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Semi-embeddings and weakly sequential completeness of the projective tensor product

by

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Abstract. We show that if $\{P_k\}$ is a boundedly complete, unconditional Schauder decomposition of a Banach space X, then X is weakly sequentially complete whenever P_kX is weakly sequentially complete for each $k \in \mathbb{N}$. Then through semi-embeddings, we give a new proof of Lewis's result: if one of Banach spaces X and Y has an unconditional basis, then $X \widehat{\otimes} Y$, the projective tensor product of X and Y, is weakly sequentially complete whenever both X and Y are weakly sequentially complete.

1. Introduction. The semi-embeddings in Banach spaces were introduced by Lotz, Peck, and Porta (see [10]). A Banach space X is said to semi-embed into a Banach space Y if there is a one-to-one continuous linear operator from X to Y such that the image of the closed unit ball of X is closed in Y. A Banach space property is said to be *semi-embedding inherited* if it is inherited from a Banach space Y to a separable Banach space X whenever X semi-embeds into Y. Delbaen (see [1]) showed that the Radon–Nikodym property is semi-embedding inherited. Moreover, the near Radon–Nikodym property (see [7]), the analytic Radon–Nikodym property and a type I Radon–Nikodym property (see [3]), a type II Radon–Nikodym property (see [11]), the (analytic) complete continuity property, and a type II complete continuity property (see [12, 13]) are semi-embedding inherited properties.

In addition, non-containment of c_0 is also a semi-embedding inherited property due to Dowling's result: a Banach space contains no copy of c_0 if and only if it has some type I Radon–Nikodym property (see [4]).

However, reflexivity and non-containment of ℓ_1 are not semi-embedding inherited. For example, $\ell_2 \widehat{\otimes} \ell_2$ semi-embeds into $\ell_2^{\text{strong}}(\ell_2)$ (see [2]). It is known that $\ell_2^{\text{strong}}(\ell_2)$ is reflexive and does not contain a copy of ℓ_1. But $\ell_2 \widehat{\otimes} \ell_2$ contains a complemented copy of ℓ_1 (see [14, p. 23]).

Recently Bu, Diestel, Dowling, and Oja [2] showed that if U is a Banach space with a boundedly complete, 1-unconditional basis and if X is any

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Banach space, then $U \widehat{\otimes} X$, the projective tensor product of U and X, semi-embeds into an X-valued Banach sequence space $U(X)$. Then they used this semi-embedding to show that several types of Radon–Nikodym properties are inherited from Banach spaces (one of which has an unconditional basis) to their projective tensor product. In this paper, we will also use this same semi-embedding to show that the weakly sequential completeness is inherited from $U(X)$ to $U \widehat{\otimes} X$, and then use this inheritance to give a short proof of Lewis's result [8]: the weakly sequential completeness is inherited from Banach spaces (one of which has an unconditional basis) to their projective tensor product.

For any Banach space X, its topological dual and closed unit ball will be denoted by X^* and B_X, respectively. For two Banach spaces X and Y, let $X \widehat{\otimes} Y$ denote the completion of the tensor product $X \otimes Y$ with respect to the projective tensor norm; and let $\mathcal{L}(X, Y)$ denote the space of all continuous linear operators from X to Y.

2. Semi-embeddings

Definition 1. A continuous linear operator T from a Banach space X to a Banach space Y is called a *semi-embedding* if T is one-to-one and $T(B_X)$ is closed in Y. A Banach space X is said to semi-embed into a Banach space Y if there is a semi-embedding from X to Y.

Lemma 2. *Suppose that X and Y are Banach spaces such that Y is weakly sequentially complete and there is a semi-embedding T from X to Y. If $\{x_n\}_{n=1}^{\infty}$ is a weakly Cauchy sequence in X, then there exists an $x \in X$ such that $T(x_n)$ converges to $T(x)$ weakly in Y.*

Proof. Since T is weakly-weakly continuous, $\{T(x_n)\}_{n=1}^{\infty}$ is a weakly Cauchy sequence in Y. Thus there exists a $y \in Y$ such that $T(x_n)$ converges to y weakly in Y. Let $z_n = x_n/c$, where $c = \sup_n \|x_n\| < \infty$. Then $z_n \in B_X$ and $T(z_n)$ converges to y/c weakly in Y. Since $T(B_X)$ is closed and convex in Y, $T(B_X)$ is also weakly closed in Y. Thus $y/c \in T(B_X)$. Therefore there exists a $z \in B_X$ such that $T(z) = y/c$. Let $x = cz$. Then $x \in X$ and $T(x_n)$ converges to $T(x) = y$ weakly in Y. ∎

Let U be a Banach space with a boundedly complete, 1-unconditional basis. Let $\{e_i\}_{i=1}^{\infty}$ be a normalized, boundedly complete, unconditional basis of U whose unconditional basis constant is 1, and let $\{e_i^*\}_{i=1}^{\infty}$ be the normalized biorthogonal functionals associated to the basis $\{e_i\}_{i=1}^{\infty}$, i.e.,

$$e_i^*(e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

For a Banach space X, define

$$U(X) = \Big\{\overline{x} = (x_i)_i \in X^{\mathbb{N}} : \sum_i \|x_i\| e_i \text{ converges in } U\Big\},$$

and define a norm on $U(X)$ to be

$$\|\overline{x}\|_{U(X)} = \Big\|\sum_{i=1}^{\infty} \|x_i\| e_i\Big\|_U.$$

Then $U(X)$ with this norm is a Banach space (see [2]). Moreover, by [2, Theorem 11], $U \mathbin{\widehat{\otimes}} X$ semi-embeds into $U(X)$ through the semi-embedding ψ defined as follows:

$$\psi : U \mathbin{\widehat{\otimes}} X \to U(X), \qquad z \mapsto \Big(\sum_{k=1}^{\infty} e_i^*(u_k) x_k\Big)_i, \tag{1}$$

where $\sum_{k=1}^{\infty} u_k \otimes x_k$ is a representation of z.

Lemma 3. *For each $T \in \mathcal{L}(U, X^*)$, define*

$$I_T : U \mathbin{\widehat{\otimes}} X \to \ell_1, \qquad z \mapsto (\langle \psi(z)_i, Te_i\rangle)_i.$$

Then I_T is a continuous linear operator.

Proof. For each $\varepsilon > 0$, any element $z \in U \mathbin{\widehat{\otimes}} X$ admits a representation $z = \sum_{k=1}^{\infty} u_k \otimes x_k$ such that

$$\sum_{k=1}^{\infty} \|u_k\| \cdot \|x_k\| \le \|z\|_{U \widehat{\otimes} X} + \varepsilon.$$

For each $(s_i)_i \in \ell_\infty$, define

$$v_k = \sum_{i=1}^{\infty} s_i e_i^*(u_k) e_i, \qquad k = 1, 2, \ldots.$$

Since $\{e_i\}$ is a 1-unconditional basis of U, it follows that $v_k \in U$ and

$$\|v_k\| \le \|(s_i)_i\|_{\ell_\infty} \cdot \|u_k\|, \qquad k = 1, 2, \ldots.$$

Thus

$$\begin{aligned}\Big|\sum_{i=1}^{\infty} s_i \langle \psi(z)_i, Te_i\rangle\Big| &= \Big|\sum_{i=1}^{\infty} s_i \Big\langle \sum_{k=1}^{\infty} e_i^*(u_k) x_k, Te_i\Big\rangle\Big| \\ &= \Big|\sum_{k=1}^{\infty} \Big\langle x_k, \sum_{i=1}^{\infty} s_i e_i^*(u_k) Te_i\Big\rangle\Big| = \Big|\sum_{k=1}^{\infty} \langle x_k, Tv_k\rangle\Big| \\ &= \Big|\Big\langle \sum_{k=1}^{\infty} v_k \otimes x_k, T\Big\rangle\Big| \le \|T\| \cdot \Big\|\sum_{k=1}^{\infty} v_k \otimes x_k\Big\|_{U \widehat{\otimes} X}\end{aligned}$$

$$\leq \|T\| \cdot \sum_{k=1}^{\infty} \|v_k\| \cdot \|x_k\| \leq \|T\| \cdot \sum_{k=1}^{\infty} \|(s_i)_i\|_{\ell_\infty} \cdot \|u_k\| \cdot \|x_k\|$$
$$\leq \|T\| \cdot \|(s_i)_i\|_{\ell_\infty} \cdot (\|z\|_{U\widehat{\otimes}X} + \varepsilon).$$

It follows that

$$\|(\langle \psi(z)_i, Te_i\rangle)_i\|_{\ell_1} \leq \|T\| \cdot (\|z\|_{U\widehat{\otimes}X} + \varepsilon).$$

Letting $\varepsilon \to 0$ gives

$$\|(\langle \psi(z)_i, Te_i\rangle)_i\|_{\ell_1} \leq \|T\| \cdot \|z\|_{U\widehat{\otimes}X}.$$

Therefore I_T is well defined and continuous. ∎

THEOREM 4. *The weak sequential completeness is inherited from $U(X)$ to $U \widehat{\otimes} X$.*

Proof. Suppose $U(X)$ is weakly sequentially complete. Let $\{z_n\}_{n=1}^{\infty}$ be a weakly Cauchy sequence in $U\widehat{\otimes}X$. Since ψ defined in (1) is a semi-embedding from $U\widehat{\otimes}X$ to $U(X)$, by Lemma 2, there exists a $z \in U\widehat{\otimes}X$ such that $\psi(z_n)$ converges to $\psi(z)$ weakly in $U(X)$. Next we will show that z_n converges to z weakly in $U \widehat{\otimes} X$.

Fix $T \in \mathcal{L}(U, X^*) = (U \widehat{\otimes} X)^*$. By Lemma 3, $I_T : U \widehat{\otimes} X \to \ell_1$ is continuous, and hence weakly-weakly continuous. Thus $\{I_T(z_n)\}_{n=1}^{\infty}$ is a weakly Cauchy sequence in ℓ_1, and hence a relatively weakly sequentially compact subset of ℓ_1. By the Schur property, $\{I_T(z_n)\}_{n=1}^{\infty}$ is a relatively sequentially compact subset of ℓ_1. Therefore there exists, for each $\varepsilon > 0$, an $m_1 \in \mathbb{N}$ such that

$$\sum_{i=m_1+1}^{\infty} |\langle \psi(z_n)_i, Te_i\rangle| \leq \varepsilon/3, \qquad n = 1, 2, \ldots. \tag{2}$$

Since $(\langle \psi(z)_i, Te_i\rangle)_i \in \ell_1$, there exists an $m_2 > m_1$ such that

$$\sum_{i=m_2+1}^{\infty} |\langle \psi(z)_i, Te_i\rangle| \leq \varepsilon/3. \tag{3}$$

Note that $\psi(z_n)$ converges to $\psi(z)$ weakly in $U(X)$. It follows that $\psi(z_n)_i$ converges to $\psi(z)_i$ weakly in X for each $i \in \mathbb{N}$. Thus there exists an $n_0 \in \mathbb{N}$ such that for each $n > n_0$,

$$|\langle \psi(z_n)_i - \psi(z)_i, Te_i\rangle| < \varepsilon/3m_2, \qquad i = 1, \ldots, m_2. \tag{4}$$

Now let $z_n, z \in U \widehat{\otimes} X$ have representations

$$z_n = \sum_{k=1}^{\infty} u_{k,n} \otimes x_{k,n}, \qquad z = \sum_{k=1}^{\infty} u_k \otimes x_k.$$

By (2)–(4), for each $n > n_0$,

$$\begin{aligned}
|\langle z_n - z, T\rangle| &= \Big|\sum_{k=1}^{\infty}\langle Tu_{k,n}, x_{k,n}\rangle - \sum_{k=1}^{\infty}\langle Tu_k, x_k\rangle\Big| \\
&= \Big|\sum_{k=1}^{\infty}\Big\langle \sum_{i=1}^{\infty} e_i^*(u_{k,n})Te_i, x_{k,n}\Big\rangle - \sum_{k=1}^{\infty}\Big\langle \sum_{i=1}^{\infty} e_i^*(u_k)Te_i, x_k\Big\rangle\Big| \\
&= \Big|\sum_{i=1}^{\infty}\Big\langle \sum_{k=1}^{\infty} e_i^*(u_{k,n})x_{k,n}, Te_i\Big\rangle - \sum_{i=1}^{\infty}\Big\langle \sum_{k=1}^{\infty} e_i^*(u_k)x_k, Te_i\Big\rangle\Big| \\
&= \Big|\sum_{i=1}^{\infty}\langle \psi(z_n)_i - \psi(z)_i, Te_i\rangle\Big| \\
&\le \sum_{i=1}^{m_2} |\langle \psi(z_n)_i - \psi(z)_i, Te_i\rangle| \\
&\quad + \sum_{i=m_2+1}^{\infty} |\langle \psi(z_n)_i, Te_i\rangle| + \sum_{i=m_2+1}^{\infty} |\langle \psi(z)_i, Te_i\rangle| \\
&\le \varepsilon.
\end{aligned}$$

Therefore z_n converges to z weakly in $U \mathbin{\widehat{\otimes}} X$, and hence $U \mathbin{\widehat{\otimes}} X$ is weakly sequentially complete. ∎

3. Weak sequential completeness and Schauder decompositions. Let X be a Banach space. A *Schauder decomposition* of X is a sequence $\{P_k\}$ of continuous projections on X such that $P_i \circ P_j = 0$ whenever $i \neq j$, and $x = \sum_{k=1}^{\infty} P_k x$ for each x in X (see [6] or [9, §1.g]). A Schauder decomposition $\{P_k\}$ of X is called *boundedly complete* if, whenever $\{\sum_{k=1}^{n} x_k\}_{n=1}^{\infty}$ is a bounded sequence with $x_k \in P_k X$ for each $k \in \mathbb{N}$, then $\sum_k x_k$ converges in X. A Schauder decomposition $\{P_k\}$ of X is called *unconditional* if for each $x \in X$, the series $\sum_k P_k x$ converges to x unconditionally. Let K denote the unconditional constant of the unconditional Schauder decomposition $\{P_k\}$ of X. Then for each $x \in X$ and each sequence $\{\theta_k\}_{k=1}^{\infty}$ of signs,

$$\Big\|\sum_{k=1}^{\infty}\theta_k P_k x\Big\| \le K \cdot \Big\|\sum_{k=1}^{\infty} P_k x\Big\| = K \cdot \|x\|. \tag{5}$$

Now for each $x^* \in X^*$, define

$$I_{x^*} : X \to \ell_1, \qquad x \mapsto (x^*(P_k x))_k. \tag{6}$$

Since $\sum_k P_k x$ converges unconditionally, $\sum_k x^*(P_k x)$ converges unconditionally, and hence $\sum_{k=1}^{\infty} |x^*(P_k x)| < \infty$. Thus I_{x^*} is well defined. Moreover, it follows from (5) that $\|I_{x^*}(x)\| \le K \cdot \|x\| \cdot \|x^*\|$ for each $x \in X$. Therefore I_{x^*} is continuous.

THEOREM 5. *Let $\{P_k\}$ be a boundedly complete, unconditional Schauder decomposition of a Banach space X. Then X is weakly sequentially complete whenever P_kX is weakly sequentially complete for each $k \in \mathbb{N}$.*

Proof. Let $\{x^{(n)}\}_{n=1}^{\infty}$ be a weakly Cauchy sequence in X and let $M = \sup_n \|x^{(n)}\|$. Then $\{P_k x^{(n)}\}_{n=1}^{\infty}$ is a weakly Cauchy sequence in P_kX for each $k \in \mathbb{N}$. Thus there exists, for each $k \in \mathbb{N}$, an $x_k \in P_kX$ such that

$$\text{weak-}\lim_n P_k x^{(n)} = x_k \quad \text{in } P_kX,\ k = 1, 2, \ldots. \tag{7}$$

Each $x^* \in X^*$ can be considered as a member of $(P_kX)^*$ for each $k \in \mathbb{N}$ if it is restricted to P_kX. Now for each fixed $m \in \mathbb{N}$, there exists, from (7), an $n_0 \in \mathbb{N}$ such that

$$|x^*(P_k x^{(n_0)} - x_k)| < 1/m, \quad k = 1, \ldots, m.$$

It follows from (5) that

$$\begin{aligned} \Big|x^*\Big(\sum_{k=1}^{m} x_k\Big)\Big| &\le \Big|\sum_{k=1}^{m} x^*(P_k x^{(n_0)} - x_k)\Big| + \Big|\sum_{k=1}^{m} x^*(P_k x^{(n_0)})\Big| \\ &\le 1 + \|x^*\| \cdot \Big\|\sum_{k=1}^{m} P_k x^{(n_0)}\Big\| \\ &\le 1 + K\|x^*\| \cdot \|x^{(n_0)}\| \le 1 + KM\|x^*\|. \end{aligned}$$

Thus $\{\sum_{k=1}^{m} x_k\}_{m=1}^{\infty}$ is bounded. Therefore there exists an $x \in X$ such that $x = \sum_{k=1}^{\infty} x_k$. Next we want to show that $x^{(n)}$ converges to x weakly in X.

For each fixed $x^* \in X^*$, I_{x^*} defined in (6) is continuous, and hence weakly-weakly continuous. Since $\{x^{(n)}\}_{n=1}^{\infty}$ is a weakly Cauchy sequence in X, $\{I_{x^*}(x^{(n)})\}_{n=1}^{\infty}$ is a weakly Cauchy sequence in ℓ_1, and hence relatively weakly sequentially compact. By the Schur property, $\{I_{x^*}(x^{(n)})\}_{n=1}^{\infty}$ is a relatively sequentially compact subset of ℓ_1. Thus for each $\varepsilon > 0$, there exists an $m_1 \in \mathbb{N}$ such that

$$\sum_{k=m_1+1}^{\infty} |x^*(P_k x^{(n)})| \le \varepsilon/3, \quad n = 1, 2, \ldots. \tag{8}$$

Since $x = \sum_{k=1}^{\infty} x_k$, there exists an $m_2 > m_1$ such that

$$\Big|\sum_{k=m_2+1}^{\infty} x^*(x_k)\Big| \le \varepsilon/3. \tag{9}$$

By (7) there exists an $n_0 \in \mathbb{N}$ such that for each $n > n_0$,

$$|x^*(P_k x^{(n)} - x_k)| < \varepsilon/3, \quad k = 1, \ldots, m_2. \tag{10}$$

It follows from (8)–(10) that for each $n > n_0$,

$$\begin{aligned}|x^*(x^{(n)} - x)| &= \Big|\sum_{k=1}^{\infty} x^*(P_k x^{(n)} - x_k)\Big| \\ &\le \sum_{k=1}^{m_2} |x^*(P_k x^{(n)} - x_k)| \\ &\quad + \sum_{k=m_2+1}^{\infty} |x^*(P_k x^{(n)})| + \Big|\sum_{k=m_2+1}^{\infty} x^*(x_k)\Big| \\ &\le \varepsilon.\end{aligned}$$

Therefore $x^{(n)}$ converges to x weakly in X, and hence X is weakly sequentially complete. ∎

4. Short proof of Lewis's result. For each $k \in \mathbb{N}$, define

$$P_k : U(X) \to U(X), \qquad \overline{x} \mapsto (0, \ldots, 0, \overset{(k)}{x_k}, 0, 0, \ldots). \tag{11}$$

Then by [2, Proposition 5], $\{P_k\}$ is a boundedly complete, unconditional Schauder decomposition of $U(X)$. Hence Theorem 5 yields the following.

COROLLARY 6. *$U(X)$ is weakly sequentially complete if X is weakly sequentially complete.*

Finally, by using the results obtained above, we will give a short proof of the following result of Lewis [8].

THEOREM 7. *Let X and Y be Banach spaces such that one of them has an unconditional basis. Then $X \widehat{\otimes} Y$, the projective tensor product of X and Y, is weakly sequentially complete whenever X and Y are weakly sequentially complete.*

Proof. Suppose that X has an unconditional basis. Since X is weakly sequentially complete, X does not contain a copy of c_0. By James's result [5] (or see [9, Theorem 1.c.10]), X has a boundedly complete unconditional basis. Without loss of generality, we can assume that X has a boundedly complete, 1-unconditional basis. It follows immediately from Theorem 4 and Corollary 6 that $X \widehat{\otimes} Y$ is weakly sequentially complete. ∎

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