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Some applications of Sargsyan's equiconsistency method

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Abstract. We apply techniques due to Sargsyan to reduce the consistency strength of the assumptions used to establish an indestructibility theorem for supercompactness. We then show how these and additional techniques due to Sargsyan may be employed to establish an equiconsistency for a related indestructibility theorem for strongness.

1. Introduction and preliminaries. We begin with the following definitions.

Definition 1.1 (Stewart Baldwin [7]). κ is 0-*hyperstrong* iff κ is strong. For $\alpha > 0$, κ is α-*hyperstrong* iff for any ordinal $\delta > \kappa$, there is an elementary embedding $j : V \to M$ witnessing the δ-strongness of κ (i.e., $\mathrm{cp}(j) = \kappa$, $j(\kappa) > |V_\delta|$, and $V_\delta \subseteq M$) generated by a (κ, λ)-extender for some ordinal λ such that $M \vDash$ "κ is β-hyperstrong for every $\beta < \alpha$". Finally, κ is *hyperstrong* iff κ is α-hyperstrong for every ordinal α.

Note that in [7], Baldwin constructed a canonical inner model for a hyperstrong cardinal. The author and Sargsyan showed in [5, Theorem 2] that ZFC + There exists a Woodin cardinal $\vdash$ Con(ZFC + There exists a proper class of hyperstrong cardinals) and also used the notion of hyperstrong cardinal in [5] to establish an equiconsistency for a weak form of universal indestructibility (see [5] and [4] for the relevant terminology).

The next definition is the obvious generalization of Definition 1.1 to supercompactness. It will play a key role in the proof of Theorem 2.

Definition 1.2. κ is 0-*hypercompact* iff κ is supercompact. For $\alpha > 0$, κ is α-*hypercompact* iff for any cardinal $\delta \ge \kappa$, there is an elementary embedding $j : V \to M$ witnessing the δ-supercompactness of κ (i.e., $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \delta$, and $M^\delta \subseteq M$) generated by a supercompact ultrafilter over $P_\kappa(\delta)$

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such that $M \vDash$ "κ is β-hypercompact for every $\beta < \alpha$". Finally, κ is *hypercompact* iff κ is α-hypercompact for every ordinal α.

Observe that hyperstrong and hypercompact cardinals are quite large in size with respect to strong and supercompact cardinals respectively. In particular, Definitions 1.1 and 1.2 imply that a hyperstrong cardinal is a strong limit of strong cardinals, and a hypercompact cardinal is a supercompact limit of supercompact cardinals.

We continue with the main narrative. In [1], the following theorem was proven, where indestructibility is as in Laver's sense of [13].

Theorem 1 ([1, Theorem 3]). *Let $V \vDash$ "ZFC + GCH + κ is almost huge". There is then a cardinal $\lambda > \kappa$ and a partial ordering $\mathbb{P} \in V_\lambda$ with $|\mathbb{P}| = \kappa$ such that in $V_\lambda^{\mathbb{P}} \vDash$ ZFC, the following hold:*

1. *κ is a supercompact limit of supercompact cardinals.*
2. *The strongly compact and supercompact cardinals coincide except at measurable limit points.*
3. *Every supercompact cardinal δ is indestructible under δ-directed closed forcing.*
4. *Every nonsupercompact strongly compact cardinal δ has both its strong compactness and degree of supercompactness indestructible under δ-directed closed forcing.*

The assumption of an almost huge cardinal used in the proof of Theorem 1 is of course rather strong. Thus, one may ask if it is possible to prove Theorem 1 from weaker hypotheses.

The first goal of this paper is to show that this is indeed the case. We begin by establishing the following result, whose conclusion is identical to that of Theorem 1.

Theorem 2. *Let $V \vDash$ "ZFC + GCH + κ is a hypercompact cardinal". There is then a partial ordering $\mathbb{P} \in V$ with $|\mathbb{P}| = \kappa$ such that in $V^{\mathbb{P}}$, the following hold:*

1. *κ is a supercompact limit of supercompact cardinals.*
2. *The strongly compact and supercompact cardinals coincide except at measurable limit points.*
3. *Every supercompact cardinal δ is indestructible under δ-directed closed forcing.*
4. *Every nonsupercompact strongly compact cardinal δ has both its strong compactness and degree of supercompactness indestructible under δ-directed closed forcing.*

Note that by [3, Theorem 5], ZFC + GCH + There exists an almost huge cardinal $\vdash$ Con(ZFC + GCH + There exists a proper class of hypercompact cardinals which are limits of hypercompact cardinals). This means that the hypotheses used to prove Theorem 2 represent a bona fide reduction in consistency strength from those used to prove Theorem 1 ([1]). Such a weakening of hypotheses was unattainable prior to the introduction of Sargsyan's techniques in [5].

As our methods will show, the proof of Theorem 2 actually yields the following theorem.

THEOREM 3. *Let $V \models$ "ZFC + GCH + κ is a hypercompact cardinal". There is then a partial ordering $\mathbb{P} \in V$ with $|\mathbb{P}| = \kappa$ such that in $V^{\mathbb{P}}$, the following hold:*

1. *κ is a supercompact limit of supercompact cardinals.*
2. *The strongly compact and supercompact cardinals coincide except at measurable limit points.*
3. *Every supercompact cardinal δ is indestructible under δ-directed closed forcing.*
4. *Every measurable limit of supercompact cardinals δ has its degree of supercompactness indestructible under δ-directed closed forcing* ([2]).

Thus, in the spirit of the equiconsistency proven in [5], one can ask the following

QUESTION. Is it possible to establish an equiconsistency if Theorem 3 is recast in terms of strongness?

The second goal of this paper is to provide an affirmative answer to the Question. Specifically, we also establish the following result (which follows from Theorems 4 and 5, to be stated and proved in Section 2), where for a cardinal δ exhibiting a nontrivial degree of strongness, *weak indestructibility* means indestructibility of δ's degree of strongness under partial orderings which are both $<\delta$-strategically closed and (δ, ∞)-distributive. Once again, prior to the introduction of Sargsyan's methods of [5], establishing this sort of equiconsistency would have been impossible.

THEOREM 6. *The theories "ZFC + There is a hyperstrong cardinal" and "ZFC + T_1", where T_1 is the theory composed of the statements "There is a*

([1]) Note that [3, Theorem 5] actually shows that for the notion of *enhanced supercompact cardinal* defined in [3], ZFC + GCH + There exists an almost huge cardinal $\vdash$ Con(ZFC + GCH + There exists a proper class of enhanced supercompact cardinals which are limits of enhanced supercompact cardinals). However, since any enhanced supercompact cardinal must be hypercompact, the desired reduction in consistency strength follows.

([2]) Since by Menas [15], any measurable limit of strongly compact cardinals is in fact strongly compact, every measurable limit of supercompact cardinals δ actually also has its strong compactness indestructible under δ-directed closed forcing.

strong limit of strong cardinals", "Every strong cardinal has its strongness weakly indestructible", and "Every measurable limit of strong cardinals has its degree of strongness weakly indestructible" are equiconsistent.

We conclude Section 1 with some definitions and terminology which will be found throughout the course of the paper. We use standard interval notation for intervals of ordinals. When forcing, $q \geq p$ means that *q is stronger than p*. If $\mathbb{P} \in V$ is a partial ordering and $G \subseteq V$ is V-generic over $\mathbb{P}$, then we will abuse notation somewhat and use $V[G]$ and $V^{\mathbb{P}}$ interchangeably to denote the generic extension. We also abuse notation slightly by occasionally writing x when we mean $\dot{x}$ or $\check{x}$, especially for ground model objects and variants of the generic object. If $\mathbb{P}$ is a partial ordering and κ is a cardinal, $\mathbb{P}$ is *κ-directed closed* if for every cardinal $\delta < \kappa$ and every directed set $\langle p_\alpha : \alpha < \delta \rangle$ of elements of $\mathbb{P}$, there is an upper bound $p \in \mathbb{P}$. $\mathbb{P}$ is *κ-strategically closed* if in the two-person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha \leq \kappa \rangle$, where player I plays odd stages and player II plays even stages (choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued. $\mathbb{P}$ is *$\prec\kappa$-strategically closed* if in the two-person game in which the players construct an increasing sequence $\langle p_\alpha : \alpha < \kappa \rangle$, where player I plays odd stages and player II plays even stages (choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued. $\mathbb{P}$ is *$<\kappa$-strategically closed* if $\mathbb{P}$ is δ-strategically closed for every cardinal $\delta < \kappa$. Finally, $\mathbb{P}$ is *(κ, ∞)-distributive* if for every sequence $\langle D_\alpha : \alpha < \kappa \rangle$ of dense open subsets of $\mathbb{P}$, $\bigcap_{\alpha<\kappa} D_\alpha$ is also a dense open subset of $\mathbb{P}$. Note that since forcing with a partial ordering which is (κ, ∞)-distributive adds no new subsets of κ, the measurability of any measurable cardinal κ (or equivalently, its $\kappa+1$-strongness) is automatically indestructible under such partial orderings.

2. The proofs of Theorems 2 and 6. We turn now to the proofs of Theorems 2 and 6 and a brief discussion as to why the proof of Theorem 2 also yields a proof of Theorem 3. Key to the proofs of these theorems are techniques developed by Sargsyan, which were used to prove the main theorem (Theorem 1) of [5].

Proof of Theorem 2. Let $V \vDash$ "ZFC + GCH + κ is a hypercompact cardinal". Without loss of generality, by truncating the universe if necessary, we assume in addition that $V \vDash$ "No cardinal $\delta > \kappa$ is measurable".

We start by giving a very slight variant of the definition of the partial ordering $\mathbb{P}$ used in the proof of [1, Theorem 3], quoting verbatim from that article when appropriate. Suppose $\gamma < \delta < \kappa$ are such that γ is regular and δ is supercompact. $\mathbb{P}_{\gamma,\delta}$ is defined to be a modification of Laver's indestructibility partial ordering of [13]. More specifically, $\mathbb{P}_{\gamma,\delta}$ is an Easton support iteration of length δ defined in the style of [13] with the following properties:

1. Every stage at which a nontrivial forcing is done is a ground model measurable cardinal.
2. The least stage at which a nontrivial forcing is done can be chosen to be an arbitrarily large measurable cardinal in (γ, δ).
3. At a stage α when a nontrivial forcing $\mathbb{Q}$ is done, $\mathbb{Q} = \mathbb{Q}^0 * \dot{\mathbb{Q}}^1$, where $\mathbb{Q}^0$ is α-directed closed, and $\dot{\mathbb{Q}}^1$ is a term for the forcing adding a nonreflecting stationary set of ordinals of cofinality γ to some cardinal $\beta > \alpha$.

By its definition, $\mathbb{P}_{\gamma,\delta}$ is a γ-directed closed partial ordering of rank $\delta + 1$ with $|\mathbb{P}_{\gamma,\delta}| = \delta$. By [6, Lemma 13, pp. 2028–2029] (see also the proof of the Theorem of [2]), $V^{\mathbb{P}_{\gamma,\delta}} \models$ "There are no strongly compact cardinals in the interval (γ, δ) since unboundedly many cardinals in (γ, δ) contain nonreflecting stationary sets of ordinals of cofinality γ + δ is an indestructible supercompact cardinal". This has as a consequence that $V^{\mathbb{P}_{\gamma,\delta}} \models$ "Any partial ordering not adding bounded subsets to δ preserves that there are no strongly compact cardinals in the interval (γ, δ)".

Let $\langle \delta_\alpha : \alpha < \kappa \rangle$ enumerate the V-supercompact cardinals below κ together with their measurable limits. We define now an Easton support iteration $\mathbb{P} = \langle \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle : \alpha < \kappa \rangle$ of length κ as follows:

1. $\mathbb{P}_1 = \mathbb{P}_0 * \dot{\mathbb{Q}}_0$, where $\mathbb{P}_0$ is the partial ordering for adding a Cohen subset to ω, and $\dot{\mathbb{Q}}_0$ is a term for $\mathbb{P}_{\aleph_2,\delta_0}$.
2. If δ_α is a measurable limit of supercompact cardinals and $\Vdash_{\mathbb{P}_\alpha}$ "There is a δ_α-directed closed partial ordering such that after forcing with it, δ_α is not ζ-supercompact for ζ minimal", then $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is a term for such a partial ordering of minimal rank which destroys the ζ-supercompactness of δ_α.
3. If δ_α is a measurable limit of supercompact cardinals and case 2 above does not hold (which will mean that $\Vdash_{\mathbb{P}_\alpha}$ "δ_α is a measurable limit of supercompact cardinals whose degree of supercompactness is indestructible under δ_α-directed closed forcing and whose strong compactness is also indestructible under δ_α-directed closed forcing"), then $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is a term for the trivial partial ordering $\{\emptyset\}$.
4. If δ_α is not a measurable limit of supercompact cardinals, $\alpha = \beta + 1$, δ_β is a measurable limit of supercompact cardinals, and case 2 above holds for δ_β, then inductively, since a direct limit must be taken at stage β, $|\mathbb{P}_\beta| = \delta_\beta < \delta_{\beta+1} = \delta_\alpha$. This means inductively $\mathbb{P}_\beta$ has been defined so as to have rank less than δ_α, so by [1, Lemma 3.1] and the succeeding remark, $\dot{\mathbb{Q}}_\beta$ can be chosen to have rank less than δ_α. Also, by [1, Lemma 3.1] and the succeeding remark, $\zeta < \delta_\alpha$ for ζ the least

such that $V^{\mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta} = V^{\mathbb{P}_\alpha} \models$ "δ_β is not ζ-supercompact". (Note that [1, Lemma 3.1] and the succeeding remark say that if $\mathbb{P}^*$ is a partial ordering, κ^* is supercompact, $|\mathbb{P}^*| < \kappa^*$, and $\dot{\mathbb{Q}}$ and γ are such that $\Vdash_{\mathbb{P}^* * \dot{\mathbb{Q}}}$ "κ^* is not γ-supercompact", then $\dot{\mathbb{Q}}$ and γ can be chosen so that the rank of $\dot{\mathbb{Q}}$ is below κ^* and $\gamma < \kappa^*$.) Let $\dot{\gamma}_\alpha$ be such that $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\gamma}_\alpha = \delta_\beta^+$", and let $\sigma \in (\delta_\beta, \delta_\alpha)$ be the least measurable cardinal (in either V or $V^{\mathbb{P}_\alpha}$) such that $\Vdash_{\mathbb{P}_\alpha}$ "$\sigma > \max(\dot{\gamma}_\alpha, \dot{\zeta}, \mathrm{rank}(\dot{\mathbb{Q}}_\beta))$". Then $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is a term for $\mathbb{P}_{\gamma_\alpha, \delta_\alpha}$ defined so that σ is below the least stage at which, in the definition of $\mathbb{P}_{\gamma_\alpha,\delta_\alpha}$, a nontrivial forcing is done.

5. If δ_α is not a measurable limit of supercompact cardinals and case 4 does not hold, then $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where for $\gamma_\alpha = (\bigcup_{\beta<\alpha} \delta_\beta)^+$, $\dot{\mathbb{Q}}_\alpha$ is a term for $\mathbb{P}_{\gamma_\alpha,\delta_\alpha}$.

We observe that [1, Lemma 3.1] and the succeeding remark remain true if "supercompact" is replaced by "strong". To see this, we use the notation found in case 4 above. Assume that $\Vdash_{\mathbb{P}^* * \dot{\mathbb{Q}}}$ "κ^* is not γ-strong". Let λ be sufficiently large with $j : V \to M$ an elementary embedding witnessing the λ-strongness of κ^* such that in M, $\Vdash_{\mathbb{P}^* * \dot{\mathbb{Q}}}$ "κ^* is not γ-strong". By reflection, since cp(j) $= \kappa^*$ and $|\mathbb{P}^*| < \kappa^*$, there must be $\mathbb{Q}^*$ having rank below κ^* and $\gamma^* < \kappa^*$ such that $\Vdash_{\mathbb{P}^* * \dot{\mathbb{Q}}^*}$ "κ^* is not γ^*-strong".

The arguments of [1, Lemmas 4.1–4.2] and the remark immediately following the proof of Lemma 4.1 literally unchanged show that in $V^{\mathbb{P}}$, the following hold:

1. κ is a limit of supercompact cardinals.
2. The strongly compact and supercompact cardinals below κ coincide except at measurable limit points.
3. Every supercompact cardinal $\delta < \kappa$ is indestructible under δ-directed closed forcing.
4. Every nonsupercompact strongly compact cardinal $\delta < \kappa$ has both its strong compactness and degree of supercompactness indestructible under δ-directed closed forcing.

LEMMA 2.1. *$V^{\mathbb{P}} \models$ "κ is an indestructible supercompact cardinal".*

Proof. We use ideas found in the proof of [5, Lemma 1.4]. We proceed inductively, taking as our inductive hypothesis that if $\alpha \geq 0$ is an ordinal and $N \subseteq V$ is such that either $N = V$ or for some $\lambda \geq \kappa$, N is the transitive collapse of $V^{P_\kappa(\lambda)}/\mathcal{U}$ for some supercompact ultrafilter $\mathcal{U}$ over $P_\kappa(\lambda)$ and $N \models$ "κ is α-hypercompact", then $N^{\mathbb{P}} \models$ "The $\kappa^{+\alpha}$-supercompactness of κ is indestructible under κ-directed closed forcing". We assume the inductive hypothesis is true for $\beta < \alpha$. If it is false at α, then let N and $\mathbb{Q}' \in N^{\mathbb{P}}$

which is κ-directed closed and of minimal rank δ be such that $N^{\mathbb{P}*\dot{\mathbb{Q}}'} \models$ "κ is not $\kappa^{+\alpha}$-supercompact". For the sake of simplicity, we assume without loss of generality that $N = V$. Choose λ to be sufficiently large, e.g., suppose λ is the least strong limit cardinal greater than $\max(|\mathrm{TC}(\mathbb{P} * \dot{\mathbb{Q}}')|, \delta, \kappa^{+\alpha})$. Let $j : V \to M$ be an elementary embedding witnessing the λ-supercompactness of κ generated by a supercompact ultrafilter over $P_\kappa(\lambda)$ such that $M \models$ "κ is β-hypercompact for every $\beta < \alpha$". Because $M^\lambda \subseteq M$, the definition of $\mathbb{P}$ implies that $j(\mathbb{P}) = \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$, where $\mathbb{Q} \in (V_\delta)^{M^{\mathbb{P}}} = (V_\delta)^{V^{\mathbb{P}}}$ and $M^{\mathbb{P}*\dot{\mathbb{Q}}} \models$ "κ is not $\kappa^{+\alpha}$-supercompact". Another appeal to the closure properties of M shows that $V^{\mathbb{P}*\dot{\mathbb{Q}}} \models$ "κ is not $\kappa^{+\alpha}$-supercompact" as well.

We complete the proof of Lemma 2.1 by showing that $V^{\mathbb{P}*\dot{\mathbb{Q}}} \models$ "κ is $\kappa^{+\alpha}$-supercompact", a contradiction. To see that this is the case, let G be V-generic over $\mathbb{P}$ and let H be $V[G]$-generic over $\mathbb{Q}$. Observe that since $M \models$ "No cardinal $\delta > \kappa$ is measurable", the least ordinal at which $\dot{\mathbb{R}}$ is forced to do nontrivial forcing is well above λ. Therefore, standard arguments, as mentioned, e.g., in [13], prove that j lifts in $V[G][H][H'][H'']$ to $j : V[G][H] \to M[G][H][H'][H'']$, where H'' contains a master condition for $j''H$ and $H' * H''$ is both $V[G][H]$- and $M[G][H]$-generic over $\mathbb{R} * j(\dot{\mathbb{Q}})$, a partial ordering which is λ-directed closed in both $V[G][H]$ and $M[G][H]$. We consequently see that $\mathcal{U} \in V[G][H][H'][H'']$ given by $x \in \mathcal{U}$ iff $\langle j(\gamma) : \gamma < \kappa^{+\alpha}\rangle \in j(x)$ is an ultrafilter over $(P_\kappa(\kappa^{+\alpha}))^{V[G][H]}$ witnessing the $\kappa^{+\alpha}$-supercompactness of κ, which, by the closure properties of $\mathbb{R} * j(\dot{\mathbb{Q}})$ in both $M[G][H]$ and $V[G][H]$, is a member of $V[G][H]$ as well. This contradiction completes the proof of Lemma 2.1. ∎

Since $\mathbb{P}$ may be defined so that $|\mathbb{P}| = \kappa$, the Lévy–Solovay results [14] show that $V^{\mathbb{P}} \models$ "No cardinal $\delta > \kappa$ is measurable". Hence, Lemma 2.1 completes the proof of Theorem 2. ∎

Proof of Theorem 3. We know of course that in $V^{\mathbb{P}}$, the strongly compact and supercompact cardinals coincide except at measurable limit points, every nonsupercompact strongly compact cardinal δ has its degree of supercompactness indestructible under δ-directed closed forcing, and every supercompact cardinal δ is indestructible under δ-directed closed forcing. Also, Menas' result of [15] tells us that every measurable limit of supercompact cardinals is in fact strongly compact, and that if $\alpha < \delta$ and δ is the αth measurable limit of supercompact cardinals, then δ is strongly compact but is not supercompact. Consequently, there are nonsupercompact strongly compact cardinals present in $V^{\mathbb{P}}$, and the only such cardinals are the measurable limits of supercompact cardinals. Hence, $V^{\mathbb{P}} \models$ "Every measurable limit of supercompact cardinals has its degree of supercompactness indestructible under δ-directed closed forcing", i.e., $V^{\mathbb{P}}$ is a model for the conclusions of Theorem 3. ∎

Proof of Theorem 6. As in [5], for clarity of exposition, we split the presentation of this proof into two distinct components. We begin with our forcing construction, i.e., we first prove the following result.

THEOREM 4. *Let T_1 be the theory composed of the statements "There is a strong limit of strong cardinals", "Every strong cardinal has its strongness weakly indestructible", and "Every measurable limit of strong cardinals has its degree of strongness weakly indestructible". Then* Con(*ZFC + There is a hyperstrong cardinal*) $\Rightarrow$ Con(*ZFC* + T_1).

Proof. Let $V \models$ "ZFC + κ is a hyperstrong cardinal". By [7, Theorem 3.12], it is also possible to assume that $V \models$ GCH. As in the proof of Theorem 2, by truncating the universe if necessary, we once again assume that $V \models$ "No cardinal $\delta > \kappa$ is measurable".

The partial ordering $\mathbb{P}$ used in the proof of Theorem 4 will be the partial ordering used in the proof of Theorem 2 recast in terms of strongness. Suppose $\gamma < \delta < \kappa$ are such that γ is regular and δ is strong. By the proof of [12, Theorem 4.10], there is a $<\gamma$-strategically closed, (γ, ∞)-distributive partial ordering $\mathbb{P}_{\gamma,\delta} \in V$ of rank $\delta + 1$ with $|\mathbb{P}_{\gamma,\delta}| = \delta$ such that $V^{\mathbb{P}_{\gamma,\delta}} \models$ "δ is a weakly indestructible strong cardinal". $\mathbb{P}_{\gamma,\delta}$ is a slight variant of Hamkins' partial ordering of [12, Theorem 4.10]. As in [12], it is defined as an Easton support iteration of length δ, with the difference from [12] that nontrivial forcing takes place only at stages $\sigma > \gamma$ with component partial orderings which are (at least) both $<\sigma$-strategically closed and (σ, ∞)-distributive. The lifting arguments used in the proof of [12, Theorem 4.10], which will be given in the proof of Lemma 2.4, then show that $\mathbb{P}_{\gamma,\delta}$ is as desired.

Let $\langle \delta_\alpha : \alpha < \kappa \rangle$ enumerate the V-strong cardinals below κ together with their measurable limits. We define now an Easton support iteration $\mathbb{P} = \langle \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle : \alpha < \kappa \rangle$ of length κ as follows:

1. $\mathbb{P}_1 = \mathbb{P}_0 * \dot{\mathbb{Q}}_0$, where $\mathbb{P}_0$ is the partial ordering for adding a Cohen subset to ω, and $\dot{\mathbb{Q}}_0$ is a term for $\mathbb{P}_{\aleph_2, \delta_0}$.
2. If δ_α is a measurable limit of strong cardinals (meaning that $\delta_\alpha = \sup_{\beta < \alpha} \delta_\beta$) and $\Vdash_{\mathbb{P}_\alpha}$ "There is a $<\delta_\alpha$-strategically closed, (δ_α, ∞)-distributive partial ordering such that after forcing with it, δ_α is not ζ-strong for ζ minimal", then $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is a term for such a partial ordering of minimal rank which destroys the ζ-strongness of δ_α.
3. If δ_α is a measurable limit of strong cardinals and case 2 above does not hold (which will mean that $\Vdash_{\mathbb{P}_\alpha}$ "δ_α is a measurable limit of strong cardinals whose degree of strongness is indestructible under $<\delta_\alpha$-strategically closed, (δ_α, ∞)-distributive partial orderings"), then $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is a term for the trivial partial ordering $\{\emptyset\}$.

4. If δ_α is not a measurable limit of strong cardinals, $\alpha = \beta + 1$, δ_β is a measurable limit of strong cardinals, and case 2 above holds for δ_β, then inductively, since a direct limit must be taken at stage β, $|\mathbb{P}_\beta| = \delta_\beta < \delta_{\beta+1} = \delta_\alpha$. This means inductively $\mathbb{P}_\beta$ has been defined so as to have rank less than δ_α, so by [1, Lemma 3.1] and the succeeding remark (which as we have previously observed remain valid if "supercompact" is replaced by "strong"), $\dot{\mathbb{Q}}_\beta$ can be chosen to have rank less than δ_α. Also, by [1, Lemma 3.1] and the succeeding remark, $\zeta < \delta_\alpha$ for ζ the least such that $V^{\mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta} = V^{\mathbb{P}_\alpha} \models$ "δ_β is not ζ-strong". Let $\gamma_\alpha = \delta_\beta^+$, and let $\sigma \in (\delta_\beta, \delta_\alpha)$ be the least measurable cardinal (in either V or $V^{\mathbb{P}_\alpha}$) such that $\Vdash_{\mathbb{P}_\alpha}$ "$\sigma > \max(\gamma_\alpha, \dot{\zeta}, \mathrm{rank}(\dot{\mathbb{Q}}_\beta))$" (3). Then $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is a term for $\mathbb{P}_{\gamma_\alpha,\delta_\alpha}$ defined such that σ is below the least stage at which, in the definition of $\mathbb{P}_{\gamma_\alpha,\delta_\alpha}$, a nontrivial forcing is done.
5. If δ_α is not a measurable limit of strong cardinals and case 4 does not hold, then $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where for $\gamma_\alpha = (\bigcup_{\beta<\alpha} \delta_\beta)^+$, $\dot{\mathbb{Q}}_\alpha$ is a term for $\mathbb{P}_{\gamma_\alpha,\delta_\alpha}$.

LEMMA 2.2. $V^{\mathbb{P}} \models$ *"There are κ many strong cardinals $\delta < \kappa$". In addition, $V^{\mathbb{P}} \models$ "Every strong cardinal $\delta < \kappa$ which is not a limit of strong cardinals has its strongness weakly indestructible".*

Proof. Suppose $V \models$ "$\delta < \kappa$ is a strong cardinal which is not a limit of strong cardinals". This means we can let $\alpha < \kappa$ be such that $\delta = \delta_\alpha$ and $\delta \neq \sup_{\beta<\alpha} \delta_\beta$. Write $\mathbb{P} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha * \dot{\mathbb{R}} = \mathbb{P}_{\alpha+1} * \dot{\mathbb{R}}$. By the definition of $\mathbb{P}$, $\Vdash_{\mathbb{P}_{\alpha+1}}$ "δ is a weakly indestructible strong cardinal and $\dot{\mathbb{R}}$ is $<\delta$-strategically closed and (δ, ∞)-distributive", from which it immediately follows that $V^{\mathbb{P}_{\alpha+1} * \dot{\mathbb{R}}} = V^{\mathbb{P}} \models$ "δ is a weakly indestructible strong cardinal". Thus, the proof of Lemma 2.2 will be complete once we have shown that $V^{\mathbb{P}} \models$ "Any strong cardinal $\delta < \kappa$ which is not a limit of strong cardinals is such that for some $\alpha < \kappa$, $\delta = \delta_\alpha$ and $\delta \neq \sup_{\beta<\alpha} \delta_\beta$".

To do this, suppose $V^{\mathbb{P}} \models$ "$\delta < \kappa$ is a strong cardinal which is not a limit of strong cardinals". Write $\mathbb{P} = \mathbb{P}' * \dot{\mathbb{P}}''$, where $|\mathbb{P}'| = \omega$, $\mathbb{P}'$ is nontrivial, and $\Vdash_{\mathbb{P}'}$ "$\dot{\mathbb{P}}''$ is $\aleph_1$-strategically closed". By Hamkins' Gap Forcing Theorem of [9, 10], this factorization tells us that $V \models$ "δ is a strong cardinal", from which we immediately infer that $\delta = \delta_\alpha$ for some $\alpha < \kappa$. If $\delta = \sup_{\beta<\alpha} \delta_\beta$, then we have that $\delta = \sup_{\beta<\alpha} \delta_{\beta+1}$. Since by the first paragraph of the proof of this lemma, for any $\beta < \alpha$, $V^{\mathbb{P}} \models$ "$\delta_{\beta+1}$ is a strong cardinal",

(3) As opposed to the proof of Theorem 2, it is possible to take $\gamma_\alpha = \delta_\beta^+$ instead of just having $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\gamma}_\alpha = \delta_\beta^+$". This is because $\Vdash_{\mathbb{P}_\beta}$ "$\dot{\mathbb{Q}}_\beta$ is (δ_β, ∞)-distributive", which means that forcing with $\mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta = \mathbb{P}_{\beta+1} = \mathbb{P}_\alpha$ preserves δ_β^+.

$V^{\mathbb{P}} \vDash$ "δ is a limit of strong cardinals". This contradiction completes the proof of Lemma 2.2. ∎

Lemma 2.3. *$V^{\mathbb{P}} \vDash$ "If $\delta < \kappa$ is a measurable limit of strong cardinals, then δ's degree of strongness is weakly indestructible".*

Proof. We follow the proof of [1, Lemma 4.1]. By the factorization of $\mathbb{P}$ given in the second paragraph of the proof of Lemma 2.2 and the results of [9, 10], any strong cardinal in $V^{\mathbb{P}}$ had to have been strong in V, and δ must be in V a measurable limit of strong cardinals. This means that $\delta = \delta_\alpha$ for some limit ordinal $\alpha < \kappa$ and $\delta = \sup_{\beta<\alpha} \delta_\beta$.

If $V^{\mathbb{P}} \vDash$ "δ_α's degree of strongness is not weakly indestructible", then let ζ smallest and $\mathbb{Q} \in V^{\mathbb{P}}$ of minimal rank be such that $V^{\mathbb{P}} \vDash$ "$\mathbb{Q}$ is $<\delta_\alpha$-strategically closed and (δ_α, ∞)-distributive", $V^{\mathbb{P}} \vDash$ "δ_α is ζ-strong", yet $V^{\mathbb{P}*\dot{\mathbb{Q}}} \vDash$ "δ_α is not ζ-strong". Write $\mathbb{P} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha * \dot{\mathbb{Q}}_{\alpha+1} * \dot{\mathbb{R}} = \mathbb{P}_{\alpha+1} * \dot{\mathbb{Q}}_{\alpha+1} * \dot{\mathbb{R}} = \mathbb{P}_{\alpha+1} * \dot{\mathbb{P}}_{\gamma_{\alpha+1},\delta_{\alpha+1}} * \dot{\mathbb{R}} = \mathbb{P}_{\alpha+2} * \dot{\mathbb{R}}$. As in the first paragraph of the proof of Lemma 2.2, the definition of $\mathbb{P}$ ensures that $\Vdash_{\mathbb{P}_{\alpha+2}}$ "$\delta_{\alpha+1}$ is a weakly indestructible strong cardinal and $\dot{\mathbb{R}}$ is $<\delta_{\alpha+1}$-strategically closed and $(\delta_{\alpha+1}, \infty)$-distributive". Hence, $V^{\mathbb{P}_{\alpha+2}*\dot{\mathbb{R}}} = V^{\mathbb{P}} \vDash$ "$\delta_{\alpha+1}$ is a weakly indestructible strong cardinal", so by [1, Lemma 3.1] and the succeeding remark (applied to strong cardinals), $\mathbb{Q}, \zeta \in (V_{\delta_{\alpha+1}})^{V^{\mathbb{P}}}$. Therefore, the preceding tells us $\mathbb{Q} \in V^{\mathbb{P}_{\alpha+2}} = V^{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha * \dot{\mathbb{P}}_{\gamma_{\alpha+1},\delta_{\alpha+1}}}$ and $V^{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha * \dot{\mathbb{P}}_{\gamma_{\alpha+1},\delta_{\alpha+1}} * \dot{\mathbb{Q}}} = V^{\mathbb{P}_\alpha * \dot{\mathbb{Q}}^*} \vDash$ "δ_α is not ζ-strong". Since $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}^*$ is $<\delta_\alpha$-strategically closed and (δ_α, ∞)-distributive", we must be in case 4 at stage $\alpha + 2$ of the definition of $\mathbb{P}$. This means that for some $\zeta' \le \zeta$, $V^{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha} = V^{\mathbb{P}_{\alpha+1}} \vDash$ "δ_α is not ζ'-strong", and consequently, $V^{\mathbb{P}} \vDash$ "δ_α is not ζ'-strong"as well. Since $V^{\mathbb{P}} \vDash$ "δ_α is ζ-strong", this is a contradiction. This proves Lemma 2.3. ∎

Lemma 2.4. *$V^{\mathbb{P}} \vDash$ "κ is a weakly indestructible strong cardinal".*

Proof. We follow the proofs of Lemma 2.1 and [5, Lemma 1.4], quoting verbatim as appropriate. We proceed inductively, taking as our inductive hypothesis that if $\alpha \ge 1$ is an ordinal and $N \subseteq V$ is such that either $N = V$ or for some λ, N is the transitive collapse of $\mathrm{Ult}(V, \mathcal{E})$ where $\mathcal{E}$ is a (κ, λ)-extender and $N \vDash$ "κ is α-hyperstrong", then $N^{\mathbb{P}} \vDash$ "The $\kappa + \alpha$-strongness of κ is weakly indestructible". For $\alpha = 1$, this amounts to showing that if $N \vDash$ "κ is 1-hyperstrong", then $N^{\mathbb{P}} \vDash$ "The $\kappa + 1$-strongness of κ, i.e., the measurability of κ, is weakly indestructible". To see that this is indeed the case, let $\mu \in N$ be a normal measure over κ such that for $j_\mu : N \to M_\mu$ the ultrapower embedding via μ, $M_\mu \vDash$ "κ is not measurable". Note that by the fact that $V_\kappa \in N$ and the definition of $\mathbb{P}$, $j_\mu(\mathbb{P}) = \mathbb{P} * \dot{\mathbb{Q}}'$, where the first ordinal at which $\dot{\mathbb{Q}}'$ is forced to do nontrivial forcing is above κ^+, and $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}'$ is $\prec\kappa^+$-strategically closed". Since $N \vDash$ GCH, standard arguments imply that j_μ lifts in N to $j_\mu : N^{\mathbb{P}} \to M_\mu^{j_\mu(\mathbb{P})}$. (An outline of these arguments is as follows. Let G be N-generic over $\mathbb{P}$. Since $\mathbb{P}$ is κ-c.c., $M_\mu[G]$ remains κ-closed with respect to $N[G]$. Because $N \models \mathrm{GCH}$ and M_μ is given by an ultrapower embedding, we may let $\langle D_\beta : \beta < \kappa^+ \rangle \in N[G]$ enumerate the dense open subsets of $\mathbb{Q}'$ present in $M_\mu[G]$. As in the construction of the generic object H' given later in the proof of this lemma, it is possible to use the $\prec\kappa^+$-strategic closure of $\mathbb{Q}'$ in both $M_\mu[G]$ and $N[G]$ to build in $N[G]$ an $M_\mu[G]$-generic object G' over $\mathbb{Q}'$. Since $j_\mu'' G \subseteq G * G'$, j_μ lifts to $j_\mu : N[G] \to M_\mu[G][G']$.) From this, it follows that $N^{\mathbb{P}} \models$ "κ is measurable". Since the measurability of κ is weakly indestructible, we have established the base case of our induction.

We now assume that $\alpha > 1$ is an arbitrary (successor or limit) ordinal. If our inductive hypothesis is false at α, then let N and $\mathbb{Q}' \in N^{\mathbb{P}}$ of minimal rank δ which is $<\kappa$-strategically closed and (κ, ∞)-distributive be such that $N^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is not $\kappa + \alpha$-strong". For the sake of simplicity, we assume without loss of generality that $N = V$. Choose λ to be sufficiently large, e.g., suppose λ is the least strong limit cardinal above $\max(|\mathrm{TC}(\mathbb{P} * \dot{\mathbb{Q}})|, \delta, \kappa + \alpha)$ having cofinality κ. Let $j : V \to M$ be an elementary embedding witnessing the λ-strongness of κ generated by a (κ, λ)-extender such that $M \models$ "κ is β-hyperstrong for every $\beta < \alpha$". By the choice of j and M, $\mathbb{Q}' \in M^{\mathbb{P}}$. Because $V_\lambda \subseteq M$, the definition of $\mathbb{P}$ implies that $j(\mathbb{P}) = \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$, where $\mathbb{Q} \in (V_\delta)^{M^{\mathbb{P}}} = (V_\delta)^{V^{\mathbb{P}}}$ and $M^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is not $\kappa + \alpha$-strong". Another appeal to the fact that $V_\lambda \subseteq M$ yields $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is not $\kappa + \alpha$-strong" as well.

We now show that the embedding j lifts in $V^{\mathbb{P} * \dot{\mathbb{Q}}}$ to $j : V^{\mathbb{P} * \dot{\mathbb{Q}}} \to M^{j(\mathbb{P} * \dot{\mathbb{Q}})}$. The methods for doing this are quite similar to those given in the proof of [12, Theorem 4.10] (as well as elsewhere). For the benefit of readers, we give the argument here as well, taking the liberty to quote freely from [12, Theorem 4.10] and [5, Lemma 1.4]. Since j is an extender embedding, we have $M = \{j(f)(a) : a \in [\lambda]^{<\omega},\ f \in V, \text{ and } \mathrm{dom}(f) = [\kappa]^{|a|}\}$. Because $V_\lambda \subseteq M$ and $V \models$ "No cardinal $\delta > \kappa$ is measurable", we may write $j(\mathbb{P})$ as $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$, where the first ordinal at which $\dot{\mathbb{R}}$ is forced to do nontrivial forcing is above λ. Since λ has been chosen to have cofinality κ, we may assume that $M^\kappa \subseteq M$. This means that if G is V-generic over $\mathbb{P}$ and H is $V[G]$-generic over $\mathbb{Q}$, then $\mathbb{R}$ is $\prec\kappa^+$-strategically closed in both $V[G][H]$ and $M[G][H]$, and $\mathbb{R}$ is λ-strategically closed in $M[G][H]$.

As in [12] and [5], by using a suitable coding that allows us to identify finite subsets of λ with elements of λ, by the definition of M, there must be some $\alpha_0 < \lambda$ and a function g such that $\dot{\mathbb{Q}} = j(g)(\alpha_0)$. Let $N^* = \{i_{G*H}(\dot{z}) : \dot{z} = j(f)(\kappa, \alpha_0, \lambda) \text{ for some function } f \in V\}$. It is easy to verify that $N^* \prec M[G][H]$, that N^* is closed under κ sequences in $V[G][H]$, and that κ, α_0, λ, $\mathbb{Q}$, and $\mathbb{R}$ are all elements of N^*. Further, since $\mathbb{R}$ is $j(\kappa)$-c.c.

in $M[G][H]$ and there are only $2^\kappa = \kappa^+$ many functions $f : [\kappa]^3 \to V_\kappa$ in V, there are at most κ^+ many dense open subsets of $\mathbb{R}$ in N^*. Therefore, since $\mathbb{R}$ is $\prec\kappa^+$-strategically closed in both $M[G][H]$ and $V[G][H]$, we can build $H' \subseteq \mathbb{R}$ in $V[G][H]$ as follows. Let $\langle D_\sigma : \sigma < \kappa^+\rangle$ enumerate in $V[G][H]$ the dense open subsets of $\mathbb{R}$ present in N^* so that every dense open subset of $\mathbb{R}$ occurring in N^* appears at an odd stage at least once in the enumeration. If σ is an odd ordinal, $\sigma = \tau + 1$ for some τ. Player I picks $p_\sigma \in D_\sigma$ extending q_τ (initially, q_0 is the empty condition), and player II responds by picking $q_\sigma \geq p_\sigma$ according to a fixed strategy $\mathcal{S}$ (so $q_\sigma \in D_\sigma$). If σ is a limit ordinal, player II uses $\mathcal{S}$ to pick q_σ extending each $q \in \langle q_\gamma : \gamma < \sigma\rangle$. By the $\prec\kappa^+$-strategic closure of $\mathbb{R}$ in $V[G][H]$, player II's strategy can be assumed to be a winning one, so $\langle q_\sigma : \sigma < \kappa^+\rangle$ can be taken as an increasing sequence of conditions with $q_\sigma \in D_\sigma$ for $\sigma < \kappa^+$.

Let $H' = \{p \in \mathbb{R} : \exists\sigma < \kappa^+\ [q_\sigma \geq p]\}$. We show now that H' is actually $M[G][H]$-generic over $\mathbb{R}$. If D is a dense open subset of $\mathbb{R}$ in $M[G][H]$, then $D = i_{G*H}(\dot{D})$ for some name $\dot{D} \in M$. Consequently, $\dot{D} = j(f)(\kappa, \kappa_1, \ldots, \kappa_n)$ for some function $f \in V$ and $\kappa < \kappa_1 < \cdots < \kappa_n < \lambda$. Let $\overline{D}$ be a name for the intersection of all $i_{G*H}(j(f)(\kappa, \alpha_1, \ldots, \alpha_n))$, where $\kappa < \alpha_1 < \cdots < \alpha_n < \lambda$ is such that $j(f)(\kappa, \alpha_1, \ldots, \alpha_n)$ yields a name for a dense open subset of $\mathbb{R}$. Since this name can be given in M and $\mathbb{R}$ is λ-strategically closed in $M[G][H]$ and therefore (λ, ∞)-distributive in $M[G][H]$, $\overline{D}$ is a name for a dense open subset of $\mathbb{R}$ which is definable without the parameters $\kappa_1, \ldots, \kappa_n$. Hence, by its definition, $i_{G*H}(\overline{D}) \in N^*$. Thus, since H' meets every dense open subset of $\mathbb{R}$ present in N^*, $H' \cap i_{G*H}(\overline{D}) \neq \emptyset$, so since $\overline{D}$ is forced to be a subset of $\dot{D}$, $H' \cap i_{G*H}(\dot{D}) \neq \emptyset$. This means H' is $M[G][H]$-generic over $\mathbb{R}$, so in $V[G][H]$, as $j''G \subseteq G * H * H'$, j lifts to $j : V[G] \to M[G][H][H']$ via the definition $j(i_G(\tau)) = i_{G*H*H'}(j(\tau))$.

It remains to lift j through the forcing $\mathbb{Q}$ while working in $V[G][H]$. To do this, it suffices to show that $j''H \subseteq j(\mathbb{Q})$ generates an $M[G][H][H']$-generic object H'' over $j(\mathbb{Q})$. Given a dense open subset $D \subseteq j(\mathbb{Q})$ with $D \in M[G][H][H']$, $D = i_{G*H*H'}(\dot{D})$ for some name $\dot{D} = j(\vec{D})(a)$, where $a \in [\lambda]^{<\omega}$ and $\vec{D} = \langle D_\sigma : \sigma \in [\kappa]^{|a|}\rangle$ is a function. We may assume that every D_σ is a dense open subset of $\mathbb{Q}$. Since $\mathbb{Q}$ is (κ, ∞)-distributive, it follows that $D' = \bigcap_{\sigma\in[\kappa]^{|a|}} D_\sigma$ is also a dense open subset of $\mathbb{Q}$. As $j(D') \subseteq D$ and $H \cap D' \neq \emptyset$, $j''H \cap D \neq \emptyset$. Thus, $H'' = \{p \in j(\mathbb{Q}) : \exists q \in j''H\ [q \geq p]\}$ is our desired generic object, and j lifts in $V[G][H]$ to $j : V[G][H] \to M[G][H][H'][H'']$. This final lifted version of j is λ-strong since $V_\lambda \subseteq M$, meaning $(V_\lambda)^{V[G][H]} \subseteq M[G][H] \subseteq M[G][H][H'][H'']$. Therefore, since $V[G][H] \models$ "$\lambda > \kappa + \alpha$ is a strong limit cardinal", $V[G][H] \models$ "κ is $\kappa + \alpha$-strong". This contradiction completes our induction and the proof of Lemma 2.4. ∎

Lemmas 2.2–2.4 imply that in $V^{\mathbb{P}}$, the following hold:

1. κ is a strong limit of strong cardinals.
2. Every strong cardinal $\delta \leq \kappa$ has its strongness weakly indestructible.
3. Every measurable limit of strong cardinals $\delta \leq \kappa$ has its degree of strongness weakly indestructible.

Since $\mathbb{P}$ may be defined so that $|\mathbb{P}| = \kappa$, as before, the results of [14] show that $V^{\mathbb{P}} \vDash$ "No cardinal $\delta > \kappa$ is measurable". This means that $V^{\mathbb{P}}$ witnesses the conclusions of Theorem 4. ∎

Having completed the proof of Theorem 4, we present now the inner model portion of our argument. Specifically, we establish the following result.

THEOREM 5. *Let T_2 be the theory composed of the statements "There is a strong limit of strong cardinals" and "Every measurable limit of strong cardinals has its degree of strongness weakly indestructible". Then* Con(*ZFC + T_2*) $\Rightarrow$ Con(*ZFC + There is a hyperstrong cardinal*).

Proof. We follow very closely the proof of [5, Theorem 4], frequently quoting verbatim when appropriate. We argue using standard core model techniques exposited in [16] and [17]. We are done if there is an inner model with a hyperstrong cardinal, so we assume without loss of generality that this is not the case.

Suppose that V is a model of ZFC in which the following hold:

1. κ is a strong limit of strong cardinals.
2. Every measurable limit of strong cardinals has its degree of strongness weakly indestructible.

Let $\lambda > \kappa$ be an arbitrary strong limit cardinal. We now observe that if $j : V \to M$ is an elementary embedding witnessing the λ-strongness of κ generated by a (κ, λ)-extender $\mathcal{E}$, then $M \vDash$ "κ is $(\kappa + 2)$-strong". Since $\text{cp}(j) = \kappa$, for any $\gamma < \kappa$ such that $V \vDash$ "γ is a strong cardinal", $M \vDash$ "$j(\gamma) = \gamma$ is a strong cardinal". The previous two sentences therefore immediately imply that $M \vDash$ "κ is a measurable limit of strong cardinals". Hence, by elementarity, $M \vDash$ "The $(\kappa + 2)$-strongness of κ is weakly indestructible". Consequently, for $\delta > \kappa^+$ an arbitrary cardinal and $\mathbb{P}_\delta = (\text{Coll}(\kappa^+, \delta))^M$, $M^{\mathbb{P}_\delta} \vDash$ "κ is $(\kappa + 2)$-strong". Since any subset of δ may now be coded by a subset of κ^+, this means that there is actually a (κ, κ^{++})-extender $\mathcal{F} \in M^{\mathbb{P}_\delta}$ such that all subsets of δ are captured in $\text{Ult}(M^{\mathbb{P}_\delta}, \mathcal{F})$. By downwards absoluteness to the core model $(K)^{M^{\mathbb{P}_\delta}}$, this last fact is true in $(K)^{M^{\mathbb{P}_\delta}}$ as well, i.e., in $(K)^{M^{\mathbb{P}_\delta}}$, for some γ, there is a (κ, γ)-extender $\mathcal{F}^* = \mathcal{F} \restriction (K)^{M^{\mathbb{P}_\delta}}$ such that all subsets of δ are captured in $\text{Ult}((K)^{M^{\mathbb{P}_\delta}}, \mathcal{F}^*)$. By the absoluteness of the core model under set forcing, in the core model $(K)^M = (K)^{M^{\mathbb{P}_\delta}}$, $\mathcal{F}^*$ is a (κ, γ)-extender witnessing that all subsets of δ are captured in the relevant target model. Since $\delta > \kappa$ was arbitrary, this just means that $(K)^M \vDash$ "κ is a strong cardinal".

Let $K = (K)^V$. We show that $K \vDash$ "κ is 1-hyperstrong". To do this, take once again λ, $\mathcal{E}$, j, and M as in the preceding paragraph. Let $\mathcal{E}^* = \mathcal{E} \restriction K$, with $i : K \to N$ the λ-strongness embedding generated by $\mathcal{E}^*$ and $\ell : N \to (K)^M$ the associated factor elementary embedding whose critical point is greater than κ. It is then the case that $N \vDash$ "κ is a strong cardinal", since by elementarity, $N \vDash$ "κ is a strong cardinal" iff $(K)^M \vDash$ "$\ell(\kappa)$ is a strong cardinal", i.e., iff $(K)^M \vDash$ "κ is a strong cardinal". Thus, for any $\lambda > \kappa$ which is a strong limit cardinal, there is an elementary embedding witnessing the λ-strongness of κ in K generated by a (κ, λ)-extender such that in the target model, κ is a strong cardinal.

Now that we know that $K \vDash$ "κ is 1-hyperstrong", we are able to proceed inductively. Specifically, we assume that for $\lambda > \kappa$ having been chosen to be a strong limit cardinal, $j : V \to M$ an elementary embedding witnessing the λ-strongness of κ generated by a (κ, λ)-extender $\mathcal{E}$, and ρ either a successor or limit ordinal, $(K)^M \vDash$ "κ is α-hyperstrong for every $\alpha < \rho$". The proof given in the preceding paragraph, with "κ is α-hyperstrong for every $\alpha < \rho$" replacing "κ is a strong cardinal", then shows that $K \vDash$ "κ is ρ-hyperstrong". As ρ was arbitrary, this completes the proof of Theorem 5. ■

Since Con(ZFC + T_1) $\Rightarrow$ Con(ZFC + T_2), the proofs of Theorems 4 and 5 complete the proof of Theorem 6. ■

3. Concluding remarks. In conclusion to this paper, we make several remarks. We begin by conjecturing that, in analogy to Theorem 6, the conclusions of Theorems 2 and 3 are actually equiconsistent with the existence of a hypercompact cardinal. Of course, since inner model theory for supercompactness is still in its infancy, an attempt at establishing this conjecture is not yet in sight.

We also ask whether it is possible to prove a version of Theorem 6 for the kind of indestructibility first described by Gitik and Shelah in [8]. The proofs of Lemmas 2.2 and 2.3 seem to suggest the use of a version of the Gap Forcing Theorem for Prikry iterations, something which has yet to be demonstrated.

Finally, we mention that in [11], Hamkins introduced the concept of *tall cardinal*, whose definition we now recall.

Definition 3.1. κ is *α-tall* iff there is an elementary embedding $j : V \to M$ with $\mathrm{cp}(j) = \kappa$ such that $j(\kappa) > \alpha$ and $M^\kappa \subseteq M$; κ is *tall* iff κ is α-tall for every ordinal α.

Hamkins also presented in [11] the thesis that "tall is to strong as strongly compact is to supercompact". In light of this, we finish by asking whether the theories "ZFC + There is a hyperstrong cardinal" and "ZFC + T_3", where T_3 is the theory composed of the statements "There is a strong limit of strong cardinals", "The strong and tall cardinals coincide except at measurable limit points", "Every strong cardinal is weakly indestructible", and "Every nonstrong tall cardinal has its degree of strongness weakly indestructible" are equiconsistent ([4]). Although the technology for dealing with tall cardinals is also still in its early stages of development, we conjecture that there is an affirmative answer to this question.

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([4]) Hamkins has shown in [11] that a tall cardinal δ must have its tallness indestructible under (δ, ∞)-distributive forcing. This means that weak indestructibility for tallness, in the sense of a tall cardinal δ having its tallness indestructible under partial orderings which are both $<\delta$-strategically closed and (δ, ∞)-distributive, is always true.

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