COLLOQUIUM MATHEMATICUM

VOL. 146 2017 NO. 1

A NON-COMMUTATIVE BANACH ALGEBRA WHOSE MAXIMAL COMMUTATIVE SUBALGEBRAS ARE ALL MUTUALLY ISOMORPHIC

BY

WIESŁAW ŻELAZKO (Warszawa)

Abstract. The algebra as in the title is constructed as a quotient of the semigroup algebra of the discrete free semigroup with a countable set of generators.

Construction. All the algebras considered here are either real or complex. Let $(e_i)_{i=1}^{\infty}$ be a sequence of variables. Denote by S the (non-unital) free semigroup generated by these variables and provide it with the discrete topology. Denote by S_n the subset of S consisting of all products $\sigma = e_{i_1} \dots e_{i_n}$ of not necessarily different variables.

The unital free algebra in the variables e_i consists of all elements of the form

$$x = \alpha_0(x)e + \sum_{n=1}^{\infty} \sum_{\sigma \in S_n} \alpha_\sigma(x)\sigma, \tag{1}$$

where e is the unit element ($ex = xe = x$ for all x), and only finitely many scalars $\alpha_0(x), \alpha_\sigma(x)$ are non-zero. The product of $\sigma_1 = e_{i_1} \dots e_{i_m}$ and $\sigma_2 = e_{j_1} \dots e_{j_n}$ is given by $\sigma_1\sigma_2 = e_{i_1} \dots e_{i_m} e_{j_1} \dots e_{j_n}$. It defines the convolution product on the above algebra.

The maximal norm (l_1-norm) of an element x of the form (1) is given by

$$\|x\| = |\alpha_0(x)| + \sum_{n=1}^{\infty} \sum_{\sigma \in S_n} |\alpha_\sigma(x)|. \tag{2}$$

It is easy to see that the norm $\|\cdot\|$ is unital ($\|e\| = 1$) and submultiplicative ($\|x_1x_2\| \le \|x_1\|\,\|x_2\|$), and that all linear functionals $x \mapsto \alpha_\sigma(x)$ and $x \mapsto \alpha_0(x)$ are continuous.

2010 *Mathematics Subject Classification*: Primary 46H05; Secondary 46H10; 47L10.
Key words and phrases: Banach algebras, maximal commutative subalgebras, algebras of operators.
Received 6 November 2015; revised 22 January 2016.
Published online 15 July 2016.

DOI: 10.4064/cm6811-2-2016

Denote by A the completion of the above algebra in the norm (2). It is a non-commutative unital Banach algebra (the convolution multiplication extends by continuity to the whole of A). Its elements are also of the form (1), but the summation is this time infinite. The algebra A is in fact a well known object called the semigroup algebra $l_1(S')$ (see [1], in particular pp. 159–160), where S' denotes the semigroup S with unit e adjoined ($S' = S \cup \{e\}$ with $e\sigma = \sigma e = \sigma, e^2 = e$). The only difference is that the elements of A, treated as a free algebra, are of the form (1), while the elements of $l_1(S')$ are functions on S' given by $\sigma \mapsto \alpha_\sigma(x)$. In this paper it will be convenient to treat A as a free algebra. Its only maximal ideal X, which is the two-sided ideal of elements x for which $\alpha_0(x) = 0$, is topologically and algebraically isomorphic to $l_1(S)$. We shall use the following direct sum decomposition of the Banach space X:

$$X = X_1 \oplus X_2 \oplus X_3,$$

where the elements of X_1 are of the form

$$x = \sum_{i=1}^{\infty} \alpha_i(x) e_i,$$

the elements of X_2 are of the form

$$x = \sum_{i,j=1}^{\infty} \alpha_{i,j}(x) e_i e_j, \tag{3}$$

and

$$X_3 = \bigcap \{\ker \alpha_\sigma : \sigma \in S_1 \cup S_2\}.$$

Clearly X_1, X_2, X_3 are closed linear subspaces of X.

Denote by Y the closed linear subspace of X_2 consisting of those elements of the form (3) for which $\alpha_{i,j}(x) = \alpha_{j,i}(x)$ for all i, j, i.e.

$$x = \sum_{i,j=1}^{\infty} \beta_{i,j}(x)(e_i e_j + e_j e_i), \quad \text{where} \quad \beta_{i,j}(x) = \alpha_{i,j}(x) = \alpha_{j,i}(x).$$

It is a closed subspace of X_2. This follows from the formula

$$Y = \Big\{ x \in X_2 : x \in \bigcap_{i,j=1}^{\infty} \ker(\alpha_{i,j} - \alpha_{j,i}) \Big\}.$$

Clearly, we have

$$XY \cup YX \subset X_3,$$

and so

$$AY \cup YA \subset Y \oplus X_3.$$

Consequently, the set

$$I = Y \oplus X_3$$

is a closed two-sided ideal in A.

Let now $\mathcal{A}$ be the quotient A/I. It is a unital Banach algebra. Without loss of precision we can denote the cosets $[e_i]$ modulo I by the same letters e_i, and similarly for the unit e. Since $Y \subset I$, we have

$$e_i e_j = -e_j e_i \quad \text{for all } i, j \in \mathbb{N}. \tag{4}$$

In particular

$$e_i^2 = 0 \quad \text{for all } i \in \mathbb{N}. \tag{5}$$

Also $X_3 \subset I$ implies

$$e_i e_j e_k = 0 \quad \text{for all } i, j, k \in \mathbb{N}. \tag{6}$$

By (4) every element in $\mathcal{A}$ can be written in the form

$$x = \alpha(x)e + \sum_{i=1}^{\infty} \alpha_i(x) e_i + \sum_{1 \le i < j} \alpha_{i,j}(x) g_{i,j}, \quad \text{where} \quad g_{i,j} = e_i e_j,\ i < j.$$

Formula (6) implies

$$e_i g_{k,l} = g_{k,l} e_i = g_{k,l} g_{m,n} = 0 \quad \text{for all } i, k, l, m, n \in \mathbb{N},\ k < l,\ m < n. \tag{7}$$

Observe that by (4) and (5) every element in $\mathcal{A}$ of the form

$$x = \sum_i \alpha_i(x) e_i \tag{8}$$

satisfies

$$x^2 = 0. \tag{9}$$

This follows from the fact that for any $i, j \in \mathbb{N}$ we have

$$\alpha_i(x)\alpha_j(x)(e_i e_j + e_j e_i) = 0.$$

We shall show now that two elements of the form (8) do not commute unless one is a scalar multiple of the other. Define the *support* of such an element x to be the set $\{i \in \mathbb{N} : \alpha_i(x) \neq 0\}$. If elements $x \neq 0 \neq y$ of the form (8) have different supports then there is an index i such that $\alpha_i(x) \neq 0$ and $\alpha_i(y) = 0$. There is also an index $j \neq i$ with $\alpha_j(y) \neq 0$. Then the product xy has a non-zero coefficient of $e_i e_j$ equal to $\alpha_i(x)\alpha_j(y)$, while yx has this coefficient equal to $-\alpha_i(x)\alpha_j(y)$. Thus $xy \neq yx$. If x and y have the same supports, but are not proportional, then (after multiplying them by suitable scalars), we can assume that there are $i \neq j$ with $\alpha_i(x) = 1 = \alpha_i(y)$ and with non-zero $\alpha_j(x) \neq \alpha_j(y)$. Then the coefficient of $e_i e_j$ in xy equals $\alpha_i(x)\alpha_j(y)$, while in yx this coefficient is $-\alpha_i(x)\alpha_j(y)$, and again $xy \neq yx$. This means in particular that two elements of the form (8) which are not proportional cannot belong to the same maximal commutative subalgebra of $\mathcal{A}$.

We can now give the general form of a maximal commutative subalgebra of $\mathcal{A}$. Set first $\mathcal{A}_0 = \operatorname{span}(g_{i,j} : i, j \in \mathbb{N})$. By (7) it is a subalgebra of $\mathcal{A}$ with trivial (zero) multiplication. Fix a non-zero element x of the form (8) and set

$M_x = \mathrm{span}(\mathcal{A}_0, x)$. It is again a subalgebra of $\mathcal{A}$ with trivial multiplication. Denote by $\mathcal{A}_x$ its unitization $\{z + \alpha e : z \in M_x, \alpha \in \mathbb{K}\}$, where $\mathbb{K}$ is the field of scalars ($\mathbb{C}$ or $\mathbb{R}$). The maximality of this subalgebra follows from the fact that no non-zero element of the form (8), different from x, can be added to it without destroying commutativity. Clearly, all algebras $\mathcal{A}_x$ are mutually isomorphic, which gives the desired result.

Remarks. Note that $\mathcal{A}$ has exactly one multiplicative-linear functional $f(x) = \alpha_0(x)$, whose kernel is the only maximal ideal of $\mathcal{A}$, both as a two-sided and a one-sided ideal. An element x in $\mathcal{A}$ is invertible if and only if $f(x) \neq 0$.

It can be shown directly that the family $(\mathcal{A}_x)$ of maximal commutative subalgebras of $\mathcal{A}$ is uncountable. This also follows immediately from a simple algebraic fact: every non-commutative algebra contains an uncountable family of maximal commutative subalgebras. To see this, observe that $ab \neq ba$ implies that the family $\{a + tb : t \in \mathbb{R}\}$ consists of mutually non-commuting elements, so they must belong to different maximal commutative subalgebras. In particular, every non-commutative Banach algebra either contains uncountably many mutually non-isomorphic maximal commutative subalgebras, or at least one of these subalgebras has uncountably many copies. The author does not know of any example of a separable Banach algebra possessing uncountably many non-isomorphic maximal commutative subalgebras (a non-separable example can be constructed). The above extreme example has only one maximal commutative subalgebra which is repeated uncountably many times.

There also exists another Banach algebra of extreme character. In [3] an infinite-dimensional non-commutative Banach algebra was constructed with only one infinite-dimensional maximal commutative subalgebra, while all remaining maximal commutative subalgebras are mutually isomorphic and of dimension two. Two is the minimal possible dimension of a maximal commutative subalgebra of a non-commutative unital algebra. This example shows that an infinite-dimensional Banach algebra may have a finite-dimensional maximal commutative subalgebra. On the other hand, by a theorem of Thomas Laffey [2],

Every infinite-dimensional algebra contains an infinite-dimensional maximal commutative subalgebra.

In [4] it is shown that in $A = L(X)$, where X is a Banach space ($\dim X > 1$), every maximal commutative subalgebra occurs in uncountably many isomorphic copies. It is also shown in [4] that if X is infinite-dimensional, then A has at least countably many mutually non-isomorphic maximal commutative subalgebras.

The above is connected with the following question of A. Pełczyński (cf. [4]): Suppose that X is an infinite-dimensional Banach space. In how many ways can it be made into a commutative unital Banach algebra? It is well known that if A is a commutative Banach algebra, then under the left regular representation $x \mapsto T_x$, where $T_x y = xy$, $x, y \in A$, the algebra A becomes a maximal commutative subalgebra of $L(A)$. So all possible multiplications on X coincide with the multiplications in all mutually non-isomorphic maximal commutative subalgebras of $L(X)$, which, as Banach spaces, are isomorphic to X. The above result of [4] shows that there are at least countably many such multiplications, but perhaps it is possible to achieve uncountably many.

It would be of interest to know the answer to the following

Problem. Under which conditions on given two commutative unital Banach algebras, does there exist a (unital) Banach algebra containing these algebras as maximal commutative subalgebras?

Some conditions are necessary here: after submission of this paper the author has constructed a pair of commutative Banach algebras which cannot be maximal commutative subalgebras in any common Banach algebra.

Acknowledgments. The author is greatly indebted to the referee whose remarks made it possible to rewrite the paper in a clearer way.

REFERENCES

[1] H. G. Dales, *Banach Algebras and Automatic Continuity*, Oxford Univ. Press, 2000.
[2] T. J. Laffey, *Commutative subalgebras of infinite dimensional algebras*, Bull. London Math. Soc. 5 (1973), 312–314.
[3] W. Żelazko, *An infinite dimensional Banach algebra with all but one maximal abelian subalgebras of dimension two*, Comm. Math. 48 (2008), 99–101.
[4] W. Żelazko, *All maximal commutative subalgebras occur in $L(X)$ uncountably many times*, Funct. Approx. Comment. Math. 53 (2015), 189–192.

Wiesław Żelazko
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-656 Warszawa, Poland
E-mail: zelazko@impan.pl