G is defined as $H_G=G\times\hat{G}\times\mathbb{T}$, equipped with the group operation

$$(g_1,\chi_1,\theta_1)(g_2,\chi_2,\theta_2)=(g_1g_2,\chi_1\chi_2,\theta_1\theta_2\chi_1(g_2))$$

for all $(g_1,\chi_1,\theta_1),(g_2,\chi_2,\theta_2)\in H_G$, where $\mathbb{T}$ is the circle group. The Schrödinger representation ρ of H_G on $L^2(G)$ is given by

$$\rho(g,\chi,\theta)f=\theta\, m_\chi\tau_g f\quad\text{for } f\in L^2(G),$$

mapping H_G into the unitary group $U(L^2(G))$.

We define the Lebesgue space $L^{p,r}_{\tilde{\omega}}(H_G)$ as the space of all measurable functions $\varPsi$ on H_G with finite norm

$$\|\varPsi\|_{L^{p,r}_{\tilde{\omega}}} = \Big(\int\limits_{\hat{G}\times\mathbb{T}} \Big(\int\limits_{G} |\varPsi(g,\chi,\theta)|^p \tilde{\omega}(g,\chi,\theta)^p \, d\mu(g) \Big)^{r/p} d\hat{\mu}(\chi)\, d\theta \Big)^{1/r}.$$

The representation coefficient can be written as

$$W_{\rho,\psi}(f)(g,\chi,\theta) = \langle f, \theta m_\chi \tau_g \psi\rangle = \bar{\theta} V_\psi(f)(g,\chi) \quad \text{for all } (g,\chi,\theta) \in H_G.$$

The Wiener amalgam space with a window P, local component $L^\infty(G)$, and global component X is defined as

$$W_P(X) = \{ f \in L^\infty_{\mathrm{loc}}(G) : M_P(f) \in X \},$$

equipped with the norm $\|f\|_{W_P(X)} = \|M_P(f)\|_X$.

Let

$$\varXi_{\tilde{\omega}} = \bigcap_{s\in(0,1]} \{ \psi \in L^2(G) : W_{\rho,\psi}(\psi) \in W(L^\infty, W(L^\infty, L^s_{\tilde{\omega}})) \}.$$

The space $W(L^\infty, W(L^\infty, L^s_{\tilde{\omega}}))$ is a triple Wiener amalgam space on the Heisenberg group $H_G = G \times \hat{G} \times \mathbb{T}$. For a function F on H_G, belonging to this space means that when localized by a bounded uniform partition of unity (with local L^∞ control), the resulting sequence of local norms lies in the global component space $W(L^\infty, L^s_{\tilde{\omega}})$ on the index set. The condition $\psi \in \varXi_{\tilde{\omega}}$ guarantees that ψ is a sufficiently nice window for coorbit space theory: it ensures that the voice transform $W_{\rho,\psi}(\psi)$ has enough decay so that the analysis operator C_ψ and the synthesis operator C^*_ψ are bounded on the full scale of weighted modulation spaces $M^{p,r}_\omega(G)$ for all $0 < p, r < \infty$. The intersection over all $s \in (0,1]$ makes the condition robust enough to handle all indices via interpolation and maximal function estimates.

We now state one of our main results.

THEOREM 3.3. *Let ω be a moderate weight on $G \times \hat{G}$ and let $\tilde{\omega}$ be its extension to H_G. Assume $\phi \in \varXi_{\tilde{\omega}}$. Then, for all $0 < p, r < \infty$, the coefficient operator $C_\phi : L^2(G) \to \ell^2(\varLambda)$ extends uniquely to a continuous linear operator*

$$C_\phi : M^{p,r}_\omega(G) \to \ell^{p,r}_{\omega(\varLambda)}(\varLambda),$$

where $\omega(\varLambda)$ denotes the restriction of ω to $\varLambda$. Moreover, for any $\delta > 0$ satisfying $0 < \delta \le \min\{p, r\}$, there exists a constant $c(\delta) > 0$, independent of p and r, such that

$$\|C_\phi\|_{M^{p,r}_\omega(G) \to \ell^{p,r}_{\omega(\varLambda)}(\varLambda)} \le c(\delta).$$

Proof. Let $\varLambda \subseteq G \times \hat{G}$ be a quasi-lattice with fundamental domain U as defined previously. Choose open, relatively compact unit neighborhoods $A \subset B_G \times B_{\hat{G}}$ in $G \times \hat{G}$. Let $\{\alpha_z \otimes \beta_\xi : (z,\xi) \in \varLambda\}$ be a $B_G \times B_{\hat{G}}$-bounded uniform partition of unity such that $\alpha_z \otimes \beta_\xi \equiv 1$ on $(z,\xi)A$. Then $\{\alpha_z \otimes \beta_\xi \otimes \mathbf{1}_{\mathbb{T}}\}$ is a $B_G \times B_{\hat{G}} \times \mathbb{T}$-bounded uniform partition of unity on H_G, localized at $\Lambda \times \{1\}$, with

$$(\alpha_z \otimes \beta_\xi \otimes \mathbf{1}_{\mathbb{T}})(z,\xi,e) = 1 \quad \text{for all } (z,\xi) \in \Lambda.$$

We estimate

$$\begin{aligned}|\langle f, m_\xi \tau_z \phi\rangle| &= |V_\phi(f)(z,\xi)| = |(\alpha_z \otimes \beta_\xi \otimes \mathbf{1}_{\mathbb{T}})(z,\xi,e) \cdot W_{\rho,\phi}(f)(z,\xi,e)| \\ &\le \|(\alpha_z \otimes \beta_\xi \otimes \mathbf{1}_{\mathbb{T}}) \cdot W_{\rho,\phi}(f)\|_{L^\infty}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|C_\phi(f)\|_{\ell^{p,r}_\omega(\Lambda)} &= \|\{\langle f, m_\xi \tau_z \phi\rangle\}\|_{\ell^{p,r}_\omega(\Lambda)} \\ &\le \|\{\|(\alpha_z \otimes \beta_\xi \otimes \mathbf{1}_{\mathbb{T}}) \cdot W_{\rho,\phi}(f)\|_{L^\infty}\}\|_{\ell^{p,r}_\omega(\Lambda)}.\end{aligned}$$

One can show that the discrete space $(L^{p,r}_{\tilde{\omega}}(H_G))_d(\Lambda \times \{1\}, B_G \times B_{\hat{G}})$ is equivalent to $\ell^{p,r}_\omega(I \times J)$, where $\omega(X) : (i,j) \mapsto \omega(z_i, \xi_j)$. Hence,

$$\begin{aligned}&\|\{\|(\alpha_z \otimes \beta_\xi \otimes \mathbf{1}_{\mathbb{T}}) \cdot W_{\rho,\phi}(f)\|_{L^\infty}\}\|_{\ell^{p,r}_\omega(\Lambda)} \\ &\qquad \approx \|\{\|(\alpha_z \otimes \beta_\xi \otimes \mathbf{1}_{\mathbb{T}}) \cdot W_{\rho,\phi}(f)\|_{L^\infty}\}\|_{(L^{p,r}_{\tilde{\omega}})_d(\Lambda\times\{1\}, B_G\times B_{\hat{G}})} \\ &\qquad \approx \|W_{\rho,\phi}(f)\|_{W(L^{p,r}_{\tilde{\omega}})} \approx \|f\|_{M^{p,r}_\omega}.\end{aligned}$$

The detailed steps involving the maximal function and the properties of the partition of unity follow standard arguments in coorbit theory, as in [21]. ∎

THEOREM 3.4. *Let ω be a moderate weight on $G \times \hat{G}$ and let $\tilde{\omega}$ be its extension to H_G. Assume $\phi \in \Xi_{\tilde{\omega}}$. Then, for all $0 < p, r < \infty$, the adjoint operator $C^*_\phi : \ell^2(\Lambda) \to L^2(G)$ extends uniquely to a continuous linear operator*

$$C^*_\phi : \ell^{p,r}_{\omega(\Lambda)}(\Lambda) \to M^{p,r}_\omega(G).$$

Moreover, for any $\delta > 0$ with $0 < \delta \le \min\{p, r\}$, there exists a constant $c(\delta) > 0$, independent of p and r, such that

$$\|C^*_\phi\|_{\ell^{p,r}_{\omega(\Lambda)} \to M^{p,r}_\omega(G)} \le c(\delta).$$

Proof. The proof has a similar structure to that of Theorem 3.3, utilizing the properties of the partition of unity and the maximal function to control the norm of the synthesized signal in the modulation space. The condition $\phi \in \Xi_{\tilde{\omega}}$ ensures the necessary decay of the representation coefficient, allowing the application of coorbit theory results [21]. The detailed estimates are omitted for brevity. ∎

We summarize the previous two theorems in the following reconstruction result.

THEOREM 3.5. *Let ω be a moderate weight on $G \times \hat{G}$ and let $\tilde{\omega}$ be its extension to H_G. For all $0 < p, r < \infty$, assume that $\phi \in \Xi_{\tilde{\omega}}$ and that there* *exists a quasi-lattice $\Lambda \subset G \times \hat{G}$ such that the Gabor frame operator satisfies*

$$\Gamma_{\phi,\gamma} = \Gamma_{\gamma,\phi} = I_{L^2},$$

where γ is the Gaussian window introduced in Proposition 3.2 (*or any other window for which the dual system forms a frame*). *Then, for any $f \in M^{p,r}_\omega(G)$, we have the unconditionally convergent expansions*

$$f = \sum_{(z,\xi)\in\Lambda} \langle f, m_\xi \tau_z \gamma\rangle m_\xi \tau_z \phi = \sum_{(z,\xi)\in\Lambda} \langle f, m_\xi \tau_z \phi\rangle m_\xi \tau_z \gamma$$

in the norm of $M^{p,r}_\omega(G)$.

Proof. We first recall that by Theorems 3.3 and 3.4, the coefficient operator $C_\phi : M^{p,r}_\omega(G) \to \ell^{p,r}_\omega(\Lambda)$ and the synthesis operator $C^*_\gamma : \ell^{p,r}_\omega(\Lambda) \to M^{p,r}_\omega(G)$ are continuous linear extensions of the corresponding operators defined on $L^2(G)$. The hypothesis $\Gamma_{\phi,\gamma} = \Gamma_{\gamma,\phi} = I_{L^2}$ means that on $L^2(G)$ we have

$$C^*_\gamma C_\phi = I_{L^2}, \quad C^*_\phi C_\gamma = I_{L^2}.$$

Both compositions are bounded operators on $M^{p,r}_\omega(G)$ (as compositions of bounded operators) and coincide with the identity on a dense subspace, e.g. on $L^2(G) \cap M^{p,r}_\omega(G)$ (which contains the Schwartz space $\mathcal{S}(G)$). Hence they extend to the identity on the whole space $M^{p,r}_\omega(G)$. Consequently, for every $f \in M^{p,r}_\omega(G)$ we have

$$f = C^*_\gamma(C_\phi f) = \sum_{(z,\xi)\in\Lambda} \langle f, m_\xi \tau_z \phi\rangle m_\xi \tau_z \gamma,$$

where the series is understood to be the image under C^*_γ of the sequence $\{\langle f, m_\xi \tau_z \phi\rangle\}_{(z,\xi)\in\Lambda} \in \ell^{p,r}_\omega(\Lambda)$.

To see that the series converges unconditionally in the norm of $M^{p,r}_\omega(G)$, observe that $\ell^{p,r}_\omega(\Lambda)$ is a Banach space whose canonical basis $\{e_{z,\xi}\}_{(z,\xi)\in\Lambda}$ (where $e_{z,\xi}$ is the sequence with 1 at (z,ξ) and 0 elsewhere) is an unconditional basis. Hence every element $c \in \ell^{p,r}_\omega(\Lambda)$ can be written as $c = \sum_{(z,\xi)} c_{z,\xi} e_{z,\xi}$ with unconditional convergence. Applying the continuous linear operator C^*_γ yields

$$C^*_\gamma c = \sum_{(z,\xi)} c_{z,\xi} m_\xi \tau_z \gamma,$$

and this series also converges unconditionally because bounded operators preserve unconditional convergence. The same argument applied to the second representation gives

$$f = C^*_\phi(C_\gamma f) = \sum_{(z,\xi)\in\Lambda} \langle f, m_\xi \tau_z \gamma\rangle m_\xi \tau_z \phi,$$

with unconditional convergence in $M^{p,r}_\omega(G)$. ■

REMARK 3.6. The theorem applies, for example, to $G = \mathbb{R}^d$ with the standard lattice $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ and the Gaussian window γ. In that case, γ is the Gaussian, and ϕ can be taken as a dual window (e.g., the canonical dual obtained via the frame operator). The expansions then provide atomic decompositions for all functions in modulation spaces.

4. Wigner's approach to pseudodifferential operators. The Gabor frame theory developed in Section 3 gives us a powerful tool to analyze functions in modulation spaces via discrete expansions. In this section we introduce τ-pseudodifferential operators and show how they can be studied using these expansions. The connection is made explicit through the τ-Wigner distribution, which, when combined with the boundedness results of Theorems 3.3 and 3.4, allows us to transfer properties of the symbol to the operator.

Let H be an open subgroup of a locally compact abelian Hausdorff group G, and let $\tau : G \to H$ be a topological isomorphism. We assume the Haar measure on H is induced by that on G. Since τ is a topological isomorphism, there exists a constant $c > 0$ such that

$$\int_G f(g)\,d\mu(g) = c\int_G f(\tau^{-1}(g))\,d\mu(g) \quad \text{for all } f \in L^1(G).$$

Indeed, define the linear functional $J_\mu : C_c(H) \to \mathbb{C}$ by $J_\mu(f) = I_\mu(f \circ \tau)$, where $I_\mu(f) = \int_G f(g)\,d\mu(g)$. This functional is translation-invariant, and since the Haar integral is unique up to a positive constant, we conclude that

$$\int_G |f(\tau(g))|\,d\mu(g) \le \tilde{c}\int_G |f(g)|\,d\mu(g)$$

for some $\tilde{c} > 0$ and all $f \in L^1(G)$.

DEFINITION 4.1. The *cross τ-ambiguity function* Am_τ is defined by

$$\mathrm{Am}_\tau(f,\psi)(h,\chi) = \int_G f(g\tau(h))\overline{\psi(g(\tau(h))^{-1})\chi(g)}\,d\mu(g)$$

for all $h \in G$, $\chi \in \hat{G}$. The *cross τ-Wigner distribution* W_τ is defined by

$$W_\tau(f,\psi)(g,\chi) = \int_G f(g\tau(h))\overline{\psi(g(\tau(h))^{-1})\chi(h)}\,d\mu(h)$$

for all $g \in G$, $\chi \in \hat{G}$.

These distributions enjoy the following properties:

- For any $f, \psi \in L^2(G)$, the mappings $\mathrm{Am}_\tau(f,\psi)$ and $W_\tau(f,\psi)$ are continuous on $G \times \hat{G}$.
- The bilinear mappings $(f,\psi) \mapsto \mathrm{Am}_\tau(f,\psi)$ and $(f,\psi) \mapsto W_\tau(f,\psi)$ from $L^2(G) \times L^2(G)$ to $C(G \times \hat{G})$ equipped with the L^∞-norm are continuous.

- The identity
$$W_\tau(f,\psi) = \mathcal{F}(P(\mathrm{Am}_\tau(f,\psi)))$$
holds for all $f,\psi \in L^2(G)$, where P denotes an appropriate permutation of variables and $\mathcal{F}$ denotes the Fourier transform.
- Marginal densities: For all $f, \hat{f} \in L^1(G) \cap L^2(G)$,
$$\int_G W_\tau(f,f)(g,\chi)\, d\hat{\mu}(\chi) = |f(g)|^2,$$
$$\int_G W_\tau(f,f)(g,\chi)\, d\mu(g) = |\hat{f}(\chi)|^2,$$
$$\int_G \int_G W_\tau(f,f)(g,\chi)\, d\hat{\mu}(\chi)\, d\mu(g) = \|f\|_{L^2}^2.$$

Example 4.2 (τ-Wigner on $\mathbb{R}$). Take $G = \mathbb{R}$, $H = \mathbb{R}$, and $\tau(x) = x$ (identity). Then for $f,\psi \in L^2(\mathbb{R})$,
$$W_\tau(f,\psi)(g,\chi) = \int_{\mathbb{R}} f(g+h)\overline{\psi(g-h)}e^{-2\pi i \chi h}\, dh,$$
which is the usual cross-Wigner distribution. For $\tau(x) = 2x$, one obtains a scaled version often used in the study of pseudodifferential operators with different quantizations.

Definition 4.3. Let a be a function or tempered distribution on $G \times \hat{G}$. The Kohn–Nirenberg type pseudodifferential operator $A(a) : \mathcal{S}_0(G) \to \mathcal{S}_0'(G)$ is defined by
$$A(a)(f)(g) = \int_G \chi(g)a(g,\chi)\hat{f}(\chi)\, d\hat{\mu}(\chi) \quad \text{for all } g \in G.$$
Straightforward calculation shows that this operator can also be expressed as
$$A(a)(f)(g) = \int_{G\times\hat{G}} \hat{a}(\chi,h)\, m_\chi \tau_{h^{-1}} f(g)\, d\mu(h)\, d\hat{\mu}(\chi),$$
where $\hat{a}$ denotes the Fourier transform of a.

Theorem 4.4. *Let $\hat{a}, \hat{b} \in L^1(\hat{G} \times G)$. Then*
$$A(a) \circ A(b) = A(\mathcal{F}^{-1}(\hat{a} \mathbin{\hat{\otimes}} \hat{b})),$$
where the twisted convolution $\hat{\otimes}$ is defined by
$$(\hat{a} \mathbin{\hat{\otimes}} \hat{b})(\chi,g) = \int_{G\times\hat{G}} \hat{a}(\xi,h)\hat{b}(\chi\xi^{-1}, gh^{-1})\chi(h)\overline{\xi(h)}\, d\mu(h)\, d\hat{\mu}(\xi).$$

Proof. By the Fubini theorem, we compute

$$
\begin{aligned}
&A(a) \circ A(b)(f) \\
&\quad = \int_{G\times\hat{G}} \int_{G\times\hat{G}} \hat{a}(\xi,h) m_\xi \tau_{h^{-1}} \hat{b}(\chi,g) m_\chi \tau_{g^{-1}} f \, d\mu(g)\, d\hat{\mu}(\xi)\, d\mu(h)\, d\hat{\mu}(\chi) \\
&\quad = \int_{G\times\hat{G}} \int_{G\times\hat{G}} \hat{a}(\xi,h) \hat{b}(\chi,g) \chi(h) m_\xi \tau_{h^{-1}g^{-1}} f \, d\mu(g)\, d\hat{\mu}(\xi)\, d\mu(h)\, d\hat{\mu}(\chi) \\
&\quad = \int_{G\times\hat{G}} \Big(\int_{G\times\hat{G}} \hat{a}(\xi,h) \hat{b}(\chi\xi^{-1}, gh^{-1}) \chi(h) \overline{\xi(h)}\, d\mu(h)\, d\hat{\mu}(\xi) \Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad m_\chi \tau_{g^{-1}} f \, d\mu(g)\, d\hat{\mu}(\chi) \\
&\quad = \int_{G\times\hat{G}} (\hat{a} \mathbin{\hat{\otimes}} \hat{b})(\chi,g)\, m_\chi \tau_{g^{-1}} f \, d\mu(g)\, d\hat{\mu}(\chi) \\
&\quad = A(\mathcal{F}^{-1}(\hat{a} \mathbin{\hat{\otimes}} \hat{b}))(f). \ \blacksquare
\end{aligned}
$$

We now define a τ-dependent pseudodifferential operator $L_\tau(a)$ by

$$L_\tau(a)(f) = \int_{G\times\hat{G}} \overline{\chi(\tau(h))} \hat{a}(\chi,h) \tau_{h^{-1}} m_\chi f \, d\mu(h)\, d\hat{\mu}(\chi),$$

where the Fourier transform $\hat{a}$ of the symbol a is given by

$$\hat{a}(\chi,h) = \int_{G\times\hat{G}} \overline{\chi(g)} \xi(h) a(g,\xi)\, d\mu(g)\, d\hat{\mu}(\xi).$$

PROPOSITION 4.5. *Let a be a function or tempered distribution on $G\times\hat{G}$. Then the pseudodifferential operator $L_\tau(a)$ satisfies*

$$\langle L_\tau(a)(f), \psi\rangle = \langle a, W_\tau(f,\psi)\rangle \quad \textit{for all } f,\psi \in \mathcal{S}_0(G),$$

where W_τ is the cross τ-Wigner distribution.

Proof. Since the Fourier transform can be extended to tempered distributions, using the relation $m_\chi \tau_h = \chi(h) \tau_h m_\chi$ we see that

$$
\begin{aligned}
\langle L_\tau(a)(f), \psi\rangle &= \langle T^{-1}(\tau) \mathcal{F}_2^{-1}(a), \psi \otimes \overline{f}\rangle \\
&= \langle a, \mathcal{F}_2 T(\tau)(\psi \otimes \overline{f})\rangle \\
&= \langle a, W_\tau(f,\psi)\rangle,
\end{aligned}
$$

where $\mathcal{F}_2$ denotes the partial Fourier transform in the second variable, and the transformation $T(\tau)$ is defined by

$$T(\tau)\Psi(g,h) = \Psi(g\tau(h), g(\tau(h))^{-1}). \ \blacksquare$$

Definition 4.6. Let a be a function or tempered distribution on $G \times \hat{G}$. The *τ-pseudodifferential operator* $L_\tau(a)$ *with symbol* a is defined by the duality

$$\langle L_\tau(a)(f), \psi\rangle = \langle a, W_\tau(f,\psi)\rangle \quad \text{for all } f,\psi \in \mathcal{S}_0(G).$$

For $a \in S'(G \times \hat{G})$ and $f,\psi \in \mathcal{S}_0(G)$, we also have the relation

$$\begin{aligned}\langle L_\tau(a)(f), \psi\rangle &= \int\limits_{G\times\hat{G}} \hat{a}(\chi,h)\overline{\chi(\tau(h))}\langle \tau_{h^{-1}} m_\chi f, \psi\rangle \, d\mu(h) d\hat{\mu}(\chi) \\ &= \langle \hat{a}, \overline{\mathrm{Am}_\tau(f,\psi)}\rangle.\end{aligned}$$

This shows $L_\tau(a)$ is a well-defined continuous operator from $S(G)$ to $S'(G)$.

Proposition 4.7. *Let a be a function or tempered distribution on $G \times \hat{G}$. Then*

$$\langle L_\tau(a)(f), \psi\rangle_{L^2(G)} = \langle K(a), \psi \otimes \overline{f}\rangle_{L^2(G\times G)} \quad \textit{for all } f,\psi \in \mathcal{S}_0(G),$$

where the kernel $K(a)$ is defined by

$$K(g,h) = \int\limits_{G} a(g,\chi)\overline{\chi(hg^{-1})}\, d\hat{\mu}(\chi) \quad \textit{for all } g,h \in G.$$

Recall that a weight ω is called moderate with respect to a submultiplicative weight on $G \times \hat{G}$, and the extended weight $\tilde{\omega} : H_G \to (0,\infty)$ defined by $\tilde{\omega}(g,\chi,\theta) = \omega(g,\chi)$ is left and right moderate on the Heisenberg group H_G. Using the properties of Am_τ and W_τ, we obtain the following boundedness result.

Theorem 4.8. *Let $a \in M^{\infty,1}(G \times \hat{G})$. Then the operator $L_\tau(a)$ is bounded on $M^{p,r}_{\omega}(G)$ for all $1 < p, r < \infty$ and for arbitrary moderate weights $\tilde{\omega}$.*

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Mykola Yaremenko
Kyiv, Ukraine
E-mail: math.kiev@gmail.com