Error functions, Mordell integrals and
an integral analogue of a partial theta function

by

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*Dedicated to Professor Bruce C. Berndt
on the occasion of his 75th birthday*

1. Introduction. Mordell initiated the study of the integral

$$\int_{-\infty}^{\infty} \frac{e^{ax^2+bx}}{e^{cx}+d}\, dx, \quad \operatorname{Re} a < 0,$$

in his two influential papers [33, 34]. Prior to his work, special cases of this integral had been studied, for example, by Riemann in his Nachlass [47] to derive the approximate functional equation for the Riemann zeta function, by Kronecker [27, 28] to derive the reciprocity for Gauss sums, and by Lerch [30, 31, 32]. Mordell showed that the above integral can be reduced to two standard forms, namely,

$$(1.1) \qquad \varphi(z,\tau) := \tau \int_{-\infty}^{\infty} \frac{e^{\pi i \tau x^2 - 2\pi z x}}{e^{2\pi x} - 1}\, dx,$$

$$\sigma(z,\tau) := \int_{-\infty}^{\infty} \frac{e^{\pi i \tau x^2 - 2\pi z x}}{e^{2\pi \tau x} - 1}\, dx,$$

for $\operatorname{Im} \tau > 0$, and was the first to study the properties of these integrals

with respect to modular transformations. Bellmann [5, p. 52] coined the term "Mordell integrals" for these types of integrals.

Mordell integrals play an important role in the groundbreaking Ph.D. thesis of Zwegers [50] which sheds a clear light on Ramanujan's mock theta functions. The definition of a Mordell integral $h(z, \tau)$, $\mathrm{Im}\,\tau > 0$, employed by Zwegers [50, p. 6], and now standard in the contemporary literature, is

$$h(z, \tau) := \int\limits_{-\infty}^{\infty} \frac{e^{\pi i \tau x^2 - 2\pi z x}}{\cosh \pi x} \, dx.$$

As remarked by Zwegers himself [50, p. 5], $h(z; \tau)$ is essentially the function $\varphi(z, \tau)$ defined in (1.1), i.e.,

$$(1.2) \qquad h(z, \tau) = \frac{-2i}{\tau} e^{-(\pi i \tau / 4 + \pi i z)} \varphi\left(z + \frac{\tau - 1}{2}, \tau\right).$$

Kuznetsov [29] has recently used $h(z; \tau)$ to simplify Hiary's algorithm [23] for computing the truncated theta function $\sum_{k=0}^{n} \exp(2\pi i(zk + \tau k^2))$, which in turn is used to compute $\zeta(1/2 + it)$ to within $\pm t^{-\lambda}$ in $O_\lambda(t^{1/3}(\ln t)^\kappa)$ arithmetic operations [22]. We refer the reader to a more recent article [10] and the references therein for further applications of Mordell integrals.

In [42] and [43], Ramanujan studied the integrals

$$\phi_\omega(z) := \int\limits_{0}^{\infty} \frac{\cos(\pi x z)}{\cosh(\pi x)} e^{-\pi \omega x^2} \, dx, \qquad \psi_\omega(z) := \int\limits_{0}^{\infty} \frac{\sin(\pi x z)}{\sinh(\pi x)} e^{-\pi \omega x^2} \, dx.$$

Of course, we require $\mathrm{Re}\,\omega > 0$ for the integrals to converge. If we replace ω by $-i\tau$ with $\mathrm{Im}\,\tau > 0$ and z by $2iz$, then the integral ϕ is nothing but the Mordell integral. That is,

$$h(z, \tau) = 2\phi_{-i\tau}(2iz).$$

Later, Ramanujan briefly worked on these two integrals in a two-page fragment transcribed by G. N. Watson from Ramanujan's loose papers and published along with Ramanujan's Lost Notebook [46, pp. 221–222]. See also [4, pp. 307–328] for details.

A third integral of this kind studied by Ramanujan [46, p. 198] is

$$F_\omega(z) := \int\limits_{0}^{\infty} \frac{\sin(\pi x z)}{\tanh(\pi x)} e^{-\pi \omega x^2} \, dx.$$

As before, one needs $\mathrm{Re}\,\omega > 0$ for convergence. One can easily rephrase it as

$$(1.3) \qquad F_\omega(z) = \int\limits_{-\infty}^{\infty} \frac{e^{-\pi \omega x^2} \sin(\pi x z)}{e^{2\pi x} - 1} \, dx = \int\limits_{-\infty}^{\infty} \frac{e^{-\pi \omega x^2 + 2\pi x} \sin(\pi x z)}{e^{2\pi x} - 1} \, dx.$$

For $\operatorname{Im}\tau > 0$, this integral F is connected to the integral $\varphi(z,\tau)$ in (1.1) (and hence to the Mordell integral $h(z,\tau)$) via

$$F_{-i\tau}(2iz) = \frac{1}{2i\tau}(\varphi(z,\tau) - \varphi(-z,\tau)).$$

Thus Mordell integrals pervade Ramanujan's papers and his Lost Notebook. In a further support to this claim, we refer the readers to Andrews' interesting paper [2].

Berndt and Xu [9] have proved all of the properties of $F_\omega(z)$ claimed by Ramanujan in the Lost Notebook. Following Ramanujan, we assume $\omega > 0$. Supposing a certain result holds for $\omega > 0$, it is clear that by analytic continuation, one may be able to extend it to complex values of ω in a certain region containing the positive real line. Among the various properties claimed by Ramanujan is the transformation

$$(1.4) \qquad F_\omega(z) = \frac{-i}{\sqrt{\omega}}e^{-\frac{\pi z^2}{4\omega}}F_{1/\omega}\left(\frac{iz}{\omega}\right).$$

In view of this property, Berndt and Xu call $F_\omega(z)$ an integral analogue of a theta function.

Assume $\alpha > 0$, let $\omega = \alpha^2$ and replace z by $\alpha z/\sqrt{\pi}$ in (1.4). Using (1.3), one sees that the above transformation translates into the following identity:

For $\alpha, \beta > 0$ such that $\alpha\beta = 1$,

$$(1.5) \qquad \sqrt{\alpha}\, e^{z^2/8} \int_{-\infty}^{\infty} \frac{e^{-\pi\alpha^2 x^2}\sin(\sqrt{\pi}\,\alpha xz)}{e^{2\pi x} - 1}\, dx$$

$$= \sqrt{\beta}e^{-z^2/8} \int_{-\infty}^{\infty} \frac{e^{-\pi\beta^2 x^2}\sinh(\sqrt{\pi}\,\beta xz)}{e^{2\pi x} - 1}\, dx.$$

Now consider the integral

$$(1.6) \qquad \int_{0}^{\infty} \frac{e^{-\pi\alpha^2 x^2}\sin(\sqrt{\pi}\,\alpha xz)}{e^{2\pi x} - 1}\, dx.$$

In analogy with the partial theta function which is the same as a theta function but summed over only half of the integer lattice, we call the above integral an integral analogue of partial theta function. Among other things, partial theta functions play a significant role in proving many non-trivial theorems in the theory of partitions [3, Section 7]. So it is natural to study the above integral analogue of the partial theta function both for its own sake, and from the point of view of finding its applications. Indeed, our theorems on these integral analogues help us evaluate some non-trivial integrals involving hypergeometric functions. They, along with the results of Ramanujan that are proved in [9], allow us to anticipate a rich theory of

integral analogues of theta functions similar to Jacobi's theory of classical theta functions.

Note that unlike $F_\omega(z)$, the integral in (1.6) cannot be expressed solely in terms of Mordell integrals, which makes them further worth studying. Also unlike the transformation formula for the Jacobi theta function which trivially gives the transformation formula for the corresponding partial theta function [49, p. 22, (2.6.3)], the transformation in (1.5) does not give rise to a corresponding transformation for the integral in (1.6). (It is easy to check that the integrands in (1.5) are not even functions of x.) Nevertheless, the primary goal of this paper is to prove a new and interesting modular transformation for the integral in (1.6) which involves error functions.

The *error function* $\mathrm{erf}(z)$ and the *complementary error function* $\mathrm{erfc}(z)$, defined by [48, p. 275]

$$(1.7) \qquad \mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt$$

and

$$\mathrm{erfc}(z) = 1 - \mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt$$

respectively, are two important special functions having a number of applications in probability theory, statistics, physics and partial differential equations. In probability, they are related to the Gaussian normal distribution. Glashier [17] was the first to coin the term *Error-function* and then the term *Error-function complement* in the sequel [18]. However, his definitions are exactly opposite to the standard definitions given above and do not involve the normalization factor $2/\sqrt{\pi}$.

The imaginary error function $\mathrm{erfi}(z)$ is defined by [24, p. 32] (^{1})

$$(1.8) \qquad \mathrm{erfi}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{t^2}\, dt.$$

From (1.7) and (1.8), it is straightforward that

$$(1.9) \qquad \mathrm{erf}(iz) = i\,\mathrm{erfi}(z).$$

Below we give the transformation linking the integrals of the type in (1.6) with the error functions $\mathrm{erf}(z)$ and $\mathrm{erfi}(z)$. This transformation is of the form $G(z, \alpha) = G(iz, \beta)$ for $\alpha, \beta > 0$, $\alpha\beta = 1$ and $z \in \mathbb{C}$. It is also related to an integral involving the Riemann Ξ-function, which is defined by

$$(1.10) \qquad \Xi(t) := \xi(1/2 + it),$$

where $\xi(s)$ is Riemann's ξ-function defined by [12, p. 60]

$$(1.11) \qquad \xi(s) := \tfrac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

(^{1}) This definition differs by a factor of $2/\sqrt{\pi}$ from the definition in [24].

$\Gamma(s)$ and $\zeta(s)$ being the Gamma function and the Riemann zeta function respectively. It involves the case $p = q = 1$ of the generalized hypergeometric function [39, p. 73]

$$(1.12) \qquad {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; w) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{w^n}{n!},$$

where $(a)_n := a(a+1) \cdots (a+n-1) = \Gamma(a+n)/\Gamma(a)$. The hypergeometric function in this special case is known as Kummer's confluent hypergeometric function.

THEOREM 1.1. *Let* $z \in \mathbb{C}$, $\alpha > 0$ *and let* $\Delta(\alpha, z, (1+it)/2)$ *be defined by*

$$(1.13) \qquad \Delta(x, z, s) := \omega(x, z, s) + \omega(x, z, 1-s),$$

with

$$(1.14) \qquad \omega(x, z, s) := x^{1/2-s} e^{-z^2/8} {}_1F_1(1 - s/2; 3/2; z^2/4).$$

Let $\Xi(t)$ *be defined in* (1.10). *Then for* $\alpha\beta = 1$,

$$(1.15) \quad \sqrt{\alpha}\, e^{z^2/8} \left(\mathrm{erf}\left(\frac{z}{2}\right) - 4 \int_0^{\infty} \frac{e^{-\pi\alpha^2 x^2} \sin(\sqrt{\pi}\,\alpha xz)}{e^{2\pi x} - 1}\, dx \right)$$

$$= \sqrt{\beta}\, e^{-z^2/8} \left(\mathrm{erfi}\left(\frac{z}{2}\right) - 4 \int_0^{\infty} \frac{e^{-\pi\beta^2 x^2} \sinh(\sqrt{\pi}\,\beta xz)}{e^{2\pi x} - 1}\, dx \right)$$

$$= \frac{z}{8\pi^2} \int_0^{\infty} \Gamma\left(\frac{-1+it}{4}\right) \Gamma\left(\frac{-1-it}{4}\right) \Xi\left(\frac{t}{2}\right) \Delta\left(\alpha, z, \frac{1+it}{2}\right) dt,$$

where $\mathrm{erf}(z)$ *and* $\mathrm{erfi}(z)$ *are defined in* (1.7) *and* (1.8) *respectively.*

We prove Theorem 1.1 by evaluating the integral on the extreme right of (1.15) to be equal to the extreme left, and by exploiting the fact that when we replace α by β and z by iz in the integral, we get it back except for an extra i in front.

This transformation generalizes a formula of Ramanujan which he gave in his first letter to Hardy [44, p. XXVI] and which also appears in [40, (13)]. This formula is equivalent to the first equality in the following identity, valid for $\alpha\beta = 1$, and which is also due to Ramanujan [41]:

$$(1.16) \quad \alpha^{1/2} - 4\pi\alpha^{3/2} \int_0^{\infty} \frac{xe^{-\pi\alpha^2 x^2}}{e^{2\pi x} - 1}\, dx = \beta^{1/2} - 4\pi\beta^{3/2} \int_0^{\infty} \frac{xe^{-\pi\beta^2 x^2}}{e^{2\pi x} - 1}\, dx$$

$$= \frac{1}{4\pi\sqrt{\pi}} \int_0^{\infty} \Gamma\left(\frac{-1+it}{4}\right) \Gamma\left(\frac{-1-it}{4}\right) \Xi\left(\frac{t}{2}\right) \cos\left(\frac{1}{2} t \log\alpha\right) dt.$$

Mordell [34, p. 331] rewrote the first equality in the above formula as (^{2})

$$(1.17) \qquad \int_{-\infty}^{\infty} \frac{xe^{-\pi ix^2/\tau}}{e^{2\pi x} - 1}\, dx = (-i\tau)^{3/2} \int_{-\infty}^{\infty} \frac{xe^{-\pi i\tau x^2}}{e^{2\pi x} - 1}\, dx.$$

That (1.16) is a special case of Theorem 1.1 is not difficult to derive: for $z \neq 0$, divide both sides by z, let $z \to 0$ and note that

$$\lim_{z\to 0} \frac{\mathrm{erf}(z)}{z} = \frac{2}{\sqrt{\pi}} = \lim_{z\to 0} \frac{\mathrm{erfi}(z)}{z}.$$

A one-variable generalization of the integral on the extreme right-hand side in (1.16) was given in [13, Theorem 1.5], which in turn gave a generalization of the extreme left side. However, this general integral is not invariant under the simultaneous application of $\alpha \to \beta$ and $z \to iz$, and so a transformation formula generalizing the first equality in (1.16) could not be obtained there. This shortcoming is overcome in Theorem 1.1.

We also obtain another transformation involving error functions that is complementary to the one in Theorem 1.1.

THEOREM 1.2. *Let* $z \in \mathbb{C}$. *For* $\alpha, \beta > 0$, $\alpha\beta = 1$,

$$(1.18) \qquad \sqrt{\alpha}\, e^{z^2/8}\left(\mathrm{erf}\left(\frac{z}{2}\right) + 4 \int_{-\infty}^{0} \frac{e^{-\pi\alpha^2 x^2} \sin(\sqrt{\pi}\,\alpha xz)}{e^{2\pi x} - 1}\, dx\right)$$

$$= \sqrt{\beta}\, e^{-z^2/8}\left(\mathrm{erfi}\left(\frac{z}{2}\right) + 4 \int_{-\infty}^{0} \frac{e^{-\pi\beta^2 x^2} \sinh(\sqrt{\pi}\,\beta xz)}{e^{2\pi x} - 1}\, dx\right).$$

It is important to observe here that subtracting the corresponding sides of the first equality in (1.15) from those of (1.18) results in (1.5), thus providing a new proof of (1.5), and hence of (1.4).

Let χ denote a primitive Dirichlet character modulo q. The character analogue $\Xi(t, \chi)$ of $\Xi(t)$ is given by

$$\Xi(t, \chi) := \xi(1/2 + it, \chi),$$

where $\xi(s, \chi) := (\pi/q)^{-(s+a)/2}\Gamma\left(\frac{s+a}{2}\right)L(s, \chi)$, and $a = 0$ if $\chi(-1) = 1$ and $a = 1$ if $\chi(-1) = -1$. The functional equation of $\xi(s, \chi)$ is $\xi(1 - s, \overline{\chi}) = \epsilon(\chi)\xi(s, \chi)$, where $\epsilon(\chi) = i^a q^{1/2}/G(\chi)$ and $G(\chi) = \sum_{m=1}^{q} \chi(m)e^{2\pi im/q}$ is the Gauss sum. See [12, pp. 69–72]. For real primitive characters, we have

$$G(\chi) = \begin{cases} \sqrt{q} & \text{for } \chi \text{ even,} \\ i\sqrt{q} & \text{for } \chi \text{ odd.} \end{cases}$$

Hence the functional equation in this case reduces to $\xi(1 - s, \chi) = \xi(s, \chi)$, which also gives $\Xi(-t, \chi) = \Xi(t, \chi)$.

(^{2}) There is a misprint in Mordell's formulation of (1.17), namely, there is an extra minus sign in front of the right-hand side which should not be present.

We now give the analogues of Theorem 1.1 for real primitive characters.

THEOREM 1.3. *Let $z \in \mathbb{C}$ and let α and β be positive numbers such that $\alpha\beta = 1$. Let χ be a real primitive Dirichlet character modulo q.*

(i) *If χ is even,*

$$(1.19) \quad \sqrt{\alpha}\, e^{z^2/8} \int_0^\infty e^{-\pi\alpha^2 x^2/q} \sin\left(\frac{\sqrt{\pi}\,\alpha xz}{\sqrt{q}}\right) \frac{\sum_{r=1}^{q-1} \chi(r) e^{-2\pi rx/q}}{1 - e^{-2\pi x}}\, dx$$

$$= \sqrt{\beta}\, e^{-z^2/8} \int_0^\infty e^{-\pi\beta^2 x^2/q} \sinh\left(\frac{\sqrt{\pi}\,\beta xz}{\sqrt{q}}\right) \frac{\sum_{r=1}^{q-1} \chi(r) e^{-2\pi rx/q}}{1 - e^{-2\pi x}}\, dx$$

$$= \frac{z\sqrt{q}}{16\pi^2} \int_0^\infty \Gamma\left(\frac{3+it}{4}\right) \Gamma\left(\frac{3-it}{4}\right) \Xi\left(\frac{t}{2}, \chi\right) \Delta\left(\alpha, z, \frac{1+it}{2}\right) dt.$$

(ii) *If χ is odd,*

$$(1.20) \quad \sqrt{\alpha}\, e^{z^2/8} \int_0^\infty e^{-\pi\alpha^2 x^2/q} \cos\left(\frac{\sqrt{\pi}\,\alpha xz}{\sqrt{q}}\right) \frac{\sum_{r=1}^{q-1} \chi(r) e^{-2\pi rx/q}}{1 - e^{-2\pi x}}\, dx$$

$$= \sqrt{\beta}\, e^{-z^2/8} \int_0^\infty e^{-\pi\beta^2 x^2/q} \cosh\left(\frac{\sqrt{\pi}\,\beta xz}{\sqrt{q}}\right) \frac{\sum_{r=1}^{q-1} \chi(r) e^{-2\pi rx/q}}{1 - e^{-2\pi x}}\, dx$$

$$= \frac{1}{16\pi^{3/2}} \int_0^\infty \Gamma\left(\frac{1+it}{4}\right) \Gamma\left(\frac{1-it}{4}\right) \Xi\left(\frac{t}{2}, \chi\right) \nabla\left(\alpha, z, \frac{1+it}{2}\right) dt.$$

Note that the sums inside the above integrals are Gauss sums of purely imaginary arguments.

The first equality in (1.16) can be rewritten for $\alpha\beta = \pi^2$ as

$$(1.21) \quad \alpha^{-1/4}\left(1 + 4\alpha \int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x} - 1}\, dx\right) = \beta^{-1/4}\left(1 + 4\beta \int_0^\infty \frac{xe^{-\beta x^2}}{e^{2\pi x} - 1}\, dx\right).$$

In [45, Volume 2, p. 268], Ramanujan gives an elegant approximation to the above expressions:

Let $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$. Define

$$I(\alpha) := \alpha^{-1/4}\left(1 + 4\alpha \int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x} - 1}\, dx\right).$$

Then

$$(1.22) \quad I(\alpha) = \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{2}{3}\right)^{1/4}, \quad \text{``nearly''.}$$

As mentioned by Berndt and Evans [8], Ramanujan frequently used the words "nearly" or "very nearly" at the end of his asymptotic expansions

and approximations. The above approximation is very good for values of α that are either very small or very large. A proof of the above fact was given in [8], where as an intermediate result, the following asymptotic expansion for $I(\alpha)$ as $\alpha \to 0$ was first obtained:

$$I(\alpha) \sim \frac{1}{\alpha^{1/4}} + \frac{\alpha^{3/4}}{6} - \frac{\alpha^{7/4}}{60} + \cdots .$$

Observe that for $\alpha\beta = \pi^2$ and $z \neq 0$, the first equality in Theorem 1.1 can be rewritten as follows:

$$
(1.23) \qquad I(z, \alpha) := \frac{\sqrt{\pi}}{z} \alpha^{-1/4} e^{z^2/8} \operatorname{erf}\left(\frac{z}{2}\right)
$$
$$
+ \frac{4}{z} \alpha^{1/4} e^{-z^2/8} \int_0^\infty \frac{e^{-\alpha x^2} \sinh(\sqrt{\alpha}\, xz)}{e^{2\pi x} - 1}\, dx
$$
$$
= \frac{\sqrt{\pi}}{z} \beta^{-1/4} e^{-z^2/8} \operatorname{erfi}\left(\frac{z}{2}\right)
$$
$$
+ \frac{4}{z} \beta^{1/4} e^{z^2/8} \int_0^\infty \frac{e^{-\beta x^2} \sin(\sqrt{\beta}\, xz)}{e^{2\pi x} - 1}\, dx =: I(iz, \beta),
$$

of which (1.21) is the special case when $z \to 0$. The following general asymptotic expansion holds for the two sides in the above identity as $\alpha \to 0$, or equivalently as $\beta \to \infty$.

THEOREM 1.4. *Fix $z \in \mathbb{C}$. As $\alpha \to 0$,*

$$
(1.24) \qquad I(z, \alpha) \sim
$$
$$
\frac{-2}{\sqrt{\pi}} \alpha^{-1/4} e^{z^2/8} \sum_{m=0}^\infty \left(\frac{-\alpha}{\pi^2}\right)^m \zeta(2m) \Gamma\left(m + \frac{1}{2}\right) {}_1F_1\left(m + \frac{1}{2}; \frac{3}{2}; \frac{-z^2}{4}\right).
$$

That is,

$$
I(z, \alpha) \sim \frac{\sqrt{\pi}}{z} e^{z^2/8} \operatorname{erf}\left(\frac{z}{2}\right) \alpha^{-1/4} + \frac{e^{-z^2/8}}{6} \alpha^{3/4} + \frac{(z^2 - 6)e^{-z^2/8}}{360} \alpha^{7/4}
$$
$$
+ \frac{(60 - 20z^2 + z^4)e^{-z^2/8}}{15120} \alpha^{11/4} + \cdots .
$$

Note that both sides of (1.23) are even functions of z. If we successively differentiate (1.23) n times with respect to z and then let $z \to 0$, we do not get anything interesting for odd n. However for n even, two different behaviors are noted.

First, $n \equiv 0 \pmod 4$, i.e., $n = 4k$, $k \in \mathbb{N} \cup \{0\}$, gives the following transformation of the form $H_k(\alpha) = H_k(\beta)$.

THEOREM 1.5. *Let $\alpha\beta = \pi^2$. Then for a non-negative integer k,*

$$(1.25) \quad \alpha^{-1/4} {}_2F_1\left(-2k, 1; \frac{3}{2}; 2\right) + 4\alpha^{3/4} \int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x} - 1} {}_1F_1\left(-2k; \frac{3}{2}; 2\alpha x^2\right) dx$$

$$= \beta^{-1/4} {}_2F_1\left(-2k, 1; \frac{3}{2}; 2\right) + 4\beta^{3/4} \int_0^\infty \frac{xe^{-\beta x^2}}{e^{2\pi x} - 1} {}_1F_1\left(-2k; \frac{3}{2}; 2\beta x^2\right) dx.$$

Ramanujan's approximation in (1.22) is a special case, when $k = 0$, of the following result:

THEOREM 1.6. *Let k be a non-negative integer. Both sides of (1.25) are approximated by*

$$(1.26) \quad \alpha^{-1/4} {}_2F_1\left(-2k, 1; \frac{3}{2}; 2\right) + 4\alpha^{3/4} \int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x} - 1} {}_1F_1\left(-2k; \frac{3}{2}; 2\alpha x^2\right) dx$$

$$= {}_2F_1\left(-2k, 1; \frac{3}{2}; 2\right) \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{2}{3 \cdot {}_2F_1(-2k, 1; 3/2; 2)}\right)^{1/4}, \text{``nearly''}.$$

Again, the above right side is a very good approximation of the left side for the values of α that are either very small or very large.

When $n \equiv 2 \pmod{4}$, i.e., $n = 4k + 2, k \in \mathbb{N} \cup \{0\}$, we get a transformation of the form $J_k(\alpha) = -J_k(\beta)$ given below.

THEOREM 1.7. *Let $\alpha\beta = \pi^2$ and let k be a non-negative integer. Then*

$$(1.27) \quad J_k(\alpha) := \alpha^{-1/4} {}_2F_1\left(-2k - 1, 1; \frac{3}{2}; 2\right)$$

$$+ 4\alpha^{3/4} \int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x} - 1} {}_1F_1\left(-2k - 1; \frac{3}{2}; 2\alpha x^2\right) dx$$

$$= -\beta^{-1/4} {}_2F_1\left(-2k - 1, 1; \frac{3}{2}; 2\right)$$

$$- 4\beta^{3/4} \int_0^\infty \frac{xe^{-\beta x^2}}{e^{2\pi x} - 1} {}_1F_1\left(-2k - 1; \frac{3}{2}; 2\beta x^2\right) dx =: -J_k(\beta).$$

In particular, $J_k(\pi) = 0$, which results in a beautiful exact evaluation of the integral in (1.27).

COROLLARY 1.8. *For any non-negative integer k,*

$$(1.28) \quad \int_0^\infty \frac{xe^{-\pi x^2}}{e^{2\pi x} - 1} {}_1F_1\left(-2k - 1; \frac{3}{2}; 2\pi x^2\right) dx = -\frac{1}{4\pi} {}_2F_1\left(-2k - 1, 1; \frac{3}{2}; 2\right).$$

Results corresponding to the ones in Theorem 1.5–Corollary 1.8 that can be obtained by writing Theorem 1.2 in an alternative form (see (4.12) below)

are collectively stated in Theorem 5.1 at the end of Section 5. When we combine the results from Theorem 1.5–Corollary 1.8 with those in Theorem 5.1, we obtain the following interesting theorem.

THEOREM 1.9. *Let* α, β *be positive numbers such that* $\alpha\beta = \pi^2$ *and let* k *be any non-negative integer. Then*

$$\alpha^{3/4} \int_{-\infty}^{\infty} \frac{xe^{-\alpha x^2}}{e^{2\pi x} - 1} \, {}_1F_1\left(-2k; \frac{3}{2}; 2\alpha x^2\right) dx$$

$$= \beta^{3/4} \int_{-\infty}^{\infty} \frac{xe^{-\beta x^2}}{e^{2\pi x} - 1} \, {}_1F_1\left(-2k; \frac{3}{2}; 2\beta x^2\right) dx,$$

$$(1.29) \quad \alpha^{3/4} \int_{-\infty}^{\infty} \frac{xe^{-\alpha x^2}}{e^{2\pi x} - 1} \, {}_1F_1\left(-2k; \frac{3}{2}; 2\alpha x^2\right) dx$$

$$= \frac{1}{2} \cdot {}_2F_1\left(-2k, 1; \frac{3}{2}; 2\right) \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{2}{3 \cdot {}_2F_1(-2k, 1; 3/2; 2)}\right)^{1/4}, \text{ ``nearly''},$$

$$(1.30) \quad \alpha^{3/4} \int_{-\infty}^{\infty} \frac{xe^{-\alpha x^2}}{e^{2\pi x} - 1} \, {}_1F_1\left(-2k - 1; \frac{3}{2}; 2\alpha x^2\right) dx$$

$$= -\beta^{3/4} \int_{-\infty}^{\infty} \frac{xe^{-\beta x^2}}{e^{2\pi x} - 1} \, {}_1F_1\left(-2k - 1; \frac{3}{2}; 2\beta x^2\right) dx.$$

In particular when $\alpha = \beta = \pi$, *we have*

$$(1.31) \quad \int_{-\infty}^{\infty} \frac{xe^{-\pi x^2}}{e^{2\pi x} - 1} \, {}_1F_1\left(-2k - 1; \frac{3}{2}; 2\pi x^2\right) dx = 0.$$

In [41], Ramanujan considered two integrals, one being that on the extreme right of (1.16), and the second one given by
$$(1.32)$$
$$\int_0^{\infty} \Gamma\left(\frac{z - 1 + it}{4}\right) \Gamma\left(\frac{z - 1 - it}{4}\right) \Xi\left(\frac{t + iz}{2}\right) \Xi\left(\frac{t - iz}{2}\right) \frac{\cos\left(\frac{1}{2}t\log\alpha\right) dt}{t^2 + (z + 1)^2}.$$

Oloa [35, Eq. 1.5] found the asymptotic expansion ([3]) of the special case of this integral when $z = 0$, namely, as $\alpha \to \infty$ (with γ the Euler constant),

$$\frac{1}{\pi^{3/2}} \int_0^{\infty} \Xi^2\left(\frac{1}{2}t\right) \left|\Gamma\left(\frac{-1 + it}{4}\right)\right|^2 \frac{\cos\left(\frac{1}{2}t\log\alpha\right)}{1 + t^2} \, dt$$

$$\sim \frac{1}{2} \frac{\log\alpha}{\sqrt{\alpha}} + \frac{1}{2\sqrt{\alpha}}(\log 2\pi - \gamma) + \frac{\pi^2}{72\alpha^{3/2}} - \frac{\pi^4}{10800\alpha^{7/2}} + \cdots.$$

————————————

([3]) There is a misprint in this asymptotic expansion given in Oloa's paper. The minus sign in front of the second expression on the right should be a plus.

In the following theorem, we obtain the asymptotic expansion of the general integral (1.32) as $\alpha \to \infty$.

THEOREM 1.10. *Fix z such that* $-1 < \operatorname{Re} z < 1$. *As* $\alpha \to \infty$,

(1.33)
$$\frac{1}{\pi^{(z+3)/2}} \int_0^\infty \Gamma\left(\frac{z-1+it}{4}\right) \Gamma\left(\frac{z-1-it}{4}\right) \Xi\left(\frac{t+iz}{2}\right) \Xi\left(\frac{t-iz}{2}\right) \frac{\cos\left(\frac{1}{2} t \log \alpha\right) dt}{t^2 + (z+1)^2}$$
$$\sim -\frac{\Gamma(z)\zeta(z)\alpha^{(z-1)/2}}{(2\pi)^z} - \frac{\Gamma(z+1)\zeta(z+1)}{2\alpha^{(z+1)/2}(2\pi)^z}$$
$$+ 2\alpha^{(z+1)/2} \sum_{m=0}^\infty \frac{(-1)^m}{(2\pi\alpha)^{2m+z+2}} \Gamma(2m+2+z)\zeta(2m+2)\zeta(2m+z+2).$$

This paper is organized as follows. In Section 2, we state some preliminary theorems and lemmas. Section 3 contains proofs of Theorems 1.1 and 1.3. In Section 4, we prove Theorems 1.2 and 1.4. The analogues of Theorem 1.4 corresponding to the second error function transformation and to Ramanujan's transformation (1.5) are also given in that section. Section 5 is devoted to proving Theorem 1.5–Corollary 1.8 and their analogues. We prove Theorem 1.10 in Section 6. Finally, Section 7 is reserved for some concluding remarks and open problems.

2. Nuts and bolts. Let f be an even function of t of the form $f(t) = \phi(it)\phi(-it)$, where ϕ is analytic in t as a function of a real variable. Using the functional equation for $\zeta(s)$ in the form $\xi(s) = \xi(1-s)$, where $\xi(s)$ is defined in (1.11), it is easy to obtain the following line integral representation for the integral on the left side below, of which the integral on the extreme right of (1.15) is a special case:

(2.1) $$\int_0^\infty f\left(\frac{t}{2}\right) \Xi\left(\frac{t}{2}\right) \Delta\left(\alpha, z, \frac{1+it}{2}\right) dt$$
$$= \frac{2}{i} \int_{1/2-i\infty}^{1/2+i\infty} \phi\left(s - \frac{1}{2}\right) \phi\left(\frac{1}{2} - s\right) \xi(s)\omega(\alpha, z, s) \, ds,$$

whenever the integral on the left converges. Here $\Delta(x, z, s)$ and $\omega(x, z, s)$ are as defined in (1.13) and (1.14). Analogous to these, define

(2.2) $$\nabla(x, z, s) := \rho(x, z, s) + \rho(x, z, 1-s),$$

where
$$\rho(x, z, s) := x^{1/2-s} e^{-z^2/8} {}_1F_1\left(\frac{1-s}{2}; \frac{1}{2}; \frac{z^2}{4}\right).$$

Then for χ a real primitive character modulo q, the following formulas can be similarly proved:

$$\int_0^\infty f\left(\frac{t}{2}\right) \Xi\left(\frac{t}{2}, \chi\right) \nabla\left(\alpha, z, \frac{1+it}{2}\right) dt$$

$$= \frac{2}{i} \int_{1/2-i\infty}^{1/2+i\infty} \phi\left(s - \frac{1}{2}\right) \phi\left(\frac{1}{2} - s\right) \xi(s, \chi) \rho(\alpha, z, s)\, ds,$$

(2.3)

$$\int_0^\infty f\left(\frac{t}{2}\right) \Xi\left(\frac{t}{2}, \chi\right) \Delta\left(\alpha, z, \frac{1+it}{2}\right) dt$$

$$= \frac{2}{i} \int_{1/2-i\infty}^{1/2+i\infty} \phi\left(s - \frac{1}{2}\right) \phi\left(\frac{1}{2} - s\right) \xi(s, \chi) \omega(\alpha, z, s)\, ds,$$

whenever the integrals on the left-hand sides converge. Note that for $\alpha\beta = 1$,

(2.4)

$$\Delta\left(\alpha, z, \frac{1+it}{2}\right) = \Delta\left(\beta, iz, \frac{1+it}{2}\right),$$

$$\nabla\left(\alpha, z, \frac{1+it}{2}\right) = \nabla\left(\beta, iz, \frac{1+it}{2}\right),$$

both of which can be proved using Kummer's first transformation for $_1F_1(a; c; w)$ [39, p. 125, (2)] given by

(2.5)
$$_1F_1(a; c; w) = e^w \, _1F_1(c - a; c; -w).$$

The formulas in (2.4) render the integrals on the left-hand sides of (2.1) and (2.3) invariant under the simultaneous replacement of α by β and z by iz, and hence, as a by-product of the evaluation of these integrals, we obtain identities of the form $G(z, \alpha) = G(iz, \beta)$ and $G(z, \alpha, \chi) = G(iz, \beta, \chi)$.

In proving Theorem 1.3, we make use of the following special case [7, Theorem 2.1] of a result due to Berndt [6, Theorem 10.1]:

THEOREM 2.1. *Let $x > 0$ and let $K_\nu(z)$ be the modified Bessel function of order ν. If χ is even with period q and $\operatorname{Re}\nu \geq 0$, then*

$$\sum_{n=1}^\infty \chi(n) n^\nu K_\nu\left(\frac{2\pi n x}{q}\right)$$

$$= \frac{\pi^{1/2}}{2xG(\overline{\chi})}\left(\frac{qx}{\pi}\right)^{\nu+1} \Gamma\left(\nu + \frac{1}{2}\right) \sum_{n=1}^\infty \overline{\chi}(n)(n^2 + x^2)^{-\nu-1/2};$$

if χ is odd with period k and $\operatorname{Re}\nu > -1$, then

$$(2.6) \qquad \sum_{n=1}^{\infty} \chi(n) n^{\nu+1} K_{\nu}\left(\frac{2\pi nx}{q}\right)$$

$$= \frac{i\pi^{1/2}(qx/\pi)^{\nu+2}}{2x^2 G(\overline{\chi})} \Gamma\left(\nu + \frac{3}{2}\right) \sum_{n=1}^{\infty} \overline{\chi}(n) n(n^2 + x^2)^{-\nu-3/2}.$$

The following two lemmas, given in [19, p. 503, (3.952.7)] and [16, pp. 318, 320, (10), (30)] respectively, will also be employed in the proof of Theorem 1.3.

LEMMA 2.2. *For $c = \operatorname{Re} s > -1$ and $\operatorname{Re} a > 0$, we have*

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{b}{2} a^{-1/2-s/2} e^{-\frac{b^2}{4a}} \Gamma\left(\frac{s+1}{2}\right) {}_1F_1\left(1-\frac{s}{2}; \frac{3}{2}; \frac{b^2}{4a}\right) x^{-s} \, ds = e^{-ax^2} \sin bx.$$

LEMMA 2.3. *For $c = \operatorname{Re} s > 0$ and $\operatorname{Re} a > 0$, we have*

$$(2.7) \qquad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{2} a^{-s/2} \Gamma\left(\frac{s}{2}\right) e^{-\frac{b^2}{4a}} {}_1F_1\left(\frac{1-s}{2}; \frac{1}{2}; \frac{b^2}{4a}\right) x^{-s} \, ds = e^{-ax^2} \cos bx.$$

We note that [13, (2.10)]

$$ {}_1F_1(1/4 - \lambda; 1/2; z^2/4) \sim e^{z^2/8} \cos(\sqrt{\lambda}\, z) $$

as $|\lambda| \to \infty$ and $|\arg(\lambda z)| < 2\pi$, and the Stirling's formula for $\Gamma(s)$, $s = \sigma + it$, in a vertical strip $\alpha \le \sigma \le \beta$ given by

$$(2.8) \qquad |\Gamma(s)| = (2\pi)^{1/2} |t|^{\sigma-1/2} e^{-\pi|t|/2}(1 + O(1/|t|)),$$

as $|t| \to \infty$ give convergence of the integrals on the extreme right-hand sides of (1.15), (1.19) and (1.20). If $F(s)$ and $G(s)$ denote the Mellin transforms of $f(x)$ and $g(x)$ respectively, and s with $\operatorname{Re} s = c$ lies in a common strip where both F and G are analytic, then a variant of Parseval's formula [38, p. 83, (3.1.13)] gives

$$(2.9) \qquad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) G(s) w^{-s} \, ds = \int_{0}^{\infty} f(x) g\left(\frac{w}{x}\right) \frac{dx}{x}.$$

Watson's lemma [36, p. 71] is given by

LEMMA 2.4. *If $q(t)$ is a function of the positive real variable t such that*

$$q(t) \sim \sum_{s=0}^{\infty} a_s t^{(s+\lambda-\mu)/\mu} \qquad (t \to 0)$$

for positive constants λ and μ, then

$$(2.10) \qquad \int_{0}^{\infty} e^{-xt} q(t) \, dt \sim \sum_{s=0}^{\infty} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{x^{(s+\lambda)/\mu}} \qquad (x \to \infty),$$

provided that this integral converges throughout its range for all sufficiently large x.

The above result also holds [48, p. 32] for complex λ with $\mathrm{Re}\,\lambda > 0$, and for $x \in \mathbb{C}$ with the integral being convergent for all sufficiently large values of $\mathrm{Re}\,x$.

3. The first error function transformation and its character analogues. We begin by proving the first error function transformation given in Theorem 1.1 and then proceed to a proof of its character analogues in Theorem 1.3.

Proof of Theorem 1.1. Let $\phi(s) = \Gamma(-1/4 + s/2)$ and

$$J(z,\alpha) = \int\limits_0^\infty \Gamma\left(\frac{-1+it}{4}\right)\Gamma\left(\frac{-1-it}{4}\right)\Xi\left(\frac{t}{2}\right)\Delta\left(\alpha, z, \frac{1+it}{2}\right) dt.$$

Use (1.10), (1.11), (2.1), the functional equation for $\Gamma(s)$ and the reflection formula to see that

$$J(z,\alpha) = \frac{2\sqrt{\alpha}\, e^{-z^2/8}}{i} \int\limits_{1/2-i\infty}^{1/2+i\infty} \Gamma\left(\frac{s+1}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma\left(1+\frac{s}{2}\right)\zeta(s)$$

$$\times {}_1F_1\left(1-\frac{s}{2}; \frac{3}{2}; \frac{z^2}{4}\right)(\sqrt{\pi}\,\alpha)^{-s}\, ds$$

$$= -\frac{4\sqrt{\alpha}\, e^{-z^2/8}}{i} \int\limits_{1/2-i\infty}^{1/2+i\infty} \frac{\pi(\sqrt{\pi}\,\alpha)^{-s}}{\sin\left(\frac{1}{2}\pi s\right)}\Gamma\left(\frac{s+1}{2}\right)\zeta(s)$$

$$\times {}_1F_1\left(1-\frac{s}{2}; \frac{3}{2}; \frac{z^2}{4}\right) ds.$$

Now in order to use the series representation for $\zeta(s)$, we shift the line of integration from $\mathrm{Re}\,s = 1/2$ to $\mathrm{Re}\,s = 1+\delta$, where $0 < \delta < 1$. Consider a positively oriented rectangular contour with sides $[1/2 + iT, 1/2 - iT]$, $[1/2 - iT, 1+\delta - iT]$, $[1+\delta - iT, 1+\delta + iT]$ and $[1+\delta + iT, 1/2 + iT]$, where T is any positive real number. We have to consider the contribution of the pole of order 1 of the integrand (due to $\zeta(s)$). Using the residue theorem, noting that by (2.8) the integrals along the horizontal line segments tend to zero as $T \to \infty$, and then interchanging the order of summation and integration [48, p. 30, Theorem 2.1], which is valid since the series representation of $\zeta(s)$, namely $\sum_{n=1}^{\infty} n^{-s}$, is absolutely and uniformly convergent in $\mathrm{Re}\,s \geq 1+\epsilon$ for any $\epsilon > 0$, we find that

$$J(z,\alpha) = -\frac{4\sqrt{\alpha}\,e^{-z^2/8}}{i}\Bigg(\sum_{n=1}^{\infty}\int_{1+\delta-i\infty}^{1+\delta+i\infty}\frac{\pi}{\sin\left(\frac{1}{2}\pi s\right)}\Gamma\left(\frac{s+1}{2}\right)$$

$$\times\; {}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right)(\sqrt{\pi}\,\alpha n)^{-s}\,ds - 2\pi i L\Bigg),$$

where

$$L = \lim_{s\to 1}(s-1)\zeta(s)\frac{\pi}{\sin\left(\frac{1}{2}\pi s\right)}\Gamma\left(\frac{s+1}{2}\right){}_1F_1\left(1-\frac{s}{2};\frac{3}{2};\frac{z^2}{4}\right)(\sqrt{\pi}\alpha)^{-s}.$$

It is easy to see that

$$L = \frac{\sqrt{\pi}}{\alpha}\,{}_1F_1\left(\frac{1}{2};\frac{3}{2};\frac{z^2}{4}\right).$$

Now using the fact [38, p. 98] that for $0 < c = \operatorname{Re} s < 2$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi}{\sin\left(\frac{1}{2}\pi s\right)}x^{-s}\,ds = \frac{2}{(1+x^2)},$$

combined with the special case when $b = z \neq 0$, $a = 1$ of Lemma 2.2, and (2.9), we see that

$$J(z,\alpha) =$$

$$-8\pi\sqrt{\alpha}\,e^{-z^2/8}\left(\frac{4e^{z^2/4}}{z}\sum_{n=1}^{\infty}\int_{0}^{\infty}\frac{e^{-x^2}\sin xz}{1+\left(\frac{\sqrt{\pi}\alpha n}{x}\right)^2}\frac{dx}{x} - \frac{\sqrt{\pi}}{\alpha}\,{}_1F_1\left(\frac{1}{2};\frac{3}{2};\frac{z^2}{4}\right)\right).$$

Employ the change of variable $x \to \sqrt{\pi}\,\alpha x$ and (2.5) to see that

$$(3.1)\qquad J(z,\alpha) = -8\pi\sqrt{\alpha}\,e^{z^2/8}\left(\frac{4e^{z^2/4}}{z}\sum_{n=1}^{\infty}\int_{0}^{\infty}\frac{xe^{-\pi\alpha^2 x^2}\sin\left(\sqrt{\pi}\,\alpha xz\right)}{n^2+x^2}\,dx\right.$$

$$\left. -\frac{\sqrt{\pi}}{\alpha}\,{}_1F_1\left(1;\frac{3}{2};\frac{-z^2}{4}\right)\right).$$

Now for $t \neq 0$ [11, p. 191],

$$(3.2)\qquad \sum_{n=1}^{\infty}\frac{1}{t^2+n^2} = \frac{\pi}{t}\left(\frac{1}{e^{2\pi t}-1}-\frac{1}{2\pi t}+\frac{1}{2}\right).$$

Since the above sum is uniformly convergent on any compact interval in $(0,\infty)$, interchanging the order of summation and integration in (3.1) using [48, p. 30, Theorem 2.1] and then substituting (3.2) and simplifying, we observe that

$$(3.3) \qquad J(z,\alpha) = -8\pi\sqrt{\alpha}\, e^{z^2/8} \Bigg\{ \frac{4\pi}{z} \int_0^\infty \frac{e^{-\pi\alpha^2 x^2} \sin(\sqrt{\pi}\,\alpha x z)}{e^{2\pi x} - 1}\, dx$$

$$- \frac{2}{z} \int_0^\infty \frac{e^{-\pi\alpha^2 x^2} \sin(\sqrt{\pi}\,\alpha x z)}{x}\, dx + \frac{2\pi}{z} \int_0^\infty e^{-\pi\alpha^2 x^2} \sin(\sqrt{\pi}\,\alpha x z)\, dx$$

$$- \frac{\sqrt{\pi}}{\alpha}\, {}_1F_1\left(1; \frac{3}{2}; \frac{-z^2}{4}\right) \Bigg\}.$$

However from [19, p. 488, formula 3.896, no. 3],

$$(3.4) \qquad \int_0^\infty e^{-\pi\alpha^2 x^2} \sin(\sqrt{\pi}\,\alpha x z)\, dx = \frac{z}{2\sqrt{\pi}\,\alpha}\, {}_1F_1\left(1; \frac{3}{2}; \frac{-z^2}{4}\right),$$

and by [19, p. 503, formula 3.952, no. 7 and p. 889, formula 8.253, no. 1] (see also [18, p. 421]),

$$(3.5) \qquad \int_0^\infty \frac{e^{-\pi\alpha^2 x^2} \sin\left(\sqrt{\pi}\,\alpha x z\right)}{x}\, dx = \frac{\pi}{2}\, \mathrm{erf}\left(\frac{z}{2}\right).$$

Thus, substituting (3.4) and (3.5) in (3.3) and simplifying, we finally arrive at

$$(3.6) \qquad \frac{1}{8\pi^2} J(z,\alpha)$$

$$= \frac{\sqrt{\alpha}\, e^{z^2/8}}{z} \left(\mathrm{erf}\left(\frac{z}{2}\right) - 4 \int_0^\infty \frac{e^{-\pi\alpha^2 x^2} \sin(\sqrt{\pi}\,\alpha x z)}{e^{2\pi x} - 1}\, dx \right).$$

Using (1.9), it is clear that simultaneously replacing α by β and z by iz in (3.6) and employing (2.4) and (1.9) give (1.15) since $J(z,\alpha)$ is invariant. ∎

Proof of Theorem 1.3. We prove the theorem only for odd real χ. The case when χ is even and real is similar. Let $\phi(s) = \Gamma(1/4 + s/2)$ and

$$P(z,\alpha,\chi) = \int_0^\infty \Gamma\left(\frac{1+it}{4}\right) \Gamma\left(\frac{1-it}{4}\right) \Xi\left(\frac{t}{2}, \chi\right) \nabla\left(\alpha, z, \frac{1+it}{2}\right) dt.$$

Using the first equality in (2.3), we see that

$$P(z,\alpha,\chi) = \frac{2\sqrt{\alpha q}\, e^{-z^2/8}}{i\sqrt{\pi}} \int_{1/2-i\infty}^{1/2+i\infty} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{1-s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) L(s,\chi)$$

$$\times\, {}_1F_1\left(\frac{1-s}{2}; \frac{1}{2}; \frac{z^2}{4}\right) \left(\frac{\sqrt{\pi}\,\alpha}{\sqrt{q}}\right)^{-s} ds$$

$$= \frac{2\sqrt{\alpha q}\, e^{-z^2/8}}{i\sqrt{\pi}} \int\limits_{1/2-i\infty}^{1/2+i\infty} \frac{\pi}{\cos\left(\frac{1}{2}\pi s\right)} \Gamma\left(\frac{s}{2}\right) L(s,\chi)$$

$$\times\ {}_1F_1\left(\frac{1-s}{2}; \frac{1}{2}; \frac{z^2}{4}\right) \left(\frac{\sqrt{\pi}\,\alpha}{\sqrt{q}}\right)^{-s} ds,$$

where in the last step, we have used a different version of the reflection formula, namely, $\Gamma(1/2+z)\Gamma(1/2-z) = \pi/\cos\pi z$ for $z-1/2 \notin \mathbb{Z}$. As before, shift the line of integration from $\mathrm{Re}\, s = 1/2$ to $\mathrm{Re}\, s = 1+\delta$, $0 < \delta < 2$, employ the residue theorem and take into account the contribution from the pole of order 1 at $s = 1$ of the integrand (due to $\cos(\frac{1}{2}\pi s)$). This gives

$$(3.7)\quad P(z,\alpha,\chi) = \frac{2\sqrt{\alpha q}\, e^{-z^2/8}}{i\sqrt{\pi}} \left(\sum_{n=1}^{\infty} \chi(n) \int\limits_{1+\delta-i\infty}^{1+\delta+i\infty} \frac{\pi}{\cos\left(\frac{1}{2}\pi s\right)} \Gamma\left(\frac{s}{2}\right) \right.$$

$$\left. \times\ {}_1F_1\left(\frac{1-s}{2}; \frac{1}{2}; \frac{z^2}{4}\right) \left(\frac{\sqrt{\pi}\,\alpha n}{\sqrt{q}}\right)^{-s} ds - 2\pi i L_1 \right),$$

where

$$(3.8)\quad L_1 = \lim_{s\to 1} \frac{(s-1)\pi\Gamma(s/2)L(s,\chi)}{\cos\left(\frac{1}{2}\pi s\right)} {}_1F_1\left(\frac{1-s}{2}; \frac{1}{2}; \frac{z^2}{4}\right) \left(\frac{\sqrt{\pi}\,\alpha}{\sqrt{q}}\right)^{-s}$$

$$= -\frac{2\sqrt{q}}{\alpha} L(1,\chi).$$

Also replacing s by $(s+1)/2$, x by x^2 in the formula [38, p. 91, (3.3.10)]

$$\frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \frac{x^{-s}}{\sin\pi s}\, ds = \frac{1}{\pi(1+x)},$$

and simplifying, we see that for $-1 < \mathrm{Re}\, s < 1$,

$$\frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \frac{\pi}{\cos\left(\frac{1}{2}\pi s\right)} x^{-s}\, ds = \frac{2x}{1+x^2}.$$

Another application of the residue theorem yields, for $0 < c < 1$,

$$(3.9)\quad \int\limits_{1+\delta-i\infty}^{1+\delta+i\infty} \frac{\pi}{\cos\left(\frac{1}{2}\pi s\right)} \Gamma\left(\frac{s}{2}\right) {}_1F_1\left(\frac{1-s}{2}; \frac{1}{2}; \frac{z^2}{4}\right) \left(\frac{\sqrt{\pi}\,\alpha n}{\sqrt{q}}\right)^{-s} ds$$

$$= \int\limits_{c-i\infty}^{c+i\infty} \frac{\pi}{\cos\left(\frac{1}{2}\pi s\right)} \Gamma\left(\frac{s}{2}\right) {}_1F_1\left(\frac{1-s}{2}; \frac{1}{2}; \frac{z^2}{4}\right) \left(\frac{\sqrt{\pi}\,\alpha n}{\sqrt{q}}\right)^{-s} ds - \frac{4\pi i\sqrt{q}}{\alpha n}$$

$$= 2\pi i \left(4e^{z^2/4} \int\limits_{0}^{\infty} \frac{n}{x^2+n^2} e^{-\pi\alpha^2 x^2/q} \cos\left(\frac{\sqrt{\pi}\,\alpha x z}{\sqrt{q}}\right) dx - \frac{2\sqrt{q}}{\alpha n} \right),$$

where in the last step we used Lemma 2.3 with $a = 1$, $x = \sqrt{\pi}\,\alpha n/\sqrt{q}$ and $b = z$, and (2.9), followed by a change of variable $x \mapsto \sqrt{\pi}\,\alpha x/\sqrt{q}$. Now substitute (3.9) and (3.8) in (3.7) and simplify to obtain

(3.10) $P(z, \alpha, \chi)$

$$= 16\sqrt{\pi\alpha q}\,e^{z^2/8} \int_0^\infty \left(\sum_{n=1}^\infty \frac{n\chi(n)}{x^2 + n^2} \right) e^{-\pi\alpha^2 x^2/q} \cos\left(\frac{\sqrt{\pi}\,\alpha xz}{\sqrt{q}} \right) dx.$$

Now use (2.6) with χ real and $\nu = -1/2$ to see that

(3.11) $$\sum_{n=1}^\infty \frac{n\chi(n)}{x^2 + n^2} = \frac{\pi}{\sqrt{q}} \sum_{n=1}^\infty \chi(n) e^{-2\pi n x/q}.$$

Employing (3.11) in (3.10), we get

(3.12) $P(z, \alpha, \chi)$

$$= 16\sqrt{\pi^3\alpha}\,e^{z^2/8} \int_0^\infty e^{-\pi\alpha^2 x^2/q} \cos\left(\frac{\sqrt{\pi}\,\alpha xz}{\sqrt{q}} \right) \sum_{n=1}^\infty \chi(n) e^{-2\pi n x/q}\, dx.$$

Writing $n = mq + r$, $0 \le m < \infty$, $0 \le r \le q - 1$, and noting that χ is periodic with period q, we have

(3.13) $$\sum_{n=1}^\infty \chi(n) e^{-2\pi n x/q} = \frac{\sum_{r=0}^{q-1} \chi(r) e^{-2\pi r x/q}}{1 - e^{-2\pi x}}.$$

Finally, (3.13) along with (3.12) gives

$$\frac{1}{16\pi^{3/2}} P(z, \alpha, \chi)$$

$$= \sqrt{\alpha}\,e^{z^2/8} \int_0^\infty e^{-\pi\alpha^2 x^2/q} \cos\left(\frac{\sqrt{\pi}\,\alpha xz}{\sqrt{q}} \right) \frac{\sum_{r=1}^{q-1} \chi(r) e^{-2\pi r x/q}}{1 - e^{-2\pi x}}\, dx.$$

This gives (1.20) as $P(z, \alpha, \chi)$ is invariant under the simultaneous application of the maps $\alpha \mapsto \beta$ and $z \mapsto iz$, which can be seen from (2.4). ∎

4. The second error function transformation and an asymptotic expansion. We first establish the second error function transformation given in Theorem 1.2, and then the asymptotic expansion from Theorem 1.4.

Proof of Theorem 1.2. Note that from (3.4) and [19, p. 889, formula 8.253, no. 1],

(4.1) $$\int_0^\infty e^{-\pi\alpha^2 x^2} \sin\left(\sqrt{\pi}\,\alpha xz\right) dx = \frac{1}{2\alpha} e^{-z^2/4} \operatorname{erfi}\left(\frac{z}{2} \right).$$

Also,

$$\sqrt{\alpha}\, e^{z^2/8}\left(\operatorname{erf}\left(\frac{z}{2}\right) + 4\int\limits_{-\infty}^{0} \frac{e^{-\pi\alpha^2 x^2}\sin(\sqrt{\pi}\,\alpha x z)}{e^{2\pi x} - 1}\, dx\right)$$

$$= \sqrt{\alpha}\, e^{z^2/8}\left(\operatorname{erf}\left(\frac{z}{2}\right) + 4\int\limits_{0}^{\infty} e^{-\pi\alpha^2 x^2}\sin\left(\sqrt{\pi}\,\alpha x z\right) dx\right.$$

$$\left. + 4\int\limits_{0}^{\infty} \frac{e^{-\pi\alpha^2 x^2}\sin(\sqrt{\pi}\,\alpha x z)}{e^{2\pi x} - 1}\, dx\right),$$

where in the first step, we replaced x by $-x$ in the integral, and then simplified it using $e^{2\pi x}/(e^{2\pi x} - 1) = 1 + 1/(e^{2\pi x} - 1)$. Now use (4.1) and the first error function transformation in (1.15) to replace the second integral in the above equation to obtain

$$(4.2) \quad \sqrt{\alpha}\, e^{z^2/8}\left(\operatorname{erf}\left(\frac{z}{2}\right) + 4\int\limits_{-\infty}^{0} \frac{e^{-\pi\alpha^2 x^2}\sin(\sqrt{\pi}\,\alpha x z)}{e^{2\pi x} - 1}\, dx\right)$$

$$= \sqrt{\alpha}\, e^{z^2/8}\left(2\operatorname{erf}\left(\frac{z}{2}\right) + \frac{1}{\alpha}e^{-z^2/4}\operatorname{erfi}\left(\frac{z}{2}\right)\right.$$

$$\left. + \frac{4}{\alpha}e^{-z^2/4}\int\limits_{0}^{\infty} \frac{e^{-\pi\beta^2 x^2}\sinh(\sqrt{\pi}\,\beta x z)}{e^{2\pi x} - 1}\, dx\right)$$

$$= \sqrt{\beta}\, e^{-z^2/8}\left(\operatorname{erfi}\left(\frac{z}{2}\right) + \frac{2}{\beta}e^{z^2/4}\operatorname{erf}\left(\frac{z}{2}\right) + 4\int\limits_{0}^{\infty} \frac{e^{-\pi\beta^2 x^2}\sinh(\sqrt{\pi}\,\beta x z)}{e^{2\pi x} - 1}\, dx\right),$$

where we have used $\alpha\beta = 1$ to simplify the last step. Now replace α by β and z by iz in (4.1), and use (1.9) to obtain

$$(4.3) \qquad \int\limits_{0}^{\infty} e^{-\pi\beta^2 x^2}\sinh(\sqrt{\pi}\,\beta x z)\, dx = \frac{1}{2\beta}e^{z^2/4}\operatorname{erf}\left(\frac{z}{2}\right).$$

Finally, use (4.3) to simplify the extreme right of (4.2), thereby obtaining (1.18) and thus completing the proof. ∎

Proof of Theorem 1.4. By a change of variable $x^2 = t$,

$$\int\limits_{0}^{\infty} \frac{e^{-\beta x^2}\sin(\sqrt{\beta}\,xz)}{e^{2\pi x} - 1}\, dx = \int\limits_{0}^{\infty} \frac{e^{-\beta t}\sin(z\sqrt{\beta t})}{2\sqrt{t}(e^{2\pi\sqrt{t}} - 1)}\, dt.$$

Let

$$f(t, z) = \frac{\sin(z\sqrt{\beta t})}{2\sqrt{t}(e^{2\pi\sqrt{t}} - 1)}.$$

First consider, for $|t| < 1$,

$$\frac{2\pi t \sin(at)}{e^{2\pi t} - 1} = \sum_{m=0}^{\infty} \frac{B_m (2\pi)^m t^m}{m!} \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1} t^{2n+1}}{(2n+1)!}$$

$$= \sum_{j=1}^{\infty} \Bigg(\sum_{m+2n+1=j} \frac{B_m (2\pi)^m}{m!} \frac{(-1)^n a^{2n+1}}{(2n+1)!} \Bigg) t^j$$

$$= \sum_{j=1}^{\infty} \Bigg(\sum_{\substack{k=0 \\ k \text{ even}}}^{j-1} \frac{B_{j-1-k}(2\pi)^{j-1-k}}{(j-1-k)!} \frac{(-1)^{k/2} a^{k+1}}{(k+1)!} \Bigg) t^j$$

$$= t \sum_{j=0}^{\infty} \Bigg(\sum_{\substack{k=0 \\ k \text{ even}}}^{j} \frac{B_{j-k}(2\pi)^{j-k}}{(j-k)!} \frac{(-1)^{k/2} a^{k+1}}{(k+1)!} \Bigg) t^j.$$

Replacing t by $\sqrt{t}$ and $a = z\sqrt{\beta}$, we get for $|t| < 1$,

$$\frac{\sin(z\sqrt{\beta t})}{2\sqrt{t}(e^{2\pi\sqrt{t}} - 1)} = \frac{1}{4\pi} \sum_{j=0}^{\infty} \Bigg(\sum_{\substack{k=0 \\ k \text{ even}}}^{j} \frac{B_{j-k}(2\pi)^{j-k}}{(j-k)!} \frac{(-1)^{k/2}(z\sqrt{\beta})^{k+1}}{(k+1)!} \Bigg) t^{(j-1)/2}.$$

Thus, as $t \to 0+$,

$$f(t,z) \sim \frac{1}{4\pi} \sum_{j=0}^{\infty} \Bigg(\sum_{\substack{k=0 \\ k \text{ even}}}^{j} \frac{B_{j-k}(2\pi)^{j-k}}{(j-k)!} \frac{(-1)^{k/2}(z\sqrt{\beta})^{k+1}}{(k+1)!} \Bigg) t^{(j-1)/2}.$$

Hence by Watson's lemma, as $\beta \to \infty$,

$$(4.4) \qquad \int_0^{\infty} \frac{e^{-\beta t} \sin(z\sqrt{\beta t})}{2\sqrt{t}(e^{2\pi\sqrt{t}} - 1)} \, dt$$

$$\sim \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{j+1}{2}\right)}{\beta^{(j+1)/2}} \sum_{\substack{k=0 \\ k \text{ even}}}^{j} \frac{B_{j-k}(2\pi)^{j-k-1}}{2(j-k)!} \frac{(-1)^{k/2}(z\sqrt{\beta})^{k+1}}{(k+1)!}.$$

From (4.4) and the notation in (1.23), we find that

$$(4.5) \qquad I(iz, \beta) \sim \frac{\sqrt{\pi}}{z} \beta^{-1/4} e^{-z^2/8} \operatorname{erfi}\left(\frac{z}{2}\right) + \sum_{j=0}^{\infty} I_{j,z},$$

where

$$I_{j,z} = \frac{4}{z} \beta^{1/4} e^{z^2/8} \frac{\Gamma\left(\frac{j+1}{2}\right)}{\beta^{(j+1)/2}} \sum_{\substack{k=0 \\ k \text{ even}}}^{j} \frac{B_{j-k}(2\pi)^{j-k-1}}{2(j-k)!} \frac{(-1)^{k/2}(z\sqrt{\beta})^{k+1}}{(k+1)!}.$$

We first evaluate $I_{j,z}$ when j is odd, say $j = 2n + 1$. Since all of the odd-indexed Bernoulli numbers except B_1 are zero and $B_1 = -1/2$, only the last term, namely $j = 2n + 1$ and $k = 2n$, contributes to the sum giving

$$(4.6) \qquad I_{2n+1,z} = \frac{(-1)^{n+1}n!}{(2n+1)!} z^{2n} e^{z^2/8} \beta^{-1/4}.$$

Now let j be even, say $j = 2n$. Then using the fact [48, p. 5, (1.14)] that

$$\frac{(-1)^{m-1}2^{2m-1}\pi^{2m}}{(2m)!} B_{2m} = \zeta(2m)$$

in the second step below, we see that

$$(4.7) \qquad \sum_{n=0}^{\infty} I_{2n,z}$$

$$= \frac{2}{z}\beta^{1/4}e^{z^2/8} \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\beta^{n+1/2}} \sum_{m=0}^{n} \frac{B_{2n-2m}(2\pi)^{2n-2m-1}}{(2n-2m)!} \frac{(-1)^m(z\sqrt{\beta})^{2m+1}}{(2m+1)!}$$

$$= \frac{2}{z}\beta^{1/4}e^{z^2/8} \sum_{n=0}^{\infty} \frac{(-1)^n(z\sqrt{\beta})^{2n+1}}{\beta^{n+1/2}} \Gamma\left(n+\frac{1}{2}\right)$$

$$\times \sum_{m=0}^{n} \frac{B_{2m}(2\pi)^{2m-1}}{(2m)!} \frac{(-1)^m(z\sqrt{\beta})^{-2m}}{(2n-2m+1)!}$$

$$= \frac{-2}{\pi z}\beta^{1/4}e^{z^2/8} \sum_{n=0}^{\infty} (-1)^n z^{2n+1} \Gamma\left(n+\frac{1}{2}\right) \sum_{m=0}^{n} \frac{\zeta(2m)}{(z\sqrt{\beta})^{2m}(2n-2m+1)!}$$

$$= \frac{-2}{\pi}\beta^{1/4}e^{z^2/8} \sum_{m=0}^{\infty} \frac{\zeta(2m)}{(z\sqrt{\beta})^{2m}} \sum_{n=m}^{\infty} \frac{(-1)^n z^{2n}}{(2n-2m+1)!} \Gamma\left(n+\frac{1}{2}\right)$$

$$= \frac{-2}{\pi}\beta^{1/4}e^{z^2/8} \sum_{m=0}^{\infty} \frac{(-1)^m\zeta(2m)}{\beta^m} \Gamma\left(m+\frac{1}{2}\right) {}_1F_1\left(m+\frac{1}{2}; \frac{3}{2}; \frac{-z^2}{4}\right).$$

From (4.5)–(4.7), we obtain the asymptotic expansion of $I(iz, \beta)$ as $\beta \to \infty$:

$$(4.8) \qquad I(iz, \beta) \sim \left(\frac{\sqrt{\pi}}{z} e^{-z^2/8} \operatorname{erfi}\left(\frac{z}{2}\right) - e^{z^2/8} \sum_{m=0}^{\infty} \frac{(-1)^m m! z^{2m}}{(2m+1)!}\right)\beta^{-1/4}$$

$$- \frac{2}{\pi}\beta^{1/4}e^{z^2/8} \sum_{m=0}^{\infty} \frac{(-1)^m\zeta(2m)}{\beta^m} \Gamma\left(m+\frac{1}{2}\right) {}_1F_1\left(m+\frac{1}{2}; \frac{3}{2}; \frac{-z^2}{4}\right).$$

Note that $\operatorname{erf}(z)$ has the following Taylor series expansion, which is valid for all $z \in \mathbb{C}$ [37, p. 162, 7.6.1]:

(4.9)
$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}.$$

From (1.9),

(4.10)
$$\operatorname{erfi}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{n!(2n+1)}.$$

It is now easy to see that

(4.11)
$$\sum_{m=0}^{\infty} \frac{(-1)^m m! z^{2m}}{(2m+1)!} = \frac{\sqrt{\pi}}{z} e^{-z^2/4} \operatorname{erfi}\left(\frac{z}{2}\right).$$

Substituting (4.11) in (4.8), we arrive at

$$I(iz,\beta) \sim -\frac{2}{\pi} \beta^{1/4} e^{z^2/8} \sum_{m=0}^{\infty} \frac{(-1)^m \zeta(2m)}{\beta^m} \Gamma\left(m+\frac{1}{2}\right) {}_1F_1\left(m+\frac{1}{2};\frac{3}{2};\frac{-z^2}{4}\right).$$

Since $\alpha\beta = \pi^2$, by using (1.23) this also gives the asymptotic expansion of $I(z,\alpha)$ as $\alpha \to 0$ as claimed in (1.24). ∎

The second error function transformation in Theorem 1.2 can be rephrased for $\alpha\beta = \pi^2$ and $z \neq 0$ as follows:

(4.12) $K(z,\alpha)$

$$:= \frac{\sqrt{\pi}}{z} \alpha^{-1/4} e^{z^2/8} \operatorname{erf}\left(\frac{z}{2}\right) - \frac{4}{z} \alpha^{1/4} e^{-z^2/8} \int_{-\infty}^{0} \frac{e^{-\alpha x^2} \sinh(\sqrt{\alpha}\, xz)}{e^{2\pi x} - 1} \, dx$$

$$= \frac{\sqrt{\pi}}{z} \beta^{-1/4} e^{-z^2/8} \operatorname{erfi}\left(\frac{z}{2}\right) - \frac{4}{z} \beta^{1/4} e^{z^2/8} \int_{-\infty}^{0} \frac{e^{-\beta x^2} \sin(\sqrt{\beta}\, xz)}{e^{2\pi x} - 1} \, dx$$

$$=: K(iz,\beta).$$

The analogue of Theorem 1.4 for the above identity is given below. The details are similar to those in the proof of Theorem 1.4, and hence not provided.

THEOREM 4.1. *Fix $z \in \mathbb{C}$. As $\alpha \to 0$,*

$$K(z,\alpha) \sim \frac{2}{\sqrt{\pi}} \alpha^{-1/4} e^{z^2/8} \sum_{m=0}^{\infty} \left(\frac{-\alpha}{\pi^2}\right)^m \zeta(2m) \Gamma\left(m+\frac{1}{2}\right) {}_1F_1\left(m+\frac{1}{2};\frac{3}{2};\frac{-z^2}{4}\right).$$

Combining Theorems 1.4 and 4.1 leads us to the following asymptotic expansion of the integral analogue of a theta function.

THEOREM 4.2. *Fix $z \in \mathbb{C}$. As $\alpha \to 0$,*

$$\int_{-\infty}^{\infty} \frac{e^{-\alpha x^2} \sinh(\sqrt{\alpha}\, xz)}{e^{2\pi x} - 1}\, dx$$

$$\sim \frac{-z}{\sqrt{\pi \alpha}} e^{z^2/4} \sum_{m=0}^{\infty} \left(\frac{-\alpha}{\pi^2}\right)^m \zeta(2m) \Gamma\left(m + \frac{1}{2}\right) {}_1F_1\left(m + \frac{1}{2}; \frac{3}{2}; \frac{-z^2}{4}\right).$$

5. Generalization of Ramanujan's approximation and integral identities involving hypergeometric functions. The results from this section follow from successively differentiating the first error function transformation in the form (1.23) with respect to z. The presence of negative powers of the exponential function in the integrands on either side justifies differentiation under the integral sign. As mentioned in the introduction, differentiating (1.23) n times where n is odd just gives $0 = 0$. However when n is even, two different behaviors are observed according as $n \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$.

5.1. The case $n \equiv 0 \pmod 4$

Proof of Theorem 1.5. Let $n = 4k$ for a non-negative integer k. Let

(5.1)
$$G(z, \alpha) := \frac{\sqrt{\pi}}{z} \alpha^{-1/4} e^{z^2/8} \operatorname{erf}\left(\frac{z}{2}\right) + \frac{4}{z} \alpha^{1/4} e^{-z^2/8} \int_0^{\infty} \frac{e^{-\alpha x^2} \sinh(\sqrt{\alpha}\, xz)}{e^{2\pi x} - 1}\, dx$$

and let

(5.2)
$$H_k(\alpha) := \frac{d^{4k}}{dz^{4k}} G(z, \alpha)\bigg|_{z=0}.$$

Using (1.23) we are led to a transformation of the form $H_k(\alpha) = H_k(\beta)$ since

(5.3)
$$H_k(\alpha) = \frac{d^{4k}}{dz^{4k}} G(z, \alpha)\bigg|_{z=0} = \frac{d^{4k}}{d(iz)^{4k}} G(iz, \beta)\bigg|_{z=0} = H_k(\beta).$$

Multiply the power series expansion of $e^{z^2/8}$ by that of $\operatorname{erf}(z)$ given in (4.9), and extract the coefficient of z^{4k} in the product. This gives

$$\frac{d^{4k}}{dz^{4k}} \frac{e^{z^2/8}}{z} \operatorname{erf}\left(\frac{z}{2}\right)\bigg|_{z=0} = \frac{2(4k)!}{\sqrt{\pi}} \sum_{m=0}^{2k} \frac{(-1)^m}{2^{4k+2m+1} m! (2k-m)! (4k-2m+1)}$$

$$= \frac{(4k)!}{2^{4k} \sqrt{\pi}(2k)!(4k+1)} {}_2F_1\left(-\frac{1}{2} - 2k, -2k; \frac{1}{2} - 2k; \frac{1}{2}\right).$$

In [48, p. 113, (5.12)], we find the following hypergeometric transformation, valid for $|\arg(1 - z)| < \pi$:

$$(5.4) \quad {}_2F_1(a, b; c; z)$$
$$= \frac{\Gamma(c)\Gamma(b - a)}{\Gamma(b)\Gamma(c - a)}(1 - z)^{-a} \, {}_2F_1\left(a, c - b; a - b + 1; \frac{1}{1 - z}\right)$$
$$+ \frac{\Gamma(c)\Gamma(a - b)}{\Gamma(a)\Gamma(c - b)}(1 - z)^{-b} \, {}_2F_1\left(b, c - a; b - a + 1; \frac{1}{1 - z}\right).$$

Use this transformation with $z = 1/2$, $a = -1/2 - 2k$, $b = -2k$, $c = 1/2 - 2k$ to obtain

$$ {}_2F_1\left(-\frac{1}{2} - 2k, -2k; \frac{1}{2} - 2k; \frac{1}{2}\right) = \frac{(4k + 1)}{2^{2k}} \, {}_2F_1\left(-2k, 1; \frac{3}{2}; 2\right).$$

This gives

$$(5.5) \quad \frac{d^{4k}}{dz^{4k}} \frac{e^{z^2/8}}{z} \operatorname{erf}\left(\frac{z}{2}\right)\Bigg|_{z=0} = \frac{(4k)!}{2^{6k}\sqrt{\pi}(2k)!} \, {}_2F_1\left(-2k, 1; \frac{3}{2}; 2\right).$$

Similarly multiplying the power series expansion of $e^{-z^2/8}$ by that of $\sinh(\sqrt{\alpha}\, xz)$ and extracting the coefficient of z^{4k} in the product, we get

$$\frac{d^{4k}}{dz^{4k}} \frac{e^{-z^2/8}}{z} \sinh(\sqrt{\alpha}\, xz)\Bigg|_{z=0} = (4k)! \sum_{m=0}^{2k} \frac{(-1)^m (\sqrt{\alpha}\, x)^{4k-2m+1}}{8^m m!(4k - 2m + 1)!}$$
$$= \frac{(4k)!\sqrt{\alpha}\, x}{2^{6k}} \sum_{m=0}^{2k} \frac{(-8)^m (\sqrt{\alpha}\, x)^{2m}}{(2k - m)!(2m + 1)!},$$

where we have replaced m by $2k - m$ in the last step. It can be seen that

$$\sum_{m=0}^{2k} \frac{(-8)^m (\sqrt{\alpha}\, x)^{2m}}{(2k - m)!(2m + 1)!} = \frac{1}{(2k)!} \, {}_1F_1\left(-2k; \frac{3}{2}; 2\alpha x^2\right).$$

Thus

$$(5.6) \quad \frac{d^{4k}}{dz^{4k}} \frac{e^{-z^2/8}}{z} \sinh(\sqrt{\alpha}\, xz)\Bigg|_{z=0} = \frac{(4k)!\sqrt{\alpha}\, x}{2^{6k}(2k)!} \, {}_1F_1\left(-2k; \frac{3}{2}; 2\alpha x^2\right).$$

Hence from (5.1), (5.2), (5.5) and (5.6), we obtain

$$H_k(\alpha) = \frac{(4k)!}{2^{6k}(2k)!}\left\{\alpha^{-1/4} \, {}_2F_1\left(-2k, 1; \frac{3}{2}; 2\right)\right.$$
$$\left. + 4\alpha^{3/4} \int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x} - 1} \, {}_1F_1\left(-2k; \frac{3}{2}; 2\alpha x^2\right) dx\right\},$$

which, combined with (5.3), proves Theorem 1.5. ∎

Next, we prove a generalization of Ramanujan's approximation in (1.22).

Proof of Theorem 1.6. Let $H_k(\beta)$ be as defined in (5.2), and consider the integral

$$(5.7) \qquad \int_0^\infty \frac{xe^{-\beta x^2}}{e^{2\pi x} - 1} \, {}_1F_1\left(-2k; \frac{3}{2}; 2\beta x^2\right) dx.$$

Employing the change of variable $x = \sqrt{t}$, using the series definition of ${}_1F_1$ and interchanging the order of summation and integration, we see that

$$\int_0^\infty \frac{xe^{-\beta x^2}}{e^{2\pi x} - 1} \, {}_1F_1\left(-2k; \frac{3}{2}; 2\beta x^2\right) dx = \frac{1}{2} \sum_{m=0}^{2k} \frac{(-2k)_m(2\beta)^m}{(3/2)_m m!} \int_0^\infty \frac{e^{-\beta t} t^m \, dt}{e^{2\pi\sqrt{t}} - 1}.$$

Now use the generating function for Bernoulli numbers to obtain, for $|t| < 1$,

$$\frac{t^m}{e^{2\pi\sqrt{t}} - 1} = \sum_{j=0}^\infty \frac{B_j(2\pi)^{j-1}}{j!} t^{(j-1+2m)/2}.$$

Using Watson's lemma from (2.10), we find that as $\beta \to \infty$,

$$H_k(\beta) \sim \beta^{-1/4} \, {}_2F_1\left(-2k, 1; \frac{3}{2}; 2\right)$$

$$+ 2\beta^{3/4} \sum_{m=0}^{2k} \frac{(-2k)_m(2\beta)^m}{(3/2)_m m!} \sum_{j=0}^\infty \frac{B_j(2\pi)^{j-1}}{j!\beta^{(j+2m+1)/2}} \Gamma\left(\frac{j+2m+1}{2}\right).$$

Since $H_k(\alpha) = H_k(\beta)$, using $\beta = \pi^2/\alpha$ yields for $\alpha \to 0$,

$$H_k(\alpha) \sim \frac{\alpha^{1/4}}{\sqrt{\pi}} \, {}_2F_1\left(-2k, 1; \frac{3}{2}; 2\right)$$

$$+ \frac{2}{\sqrt{\pi}} \alpha^{-1/4} \sum_{m=0}^{2k} \frac{(-2k)_m 2^m}{(3/2)_m m!} \sum_{j=0}^\infty \frac{B_j 2^{j-1} \alpha^{j/2}}{j!} \Gamma\left(\frac{j+2m+1}{2}\right)$$

$$= \frac{\alpha^{-1/4}}{\sqrt{\pi}} \sum_{\substack{j=0 \\ j\neq 1}}^\infty \frac{B_j 2^j \alpha^{j/2}}{j!} \Gamma\left(\frac{j+1}{2}\right) {}_2F_1\left(-2k, \frac{1+j}{2}; \frac{3}{2}; 2\right)$$

$$= \alpha^{-1/4} \, {}_2F_1\left(\frac{1}{2}, -2k; \frac{3}{2}; 2\right) + \frac{\alpha^{3/4}}{6}$$

$$+ \frac{\alpha^{-1/4}}{\sqrt{\pi}} \sum_{j=3}^\infty \frac{B_j 2^j \alpha^{j/2}}{j!} \Gamma\left(\frac{j+1}{2}\right) {}_2F_1\left(-2k, \frac{1+j}{2}; \frac{3}{2}; 2\right).$$

We now find a simpler function, namely the one claimed on the right-hand side of (1.26), that is "nearly" equal to $H_k(\alpha)$ when α is very small in

the sense that the asymptotic expansion of this simpler function as $\alpha \to 0$ matches the first two terms in those of $H_k(\alpha)$. Note that such a function should preserve the invariance under replacing α by β and vice versa. In order to match the leading term in the asymptotic expansion, we raise $1/\alpha$ to the power $1/4$ and have its coefficient as $_2F_1(1/2, -2k; 3/2; 2)$, which is equal to $_2F_1(-2k, 1; 3/2; 2)$ by Pfaff's transformation [48, p. 110, (5.5)]

$$_2F_1(a, b; c; z) = (1 - z)^{-b} \, _2F_1\left(c - a, b; c; \frac{z}{z - 1}\right).$$

Since the approximating function has to be symmetric, we need to raise $1/\beta$ along with $1/\alpha$ to the power $1/4$. So the function we are seeking assumes the form

$$_2F_1\left(-2k, 1; \frac{3}{2}; 2\right)\left(\frac{1}{\alpha} + \frac{1}{\beta} + c(k)\right)^{1/4},$$

where $c(k)$ is some constant depending on only k. Since α is very small, the main contribution in the asymptotic expansion comes from $1/\alpha$ and $c(k)$ but not from $1/\beta$. Thus, the next term in the Taylor series of this function is

$$\frac{\alpha^{3/4}}{4} \, _2F_1\left(-2k, 1; \frac{3}{2}; 2\right)c(k).$$

As we want this to be equal to $\alpha^{3/4}/6$, it is clear that

$$c(k) = \frac{2}{3 \cdot \, _2F_1(-2k, 1; 3/2; 2)}.$$

Thus the required function is

$$_2F_1\left(-2k, 1; \frac{3}{2}; 2\right)\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{2}{3 \cdot \, _2F_1(-2k, 1; 3/2; 2)}\right)^{1/4}.$$

This completes the proof of (1.26). ∎

5.2. The case $n \equiv 2 \pmod 4$

Proof of Theorem 1.7. Now let $n = 4k + 2$, where k is again any non-negative integer. Let

$$J_k(\alpha) := \frac{d^{4k+2}}{dz^{4k+2}}G(z, \alpha)\Big|_{z=0},$$

where $G(z, \alpha)$ is defined in (5.1). This time (1.23) gives us a transformation of the form $J_k(\alpha) = -J_k(\beta)$ since

$$J_k(\alpha) = \frac{d^{4k+2}}{dz^{4k+2}}G(z,\alpha)\Big|_{z=0} = \frac{d^{4k+2}}{dz^{4k+2}}G(iz,\beta)\Big|_{z=0}$$

$$= -\frac{d^{4k+2}}{d(iz)^{4k+2}}G(iz,\beta)\Big|_{z=0} = -J_k(\beta).$$

The proof is now similar to that of Theorem 1.5, and so we state only the important steps. Let us start with the fact that

$$(5.8) \qquad -J_k(\beta) = \frac{d^{4k+2}}{dz^{4k+2}}G(iz,\beta)\Big|_{z=0}.$$

Multiply the power series expansion of $e^{-z^2/8}$ with that of $\operatorname{erfi}(z/2)$ given in (4.10), and extract the coefficient of z^{4k+2} in the product. Then identifying the coefficient as a hypergeometric function and using the transformation (5.4) with $z = 1/2$, $a = -3/2 - 2k$, $b = -1 - 2k$, $c = -1/2 - 2k$ to simplify this hypergeometric function results in

$$\frac{d^{4k+2}}{dz^{4k+2}}\frac{e^{-z^2/8}}{z}\operatorname{erfi}\left(\frac{z}{2}\right)\Big|_{z=0} = \frac{(4k+2)!}{2^{6k+3}\sqrt{\pi}(2k+1)!}\,_2F_1\left(-2k-1,1;\frac{3}{2};2\right).$$

Similarly,

$$(5.9) \qquad \frac{d^{4k+2}}{dz^{4k+2}}\frac{e^{z^2/8}}{z}\sin(\sqrt{\beta}xz)\Big|_{z=0}$$

$$= -(4k+2)!\sum_{m=0}^{2k+1}\frac{(-1)^m(\sqrt{\beta}\,x)^{4k-2m+3}}{8^m m!(4k-2m+3)!}$$

$$= \frac{(4k+2)!\sqrt{\beta}\,x}{2^{6k+3}}\sum_{m=0}^{2k+1}\frac{(-8)^m(\sqrt{\beta}\,x)^{2m}}{(2k+1-m)!(2m+1)!}$$

$$= \frac{(4k+2)!\sqrt{\beta}\,x}{2^{6k+3}(2k+1)!}\,_1F_1\left(-2k-1;\frac{3}{2};2\beta x^2\right).$$

From (5.8) and (5.9),

$$-J_k(\beta) = \frac{(4k+2)!}{2^{6k+3}(2k+1)!}\left\{\beta^{-1/4}\,_2F_1\left(-2k-1,1;\frac{3}{2};2\right)\right.$$

$$\left. + 4\beta^{3/4}\int_0^\infty \frac{xe^{-\beta x^2}}{e^{2\pi x}-1}\,_1F_1\left(-2k-1;\frac{3}{2};2\beta x^2\right)dx\right\}.$$

This proves Theorem 1.7. ∎

As remarked in the introduction, results corresponding to the ones in Theorem 1.5–Corollary 1.8 can be obtained using similar techniques through

(4.12). These results are collectively put in the following theorem. We refrain from giving the proof since the details are similar to those of Theorem 1.5–Corollary 1.8.

THEOREM 5.1. *Let* α, β *be two positive numbers such that* $\alpha\beta = \pi^2$. *Then*

$$\alpha^{-1/4} \, _2F_1\left(-2k, 1; \frac{3}{2}; 2\right) - 4\alpha^{3/4} \int_{-\infty}^{0} \frac{xe^{-\alpha x^2}}{e^{2\pi x} - 1} \, _1F_1\left(-2k; \frac{3}{2}; 2\alpha x^2\right) dx$$

$$= \beta^{-1/4} \, _2F_1\left(-2k, 1; \frac{3}{2}; 2\right) - 4\beta^{3/4} \int_{-\infty}^{0} \frac{xe^{-\beta x^2}}{e^{2\pi x} - 1} \, _1F_1\left(-2k; \frac{3}{2}; 2\beta x^2\right) dx,$$

$$\alpha^{-1/4} \, _2F_1\left(-2k, 1; \frac{3}{2}; 2\right) - 4\alpha^{3/4} \int_{-\infty}^{0} \frac{xe^{-\alpha x^2}}{e^{2\pi x} - 1} \, _1F_1\left(-2k; \frac{3}{2}; 2\alpha x^2\right) dx$$

$$= - \, _2F_1\left(-2k, 1; \frac{3}{2}; 2\right)\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{2}{3 \cdot \, _2F_1(-2k, 1; \frac{3}{2}; 2)}\right)^{1/4}, \text{ ``nearly''},$$

$$-\alpha^{-1/4} \, _2F_1\left(-2k - 1, 1; \frac{3}{2}; 2\right)$$

$$+ 4\alpha^{3/4} \int_{-\infty}^{0} \frac{xe^{-\alpha x^2}}{e^{2\pi x} - 1} \, _1F_1\left(-2k - 1; \frac{3}{2}; 2\alpha x^2\right) dx$$

$$= \beta^{-1/4} \, _2F_1\left(-2k - 1, 1; \frac{3}{2}; 2\right)$$

$$- 4\beta^{3/4} \int_{-\infty}^{0} \frac{xe^{-\beta x^2}}{e^{2\pi x} - 1} \, _1F_1\left(-2k - 1; \frac{3}{2}; 2\beta x^2\right) dx.$$

In particular when $\alpha = \beta = \pi$,

$$(5.10) \quad \int_{-\infty}^{0} \frac{xe^{-\pi x^2}}{e^{2\pi x} - 1} \, _1F_1\left(-2k - 1; \frac{3}{2}; 2\pi x^2\right) dx = \frac{1}{4\pi} \, _2F_1\left(-2k - 1, 1; \frac{3}{2}; 2\right).$$

6. Generalization of an asymptotic expansion of Oloa. We first explain how the integral in (1.32) is related to the one on the right extreme of (1.16). Write the latter integral as

$$(6.1) \quad \int_{0}^{\infty} (1 + t^2)\Gamma\left(\frac{-1 + it}{4}\right)\Gamma\left(\frac{-1 - it}{4}\right)\frac{\Xi(t/2)}{1 + t^2} \cos\left(\frac{1}{2}t \log \alpha\right) dt.$$

If we now square the expression $\frac{\Xi(t/2)}{1+t^2}$ in (6.1), then as discussed in [14], this amounts to squaring the functional equation of the Riemann zeta function,

and moreover the squared expression admits a generalization

$$\frac{\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)}{(t^2+(z+1)^2)(t^2+(z-1)^2)}.$$

This is what Ramanujan may have had at the back of his mind when he worked [41, Section 5] with the generalization

$$\int\limits_0^\infty (t^2+(z-1)^2)\Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)$$

$$\times\frac{\Xi\left(\frac{t+iz}{2}\right)\Xi\left(\frac{t-iz}{2}\right)}{(t^2+(z+1)^2)(t^2+(z-1)^2)}\cos\left(\frac{1}{2}t\log\alpha\right)dt$$

of (6.1), which upon simplification gives (1.32). Ramanujan [41] obtained a transformation formula associated with this integral. In [14], Moll and one of the present authors found the following new representation of this transformation, which generalizes a transformation of Koshliakov [25, (6)], [26, (21), (27)].

Assume $-1 < \operatorname{Re} z < 1$. *Define*

$$\Omega(x,z) =$$

$$2\sum_{n=1}^\infty \sigma_{-z}(n)n^{z/2}\left(e^{\pi iz/4}K_z(4\pi e^{\pi i/4}\sqrt{nx})+e^{-\pi iz/4}K_z(4\pi e^{-\pi i/4}\sqrt{nx})\right),$$

where $\sigma_{-z}(n) = \sum_{d|n} d^{-z}$. *Then for* $\alpha,\beta > 0$ *with* $\alpha\beta = 1$,

$$(6.2)\quad \alpha^{(z+1)/2}\int\limits_0^\infty e^{-2\pi\alpha x}x^{z/2}\left(\Omega(x,z)-\frac{1}{2\pi}\zeta(z)x^{z/2-1}\right)dx$$

$$= \beta^{(z+1)/2}\int\limits_0^\infty e^{-2\pi\beta x}x^{z/2}\left(\Omega(x,z)-\frac{1}{2\pi}\zeta(z)x^{z/2-1}\right)dx$$

$$= \frac{1}{2\pi^{(z+5)/2}}\int\limits_0^\infty \Gamma\left(\frac{z-1+it}{4}\right)\Gamma\left(\frac{z-1-it}{4}\right)\Xi\left(\frac{t+iz}{2}\right)$$

$$\times \Xi\left(\frac{t-iz}{2}\right)\frac{\cos\left(\frac{1}{2}t\log\alpha\right)dt}{(t^2+(z+1)^2)}.$$

Theorem 1.10 is now proved using (6.2).

Proof of Theorem 1.10. We obtain the asymptotic expansion of the integral indirectly by obtaining the same for the integral on the left extreme

of (6.2). Let

$$g(t, z) := t^{z/2}\left(\Omega(t, z) - \frac{1}{2\pi}\zeta(z)t^{z/2-1}\right).$$

We use the following identity established in [14, Proposition 6.1]:

$$\Omega(t, z) = -\frac{\Gamma(z)\zeta(z)}{(2\pi\sqrt{t})^z} + \frac{t^{z/2-1}}{2\pi}\zeta(z) - \frac{t^{z/2}}{2}\zeta(z+1) + \frac{t^{z/2+1}}{\pi}\sum_{n=1}^{\infty}\frac{\sigma_{-z}(n)}{n^2+t^2}.$$

Since for $|t| < 1$,

$$\sum_{n=1}^{\infty}\frac{\sigma_{-z}(n)}{n^2+t^2} = \sum_{n=1}^{\infty}\frac{\sigma_{-z}(n)}{n^2}\frac{1}{1+\left(\frac{t}{n}\right)^2}$$

$$= \sum_{n=1}^{\infty}\frac{\sigma_{-z}(n)}{n^2}\sum_{m=0}^{\infty}(-1)^m\left(\frac{t}{n}\right)^{2m} = \sum_{m=0}^{\infty}(-1)^m t^{2m}\sum_{n=1}^{\infty}\frac{\sigma_{-z}(n)}{n^{2m+2}}$$

$$= \sum_{m=0}^{\infty}(-1)^m\zeta(2m+2)\zeta(2m+2+z)t^{2m},$$

we see that if

$$h(t, z) := g(t, z) + \frac{\Gamma(z)\zeta(z)}{(2\pi)^z} + \frac{t^z}{2}\zeta(z+1),$$

then

$$h(t, z) = \sum_{m=0}^{\infty}\frac{(-1)^m}{\pi}\zeta(2m+2)\zeta(2m+2+z)t^{2m+z+1},$$

so also as $t \to 0$,

$$h(t, z) \sim \sum_{m=0}^{\infty}\frac{(-1)^m}{\pi}\zeta(2m+2)\zeta(2m+2+z)t^{2m+z+1}.$$

We now apply Lemma 2.4 with $\lambda = (z+2)/2$ and $\mu = 1/2$. The condition $-1 < \operatorname{Re} z < 1$ guarantees that $\operatorname{Re}\lambda > 0$, which is necessary as remarked after Lemma 2.4. Then as $\alpha \to \infty$,

$$(6.3) \quad \int_0^{\infty} e^{-2\pi\alpha t}h(t, z)\,dt$$

$$\sim \sum_{m=0}^{\infty}\frac{(-1)^m}{\pi(2\pi\alpha)^{2m+z+2}}\Gamma(2m+2+z)\zeta(2m+2)\zeta(2m+2+z).$$

Note that

$$(6.4) \qquad \int\limits_0^\infty e^{-2\pi\alpha t} h(t,z)\, dt$$

$$= \int\limits_0^\infty e^{-2\pi\alpha t} g(t,z)\, dt + \frac{\Gamma(z)\zeta(z)}{(2\pi)^z} \int\limits_0^\infty e^{-2\pi\alpha t}\, dt + \frac{\zeta(z+1)}{2} \int\limits_0^\infty e^{-2\pi\alpha t} t^z\, dt$$

$$= \int\limits_0^\infty e^{-2\pi\alpha t} g(t,z)\, dt + \frac{\Gamma(z)\zeta(z)}{\alpha(2\pi)^{z+1}} + \frac{\zeta(z+1)\Gamma(z+1)}{2(2\pi\alpha)^{z+1}}.$$

From (6.3), (6.4) and (6.2), one now obtains (1.33) after simplification. ∎

7. Concluding remarks and some open questions

1. In this paper, we found two new transformations involving error functions, namely the ones in Theorems 1.1 and 1.2, which when combined give Ramanujan's transformation (1.5) (or equivalently (1.4)) for an integral analogue of theta functions, thus giving a better understanding of Ramanujan's transformation. Also, the results in Theorem 1.9 could have been obtained directly without having to resort to Theorems 1.5–Corollary 1.8 and Theorem 5.1. However, obtaining Theorem 1.9 from these theorems is useful since they give us many interesting results which otherwise might not have been revealed. For example, one could have proved (1.31) directly through (iii) of Theorem 1.9. However, proving it through (1.28) and (5.10) gives those non-trivial integral evaluations in addition.

In light of the existence of the integral on the extreme right side of (1.15) which equals two sides of the first error function transformation, it is natural to ask if a similar integral exists for the second error function transformation in (1.2). We have been unable to find such an integral, and so we leave it as an open problem. However, it is important to state here the difficulty in finding this integral, if it exists.

If we reverse the steps used in proving the equality of the extreme sides of the transformation (1.15) in Theorem 1.1, we notice that a crucial step was to use the integral representation for the error function given in (3.5). However, employing the same method to the left-hand side of (1.2) does not help, as the error function there does not cancel with the integral

$$\frac{2}{\pi} \int\limits_0^\infty \frac{e^{-\pi\alpha^2 x^2 + 2\pi x} \sin\left(\sqrt{\pi}\,\alpha x z\right)}{x}\, dx.$$

Another reason why this is a difficult problem is that while the Mellin transform of $e^{-ax^2} \sin bx$ is essentially just a $_1F_1$ (see Lemma 2.2), that

of $e^{-ax^2-cx} \sin bx$ involves parabolic cylinder functions [19, p. 503, formula 3.953, no. 1].

We now explain the significance of this integral, provided it exists. As remarked in the introduction, subtracting the first error function transformation in (1.15) from the second given in (1.18) leads to Ramanujan's transformation (1.5) for what is called an integral analogue of the Jacobi theta function. The corresponding transformation for the Jacobi theta function, which has an integral involving $\Xi(t)$ equal to it [13, Theorem 1.2], is

$$(7.1) \quad \sqrt{\alpha}\left(\frac{e^{-z^2/8}}{2\alpha} - e^{z^2/8}\sum_{n=1}^{\infty} e^{-\pi\alpha^2 n^2}\cos(\sqrt{\pi}\,\alpha n z)\right)$$

$$= \sqrt{\beta}\left(\frac{e^{z^2/8}}{2\beta} - e^{-z^2/8}\sum_{n=1}^{\infty} e^{-\pi\beta^2 n^2}\cos(i\sqrt{\pi}\,\beta n z)\right)$$

$$= \frac{1}{\pi}\int_0^{\infty} \frac{\Xi(t/2)}{1+t^2}\nabla\left(\alpha, z, \frac{1+it}{2}\right)dt,$$

where the ∇ function is defined in (2.2).

The equality of the extreme left and right sides of the special case $z = 0$ of the above identity was used by Hardy [20, (2)] to prove that infinitely many non-trivial zeros of $\zeta(s)$ lie on the critical line $\operatorname{Re} s = 1/2$. Thus if an integral involving $\Xi(t)$ equal to both sides of (1.5) is found, then this integral analogue of Hardy's formula may be used to obtain more information on the non-trivial zeros of $\zeta(s)$. However, this requires us to first obtain an integral involving $\Xi(t)$ equal to the two sides of (1.18).

Further, since our results involve an extra parameter z, it may be important to see what information about $\zeta(s)$, or some generalization of it, could be extracted from them. It would also be worth further studying these two error function transformations from the point of view of further applications in analytic number theory.

REMARK. Hardy [21] conjectured that Ramanujan's formula (1.16) may also be used to prove the infinitude of the zeros of $\zeta(s)$ on the critical line. However, we believe that it is not this formula but rather the special case $z = 0$ of the identity which has an integral involving $\Xi(t)$ equal to both sides of (1.5) that may lead to the existence of infinitely many zeros.

2. Consider the transformation in (1.16) and its equivalent version (1.17) given by Mordell. Let $q = e^{i\pi w}$, $\operatorname{Im} w > 0$, and let $\Lambda(w) := \sum_{n=1}^{\infty} F(n)q^n$, where $F(D)$ denotes the number of classes of positive definite binary quadratic forms $ax^2 + 2hxy + by^2$ with a, b not both even, and determinant $-D$.

Then Mordell [34, (2.18)] proved

$$(7.2) \quad \int_0^\infty \frac{xe^{\pi iwx^2}}{e^{2\pi x} - 1} \, dx$$

$$= -\frac{i}{4\pi w} - \Lambda(w) + \frac{\sqrt{-iw}}{w^2} \Lambda\left(-\frac{1}{w}\right) + \frac{1}{8}\left(\sum_{n=-\infty}^\infty e^{i\pi n^2 w}\right)^3,$$

so that with $\alpha^2 = -iw$, we have for $\operatorname{Re}\alpha^2 > 0$,

$$\int_0^\infty \frac{xe^{-\pi\alpha^2 x^2}}{e^{2\pi x} - 1} \, dx = \frac{-1}{4\pi\alpha^2} - \sum_{n=1}^\infty F(n)e^{-\pi n\alpha^2} - \frac{1}{\alpha^3} \sum_{n=1}^\infty F(n)e^{-\pi n/\alpha^2}$$

$$+ \frac{1}{8}\left(\sum_{n=-\infty}^\infty e^{-\pi\alpha^2 n^2}\right)^3.$$

It will be interesting to see whether the above result admits a generalization when we work with the integral in (1.15).

3. For a fixed $z \in \mathbb{C}$, consider the integral

$$(7.3) \quad \int_{-\infty}^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x} - 1} \, {}_1F_1\left(z; \frac{3}{2}; 2\alpha x^2\right) dx.$$

Using the asymptotic expansion of the confluent hypergeometric function [1, p. 508, Eq. 13.5.1], it can be seen that as $|x| \to \infty$,

$${}_1F_1\left(z; \frac{3}{2}; 2\alpha x^2\right) \sim \frac{\sqrt{\pi}}{2}\left(\frac{e^{i\pi z}(2\alpha x^2)^{-z}}{\Gamma(3/2 - z)} + \frac{e^{2\alpha x^2}(2\alpha x^2)^{z-3/2}}{\Gamma(z)}\right).$$

Note that because of the presence of $e^{2\alpha x^2}$ in the second expression of the asymptotic expansion, and since $\alpha > 0$, the only way the integral in (7.3) can converge is if this expression is annihilated by $\Gamma(z)$. This happens only when z is a non-positive integer. This leads us to consider two cases based on the parity of such z.

CASE 1: *z is a non-positive even integer.* Note that for α either very small or very large, the integral in (1.29) is nicely approximated by the expression on its right side, as in this case β is respectively very large or very small. However, the case $\alpha = \beta = \pi$ is the worst in terms of approximating this integral, i.e.

$$\int_{-\infty}^\infty \frac{xe^{-\pi x^2}}{e^{2\pi x} - 1} \, {}_1F_1\left(-2k; \frac{3}{2}; 2\pi x^2\right) dx,$$

since then α and β are not only of the same order of magnitude but in fact equal. Table 1 below shows how the integral in (1.29) is approximated by

the right side of (1.29) for some small values of α. (The calculations in this table are done for the identity obtained by dividing both sides of (1.29) by $\alpha^{3/4}$. They are performed in *Mathematica*.)

Table 1. Both sides of (1.29) (after dividing through by $\alpha^{3/4}$)

α	0.000007		1.5		2.378		9361.79	
k	LHS	RHS	LHS	RHS	LHS	RHS	LHS	RHS
1	33333.4	33333.4	0.212975	0.210775	0.1483410	0.1465060	0.00136109	0.001361096
2	24263.1	24263.1	0.162014	0.161821	0.112982	0.112883	0.000990862	0.000990862
3	19861.2	19861.2	0.135921	0.137363	0.0948065	0.0960151	0.000811187	0.000811187
4	17166.6	17166.6	0.11939	0.122057	0.0832805	0.085431	0.000701201	0.000701201
5	15309.6	15309.6	0.107718	0.111318	0.0751402	0.07799044	0.000625405	0.000625405
6	13934	13934	0.0989131	0.103239	0.0689983	0.0723852	0.000569256	0.000569256
7	12864	12864	0.091965	0.096872	0.0641517	0.0679618	0.000525582	0.000525583
8	12002	12002	0.0863014	0.0916811	0.060201	0.0643522	0.000490396	0.000490397
9	11288.8	11288.8	0.0815698	0.0873407	0.0569004	0.0613316	0.000461286	0.000461287
10	10686.4	10686.4	0.0775398	0.0836389	0.0540892	0.0587534	0.000436698	0.000436698

CASE 2: *z is a non-positive odd integer.* When $\alpha = \pi$ in the integral (7.3), we have shown in (1.31) that it is equal to zero.

Thus there is a trade-off in that (1.29) has no restriction on α (except $\alpha > 0$) but is an approximation, whereas we can exactly evaluate the integral (7.3), but only for a specific value of α, i.e., when $\alpha = \pi$. This leads us to two open questions:

QUESTION 1. Find the exact evaluation of

$$\int_0^\infty \frac{xe^{-\pi x^2}}{e^{2\pi x} - 1} \, {}_1F_1(-2k; 3/2; 2\pi x^2) \, dx$$

for $k \in \mathbb{Z}^+ \cup \{0\}$.

QUESTION 2. Find the exact evaluation of, or at least an approximation to, the integral

$$\int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x} - 1} \, {}_1F_1(-2k - 1; 3/2; 2\alpha x^2) \, dx$$

when $\alpha \neq \pi$ is a positive real number and $k \in \mathbb{Z}^+ \cup \{0\}$.

It is interesting to note that in Theorem 1.3, the integral involving $\Xi(t, \chi)$ involves the Δ function when χ is even and the ∇ function when χ is odd, which is exactly opposite to what happens in [15, Theorems 1.3–1.5]. Besides the fact that, in doing so, one can explicitly evaluate the Mellin transforms and that one *does* get what one is looking for, is there some intrinsic reason behind this reversal?

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