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Holomorphic Characterizations of Certain Classes of Distributions

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PhD THESIS

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ABSTRACT

We give two holomorphic characterizations of the space $\mathfrak{D}'_{L^p}$ of distributions for $1 < p < \infty$. Using the almost analytic continuation method we have found sufficient conditions for a function f holomorphic in $\mathbb{C} \setminus \mathbb{R}$ or even in a smaller region, a stripe along the real line, to have its boundary values in the space $\mathfrak{D}'_{L^p}(\mathbb{R})$ for $1 < p < \infty$. We will also give another characterization of holomorphic functions whose boundary values are in $\mathfrak{D}'_{L^p}$ for $1 < p < \infty$.

We are also concerned with the holomorphic representation of integrable distributions.

We present some applications of the holomorphic representation of distributions.

MSC: 46F20, 30E20.

Key words and phrases: Cauchy transform of distributions, analytic representation of distributions.

Contents

INTRODUCTION

The thesis consists of four chapters. In Chapter I we introduce some definitions and present well known facts that will prove useful in the next chapters. We discuss such integral transforms as the Cauchy, the Hilbert and the Fourier transforms. We also outline important facts concerning all these transforms. Next, we present the complex Stokes theorem which is an essential tool in one of the proofs in Chapter II. Moreover, we give definitions of the spaces $\mathfrak{D}'(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ and state some important theorems for these spaces.

Chapter II contains main results of the thesis. We give two holomorphic characterizations of the space $\mathfrak{D}'_{L^p}$ of distributions for $1 < p < \infty$ (see Theorem 2.5, p. 29 and Theorem 2.6, p. 32). The study of holomorphic characterization of distributions was inspired by the Cauchy transform. The Cauchy transform of a distribution is a generalization of the Cauchy integral of a function f holomorphic in a domain D, given by

$$Cf(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(s)}{s-z}ds = \begin{cases} f(z), & \text{for} \quad z \in D; \\ 0, & \text{for} \quad z \notin D, \end{cases}$$

where the function f is to be replaced by a distribution Λ, and the boundary ∂D of D by the real line.

In the thesis we discuss some classes of distributions for which the Cauchy transform can be defined. In cases when the Cauchy transform $C\Lambda$ of a distribution Λ exists, it is a holomorphic function in $\mathbb{C} \setminus \mathbb{R}$. Moreover, this function $C\Lambda(z)$ has got an important property. Namely,

$$\Lambda = \lim_{\varepsilon \to 0}[C\Lambda(\cdot + i\varepsilon) - C\Lambda(\cdot - i\varepsilon)],$$

in the sense of convergence in the appropriate space of distributions. In other words, for a distribution Λ we can find two holomorphic functions, such that Λ is a boundary value of their difference (see Theorem 2.4, p. 26 and Theorem 3.3, p. 39).

For example take the Dirac δ. This is an integrable distribution (the definition of the space of integrable distributions can be found in Chapter III). For each function $\varphi \in \mathfrak{D}_{L^\infty}$ we have $\delta(\varphi) = \varphi(0)$. Due to the Cauchy transform the distribution δ can be interpreted as:

$$\delta = \lim_{y \to 0}\frac{-1}{2\pi i}\Big[\frac{1}{\cdot + iy} - \frac{1}{\cdot - iy}\Big] \qquad \text{(see p. 40).}$$

The function $f(z) = \frac{-1}{2\pi i z}$ is a holomorphic function in $\mathbb{C} \setminus \mathbb{R}$.

This holomorphic representation, sometimes referred to as a boundary value representation, has got many applications, eg. to differential equations. The main advantage is that many operations on distributions can be replaced by operations on corresponding holomorphic functions. Some important applications of holomorphic representation can be found in Chapter IV.

In Chapter II we have characterized the Cauchy transform of distributions in $\mathfrak{D}'_{L^p}(\mathbb{R})$, where $1 < p < \infty$. Namely, if $\Lambda \in \mathfrak{D}'_{L^p}(\mathbb{R})$, where $1 < p < \infty$, then $C\Lambda$ is a holomorphic function in $\mathbb{C} \setminus \mathbb{R}$ and there exist constants $m \in \mathbb{N}_0$ and $M > 0$ such that

$$\| C\Lambda(\cdot + iy) \|_{L^p} \leq M \frac{1}{|y|^k},$$

where

$$k = \begin{cases} 0, & \text{for} \quad |y| > 1, \\ m, & \text{for} \quad 0 < |y| \leq 1, \end{cases}$$

(see Theorem 2.3, p. 22).

The Cauchy transform is an operator from the space $\mathfrak{D}'_{L^p}(\mathbb{R})$ into the space $H(\mathbb{C} \setminus \mathbb{R})$ of functions holomorphic in $\mathbb{C} \setminus \mathbb{R}$. In the thesis we have considered the inverse problem. We have tried to distinguish those holomorphic functions in $\mathbb{C} \setminus \mathbb{R}$ that have their boundary values in the space $\mathfrak{D}'_{L^p}$. We have found sufficient conditions for a function f holomorphic in $\mathbb{C} \setminus \mathbb{R}$ or even in a smaller region, a stripe along the real line, to have its boundary values in the space $\mathfrak{D}'_{L^p}(\mathbb{R})$ for $1 < p < \infty$. Namely, using the almost analytic continuation method (see [26]) we have arrived at the following characterization (see Theorem 2.5, p. 29).

Let f be a holomorphic function in $\Psi_0 \setminus \mathbb{R}$, where $\Psi_0 = \{z : z \in \mathbb{C}, \Im z < \delta_0, \}$, $\delta_0 = \delta_0(f) > 0$ and

$$\| f(\cdot + iy) \|_{L^p} \leq M \frac{1}{|y|^k},$$

where

$$k = \begin{cases} 0, & \text{for} \quad |y| > 1, \\ m, & \text{for} \quad 0 < |y| \leq 1, \end{cases}$$

$p > 1$ and $M > 0$. Then for each $\varphi \in \mathfrak{D}_{L^q}$, where $\frac{1}{p} + \frac{1}{q} = 1$ there exists the limit

$$\langle Tf, \varphi \rangle = \lim_{y \to 0} \int_{\mathbb{R}} [f(x + iy) - f(x - iy)] \varphi(x) dx,$$

in $\mathfrak{D}'_{L^p}$ and Tf is a distribution in $\mathfrak{D}'_{L^p}$.

We will also give another characterization of holomorphic functions whose boundary values are in $\mathfrak{D}'_{L^p}$ for $1 < p < \infty$. By assuming a different condition on the growth of holomorphic functions we have obtained a condition for a function to be the Cauchy transform of a distribution in $\mathfrak{D}'_{L^p}$ for $1 < p < \infty$. This result is a generalization of the theorem known for $p = 2$ (see [17]) and for $1 < p \leq 2$ (see [32]). We have the following theorem (see Theorem 2.6, p. 32):

A function $F \in H(\Pi^+ \cup \Pi^-)$, where $\Pi^+ = \{z: \ z \in \mathbb{C}, \quad \Im z > 0\}$ and $\Pi^- = \{z: \ z \in \mathbb{C}, \quad \Im z < 0\}$, is the Cauchy transform of a distribution from $\mathfrak{D}'_{L^p}$, $1 < p < \infty$, if and only if

$$F(z) = \sum_{n=0}^{m} \frac{d^n}{dz^n} F_n(z),$$

where $F_n \in H(\Pi^+ \cup \Pi^-)$ and $\quad \sup_{y \neq 0} \int_{\mathbb{R}} |F_n(x+iy)|^p dx < \infty.$

The holomorphic characterization of distributions has a long history.

The boundary value representation for distributions in $\mathfrak{D}'(\mathbb{R})$ was considered by H. J. Bremermann and L. Durand (see [2]), H. G. Tillmann (see [36], [37]), D. Vogt (see [39]) and M. Orton (see [23]) and was also examined in a different context by Z. Szmydt and B. Ziemian in [35].

The problem of holomorphic representation for distributions in $\mathfrak{D}'_{L^p}$ was tackled by Z. Łuszczki and Z. Zielezny in [21] and R. D. Carmichael in [4], [5] and [6]. The latter, however, assumed a different growth of holomorphic functions. Also J. N. Pandey in [24] assumed a different estimation for his boundary value representation (see Theorem 6, p. 104 and Theorem 7, p. 106). In fact we have succeeded in improving the results of J. N. Pandey. In comparison with the preceding results the Theorems 2.5 and 2.6 seem to be a natural characterization. Moreover, the proofs are not complicated and do not require using advanced tools.

Chapter III is devoted to integrable distributions and their holomorphic representations. We have characterized the Cauchy transform of an integrable distribution. The Cauchy transform of integrable distributions plays a similar role as it does for the distributions in $\mathfrak{D}'_{L^p}$ for $1 < p < \infty$. Namely, an integrable distribution can be recovered from its Cauchy transform. Unfortunately, we have not been able to describe holomorphic functions which have their boundary values in $\mathfrak{D}'_{L^1}(\mathbb{R})$.

In Chapter IV we present some applications of the holomorphic representation of distributions. We show that the holomorphic representation can be applied to obtain the integral of an integrable distribution (see Theorem 4.1, p. 43 and Theorem 4.4, p. 47). We also give an example of distributions that belong to the space $\mathfrak{D}'_{L^p}$, namely, the Cauchy finite parts of an integral. We calculate the Cauchy transforms of these distributions (see formula (4.18), p. 49). Then we estimate their Cauchy transforms to illustrate the theorems obtained in Chapter II and Chapter III. Next we use the Cauchy transforms to obtain in an easy way the Fourier transforms of finite parts of an integral treated as distributions in $\mathfrak{D}'_{L^p}$ for $p > 1$ (see formula (4.19), p. 51). We also show that the holomorphic representation of distributions plays an important role in solving the Hilbert problem.

CHAPTER I

PRELIMINARY NOTIONS AND DEFINITIONS

This chapter is devoted to integral transforms such as the Cauchy transform, the Hilbert transform and the Fourier transform. The aim is to outline basic properties of these mappings acting on the spaces $L^p(\mathbb{R})$ and to prove essential theorems required in the main results of the thesis. We will also introduce some spaces of distributions and show their properties which we will use later.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By $\|\cdot\|_{L^p}$ we denote the norm in the space $L^p(\mathbb{R})$.

1. The Fourier Transform.

Definition 1.1.

The *Fourier transform* $\mathcal{F}f$ of a function $f : \mathbb{R} \to \mathbb{C}$ is a function $\hat{f}$ defined by

$$(\mathcal{F}f)(w) = \hat{f}(w) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{iwt}dt$$

provided that the integral exists.

We are mainly interested in the Fourier transform on the spaces $L^p(\mathbb{R})$. Let us recall the following properties of the Fourier transform on the space $L^1(\mathbb{R})$.

Theorem 1.1. *Let $f \in L^1(\mathbb{R})$. Then*
i) $\mathcal{F}f(w) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{iwt}dt$ is well defined for all $w \in \mathbb{R}$;
ii) $\mathcal{F}f(w)$ is uniformly continuous and bounded on $\mathbb{R}$;
iii) $\mathcal{F}f(w) \to 0$ as $|w| \to \infty$.

The Fourier transform of functions belonging to $L^2(\mathbb{R})$ does not necesserily exist in the pointwise sense. Because of that Plancherel defined the Fourier transform of functions in $L^2(\mathbb{R})$ in the following way.

Definition 1.2 (see [29], p. 200 and [38], p. 29).

Let $f \in L^2(\mathbb{R})$. Then there exists a function $\hat{f}(w) \in L^2(\mathbb{R})$ such that

$$\| \hat{f}(w) - \int_{-a}^{a} f(t)\mathrm{e}^{iwt} dt \|_{L^2} \to 0, \quad as \quad a \to \infty.$$

We write

$$\hat{f}(w) = l.i.m._{a\to\infty} \int_{-a}^{a} f(t)\mathrm{e}^{iwt} dt$$

and say that $\hat{f}$ is the *Fourier-Plancherel transform* of f.

Moreover,

$$\| f(x) - \frac{1}{2\pi} \int_{-a}^{a} \hat{f}(w)\mathrm{e}^{-iwx} dw \|_{L^2} \to 0 \quad as \quad a \to \infty.$$

The theory of the Fourier transform for $L^p(\mathbb{R})$ functions, where $1 < p \leq 2$ was developed by Titchmarsh in [38].

Theorem 1.2 (see [38], p. 96 and [24], p. 7). *Let $f \in L^p(\mathbb{R})$, $\quad 1 < p \leq 2$. Then there exists a function $\hat{f}(\xi) \in L^q(\mathbb{R})$, where $\frac{1}{p} + \frac{1}{q} = 1$ such that*

$$\| \hat{f}(\xi) - \int_{-N}^{N} f(t)\mathrm{e}^{it\xi} dt \|_{L^q} \to 0 \quad as \quad N \to \infty.$$

Furthermore

$$\| f(\xi) - \frac{1}{2\pi} \int_{-N}^{N} \hat{f}(t)\mathrm{e}^{-it\xi} dt \|_{L^p} \to 0 \quad as \quad N \to \infty.$$

The Fourier reciprocity relation holds in the sense that

$$\hat{f}(\xi) = \frac{d}{d\xi} \int_{-\infty}^{\infty} f(t) \frac{\mathrm{e}^{i\xi t} - 1}{it} dt \quad a.e.,$$

$$f(x) = \frac{d}{dx} \int_{-\infty}^{\infty} \hat{f}(\xi) \frac{\mathrm{e}^{-i\xi x} - 1}{-i\xi} d\xi \quad a.e.$$

Moreover,

$$\| \hat{f} \|_{L^q} \leq K(p) \left(\int_{-\infty}^{\infty} |f(x)|^p dx \right)^{\frac{1}{p-1}},$$

where $K(p)$ is a constant depending on p.

2. The Hilbert Transform.

Definition 1.3 (see [8], p. 341, [24]; p. 75 and [38], p. 120).

The *Hilbert transform* $(\mathcal{H}f)(x)$ of a function $f : \mathbb{R} \to \mathbb{C}$ is a function $\mathcal{H}f$ defined by

$$(\mathcal{H}f)(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{x-t}dt, \tag{1.1}$$

provided that the integral exists.

Let $f \in L^1$. Then the Hilbert transform $\mathcal{H}f$ exists a.e. provided that the integral has to be understood in the sense of the principal value of Cauchy (Pv), i.e.

$$(\mathcal{H}f)(x) = \frac{1}{\pi}(Pv)\int_{-\infty}^{\infty} \frac{f(t)}{x-t}dt = \frac{1}{\pi}\lim_{\varepsilon\to 0+}\left(\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty}\right)\frac{f(t)}{x-t}dt. \tag{1.1'}$$

The expression on the right of (1.1') is sometimes denoted by

$$\frac{1}{\pi}p.v.\int_{-\infty}^{\infty} \frac{f(t)}{x-t}dt \quad \text{or by} \quad \frac{1}{\pi}(P)\int_{-\infty}^{\infty} \frac{f(t)}{x-t}dt$$

and the function $\mathcal{H}f$ is sometimes called the *conjugate* of f.

The theory of Hilbert transform was developed by Riesz, Titchmarsh, Zygmund, Calderon and many others in the study of singular integrals. We will briefly recall some of the important properties of the Hilbert transform of functions in $L^p(\mathbb{R})$.

1. If $f \in L^p(\mathbb{R})$, $1 < p < \infty$, then $(\mathcal{H}f)(x)$ exists a.e. on $\mathbb{R}$ and it belongs to $L^p(\mathbb{R})$. The limit (1.1') exists in L^p sense. There exists $C_p > 0$ independent of f such that

$$\| \mathcal{H}f \|_{L^p} \leq C_p \| f \|_{L^p} .$$

2. If $f(t)$ and $g(t)$ belong to $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$ respectively, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_{-\infty}^{\infty} (\mathcal{H}f)(x)g(x)dx = -\int_{-\infty}^{\infty} f(x)(\mathcal{H}g)(x)dx.$$

This relation can be written as $\langle \mathcal{H}f, g\rangle = \langle f, -\mathcal{H}g\rangle$.

3. Let $\mathcal{H}$ be the operator on L^p, $1 < p < \infty$, defined by (1.1'). Then

$$(\mathcal{H}^2 f)(x) = -f(x) \quad a.e.$$

and in L^p sense. The above relation can be expressed as $\mathcal{H}^2 = -I$ on $L^p(\mathbb{R})$, for $1 < p < \infty$.

4. If $f \in L^p(\mathbb{R})$, $1 < p < \infty$, then we can define functions $u(x,y)$ and $v(x,y)$ by

$$
\begin{aligned}
u(x,y) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)y}{(t-x)^2 + y^2} dt, \\
v(x,y) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)(t-x)}{(t-x)^2 + y^2} dt.
\end{aligned} \tag{1.2}
$$

Both functions exist for $y \neq 0$ and belong to $L^p(\mathbb{R})$ when treated as functions of x for each fixed $y \neq 0$. Functions u and v are also *conjugate harmonic functions* in the region $y \neq 0$. Furthermore

$$\lim_{y \to 0^+} u(x,y) = f(x)$$

$$\lim_{y \to 0^+} v(x,y) = -\mathcal{H}f(x)$$

where the limits hold in L^p sense and a.e. These facts can be found eg. in [20], pp 142-145, [24], pp 92-93 and [38], pp 122-141.

Note that if $f \in L^1$ then the Hilbert transform $\mathcal{H}f$ is in general not a L^1-function.

3. The Cauchy Transform.

The Cauchy transform is a generalization of the Cauchy integral of a holomorphic function in a domain D, given by

$$Cf(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(s)}{s-z} ds = \begin{cases} f(z), & \text{for} \quad z \in D \\ 0, & \text{for} \quad z \notin D, \end{cases}$$

where the boundary ∂D of D, the contour of integration is to be replaced by the real line, and a holomorphic function by a function eg. of $L^p(\mathbb{R})$ or by a distribution (see [3], p. 43, [8], p. 332, [13], p. 164). It is sometimes called *a generalized Cauchy integral* or a *Cauchy representation*.

Definition 1.4.

Let $f \in L^p(\mathbb{R})$, where $1 < p < \infty$. The function $Cf : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ defined by

$$Cf(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z} dt, \quad \Im z \neq 0$$

is called the *Cauchy transform* of f.

The properties of the Cauchy transform of a function belonging to L^p are shown in the next theorem which can be derived from the results obtained by Riesz, Hille and Tamarkin (see [8], Theorems 2.1 and 3.1, [9], Theorem 2 and [20], pp 137-142). We give an elementary proof of this important theorem.

Theorem 1.3. *Let $f \in L^p(\mathbb{R})$, where $1 < p < \infty$. Then the Cauchy transform $Cf(z)$ of the function f exists and is a holomorphic function in $\mathbb{C} \setminus \mathbb{R}$. For the function Cf(z) we have*

$$\frac{d}{dz} Cf(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{(t-z)^2} dt \quad \textit{for each } \Im z \neq 0.$$

Moreover, $Cf(x+iy)$ belongs to $L^p(\mathbb{R})$ as a function in x for each fixed $y \neq 0$ and

$$\sup_{y \neq 0} \| Cf(\cdot + iy) \|_{L^p} < \infty. \tag{1.3}$$

Proof.

Let us notice that the function

$$g(t) = \frac{1}{t-z} \quad \text{is in} \quad L^p(\mathbb{R}) \quad \text{for} \quad p > 1 \quad \text{and} \quad \Im z \neq 0.$$

We will prove that

$$\frac{Cf(z+h) - Cf(z)}{h} - \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{(t-z)^2} dt \to 0 \quad \text{as} \quad h \to 0, \quad \text{for} \quad \Im z \neq 0.$$

We have

$$\frac{Cf(z+h) - Cf(z)}{h} - \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{(t-z)^2} dt = \frac{h}{2\pi i} \int_{\mathbb{R}} \frac{f(t)dt}{(t-z-h)(t-z)^2}.$$

Using the Hölder inequality we can estimate now

$$\left| \int_{\mathbb{R}} \frac{f(t)dt}{(t-z-h)(t-z)^2} \right| \leq \left(\int_{-\infty}^{\infty} |f(t)|^p dt \right)^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} \left| \frac{1}{(t-z-h)(t-z)^2} \right|^q dt \right)^{\frac{1}{q}}$$
$$\leq C \left(\int_{-\infty}^{\infty} \left| \frac{1}{(t-z-h)(t-z)^2} \right|^q dt \right)^{\frac{1}{q}}.$$

It suffices to estimate the integral

$$\int_{-\infty}^{\infty} | \frac{1}{(t-z-h)(t-z)^2} |^q dt.$$

Take now $z = x + iy$ and $h = \xi + i\eta$. By Hölder's inequality we have

$$\int_{-\infty}^{\infty} \left| \frac{1}{(t-z-h)(t-z)^2} \right|^q dt \leq$$

$$\leq \left(\int_{-\infty}^{\infty} \left(\frac{1}{(t-x)^2 + y^2} \right)^{q^2} dt \right)^{\frac{1}{q}} \left(\int_{-\infty}^{\infty} \left(\frac{1}{(t-x-\xi)^2 + (y+\eta)^2} \right)^{\frac{pq}{2}} dt \right)^{\frac{1}{p}}.$$

Take now h small such that $|h| < \delta < |y|$ and put $d = y - \delta$ if $y > 0$ and $d = |y + \delta|$ if $y < 0$. Then, since $q^2 > 1$ and $\frac{pq}{2} > \frac{1}{2}$ we have

$$\left(\int_{-\infty}^{\infty} \left(\frac{1}{(t-x)^2 + y^2} \right)^{q^2} dt \right)^{\frac{1}{q}} \leq \left(\int_{-\infty}^{\infty} \left(\frac{1}{(t-x)^2 + d^2} \right)^{q^2} dt \right)^{\frac{1}{q}} = B < \infty,$$

$$\left(\int_{-\infty}^{\infty} \left(\frac{1}{(t-x-\xi)^2 + (y+\eta)^2} \right)^{\frac{pq}{2}} dt \right)^{\frac{1}{p}} \leq \left(\int_{-\infty}^{\infty} \left(\frac{1}{(t-x-\xi)^2 + d^2} \right)^{\frac{pq}{2}} dt \right)^{\frac{1}{p}} = D < \infty.$$

Hence

$$\Big| \frac{h}{2\pi i} \int_{\mathbb{R}} \frac{f(t)dt}{(t-z-h)(t-z)^2} \Big| \leq \frac{|h|}{2\pi} C \cdot B^{\frac{1}{q}} \cdot D^{\frac{1}{q}} \to 0 \quad \text{as} \quad h \to 0.$$

Therefore $Cf(z)$ is a holomorphic function for $\Im z \neq 0$ and its derivative is equal

$$\frac{d}{dz} Cf(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{(t-z)^2} dt \quad \text{for each } \Im z \neq 0.$$

Let us notice that for any function $f \in L^p(\mathbb{R})$ the Cauchy transform $Cf(z)$ can be expressed as a sum of two functions:

$$Cf(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)(t-x)dt}{(t-x)^2 + y^2} + \frac{1}{2\pi} \int_{\mathbb{R}} \frac{f(t)ydt}{(t-x)^2 + y^2}.$$

By (1.2) we obtain that

$$Cf(x+iy) = \frac{1}{2i} v(x,y) + \frac{1}{2} u(x,y),$$

where both functions $u(x,y)$ and $v(x,y)$ belong to $L^p(\mathbb{R})$ as functions in $x \in \mathbb{R}$ for each $y \neq 0$. Hence, $Cf(\cdot + iy) \in L^p(\mathbb{R})$ for each $y \neq 0$.

To show that

$$\sup_{y \neq 0} \| Cf(\cdot + iy) \|_{L^p} < \infty$$

we can refer to Theorem 2.1 in [8] and Theorem 2 in [9] or Corollary 1, p. 145 in [20].

4. Complex Stokes' Theorem (see [10], p. 2).

Let D be a bounded open set in $\mathbb{C}$, such that the boundary ∂D consists of a finite number of C^1 Jordan curves. If $u \in C^1(\bar{D})$, then by Stokes' formula

$$\int_{\partial D} u dz = \int\int_D du \wedge dz,$$

or if we note that $du \wedge dz = \frac{\partial u}{\partial \bar{z}} d\bar{z} \wedge dz = 2i \frac{\partial u}{\partial \bar{z}} dx \wedge dy$

$$\int_{\partial D} u dz = 2i \int\int_D \frac{\partial u}{\partial \bar{z}} d\bar{z} \wedge dz.$$

5. Some Spaces of Distributions and their Properties.

Let Ω be an open set in $\mathbb{R}$. We shall denote by $\mathfrak{D}(\Omega)$ the set of all infinitely differentiable functions with compact supports contained in Ω. Obviously, $\mathfrak{D}(\Omega)$ is a vector space.

We introduce a convergence of sequences in $\mathfrak{D}(\Omega)$.

Definition 1.5 (see [22], p. 15).

We say that a *sequence* $(\varphi_\nu)_{\nu \in \mathbb{N}}$, $\varphi_\nu \in \mathfrak{D}(\Omega)$ *converges to* $\varphi \in \mathfrak{D}(\Omega)$ in $\mathfrak{D}$ if

(i) there exists a compact set $K \subset \Omega$ such that $\operatorname{supp} \varphi_\nu \subset K$ for each $\nu \in \mathbb{N}$

and

(ii) the sequence $(\varphi_\nu^{(n)})_{\nu \in \mathbb{N}}$ converges to $\varphi^{(n)}$ uniformly on K for each n $\in \mathbb{N}_0$.

Definition 1.6.

A linear complex form defined on $\mathfrak{D}(\Omega)$ continuous with respect to the above convergence is said to be a *distribution*. The set of all distributions will be denoted by $\mathfrak{D}'(\Omega)$.

If $\Omega = \mathbb{R}$ we write $\mathfrak{D}$ instead of $\mathfrak{D}(\Omega)$ and $\mathfrak{D}'$ instead of $\mathfrak{D}'(\Omega)$.

Theorem 1.4 (see [12], p. 313 and [11], p. 33). *A linear form Λ defined on $\mathfrak{D}(\Omega)$ is continuous if and only if for every compact set $K \subset \Omega$ there exist a constant $C > 0$ and $m \in \mathbb{N}_0$ such that for each $\varphi \in \mathfrak{D}(\Omega)$*

$$|\Lambda(\varphi)| \leq C \parallel \varphi \parallel_{m,K}, \tag{1.4}$$

where

$$\parallel \varphi \parallel_{m,K} = \max_{n \leq m} \sup_{x \in K} |\varphi^{(n)}(x)| \quad \text{and} \quad \operatorname{supp} \varphi \subset K.$$

Examples.

1. Regular distributions.

Let $f \in L^1(\mathbb{R})$ and define $\Lambda_f(\varphi) = \int_{\mathbb{R}} f(x)\varphi(x)dx$. In fact, it is sufficient to take $f \in L^1_{loc}$ because the supp φ is compact. Of course,

$$|\Lambda_f(\varphi)| \leq \| f \|_{L^1_{loc}} \| \varphi \|_{m,K},$$

if $\operatorname{supp}\varphi \subset K$. Therefore Λ_f is in $\mathfrak{D}'$. The distribution Λ_f is called a *regular distribution*.

2. The Dirac delta distribution.

Put $\delta(\varphi) = \varphi(0)$ for $\varphi \in \mathfrak{D}$. We have

$$|\delta(\varphi)| = |\varphi(0)| \leq \sup_{x \in K} \| \varphi \|_{0,K}$$

where $\operatorname{supp}\varphi \subset K$. Hence, δ is a distribution. The Dirac delta distribution is not a regular distribution. Any distribution that cannot be defined in terms of a locally integrable function f as in Example 1, is said to be *singular*.

We will show that δ is a singular distribution. Suppose by contradiction that δ is a regular distribution, i.e. there exists a function $f \in L^1_{loc}$ such that for each function $\varphi \in \mathfrak{D}$

$$\delta(\varphi) = \int_{\mathbb{R}} f(x)\varphi(x)dx.$$

Take a function

$$\varphi(x) := \begin{cases} \exp(x^2-1)^{-1}, & \text{for} \quad |x| < 1, \\ 0, & \text{for} \quad |x| \geq 1. \end{cases}$$

Put $\varphi_n(x) = \varphi(nx)$. Of course $\varphi \in \mathfrak{D}$. For each $n \in \mathbb{N}$ we have $\delta(\varphi_n) = \varphi(0) = \mathrm{e}^{-1}$. On the other hand

$$\delta(\varphi) = \int_{\mathbb{R}} f(x)\varphi_n(x)dx = \int_{|x|<\frac{1}{n}} f(x)\exp((nx)^2-1)^{-1}dx \leq$$

$$\leq \mathrm{e}^{-1} \int_{|x|<\frac{1}{n}} |f(x)|dx \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $\delta(\varphi_n) \to 0$ as $n \to \infty$. This contradicts the fact that $\delta(\varphi_n) = \mathrm{e}^{-1}$ for each $n \in \mathbb{N}$.

3. The Cauchy Principal Value.

The function x^{-1} is not integrable at zero and so it does not define a regular distribution; for an arbitrary function $\varphi \in \mathfrak{D}$ the integral

$$\int_{-\infty}^{\infty} \frac{\varphi(x)}{x} dx$$

in general does not converge. Nevertheless we can use it to generate a singular distribution by taking the *Cauchy Principal Value* of the divergent integrals which arise. Writing this distribution as $Pv\{x^{-1}\}$, to distinguish it from the ordinary function x^{-1}, we have

$$\begin{aligned} Pv\{x^{-1}\}(\varphi) &= Pv \int_{-\infty}^{\infty} \frac{\varphi(x)}{x} dx \\ &= \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x} dx = \lim_{\varepsilon \to 0} \left[\int_{+\varepsilon}^{\infty} \frac{\varphi(x)}{x} dx + \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x} dx \right] \\ &= \lim_{\varepsilon \to 0} \int_{+\varepsilon}^{\infty} \frac{\varphi(x) - \varphi(-x)}{x} dx. \end{aligned}$$

We will show that $Pv\{x^{-1}\}$ defines a distribution in $\mathfrak{D}'$.

Let φ be in $\mathfrak{D}$ and $\operatorname{supp} \varphi \subset [-a, a]$. We can define a function

$$\psi(x) = \begin{cases} \frac{\varphi(x) - \varphi(0)}{x} & \text{for} \quad x \neq 0, \\ \varphi'(0) & \text{for} \quad x = 0. \end{cases}$$

Clearly, ψ is continuous and $\varphi(x) = \varphi(0) + x\psi(x)$ for $x \in \mathbb{R}$. By the mean value theorem, we have $\psi(x) = \varphi'(\theta_x \cdot x), \quad 0 < \theta_x < 1$ for $x \neq 0$.

This implies that

$$|\psi(x)| \leq \sup_{x \in [-a,a]} |\varphi'(x)| \leq \| \varphi \|_{1,[-a,a]}$$

for $x \in \mathbb{R}$. Note that

$$\int_{|x|>\varepsilon} \frac{\varphi(x)}{x} dx = \int_{\varepsilon<|x|\leq a} \frac{\varphi(x)}{x} dx = \int_{\varepsilon<|x|\leq a} \psi(x) dx.$$

By the Lebesgue theorem we have

$$\Lambda(\varphi) = \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x} dx = \int_{|x|\leq a} \psi(x) dx.$$

Hence,

$$|\Lambda(\varphi)| \leq 2a \| \varphi \|_{1,[-a,a]} .$$

By Theorem 1.4, Λ is a distribution.

The Cauchy Principal Value is also denoted by $Fp\frac{1}{x}$ or by $\frac{1}{(\cdot)}$ and is sometimes called the *Cauchy finite part of an integral*.

Let Λ_α, Λ be in $\mathfrak{D}'(\Omega)$.

Definition 1.7 (see [22], p. 47).

We say that Λ_α *distributionally converges* to Λ as $\alpha \to \alpha_0$ if for each $\varphi \in \mathfrak{D}(\Omega)$ $\lim_{\alpha\to\alpha_0} \Lambda_\alpha(\varphi) = \Lambda(\varphi)$.

In particular, if $\nu \in \mathbb{N}$ then we say that $\lim_{\nu\to\infty} \Lambda_\nu = \lim_{\frac{1}{\nu}\to 0} \Lambda_\nu = \Lambda$ if for each $\varphi \in \mathfrak{D}(\Omega)$ we have $\lim_{\nu\to\infty} \Lambda_\nu(\varphi) = \Lambda(\varphi)$.

Example.

The following sequences

$$\frac{1}{2}\nu \mathrm{e}^{-\nu|x|} \quad (Picard),$$

$$\frac{1}{\pi}\frac{\nu}{1+(\nu x)^2} \quad (Cauchy),$$

$$\frac{\nu}{\sqrt{2\pi}}\mathrm{e}^{\frac{-1}{2}(\nu x)^2} \quad (Gauss),$$

where $\nu = 1, 2, ...$, are distributionally convergent to δ.

Let $\mathcal{S}$ denote the set of all $C^\infty(\mathbb{R})$ functions for which there exist constants $C_{k,n}$ such that $|x^k\varphi^{(n)}(x)| \le C_{k,n}$, for $x \in \mathbb{R}$ and $k, n \in \mathbb{N}_0$.

We introduce a family of norms in $\mathcal{S}$ by putting

$$\| \varphi \|_m = \max_{0\le k+n\le m} \sup_{x\in\mathbb{R}} |x^k\varphi^{(n)}(x)|.$$

Definition 1.8 (see [22], p. 60 and [34], p. 102).

A sequence $(\varphi_\nu)_{\nu\in\mathbb{N}}$ is said to be *convergent to* φ *in* $\mathcal{S}$, if $\| \varphi_\nu - \varphi \|_m$ tends to zero as $\nu \to \infty$ for each $m = 0, 1, 2...$

Obviously, $(\varphi_\nu)_{\nu\in\mathbb{N}}$ converges in $\mathcal{S}$ to φ as $\nu \to \infty$ if and only if $(x^n\varphi_\nu^{(m)})_{\nu\in\mathbb{N}}$ converges uniformly to $x^n\varphi^{(m)}$ in $\mathbb{R}$ for each $n, m \in \mathbb{N}_0$ (see [22], p. 60). If $\varphi_\nu, \varphi \in \mathfrak{D}$ and $(\varphi_\nu)_{\nu\in\mathbb{N}}$ converges to φ as $\nu \to \infty$ in $\mathfrak{D}$ then $(\varphi_\nu)_{\nu\in\mathbb{N}}$ converges to φ as $\nu \to \infty$ in $\mathcal{S}$.

It is easy to show that $\mathfrak{D}$ is dense in $\mathcal{S}$ (see [22], p. 60).

Definition 1.9 (see [34], p. 111).

A linear form defined on $\mathcal{S}$ and continuous with respect to the convergence in $\mathcal{S}$ is called a *tempered distribution.* The set of all tempered distributions will be denoted by $\mathcal{S}'$.

Remark 1.1.

Not every distribution on $\mathfrak{D}$ can be extended to a tempered distribution.

Example.

Let $\delta_a(\varphi) = \varphi(a)$ and define $\Lambda = \sum_{n=-\infty}^{\infty} \delta_n \mathrm{e}^{n^2}$. Then $\Lambda \in \mathfrak{D}'(\mathbb{R})$ but it does not belong to $\mathcal{S}'(\mathbb{R})$. We have

$$\Lambda(\varphi) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{n^2} \varphi(n).$$

For each $\varphi \in \mathfrak{D}$ with supp $\varphi \subset K$ we have $|\Lambda(\varphi)| \leq C \parallel \varphi \parallel_{0,K}$. Hence Λ is a distribution in $\mathfrak{D}'$ but the series diverges for $\varphi(x) = \mathrm{e}^{-x^2} \in \mathcal{S}$.

Definition 1.10.

We say that a sequence $(\Lambda_\nu) \in \mathcal{S}'$ *converges* to $\Lambda \in \mathcal{S}'$ *in* $\mathcal{S}'$ if for each $\varphi \in \mathcal{S}$ the sequence $(\Lambda_\nu(\varphi))$ converges to $\Lambda(\varphi)$.

Definition 1.11 (see [34], p. 102).

Let $\varphi \in \mathcal{S}$. The function

$$(\mathcal{F}\varphi)(\sigma) = \int_{\mathbb{R}} \mathrm{e}^{i\sigma x} \varphi(x) dx$$

is called the *Fourier transform* of φ.

Theorem 1.5 (see [22], p. 61 and [34], p. 102). *If $\varphi \in \mathcal{S}$ then its Fourier transform is also in $\mathcal{S}$.*

Definition 1.12 (see [22], p. 68 and [34], p. 135).

Let $T \in \mathcal{S}'$. The *Fourier transform of* T is given by the relation

$$\mathcal{F}T(\varphi) = T(\mathcal{F}\varphi)$$

for each function $\varphi \in \mathcal{S}$.

Theorem 1.5 (see [22], p. 69; [34], p. 137). *The Fourier transform is a continuous map on $\mathcal{S}'$.*

CHAPTER II

ELEMENTS OF $\mathfrak{D}'_{L^p}(\mathbb{R})$ AS BOUNDARY VALUES OF HOLOMORPHIC FUNCTIONS

1. Introduction.

In this chapter we give a boundary value characterization of the space $\mathfrak{D}'_{L^p}(\mathbb{R})$ for $1 < p < \infty$. Namely, we will show that every distribution that belongs to the space $\mathfrak{D}'_{L^p}(\mathbb{R})$ for $1 < p < \infty$ can be represented by analytic functions. We will also show that analytic functions fulfilling certain growth conditions have their boundary values in the space $\mathfrak{D}'_{L^p}(\mathbb{R})$. We will give two different characterizations of holomorphic function spaces whose elements have boundary values in spaces of distributions. The first is based on the almost analytic continuation method used first by Petzsche for ultradistributions (see [26]) and it requires applying complex Stokes' theorem. The second is obtained by assuming a different condition on the growth of a holomorphic function. This characterization is elementary and does not refer to the almost analytic continuation. Moreover, it improves results known for the space $\mathfrak{D}'_{L^2}(\mathbb{R})$.

Definition 2.1.

We define spaces $\mathfrak{D}_{L^q}(\mathbb{R})$ for $q \geq 1$ as follows:

$$\mathfrak{D}_{L^q}(\mathbb{R}) = \{\varphi \in C^\infty(\mathbb{R}) : \quad \|\varphi^{(n)}\|_{L^q} < \infty \quad \text{for all} \quad n \in \mathbb{N}_0\}$$

and introduce a collection of seminorms

$$\|\varphi\|_m := \left(\int_{\mathbb{R}} |\varphi^{(m)}(x)|^q dx\right)^{\frac{1}{q}}, \quad m = 0, 1, 2...$$

We say that a sequence $(\varphi_n)_{n\in\mathbb{N}}$, $\varphi_n \in \mathfrak{D}_{L^q}(\mathbb{R})$, *converges to zero in the space* $\mathfrak{D}_{L^q}(\mathbb{R})$ if for each $m \in \mathbb{N}_0$ the sequence of norms $(\|\varphi_n\|_m)_{n\in\mathbb{N}}$ converges to zero as n tends to infinity.

We will write $\mathfrak{D}_{L^q}$ instead of $\mathfrak{D}_{L^q}(\mathbb{R})$.

It is obvious that $\mathfrak{D}_{L^q}$ is a Fréchet space.

In contrast to the classical spaces L^q the functions that belong to $\mathfrak{D}_{L^q}$ have the following properties.

Lemma 2.1 (see [1], p. 172). If $f \in \mathfrak{D}_{L^q}$ $(1 \le p \le q \le \infty)$ then f is bounded in $\mathbb{R}$. Moreover, if $1 \le p < \infty$ then f tends to zero at infinity.

Lemma 2.2 (see [1], p. 172). If $1 \le p \le q \le \infty$, then $\mathfrak{D}_{L^p} \subset \mathfrak{D}_{L^q}$ with continuous imbeddings.

Both Lemmas follow from the fact that a function in $\mathfrak{D}_{L^p}$ has all its derivatives in L^p.

Definition 2.2 (see [1] p. 173).

Let $1 \le q < \infty$ and let $\frac{1}{p} + \frac{1}{q} = 1$. The set of all linear functionals continuous on the space $\mathfrak{D}_{L^q}(\mathbb{R})$ will be denoted by $\mathfrak{D}'_{L^p}(\mathbb{R})$. The elements of this set are called *generalized functions* or *distributions of the class* $\mathfrak{D}'_{L^p}(\mathbb{R})$.

We have got the following structure theorem for distributions in $\mathfrak{D}'_{L^p}$, where $1 < p \le \infty$.

Theorem 2.1 (see [33], p. 200, [1], p. 173). *$\Lambda \in \mathfrak{D}'_{L^p}(\mathbb{R})$, where $1 < p \le \infty$, if and only if there exist an $m \in \mathbb{N}$ and functions $f_n \in L^p(\mathbb{R})$ for $n \in \mathbb{N}_0$ and $n \le m$ such that for each function $\varphi \in \mathfrak{D}_{L^q}(\mathbb{R})$*

$$\Lambda(\varphi) = \sum_{n=0}^{m} (-1)^n \int_{\mathbb{R}} \varphi^{(n)}(x) f_n(x) dx. \tag{2.1}$$

Proof. An easy proof of this theorem can be found in [30] and in [1], p. 174.

In this chapter we are concerned with distributions in $\mathfrak{D}'_{L^p}(\mathbb{R})$ for $1 < p < \infty$.

2. Examples.

1. Regular Distributions.

Let $f \in L^p$, $1 < p < \infty$. We can define

$$\langle f, \varphi \rangle = \int_{-\infty}^{\infty} f(x)\varphi(x) dx, \quad \forall \varphi \in \mathfrak{D}_{L^q}.$$

The above functional is linear and continuous. Hence it defines a functional on $\mathfrak{D}_{L^q}$. Therefore it is an element of $\mathfrak{D}'_{L^p}$.

2. The Cauchy Finite Part of an Integral.

For $\varphi \in \mathfrak{D}_{L^q}$, $q > 1$ we define

$$Fp\frac{1}{(\cdot)}(\varphi) := \int_{-\infty}^{\infty} \frac{(\varphi(t) - \varphi(-t))}{2t} dt.$$

We will show that the above functional belongs to $\mathfrak{D}'_{L^p}$.

Let $\psi \in \mathfrak{D}(\mathbb{R})$ be a function such that $\psi(x) = 1$ for $x \in [-1, 1]$, and supp$\psi \subset [-2, 2]$. We have

$$Fp\frac{1}{(\cdot)} = \psi Fp\frac{1}{(\cdot)} + (1 - \psi)Fp\frac{1}{(\cdot)} = \Lambda_1 + \Lambda_2,$$

where $\Lambda_1 = \psi Fp\frac{1}{(\cdot)}$, $\Lambda_2 = (1 - \psi)Fp\frac{1}{(\cdot)}$.

Let us notice that Λ_1 is a distribution with a compact support (for the definition see Chapter IV). Therefore it belongs to $\mathfrak{D}'_{L^p}$ for $p \geq 1$ (see Remark 3.1, p. 38).

We will show that

$$\Lambda_2 = (1 - \psi)Fp\frac{1}{(\cdot)} \in L^p$$

for $p > 1$ and as a result belongs to $\mathfrak{D}'_{L^p}$.

Let us notice that $1 - \psi$ is a smooth function. In particular it is smooth in $(-2, -1) \cup (1, 2)$.

Moreover,

$$1 - \psi(x) = 0 \quad \text{for} \quad x \in [-1, 1],$$

and

$$1 - \psi(x) = 1 \quad \text{for} \quad x \in (-\infty, -2] \cup [2, \infty).$$

Therefore we have

$$\int_{\mathbb{R}} \left| \frac{1 - \psi(x)}{x} \right|^p dx = \int_{|x| \geq 2} \frac{1}{|x|^p} dx + \int_{-2}^{-1} \left| \frac{1 - \psi(x)}{x} \right|^p dx + \int_1^2 \left| \frac{1 - \psi(x)}{x} \right|^p dx \leq$$

$$\leq \frac{2}{1 - p} |x|^{-p+1} \Big|_{-\infty}^{-2} + \sup_{x \in [-2, -1]} \left| \frac{1 - \psi(x)}{x} \right|^p + \sup_{x \in [1, 2]} \left| \frac{1 - \psi(x)}{x} \right|^p < \infty$$

if $p > 1$.

Hence $\Lambda_2 \in L^p$ for $p > 1$. As a result, we have

$$Fp\frac{1}{(\cdot)} \in \mathfrak{D}'_{L^p} \quad \text{for} \quad p > 1.$$

3. The Cauchy Transform of Distributions Belonging to $\mathfrak{D}'_{L^p}$.

We will show that the function $f_z(t) = \frac{1}{t-z}$ belongs to $\mathfrak{D}_{L^q}(\mathbb{R})$ for $q > 1$ and each $z \in \mathbb{C} \setminus \mathbb{R}$.

Note that

$$\frac{\partial^n}{\partial t^n}(t-z)^{-1} = (-1)^n n!(t-z)^{-n-1}.$$

An easy computation shows that

$$\begin{aligned}\int_{\mathbb{R}} |t-x-iy|^{-q(n+1)} dt &\le |y|^{-q(n+1)} \int_{\mathbb{R}} \left(\left(\frac{t-x}{y}\right)^2 + 1 \right)^{\frac{-q}{2}(n+1)} dt \\ &= |y|^{-q(n+1)+1} \int_{\mathbb{R}} \left(\omega^2+1\right)^{\frac{-q}{2}(n+1)} d\omega \\ &= C|y|^{-q(n+1)+1},\end{aligned}$$

where $C = \int_{\mathbb{R}} \left(\omega^2+1\right)^{\frac{-q}{2}(n+1)} d\omega$. Therefore $\frac{1}{\cdot - z} \in \mathfrak{D}_{L^q}$, $1 < q < \infty$ if $\Im z \neq O$.

This implies that the symbol $C\Lambda(z) := \frac{1}{2\pi i}\Lambda_t\left(\frac{1}{t-z}\right)$ is meaningful for $\Lambda \in \mathfrak{D}'_{L^p}$, where $\Lambda_t\left(\frac{1}{t-z}\right)$ stands for the value $\Lambda(f_z)$. In the sequel we shall denote by Λ_t the value of the distribution Λ on the function in variable t.

Definition 2.3.

Let $\Lambda \in \mathfrak{D}'_{L^p}(\mathbb{R})$, where $1 < p < \infty$. The function

$$C\Lambda(z) := \frac{1}{2\pi i}\Lambda_t\left(\frac{1}{t-z}\right) \tag{2.2}$$

defined for each $z \in \mathbb{C} \setminus \mathbb{R}$ is called the *Cauchy transform* of the distribution Λ.

Theorem 2.2. *If $\Lambda \in \mathfrak{D}'_{L^p}(\mathbb{R})$, $1 < p < \infty$, then the function $C\Lambda(z)$ is a holomorphic function in $\mathbb{C} \setminus \mathbb{R}$ and*

$$\frac{d}{dz}C\Lambda(z) = \frac{1}{2\pi i}\Lambda\left(\frac{1}{(t-z)^2}\right) = \left(C\Lambda^{(1)}\right)(z),$$

where $\Lambda^{(1)}$ is called the distributional derivative of Λ.

Proof.

Fix $z \in \mathbb{C} \setminus \mathbb{R}$. Let us notice that

$$\frac{(C\Lambda(z+h) - C\Lambda(z))}{h} - \frac{1}{2\pi i}\Lambda\left(\frac{1}{(t-z)^2}\right) = \frac{1}{2\pi i}\Lambda_t((t-z)^{-2}(t-z-h)^{-1}h).$$

Since Λ belongs to $\mathfrak{D}'_{L^p}$, it suffices to prove that

$$g_{h,z}(\cdot) = (\cdot - z)^{-2}(\cdot - z - h)^{-1}h$$

belongs to $\mathfrak{D}_{L^q}$ and tends to zero as $h \to 0$ in $\mathfrak{D}_{L^q}$, where $\frac{1}{p} + \frac{1}{q} = 1$.

By the Leibniz formula we get

$$\frac{\partial^n}{\partial t^n} g_{h,z}(t) = (-1)^n n! h \sum_{\nu=1}^{n} (\nu+1)(t-z)^{-(\nu+2)}(t-z-h)^{-(n+1-\nu)}.$$

Putting $z = x + iy$ and $h = \xi + i\eta$ we get

$$\int_{\mathbb{R}} \left|(t-z)^{-(\nu+2)}(t-z-h)^{-(n+1-\nu)}\right|^q dt$$

$$\leq \int_{\mathbb{R}} \left((t-x)^2 + y^2\right)^{-\frac{(\nu+2)q}{2}} \left((t-x-\xi)^2 + (y+\eta)^2\right)^{-\frac{(n+1-\nu)q}{2}} dt := A_\nu,$$

As $h \to 0$ we can now put $d = y - \delta$ if $y > 0$ and $d = |y + \delta|$ if $y < 0$. Then, for $|h| < \delta < |y|$, by the Hölder inequality, we get:

$$A_\nu \leq \left(\int_{\mathbb{R}} \left((t-x)^2 + d^2\right)^{-\frac{(\nu+2)q^2}{2}} dt\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}} \left((t-x-\xi)^2 + d^2\right)^{-\frac{(n+1-\nu)pq}{2}} dt\right)^{\frac{1}{p}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $\nu = 0, 1, 2, \ldots.$

Note that $\frac{(\nu+2)q^2}{2} > 1$ and $\frac{(n+1-\nu)pq}{2} > \frac{1}{2}$ for $\nu = 0, 1, 2, \ldots$ Hence, we get

$$\int_{\mathbb{R}} \left((t-x)^2 + d^2\right)^{-\frac{(\nu+2)q^2}{2}} dt = B_\nu$$

and

$$\int_{\mathbb{R}} \left((t-x-\xi)^2 + d^2\right)^{-\frac{(n+1-\nu)pq}{2}} dt = C_\nu.$$

Therefore $A_\nu \leq {B_\nu}^{\frac{1}{q}} C_\nu^{\frac{1}{p}}$. Finally, we have

$$\| \partial^n g_{h,z}(\cdot) \|_{L^q} \leq n! |h| \sum_{\nu=0}^{n} (\nu+1) B_\nu^{\frac{1}{q}} C_\nu^{\frac{1}{p}}.$$

This implies that $\| \partial^n g_{h,z}(\cdot) \|_{L^q}$ tends to zero as $h \to 0$. We have shown that

$$\frac{d}{dz} C\Lambda(z) = \frac{1}{2\pi i} \Lambda \left(\frac{1}{(\cdot - z)^2} \right).$$

The equality $\frac{1}{2\pi i} \Lambda \left(\frac{1}{(\cdot - z)^2} \right) = \Lambda^{(1)}(z)$ is obvious. This finishes the proof.

Theorem 2.3 (see [21], [31]). *Let z=x+iy. The Cauchy transform* $C\Lambda(z)$ *of the distribution* $\Lambda \in \mathfrak{D}'_{L^p}(\mathbb{R})$, *where* $1 < p < \infty$, *is in* $L^p(\mathbb{R})$ *as the function of* x *for* $\Im z \neq 0$. *Moreover, there exist constants* $M > 0$ *and* $m \in \mathbb{N}_0$ *such that*

$$\|C\Lambda(\cdot + iy)\|_{L^p} \leq \frac{M}{|y|^k}, \tag{2.3}$$

where

$$k = \begin{cases} 0, & for \quad |y| > 1; \\ m, & for \quad 0 < |y| \leq 1. \end{cases}$$

Proof.

Let $\Lambda \in \mathfrak{D}'_{L^p}(\mathbb{R})$.

According to Theorem 2.1 (see p. 18) and Definition 2.3 (see p. 20) we have the following representation

$$C\Lambda(z) = \frac{1}{2\pi i} \sum_{n=0}^{m} (-1)^n \int_{\mathbb{R}} \frac{\partial^n}{\partial t^n} \left(\frac{1}{t - z} \right) f_n(t) dt.$$

It is sufficient to estimate the integral

$$\int_{\mathbb{R}} \frac{1}{(t - z)^{n+1}} f_n(t) dt \quad \text{for a fixed} \quad n.$$

By the Young inequality (see [28], p. 45), for $n \geq 1$ there exists a constant $c > 0$ such that

$$\begin{aligned} &\left\| \int_{\mathbb{R}} \frac{1}{(t - x - iy)^{n+1}} f_n(t) dt \right\|_{L^p} \leq \|g\|_{L^1} \|f_n\|_{L^p} \leq \\ &\leq c \int_{-\infty}^{\infty} \frac{dt}{(t^2 + y^2)^{\frac{n+1}{2}}} \leq c \int_{-\infty}^{\infty} \frac{dt}{\left(y^2 \left(\frac{t^2}{y^2} + 1 \right) \right)^{\frac{n+1}{2}}} \\ &\leq c \frac{1}{|y|^n} \int_{-\infty}^{\infty} \frac{ds}{(s^2 + 1)^{\frac{n+1}{2}}} \leq c \frac{1}{|y|^n} \int_{-\infty}^{\infty} \frac{ds}{(s^2 + 1)} \leq c \frac{\pi}{|y|^n}, \end{aligned}$$

where
$$g(x)=\frac{1}{(x^2+y^2)^{\frac{n+1}{2}}}$$
belongs to $L^1(\mathbb{R})$ for $n \geq 1$.

It remains to estimate the integral
$$\int_{\mathbb{R}} \frac{1}{(t-x-iy)} f_0(t)dt.$$
Let us notice that
$$\int_{\mathbb{R}} \frac{1}{(t-x-iy)} f_0(t)dt = Cf_0(z),$$
where Cf_0 denotes the Cauchy transform of f_0.

Hence, by Theorem 1.3 (see p. 10), $Cf_o(\cdot + iy) \in L^p$ and fulfils (1.3).

Therefore, there exists a constant $B>0$ such that
$$\left(\int_{\mathbb{R}} \Big| \int_{\mathbb{R}} \frac{1}{t-x-iy} f_0(t)dt \Big|^p dx\right)^{\frac{1}{p}} \leq B$$
and we obtain (2.3). This finishes the proof.

4. Boundary Value Characterization of $\mathfrak{D}'_{L^p}(\mathbb{R})$ for $1<p<\infty$.

The Cauchy transform assigns to each generalized function $\Lambda \in \mathfrak{D}'_{L^p}(\mathbb{R})$ the function $C\Lambda(z)$, which is holomorphic in $\mathbb{C}\backslash\mathbb{R}$. This function has an important property. Namely, each $\Lambda \in \mathfrak{D}'_{L^p}(\mathbb{R})$ may be reproduced by means of its Cauchy transform. In order to get this result we shall need the following Lemmas.

Lemma 2.3.

If $\varphi \in \mathfrak{D}_{L^q}$, $1 \leq q \leq \infty$ and $\psi \in L^1(\mathbb{R})$, then $\varphi * \psi$ is a smooth function and $(\varphi * \psi)^{(n)} = \varphi^{(n)} * \psi$ for all $n \in \mathbb{N}$.

Proof.

By Lemma 2.2 (see p. 18), $\mathfrak{D}_{L^q} \subset \mathfrak{D}_{L^p}$ if $1 \leq q \leq p \leq \infty$. In particular $\mathfrak{D}_{L^p} \subset \mathfrak{D}_{L^\infty}$ if $p \geq 1$.

Note that
$$\frac{1}{h}\left(\varphi * \psi(x+h) - \varphi * \psi(x)\right) = \frac{1}{h}\left(\int_{\mathbb{R}} \varphi(x+h-y)\psi(y)dy - \int_{\mathbb{R}} \varphi(x-y)\psi(y)dy\right)$$
$$= \int_{\mathbb{R}} \frac{\varphi(x+h-y)-\varphi(x-y)}{h}\psi(y)dy.$$

The function φ is in C^∞. By the mean value theorem we can express the differential quotient in the form

$$\frac{\varphi(x+h-y)-\varphi(x-y)}{h}=\frac{\partial}{\partial x}\varphi(x+\theta h-y),$$

where $0<\theta<1$ (see [22], p. 43). By the Hölder inequality

$$\Big|\int_{\mathbb{R}}\frac{\partial}{\partial x}\varphi(x+\theta h-y)\psi(y)dy\Big|<\infty.$$

Hence, by the Lebesgue theorem we obtain that

$$\lim_{h\to 0}\int_{\mathbb{R}}\left(\frac{\varphi(x+h-y)-\varphi(x-y)}{h}-\frac{\partial}{\partial x}\varphi(x-y)\right)\psi(y)dy=0.$$

Now we can apply mathematical induction for higher derivatives.

Lemma 2.4.

If $\Lambda\in\mathfrak{D}'_{L^p}(\mathbb{R})$, $1<p<\infty$, $g\in\mathfrak{D}_{L^1}$ and $\int_{\mathbb{R}}g(x)dx=1$ then

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\Lambda_{g_\varepsilon}(x)\varphi(x)dx=\Lambda(\varphi)$$

for each $\varphi\in\mathfrak{D}_{L^q}$, $\frac{1}{p}+\frac{1}{q}=1$, where $g_\varepsilon(x)=\varepsilon^{-1}g(\frac{x}{\varepsilon})$ and $\Lambda_{g_\varepsilon}(x):=\Lambda_t\left(\varepsilon^{-1}g(\frac{t-x}{\varepsilon})\right)$.

Proof.

First of all we have to show that Λ_{g_ε} is in $\mathfrak{D}_{L^p}$. By Lemma 2.2 (see p. 18), $g_\varepsilon\in\mathfrak{D}_{L^q}$ for $1\le q\le\infty$. Hence, in view of Theorem 2.1 (see p. 18), we have:

$$\Lambda_{g_\varepsilon}(x)=\sum_{n=0}^{m}(-1)^n\int_{\mathbb{R}}\frac{\partial^n}{\partial y^n}g_\varepsilon(x-y)f_n(y)dy,$$

where f_n is a fixed element of L^p for $n=0,1,..m$.

We will estimate $\Lambda_{g_\varepsilon}(x)$ and its derivatives in the L^p norm.

Let us notice that

$$\frac{\partial^n}{\partial x^n}g_\varepsilon(x-y)=(-1)^n\frac{\partial^n}{\partial y^n}g_\varepsilon(x-y).$$

By Lemma 2.3 (see p. 23), we have

$$\frac{d^k}{dx^k}\Lambda_{g_\varepsilon}(x)=\sum_{n=0}^{m}(-1)^n\int_{\mathbb{R}}\frac{\partial^k}{\partial x^k}\frac{\partial^n}{\partial y^n}g_\varepsilon(x-y)f_n(y)dy$$

$$= \sum_{n=0}^{m} \int_{\mathbb{R}} \frac{\partial^{k+n}}{\partial x^{k+n}} g_\varepsilon(x-y) f_n(y) dy.$$

By the assumption $g_\varepsilon \in \mathfrak{D}_{L^1}$ and $f_n \in L^p$. Hence, by the Young inequality (see [28], p. 45 or [11], p.117)

$$\| \frac{d^k}{dx^k} \Lambda_{g_\varepsilon} \|_{L^p} \leq \sum_{n=0}^{m} \| \frac{\partial^{k+n}}{\partial x^{k+n}} g_\varepsilon(x) \|_{L^1} \| f_n \|_{L^p} < \infty.$$

Hence, $\Lambda_{g_\varepsilon} \in \mathfrak{D}_{L^p}$ for $\varepsilon > 0$. Therefore we have:

$$\int_{\mathbb{R}} \Lambda_{g_\varepsilon}(x) \varphi(x) dx = \int_{\mathbb{R}} \varphi(x) \left(\sum_{n=0}^{m} (-1)^n \int_{\mathbb{R}} \frac{\partial^n}{\partial y^n} g_\varepsilon(x-y) f_n(y) dy \right) dx$$
$$= \sum_{n=0}^{m} (-1)^n \int_{\mathbb{R}} \varphi(x) \left(\int_{\mathbb{R}} \frac{\partial^n}{\partial y^n} g_\varepsilon(x-y) f_n(y) dy \right) dx.$$

By Lemma 2.3 (see p. 23), the last integral can be written as follows

$$\int_{\mathbb{R}} \varphi(x) \left(\frac{\partial^n}{\partial x^n} \int_{\mathbb{R}} g_\varepsilon(x-y) f_n(y) dy \right) dx.$$

Integrating by parts we obtain

$$\int_{\mathbb{R}} \varphi(x) \left(\frac{\partial^n}{\partial x^n} \int_{\mathbb{R}} g_\varepsilon(x-y) f_n(y) dy \right) dx$$

$$= (-1)^n \int_{\mathbb{R}} \frac{\partial^n}{\partial x^n} \varphi(x) \left(\int_{\mathbb{R}} g_\varepsilon(x-y) f_n(y) dy \right) dx.$$

On the other hand we have

$$\Lambda(\varphi) = \sum_{n=0}^{m} (-1)^n \int_{\mathbb{R}} f_n(x) \frac{\partial^n}{\partial x^n} \varphi(x) dx.$$

Hence we have got the following equality:

$$\int_{\mathbb{R}} \Lambda_{g_\varepsilon}(x) \varphi(x) dx - \Lambda(\varphi)$$

$$= \sum_{n=0}^{m} (-1)^n \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g_\varepsilon(x-y) f_n(y) dy - f_n(x) \right) \frac{\partial^n}{\partial x^n} \varphi(x) dx.$$

It is easy to check that the change of variable in the integral gives

$$\int_{\mathbb{R}} g_\varepsilon(x-y)f_n(y)dy - f_n(x) = \int_{\mathbb{R}} (f_n(x-\varepsilon y) - f_n(x))g(y)dy.$$

This implies that

$$\left|\int_{\mathbb{R}} \Lambda_{g_\varepsilon}(x)\varphi(x)dx - \Lambda(\varphi)\right|$$

$$\leq \left|\sum_{n=0}^{m} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (f_n(x-\varepsilon y) - f_n(x))g(y)dy\right) \frac{\partial^n}{\partial x^n}\varphi(x)dx\right|$$

$$\leq \sum_{n=0}^{m} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f_n(x-\varepsilon y) - f_n(x)||g(y)|dy\right)^p dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} \left|\frac{\partial^n}{\partial x^n}\varphi(x)\right|^q dx\right)^{\frac{1}{q}}.$$

It remains to prove that the expression

$$\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f_n(x-\varepsilon y) - f_n(x)||g(y)|dy\right)^p dx\right)^{\frac{1}{p}}$$

tends to zero as $\varepsilon \to 0$. Using Minkowski's inequality we get

$$\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f_n(x-\varepsilon y) - f_n(x)||g(y)|dy\right)^p dx\right)^{\frac{1}{p}}$$

$$\leq \int_{\mathbb{R}} |g(y)| \left(\int_{\mathbb{R}} |f_n(x-\varepsilon y) - f_n(x)|^p dx\right)^{\frac{1}{p}} dy.$$

Let us notice that for each $n = 0, 1, 2...$

$$\| f_n(\cdot - \varepsilon y) - f_n \|_{L^p} \leq 2 \| f_n \|_{L^p} \quad \text{for} \quad y \in \mathbb{R} \quad \text{and}$$

$$\| f_n(\cdot - \varepsilon y) - f_n \|_{L^p} \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{for} \quad \text{each} \quad y \in \mathbb{R}, \quad \text{(see [22], p. 77)}.$$

Hence, by the Lebesgue theorem the last integral tends to zero as $\varepsilon \to 0$. This finishes the proof.

In the next theorem we shall show that a distribution $\Lambda \in \mathfrak{D}'_{L^p}(\mathbb{R})$ for $1 < p < \infty$ can be recovered from its Cauchy transform.

Theorem 2.4 (see [15], [30]). *Let $\Lambda \in \mathfrak{D}'_{L^p}(\mathbb{R})$, where $1 < p < \infty$.*
Then for each $\varphi \in \mathfrak{D}_{L^q}(\mathbb{R})$, where $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\lim_{y \to 0} \int_{\mathbb{R}} \left[C\Lambda(x+iy) - C\Lambda(x-iy)\right] \varphi(x)dx = \Lambda(\varphi).$$

Proof.

Let $\Lambda \in \mathfrak{D}'_{L^p}(\mathbb{R})$. We know that

$$C\Lambda(x+iy) - C\Lambda(x-iy) = \Lambda_t\left(\frac{1}{\pi}\frac{y}{(t-x)^2+y^2}\right).$$

It is now sufficient to apply Lemma 2.4 (see p. 24) with $\varepsilon = y$ to $g(x) = \frac{1}{\pi}\frac{1}{x^2+1} \in \mathfrak{D}_{L^1}$.

Remark 2.1.

In the case when Λ is a function in L^p, where $1 < p < \infty$, Theorem 2.4 coincides with the well-known theorem for functions. Namely, we have

$$Cf(x+iy) - Cf(x-iy) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(t)}{t-x-iy}dt - \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(t)}{t-x+iy}dt$$

$$= \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)y}{(t-x)^2+y^2}dt.$$

The function

$$K_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2}$$

is called the *Poisson kernel*. It is well known (see Chapter I) that

$$\lim_{y\to 0}(K_y * f)(x) = \lim_{y\to 0}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)y}{(t-x)^2+y^2}dt = f(x) \quad \text{in the} \quad L^p \quad \text{sense and a.e.}$$

Hence we can recover a function $f \in L^p$ from its Cauchy transform.

For the functions in $\mathfrak{D}_{L^p}$, where $1 \le p < \infty$, we have the following property

Lemma 2.5.

Let $\psi \in L^1$ be such that $\int_{\mathbb{R}} \psi(x)dx = 1$ and let $\varphi \in \mathfrak{D}_{L^p}$, where $1 \le p < \infty$. Put $\psi_\varepsilon(x) = \varepsilon^{-1}\psi(\frac{x}{\varepsilon})$. Then $\varphi * \psi_\varepsilon$ tends to φ as $\varepsilon \to 0$ in $\mathfrak{D}_{L^p}$.

Proof.

We will show that for each $n \in \mathbb{N}_0$

$$\| (\varphi * \psi_\varepsilon)^{(n)} - \varphi^{(n)} \|_{L^p} \quad \text{converges to} \quad 0 \quad \text{as} \quad \varepsilon \to 0.$$

By Lemma 2.3 (see p.23), we have

$$(\varphi * \psi_\varepsilon)^{(n)} - \varphi^{(n)} = \varphi^{(n)} * \psi_\varepsilon - \varphi^{(n)}.$$

Hence,

$$\begin{aligned}
&\| (\varphi * \psi_\varepsilon)^{(n)} - \varphi^{(n)} \|_{L^p} = \| \varphi^{(n)} * \psi_\varepsilon - \varphi^{(n)} \|_{L^p} \\
&= \| \int_{\mathbb{R}} \frac{\partial^n}{\partial x^n} (\varphi(x-y)\psi_\varepsilon(y)dy - \frac{\partial^n}{\partial x^n}\varphi(x) \|_{L^p} \\
&= \| \int_{\mathbb{R}} \frac{\partial^n}{\partial x^n} \left(\varphi(x-\varepsilon y) - \varphi(x)\right) \psi(y)dy \|_{L^p} .
\end{aligned}$$

Hence, by Minkowski's inequality we get

$$\left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \frac{\partial^n}{\partial x^n} \left(\varphi(x-\varepsilon y) - \varphi(x)\right) \psi(y)dy \right|^p dx \right)^{\frac{1}{p}}$$

$$\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| \frac{\partial^n}{\partial x^n} \left(\varphi(x-\varepsilon y) - \varphi(x)\right) \psi(y) \right|^p dx \right)^{\frac{1}{p}} dy$$

$$\leq \| \psi \|_{L^1} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| \frac{\partial^n}{\partial x^n} \left(\varphi(x-\varepsilon y) - \varphi(x)\right) \right|^p dx \right)^{\frac{1}{p}} dy.$$

By the same reasoning as at the end of the proof of Lemma 2.4 (see p. 24) we obtain that the last integral tends to 0 as $\varepsilon \to 0$. This finishes the proof. For the proof of a similar fact in the case $\varphi \in \mathfrak{D}_{L^\infty}$ see Lemma 3.1 (see p. 36).

Let us return to distributions. It turns out that not only the Cauchy transform allows us to reproduce distributions. We can find necessary and sufficient conditions for a function ϕ holomorphic in the region $\Psi_0 \setminus \mathbb{R}$, where

$$\Psi_0 = \{z : \quad z \in \mathbb{C}, \quad |\Im z| < \delta_0, \quad \delta_0 > 0\}, \quad \delta_0 = \delta_0(\phi),$$

to have its boundary values in $\mathfrak{D}'_{L^p}(\mathbb{R})$, where $1 < p < \infty$.

First, we will introduce one more definition.

Definition 2.4 (see [26], [27]).

Let f be a holomorphic function in $\Psi_0 \setminus \mathbb{R}$, where $\Psi_0 = \{z : z \in \mathbb{C}, |\Im z| < \delta_0\}$, $\delta_0 = \delta_0(f) > 0$.

If for every $\varphi \in \mathfrak{D}_{L^q}(\mathbb{R})$ there exists the limit

$$\langle Tf, \varphi\rangle := \lim_{y \to 0} \int_{\mathbb{R}} \left[f(x+iy) - f(x-iy)\right] \varphi(x)dx$$

then we call Tf the *boundary value of* f in $\mathfrak{D}'_{L^p}(\mathbb{R})$.

Denote by H_{L^p} the *space of all functions* f holomorphic in $\Psi_0 \setminus \mathbb{R}$, which satisfy the estimate

$$\text{(2.3')} \qquad \|f(\cdot + iy)\|_{L^p} \quad \leq \frac{M}{|y|^k},$$

where

$$k = \begin{cases} 0, & \text{for} \quad |y| > 1; \\ m, & \text{for} \quad 0 < |y| \leq 1. \end{cases}$$

for some $M > 0$, some $m \in \mathbb{N}_0$ and $1 < p < \infty$.

For the functions in H_{L^p} the following Lemma is valid in an appropriate region:

Lemma 2.6 (see [38], p. 125).

Let $\Phi(z)$ be analytic, and let

$$\int_{-\infty}^{\infty} |\Phi(x+iy)|^p dx < \infty \quad (p > 1)$$

exist and be bounded for $y_1 \leq y \leq y_2$. Then, as $|x| \to \infty$, $\Phi(x+iy) \to 0$, uniformly for $y_1 + \delta \leq y \leq y_2 - \delta$.

For the next theorem we need the following notations.

Let $\Psi = \{z : \quad z \in \mathbb{C}, \quad |\Im z| < \delta, \quad \delta \in (0, \delta_0)\ \}$, where $\delta_0 = \delta_0(f)$;
$\Gamma_{\delta+} = \{z : \quad z \in \mathbb{C}, \quad \Im z = \delta\}$;
$\Gamma_{\delta-} = \{z : \quad z \in \mathbb{C}, \quad \Im z = -\delta\}$.

Theorem 2.5. *Let $f \in H_{L^p}$. Then the boundary value Tf exists and for each function $\varphi \in \mathfrak{D}_{L^q}(\mathbb{R})$*

$$\text{(2.4)} \quad \langle Tf, \varphi\rangle = \int\!\!\int_{\Psi} f(z) \frac{\partial}{\partial \bar{z}} \Phi(z) dz \wedge d\bar{z} - \int_{\Gamma_{\delta-}} f(z)\Phi(z)dz + \int_{\Gamma_{\delta+}} f(z)\Phi(z)dz,$$

where $\Phi \in C^1(\mathbb{C})$ is such a function that $\Phi|_{\mathbb{R}} = \varphi$, "$\wedge$" denotes the exterior multiplication, and

$$\text{(2.5)} \qquad \sup_{0<|y|<\delta_0} \left\{ \frac{1}{|y|^k} \left\| \frac{\partial}{\partial \bar{z}} \Phi(\cdot + iy) \right\|_{L^q}, \frac{1}{|y|^k} \|\Phi(\cdot + iy)\|_{L^q} \right\} \leq c\|\varphi\|_{\mathfrak{D}_{L^q}},$$

where $k = k(f)$ and $\|\varphi\|_{\mathfrak{D}_{L^q}} = \sup_{0 \leq m \leq k} \| \varphi \|_m$. Moreover, $Tf \in \mathfrak{D}'_{L^p}(\mathbb{R})$.

Proof.

One can construct a function Φ fulfilling the condition (2.5). For example the function obtained by Petzsche in [26] satisfies (2.5).

Take $a > 0$ and $\delta \in (0, \delta_0)$ and define the segments:

$$\begin{aligned}
\Psi_{a_+} &= \{z : \quad z \in \mathbb{C}, \quad \Im z \in (0,\delta), \quad |\Re z| < a\},\\
\Psi_{a_-} &= \{z : \quad z \in \mathbb{C}, \quad \Im z \in (-\delta,0), \quad |\Re\, z| < a\},\\
\gamma_{a,+} &= \{z : \quad z \in \mathbb{C}, \quad z = a + it, \quad 0 < t < \delta\},\\
\gamma_{a,-} &= \{z : \quad z \in \mathbb{C}, \quad z = -a + it, \quad 0 < t < \delta\},\\
\Gamma_{a,\delta+\varepsilon} &= \{z : \quad z \in \mathbb{C}, \quad z = t + i(\delta+\varepsilon), \quad |t| < a\},\\
\Gamma_{a,\varepsilon} &= \{z : \quad z \in \mathbb{C}, \quad z = t + i\varepsilon, \quad |t| < a\}.
\end{aligned}$$

Let $\varepsilon < \frac{(\delta_o - \delta)}{2}$. Using the complex Stokes theorem, we get

$$\begin{aligned}
\int\!\!\int_{\Psi_{a+}} f(x+i(y+\varepsilon))\frac{\partial}{\partial \bar z}\Phi(z)d\bar z \wedge dz &= \int_{\partial \Psi_{a+}} f(x+i(y+\varepsilon))\Phi(z)dz\\
&= -\int_{\Gamma_{a,\delta+\varepsilon}} f(x+i(y+\varepsilon))\Phi(z)dz + \int_{\Gamma_{a,\varepsilon}} f(x+i(y+\varepsilon))\Phi(z)dz\\
&+ \int_{\gamma_{a+}} f(x+i(y+\varepsilon))\Phi(z)dz - \int_{\gamma_{a-}} f(x+i(y+\varepsilon))\Phi(z)dz.
\end{aligned}$$

We will estimate the integrals along the segments γ_{a^+} and γ_{a^-}.

Since $y+\varepsilon \in (\varepsilon, \varepsilon+\delta)$ for $y \in (0,\delta)$ and $f \in H_{L^p}$, we get

$$\|f(\cdot + i(y+\varepsilon))\|_{L^p} < \frac{M}{(y+\varepsilon)^m}, \quad \text{for some} \quad m \in \mathbb{N}_0 \quad \text{and} \quad M > 0.$$

Lemma 2.6 implies that

$$\lim_{|x|\to\infty} f(x+i(y+\varepsilon)) = 0.$$

Thus, by the assumption (2.5), the integrals over γ_{a^+} and γ_{a^-} tend to zero as $a \to \infty$. As a result the integral over Ψ_{a^+} becomes the integral over Ψ_+ and we have:

$$\begin{aligned}
&\int\!\!\int_{\Psi_+} f(x+i(y+\varepsilon))\frac{\partial}{\partial \bar z}\Phi(z)d\bar z \wedge dz =\\
&\qquad = \int_{-\infty}^{+\infty} f(x+i\varepsilon)\Phi(x)dx - \int_{-\infty}^{+\infty} f(x+i(\varepsilon+\delta))\Phi(x+i\delta)dx, \quad \text{and}
\end{aligned}$$

$$\int\int_{\Psi_-} f(x-i(y+\varepsilon))\frac{\partial}{\partial\bar z}\Phi(z)d\bar z\wedge dz =$$
$$= \int_{-\infty}^{+\infty} f(x-i(\varepsilon+\delta))\Phi(x-i\delta)dx - \int_{-\infty}^{+\infty} f(x-i\varepsilon)\Phi(x)dx.$$

Now, using the Hölder inequality we obtain

$$\Big|\int\int_{\Psi_+} f(x+i(y+\varepsilon))\frac{\partial}{\partial\bar z}\Phi(z)d\bar z\wedge dz\Big|$$
$$= 2\left|\int_0^\delta \left(\int_{-\infty}^{+\infty} f(x+i(y+\varepsilon))\frac{\partial}{\partial\bar z}\Phi(x+iy)dx\right)dy\right|$$
$$\leq 2\int_0^\delta \left(\int_{-\infty}^{+\infty}\left|f(x+i(y+\varepsilon))\right|^p dx\right)^{\frac{1}{p}}\left(\int_{-\infty}^{+\infty}\left|\frac{\partial}{\partial\bar z}\Phi(x+iy)\right|^q dx\right)^{\frac{1}{q}} dy \leq$$
$$\leq 2\delta\|\varphi\|_{\mathfrak{D}_{L^q}} < \infty \quad .$$

Since the integrands converge pointwise to the corresponding integrable functions as $\varepsilon \to 0$ we obtain the formula (2.4).

We will now prove that the formula (2.4) defines a functional $Tf \in \mathfrak{D}'_{L^p}(\mathbb{R})$.

By a well-known property of locally convex spaces (see Proposition 1.1 in [18]) it is sufficient to show that there exists a constant C such that for each $\varphi \in \mathfrak{D}_{L^q}(\mathbb{R})$

$$|\langle Tf,\varphi\rangle| \leq C\|\varphi\|_{\mathfrak{D}_{L^q}}.$$

Using the Hölder inequality, (2.3') and (2.5), we get

$$\begin{aligned}
|\langle Tf,\varphi\rangle| \leq & 2\int_0^\delta \left(\int_{-\infty}^{+\infty}|f(x+iy)|^p dx\right)^{\frac{1}{p}}\left(\int_{-\infty}^{+\infty}|\frac{\partial}{\partial\bar z}\Phi(x+iy)|^q dx\right)^{\frac{1}{q}} dy\\
&+\int_{-\infty}^{+\infty}|f(x+i\delta)\Phi(x+i\delta)|dx + \int_{-\infty}^{+\infty}|f(x-i\delta)\Phi(x-i\delta)|dx\\
&\leq 2\int_0^\delta \frac{M}{|y|^k}\|\frac{\partial}{\partial\bar z}\Phi(\cdot+iy)\|_{L^p}dy\\
&+\left(\int_{-\infty}^{+\infty}|f(x+i\delta)|^p dx\right)^{\frac{1}{p}}\left(\int_{-\infty}^{+\infty}|\Phi(x+i\delta)|^q dx\right)^{\frac{1}{q}}\\
&+\left(\int_{-\infty}^{+\infty}|f(x-i\delta)|^p dx\right)^{\frac{1}{p}}\left(\int_{-\infty}^{+\infty}|\Phi(x-i\delta)|^q dx\right)^{\frac{1}{q}}\\
&\leq 2M\int_0^\delta \|\varphi\|_{\mathfrak{D}_{L^q}}dy + \frac{M}{|\delta|^k}\|\Phi(\cdot+i\delta)\|_{L^q} + \frac{M}{|\delta|^k}\|\Phi(\cdot-i\delta)\|_{L^q} \leq\\
&\leq 2M\delta\|\varphi\|_{\mathfrak{D}_{L^q}} + Mc\|\varphi\|_{\mathfrak{D}_{L^q}} + Mc\|\varphi\|_{\mathfrak{D}_{L^q}}\\
&\leq C\|\varphi\|_{\mathfrak{D}_{L^q}}.
\end{aligned}$$

This finishes the proof.

We will now show one more characterization of distributions as boundary values of holomorphic functions. This characterization does not refer to the almost analytic continuation of the real function φ to $\mathbb{C}$ as the previous theorem. We will extend results obtained in [17] for $p = 2$ and in [32] for $1 < p \leq 2$ to $1 < p < \infty$.

Let $\Pi^+ = \{z : z \in \mathbb{C}, \quad \Im z > 0\}$ and $\Pi^- = \{z : z \in \mathbb{C}, \quad \Im z < 0\}$. Denote by $H(\Pi^+ \cup \Pi^-)$ the set of all holomorphic functions in $\Pi^+ \cup \Pi^-$.

Theorem 2.6. *A function $F \in H(\Pi^+ \cup \Pi^-)$ is the Cauchy transform of a distribution Λ from $\mathfrak{D}'_{L^p}$, $1 < p < \infty$, if and only if*

$$F(z) = \sum_{n=0}^{m} \frac{d^n}{dz^n} F_n(z),$$

where $F_n \in H(\Pi^+ \cup \Pi^-)$ and

$$\sup_{y \neq 0} \int_{\mathbb{R}} |F_n(x+iy)|^p dx < \infty. \tag{2.6}$$

Proof.

Necessity.
Let $\Lambda \in \mathfrak{D}'_{L^p}(\mathbb{R})$. Then

$$C\Lambda(z) := \frac{1}{2\pi i} \Lambda_t \left(\frac{1}{t-z} \right).$$

By Theorem 2.1 (see p. 18),

$$C\Lambda(z) = \frac{1}{2\pi i} \sum_{n=0}^{m} (-1)^n \int_{\mathbb{R}} \frac{\partial^n}{\partial t^n} \left(\frac{1}{t-z} \right) f_n(t) dt,$$

where $f_n \in L^p(\mathbb{R})$.

This means that $C\Lambda(z) = \sum_{n=0}^{m} CD^n f_n(z)$, where D^n indicates the distributional derivative of f_n. By Theorem 2.2 (see p. 20), we obtain

$$C\Lambda(z) = \sum_{n=0}^{m} \frac{d^n}{dz^n} Cf_n(z).$$

Put $F_n(z) = Cf_n(z)$. By Theorem 1.3 (see p. 10), the functions F_n fulfil (2.6).

Sufficiency.

Suppose that

$$F(z) = \sum_{n=0}^{m} \frac{d^n}{dz^n} F_n(z),$$

where the functions F_n fulfil (2.6) for each $n = 0, 1, 2...$

By Theorem 2.1 in [8], there exist functions $f_n \in L^p(\mathbb{R})$ such that $F_n = Cf_n$ for each $n = 0, 1, 2..., m$.

Therefore

$$F(z) = \sum_{n=0}^{m} \frac{d^n}{dz^n} Cf_n(z) = \sum_{n=0}^{m} C(D^n f)(z).$$

This means that $F = C\Lambda$, where $\Lambda = \sum_{n=0}^{m} D^n f_n$, and $f_n \in L^p(\mathbb{R})$.

Hence, by Theorem 2.1 (see p. 18), Λ is in $\mathfrak{D}'_{L^p}(\mathbb{R})$. This finishes the proof.

Remark 2.2.

If Λ is a regular distribution in $\mathfrak{D}'_{L^p}$, where $1 < p < \infty$, then Theorem 2.6 holds with $m = 0$.

Proof.

Let Λ be a regular distribution in $\mathfrak{D}'_{L^p}$, where $1 < p < \infty$. Then for each $\varphi \in \mathfrak{D}_{L^q}$, where $\frac{1}{p} + \frac{1}{q} = 1$

$$\Lambda(\varphi) = \int_{-\infty}^{\infty} \varphi(x) f(x) dx,$$

where f is a fixed element of L^p. Hence

$$C\Lambda(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t - z} dt = Cf(z).$$

Moreover, $C\Lambda(z)$ fulfils the condition (1.3) of Theorem 1.3 (see p. 10), hence (2.6) holds with $m = 0$.

Remark 2.3.

In general the Cauchy transform $C\Lambda$ of a distribution $\Lambda \in \mathfrak{D}'_{L^p}$ does not fulfil (1.3). We shall give a corresponding example in Chapter IV.

Remark 2.4.

Denote by L^p_σ, $1 \leq p < \infty$, the space of all entire functions of exponential type $\leq \sigma$ which belong to $L^p(\mathbb{R})$ (see [20], p. 149).

We have found sufficient conditions for a function holomorphic in a stripe along the real line to have its boundary values in the space $\mathfrak{D}'_{L^p}$ for $1 < p < \infty$. In fact we can distinguish the whole family of functions which have their boundary values in $\mathfrak{D}'_{L^p}$. Namely, the following statement is true.

Proposition.

Let $F(z) \in L^p_\sigma$ *and let* $\Lambda \in \mathfrak{D}'_{L^p}$, $(1 < p < \infty)$. *Then*

$$\Lambda = \lim_{\varepsilon \to 0} \left[(C\Lambda(\cdot + i\varepsilon) + F(\cdot + i\varepsilon)) - (C\Lambda(\cdot - i\varepsilon) + F(\cdot - i\varepsilon)) \right].$$

Proof.

It is sufficient to show that for each $\varphi \in \mathfrak{D}_{L^q}$, where $\frac{1}{p} + \frac{1}{q} = 1$,

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \left[F(x + i\varepsilon) - F(x - i\varepsilon) \right] \varphi(x) dx = 0.$$

This follows directly from the Hölder inequality.

In this chapter we have shown that one distribution can be reproduced by several holomorphic functions, more precisely by a whole family of functions. Hence, the Cauchy transform of the distribution Λ need not coincide with the generating function. Therefore we can venture a question what is the difference of the Cauchy transform of Λ and an arbitrary other generating function. Our conjecture is that the difference is a function holomorphic in a stripe around the real line.

We have not discussed the space $\mathfrak{D}'_{L^\infty}$ and the holomorphic representation of the distributions in $\mathfrak{D}'_{L^\infty}$ because the function $f(t) = \frac{1}{t-z}$ is not in L^1. Therefore we have not been able to consider the Cauchy transform of these distributions and the methods used in Chapter II could not be applied.

In Chapter III we will be concerned with the case $\mathfrak{D}'_{L^1}$.

In Chapter IV we will give one more example of a distribution in $\mathfrak{D}'_{L^p}$ and we will calculate its Cauchy transform. This example will illustrate the theorems that we have obtained in Chapter II.

CHAPTER III

INTEGRABLE DISTRIBUTIONS

1. Introduction.

A natural generalization of integrable functions with respect to the Lebesgue measure are integrable distributions. In this chapter we will describe the Cauchy transform of integrable distributions. We will also show that an integrable distribution can be recovered from its Cauchy transform.

The vector space $\mathfrak{D}_{L^\infty}$ is the set of all smooth functions φ defined on $\mathbb{R}$ such that their derivatives $\varphi^{(k)}$ are bounded on $\mathbb{R}$ for $k = 0, 1, 2, ...$

Definition 3.1.

We say that a *sequence* $(\varphi_\nu)_{\nu\in\mathbb{N}}$ *converges to zero in the space* $\mathfrak{D}_{L^\infty}$ if it satisfies the following two conditions:

$$(\beta) \qquad \begin{cases} (i) \quad \text{there exist positive numbers} \quad A_k \\ \quad \text{such that} \quad |\varphi_\nu^{(k)}(x)| < A_k \quad \text{for all} \quad x \in \mathbb{R}, \quad \nu \in \mathbb{N} \\ \quad \text{and for fixed} \quad k = 0, 1, 2, ..., \\ (ii) \quad \text{the sequence} \quad (\varphi_\nu^{(k)})_{\nu\in\mathbb{N}} \quad \text{converges almost uniformly} \\ \quad \text{(i.e. uniformly on compact subsets)} \\ \quad \text{to zero on} \quad \mathbb{R} \quad \text{for each fixed} \quad k = 0, 1, 2, ... \end{cases}$$

Definition 3.2.

A linear form defined on $\mathfrak{D}_{L^\infty}(\mathbb{R})$ continuous with respect to (β)-convergence is said to be an *integrable distribution.* The space of all integrable distributions will be denoted by $\mathfrak{D}'_{L^1}(\mathbb{R})$.

For $\Lambda \in \mathfrak{D}'_{L^1}(\mathbb{R})$ we have the following theorem (see [12], p. 347)

Theorem 3.1 (*Representation Theorem for Integrable Distributions*).

Let $\Lambda \in \mathfrak{D}'_{L^1}(\mathbb{R})$. Then there exist $m \in \mathbb{N}$ and complex Borel measures μ_n for $n \leqslant m$ such that for each function $\varphi \in \mathfrak{D}_{L^\infty}(\mathbb{R})$

$$\Lambda(\varphi) = \sum_{n=0}^{m} (-1)^n \int_{\mathbb{R}} \varphi^{(n)}(x) d\mu_n(x), \tag{3.1}$$

where $\mu_n = h_n|\mu_n|$, the symbol $|\mu_n|$ denotes the total variation of μ_n and $|h_n(x)| = 1$ for $x \in \mathbb{R}$.

Remark 3.1 (see [1], p. 173).

Let $\mathfrak{D}'_{L^p}$ be the dual of $\mathfrak{D}_{L^{p'}}$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and denote by $\mathfrak{D}'_{L^1}$ the dual of $\mathfrak{D}_{L^\infty}$. Then

$$\mathfrak{D}'_{L^p} \subset \mathfrak{D}'_{L^q} \quad \forall p \leq q.$$

2. The Cauchy Transform of Integrable Distributions.

Remark 3.2.

The function $f(t) = \frac{1}{2\pi i} \frac{1}{t-z}$ is in $\mathfrak{D}_{L^\infty}$ for $\Im z \neq 0$.

Definition 3.3.

Let $\Lambda \in \mathfrak{D}'_{L^1}(\mathbb{R})$. The function

$$C\Lambda(z) := \frac{1}{2\pi i} \Lambda_t\Big(\frac{1}{t-z}\Big) \tag{3.2}$$

defined for $z \in \mathbb{C} \setminus \mathbb{R}$ is called the *Cauchy transform of the distribution* Λ.

Theorem 3.2. *If $\Lambda \in \mathfrak{D}'_{L^1}$, then $C\Lambda$ is a holomorphic function in $\mathbb{C} \setminus \mathbb{R}$ and*

$$\frac{d}{dz} C\Lambda(z) = \frac{1}{2\pi i} \Lambda\left(\frac{1}{(t-z)^2}\right) = (C\Lambda^{(1)})(z),$$

where $\Lambda^{(1)}$ is the distributional derivative of Λ.

Moreover, there exist a positive constant A and $m \in \mathbb{N}_0$ such that

$$\sup_{x \in \mathbb{R}} |C\Lambda(x+iy)| \leq \frac{A}{|y|^k},$$

with

$$k = \begin{cases} m+1, & if \quad 0 < |y| < 1, \\ 0, & if \quad |y| \geq 1, \end{cases}$$

where m comes from Theorem 3.1.

Proof.

We will first show that $C\Lambda(z)$ is holomorphic in $\mathbb{C} \setminus \mathbb{R}$. The method follows that of Theorem 2.2 (see p. 20). We have to show that

$$\lim_{h\to 0} \frac{1}{2\pi i} \Lambda_t \left[h^{-1}((t-z-h)^{-1} - (t-z)^{-1}) - (t-z)^{-2} \right] = 0.$$

It is sufficient to verify that for fixed $z \in \mathbb{C} \setminus \mathbb{R}$ and $n \in \mathbb{N}_0$,

$$\frac{\partial^n}{\partial t^n} \left[h(t-z)^{-2}(t-z-h)^{-1} \right]$$

tends uniformly to zero on $\mathbb{R}$ as $h \to 0$.

Put now

$$g_{h,z}(t) = h(t-z)^{-2}(t-z-h)^{-1}.$$

By Leibniz formula, we get

$$\frac{\partial^n}{\partial t^n} g_{h,z}(t) = (-1)^n n! h \sum_{\nu=0}^{n} (\nu+1)(t-z)^{-(\nu+2)}(t-z-h)^{-(n+1-\nu)}.$$

Taking $z = x + iy$ and $h = \xi + i\eta$ we obtain

$$\frac{\partial^n}{\partial t^n} g_{h,z}(t) = (-1)^n n! h \sum_{\nu=0}^{n} (\nu+1)((t-x)^2+y^2)^{-\frac{(\nu+2)}{2}}((t-x-\xi)^2+(y+\eta)^2)^{-\frac{(n+1-\nu)}{2}}.$$

As $h \to 0$ we can put now $d = y - \delta$ for $y > 0$ and $d = |y + \delta|$ for $y < 0$.

Then for $|h| < \delta < |y|$ and $t \in \mathbb{R}$ we have

$$((t-x)^2 + y^2)^{-\frac{(\nu+2)}{2}}((t-x-\xi)^2 + (y+\eta)^2)^{-\frac{(n+1-\nu)}{2}} \leq d^{-n-3}.$$

Finally we obtain the following estimate

$$\left| \frac{\partial^n}{\partial t^n} g_{h,z}(t) \right| \leq C|h| d^{-n-3}, \quad \text{where} \quad C = n! \sum_{\nu=0}^{n} (\nu+1).$$

Thus we have proved that $\frac{\partial^n}{\partial t^n} g_{h,z}(t)$ converges uniformly to zero with respect to $t \in \mathbb{R}$ as $h \to 0$.

Hence, it follows that

$$\frac{d}{dz} C\Lambda(z) = \frac{1}{2\pi i} \Lambda\left(\frac{1}{(t-z)^2}\right).$$

We will now estimate the function $C\Lambda(z)$.

By Theorem 3.1 (see p. 36), there exist $m \in \mathbb{N}$ and complex Borel measures μ_n for $n \leq m$ such that

$$C\Lambda(z) = \sum_{n=0}^{m} (-1)^n \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\partial^n}{\partial t^n}\left(\frac{1}{t-z}\right) d\mu_n(t).$$

Hence, by Leibniz' formula, we get

$$C\Lambda(z) = \sum_{n=0}^{m} \frac{n!}{2\pi i} \int_{\mathbb{R}} \left(\frac{1}{(t-z)^{n+1}}\right) d\mu_n(t).$$

Therefore,

$$|C\Lambda(z)| \leq \sum_{n=0}^{m} \frac{n!}{2\pi} \int_{\mathbb{R}} \left(\frac{1}{(t-x)^2+y^2}\right)^{\frac{n+1}{2}} |\mu_n|(t)dt.$$

For each $0 \leq n \leq m$, $|\mu_n|(\mathbb{R}) < \infty$. Therefore, we have the following estimate

$$\sup_{x\in\mathbb{R}} |C\Lambda(x+iy)| \leq \sum_{n=0}^{m} \frac{n!}{2\pi} \sup_{x\in\mathbb{R}} \left|\frac{1}{[(t-x)^2+y^2]^{\frac{n+1}{2}}}\right| |\mu_n|(\mathbb{R}).$$

Hence, there exists a positive constant A such that

$$\sup_{x\in\mathbb{R}} |C\Lambda(x+iy)| \leq A\frac{1}{|y|^{m+1}}.$$

This finishes the proof.

The Cauchy transform of integrable distributions plays a similar role as it does for distributions in $\mathfrak{D}'_{L^p}$, where $1 < p < \infty$. Namely, an integrable distribution is a boundary value of its Cauchy transform. Before we state this fact as a theorem for integrable distributions we have to prove the following Lemma

Lemma 3.1.

Let $\psi \in L^1$ be such that $\int_{\mathbb{R}} \psi(x)dx = 1$. If $\varphi \in \mathfrak{D}_{L^\infty}$ then $\varphi * \psi_\varepsilon$ tends to φ as $\varepsilon \to 0$ in $\mathfrak{D}_{L^\infty}$, where $\psi_\varepsilon = \varepsilon^{-1}\psi(\frac{x}{\varepsilon})$.

Proof.

Notice that by the Young inequality the condition i) of Definition 3.1 (see p. 35) is satisfied. We will show that for each compact set V and $n \in \mathbb{N}$, $(\varphi * \psi_\varepsilon)^{(n)}$ tends uniformly on V to $\varphi^{(n)}$ as $\varepsilon \to 0$. Fix $\delta > 0$ and choose a compact set $W \subset \mathbb{R}$ such that $\int_{\mathbb{R}\setminus W} |\psi(x)|dx < \delta$. By Lemma 2.3 (see p. 23), we have

$$(\varphi * \psi_\varepsilon)^{(n)} = \int_{\mathbb{R}} \frac{\partial^n}{\partial x^n}(\varphi(x-y))\psi_\varepsilon(y)dy =$$

$$\int_{\mathbb{R}} \frac{\partial^n}{\partial x^n}\left(\varphi(x-\varepsilon y)\right)\psi(y)dy.$$

Hence, we can estimate

$$\sup_{x\in V} |(\varphi * \psi_\varepsilon)^{(n)} - \varphi^{(n)}| \leq$$

$$\leq \sup_{x\in V} \left| \int_W \frac{\partial^n}{\partial x^n}\left(\varphi(x-\varepsilon y) - \varphi(x)\right)\psi(y)dy\right|$$

$$+ \sup_{x\in V} \left| \int_{\mathbb{R}\setminus W} \frac{\partial^n}{\partial x^n}\left(\varphi(x-\varepsilon y) - \varphi(x)\right)\psi(y)dy\right|$$

$$\leq \sup_{x\in V} \int_W \left|\frac{\partial^n}{\partial x^n}\left(\varphi(x-\varepsilon y) - \varphi(x)\right)\right||\psi(y)|dy$$

$$+ \sup_{x\in V} \int_{\mathbb{R}\setminus W} \left|\frac{\partial^n}{\partial x^n}\left(\varphi(x-\varepsilon y) - \varphi(x)\right)\right||\psi(y)|dy$$

$$\leq \eta \int_{\mathbb{R}} |\psi(y)|dy + 2\delta \parallel \varphi^{(n)} \parallel_{L^\infty}.$$

Note that η coming from the uniform continuity on V is sufficiently small for small ε. This finishes the proof.

Theorem 3.3. *If Λ belongs to $\mathfrak{D}'_{L^1}$ then for each φ in $\mathfrak{D}_{L^\infty}$*

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} \left[C\Lambda(x+i\varepsilon) - C\Lambda(x-i\varepsilon)\right]\varphi(x)dx = \Lambda(\varphi).$$

Proof.

Let Λ be in $\mathfrak{D}'_{L^1}$. By Theorem 3.1 (see p. 36), there exist $m \in \mathbb{N}$ and complex Borel measures μ_n for $n \leq m$ such that for each $\varphi \in \mathfrak{D}_{L^\infty}$

$$\Lambda(\varphi) = \sum_{n=0}^{m} (-1)^n \int_{\mathbb{R}} \varphi^{(n)}(x)d\mu_n(x).$$

We have to show that

$$\int_{\mathbb{R}} \left[C\Lambda(x+i\varepsilon) - C\Lambda(x-i\varepsilon)\right]\varphi(x)dx \to \sum_{n=0}^{m}(-1)^n \int_{\mathbb{R}} \varphi^{(n)}(x)d\mu_n(x) \quad \text{as} \quad \varepsilon \to 0.$$

Let us notice that by Theorem 3.1 (see p. 36),

$$\int_{\mathbb{R}} \left[C\Lambda(x+i\varepsilon) - C\Lambda(x-i\varepsilon)\right]\varphi(x)dx = \sum_{n=0}^{m}(-1)^n \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{\partial^n}{\partial t^n}\psi_\varepsilon(t-x)d\mu_n(t)\right)\varphi(x)dx,$$

where $\psi_\varepsilon(t) = \frac{1}{\pi}\frac{\varepsilon}{t^2+\varepsilon^2}$.

By the Fubini theorem and Lemma 2.3 (see p. 23),

$$\sum_{n=0}^{m}(-1)^n \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{\partial^n}{\partial t^n}\psi_\varepsilon(t-x)d\mu_n(t)\right)\varphi(x)dx =$$

$$= \sum_{n=0}^{m}(-1)^n \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{\partial^n}{\partial t^n}\psi_\varepsilon(t-x)\varphi(x)dx\right)d\mu_n(t)$$

$$= \sum_{n=0}^{m}(-1)^n \int_{\mathbb{R}} \left(\frac{\partial^n}{\partial t^n}\int_{\mathbb{R}} \psi_\varepsilon(t-x)\varphi(x)dx\right)d\mu_n(t)$$

$$= \sum_{n=0}^{m}(-1)^n \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \psi_\varepsilon(t-x)\frac{\partial^n}{\partial x^n}\varphi(x)dx\right)d\mu_n(t).$$

Hence, by Lemma 3.1 (see p. 38) and the Lebesgue theorem,

$$\lim_{\varepsilon\to 0}\sum_{n=0}^{m}(-1)^n \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \psi_\varepsilon(t-x)\frac{\partial^n}{\partial x^n}\varphi(x)dx\right)d\mu_n(t)$$

$$= \sum_{n=0}^{m}(-1)^n \int_{\mathbb{R}} \frac{\partial^n}{\partial t^n}\varphi(t)d\mu_n(t).$$

This finishes the proof.

Example.

Take the Dirac δ defined for each function $\varphi \in \mathfrak{D}_{L^\infty}$ by

$$\delta(\varphi) = \varphi(0).$$

It is easy to show that δ is an integrable distribution. We can easily calculate the Cauchy transform of this distribution. We have

$$C\delta(z) = \frac{-1}{2\pi i}\frac{1}{z} \quad \text{for each} \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Hence $C\delta(x+iy) - C\delta(x-iy) = \frac{1}{\pi}\frac{y}{x^2+y^2}$. By Theorem 3.3 (see p. 39), $\frac{1}{\pi}\frac{y}{x^2+y^2}$ tends to δ as $y \to 0$ in $\mathfrak{D}'_{L^1}$.

Unfortunately, the methods used in the proofs of Theorem 2.5 (see p. 29) and Theorem 2.6 (see p. 32) cannot be applied to integrable distributions. Moreover, the function $f(t) = \frac{1}{t-z}$ is not in $L^1(\mathbb{R})$ for $\Im z \neq 0$. That is the reason why we have not been able to distinguish functions holomorphic in $\mathbb{C} \setminus \mathbb{R}$ which have their boundary values in $\mathfrak{D}'_{L^1}$.

CHAPTER IV

APPLICATIONS OF HOLOMORPHIC REPRESENTATION

1. Application of the Holomorphic Representation of Distributions to Determining the Integral of Integrable Distributions.

A natural generalization of integrable functions with respect to the Lebesgue measure are integrable distributions, (see [33], p. 203). In this paragraph we will give a method of determining "the integral" of an integrable distribution.

If $\Lambda = f$, where $f \in L^1(\mathbb{R})$, then

$$\Lambda(1) = \int f 1 dx \tag{4.1}$$

is the Lebesgue integral of f.

It is natural to take (4.1) for the definition of the integral of Λ i.e.

$$\int \Lambda = \Lambda(1).$$

It is easily seen, that

$$\Lambda(1) = \mu_0(\mathbb{R}). \tag{4.2}$$

According to Theorem 3.1 (see p. 36) we have the following formula

$$C\Lambda(z) = \sum_{n=0}^{m} (-1)^n \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\partial^n}{\partial t^n} \left(\frac{1}{t-z} \right) d\mu_n(t), \tag{4.3}$$

for the Cauchy transform of Λ.
If $\Lambda = f$ and f is in $L^1(\mathbb{R})$, then

$$Cf(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{1}{t-z} f(t)dt.$$

It is easy to verify that

$$\int_{\mathbb{R}} \left[Cf(x+iy) - Cf(x-iy)\right] dx = \int_{-\infty}^{+\infty} f(x)dx, \tag{4.4}$$

for $y > 0$.

One can ask whether for each integrable distribution the equality

$$\int_{\mathbb{R}} \left[C\Lambda(x+iy) - C\Lambda(x-iy)\right] dx = \Lambda(1) \tag{4.5}$$

holds.

The answer to this question is affirmative.

Theorem 4.1. *If $\Lambda \in \mathfrak{D}'_{L^1}$, then*

$$\int \Lambda = \int_{\mathbb{R}} [C\Lambda(x+iy) - C\Lambda(x-iy)]dx, \quad \textit{where} \quad y > 0.$$

Proof.

Let Λ be in $\mathfrak{D}'_{L^1}$ and $y > 0$. Then we have

$$\int_{\mathbb{R}} \left[C\Lambda(x+iy) - C\Lambda(x-iy)\right] dx = \tag{4.6}$$

$$\sum_{n=0}^{m} (-1)^n \frac{1}{2\pi i} \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} \frac{\partial^n}{\partial t^n} \Big(\frac{1}{t-x-iy} - \frac{1}{t-x+iy} \Big) d\mu_n(t) \Big] dx.$$

In view of (4.2) it is enough to show that

$$\frac{1}{2\pi i} \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} \Big(\frac{1}{t-x-iy} - \frac{1}{t-x+iy} \Big) d\mu_0(t) \Big] dx = \mu_0(\mathbb{R}), \tag{4.7}$$

$$\frac{1}{2\pi i} \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} \frac{\partial^n}{\partial t^n} \Big(\frac{1}{t-x-iy} - \frac{1}{t-x+iy} \Big) d\mu_n(t) \Big] dx = 0 \quad \text{for} \quad n > 0. \tag{4.8}$$

Proof of (4.7) and (4.8).

The integral (4.7) can be written as follows

$$\int_{\mathbb{R}} \Big[\frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(t-x)^2+y^2} h_0(t)d|\mu_0|(t)\Big]dx. \tag{4.9}$$

Applying the Fubini theorem to the above integral we get

$$\int_{\mathbb{R}} \Big[\frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(t-x)^2+y^2} h_0(t)d|\mu_0|(t)\Big]dx =$$

$$\int_{\mathbb{R}} h_0(t)\Big[\frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(t-x)^2+y^2}dx\Big]d|\mu_0|(t) = \int_{\mathbb{R}} h_o(t)d|\mu_0|(t) = \mu_0(\mathbb{R}).$$

Now we go on to showing that (4.8) holds.

The integral (4.8) can be written in the form

$$\int_{\mathbb{R}} (-1)^n\frac{1}{\pi}\Big[\int_{\mathbb{R}} \frac{\partial^n}{\partial t^n}\Big(\frac{y}{(t-x)^2+y^2}\Big)h_n(t)d|\mu_n|(t)\Big]dx.$$

It is easy to check that

$$\frac{\partial^n}{\partial t^n}\Big(\frac{y}{(t-\cdot)^2+y^2}\Big) \in L^1(\mathbb{R}) \quad \text{for } t\in\mathbb{R}, \quad y>0 \quad \text{and} \quad n=0,1,2,\ldots. \tag{4.10}$$

In view of the Fubini theorem we obtain

$$\int_{\mathbb{R}} (-1)^n\frac{1}{\pi}\Big[\int_{\mathbb{R}} \frac{\partial^n}{\partial t^n}\frac{y}{(t-x)^2+y^2}h_n(t)d|\mu_n|(t)\Big]dx =$$

$$\int_{\mathbb{R}} (-1)^n h_n(t)\frac{1}{\pi}\int_{\mathbb{R}}\Big[\frac{\partial^n}{\partial t^n}\frac{y}{(t-x)^2+y^2}dx\Big]d|\mu_n|(t). \tag{4.11}$$

It is well known that

$$\lim_{h\to 0}\int_{\mathbb{R}} |f(\cdot+h)-f| = 0 \quad \text{if} \quad f\in L^1.$$

Taking into account (4.10) and the above statement, it is easy to verify that

$$\int_{\mathbb{R}} \frac{\partial^n}{\partial t^n}\frac{y}{(t-x)^2+y^2}dx = \frac{\partial^n}{\partial t^n}\int_{\mathbb{R}} \frac{y}{(t-x)^2+y^2}dx. \tag{4.12}$$

Hence we get

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{\partial^n}{\partial t^n}\frac{y}{(t-x)^2+y^2}dx = 0 \quad \text{for} \quad t\in\mathbb{R} \quad \text{and} \quad y>0.$$

This finishes the proof.

2. The Integral of Distributions with Compact Support.

If we want to speak of a support of a distribution we first have to explain what is a restriction of a distribution.

Definition 4.1 ([11], p. 41).

Let Ω be an open subset of $\mathbb{R}$ and U its open subset. Every function belonging to $\mathfrak{D}(U)$ can be considered as a function belonging to $\mathfrak{D}(\Omega)$. If $\Lambda \in \mathfrak{D}'(\Omega)$, then its restriction to $\mathfrak{D}(U)$ is the distribution $\Lambda_U \in \mathfrak{D}'(U)$ defined by

$$\Lambda_U(\varphi) = \Lambda(\varphi) \quad \text{for every} \quad \varphi \in \mathfrak{D}(U).$$

We shall call Λ_U the *restriction of* Λ to U, or the distribution *induced* on U by Λ. If $\Lambda|U = 0$ we say that Λ is zero in U.

Theorem 4.2. *If $\Lambda \in \mathfrak{D}'(U)$ and for each $x \in U$ there exists an open neighbourhood U_x such that $\Lambda|U_x = 0$, then Λ is zero in U.*

Definition 4.2 ([12], p. 318).

Let $\Lambda \in \mathfrak{D}'(\Omega)$. If U is the largest open subset of Ω in which Λ is zero, then the complement of U in Ω is called the *support* of Λ and is denoted by $\operatorname{supp}\Lambda$.

The set $\operatorname{supp}\Lambda$ is closed by definition. A point $x \in \Omega$ belongs to $\operatorname{supp}\Lambda$ if and only if for every neighbourhood V of x there exists $\varphi \in \mathfrak{D}(\Omega)$ with $\operatorname{supp}\varphi \subset V$ and such that $\Lambda(\varphi) \neq 0$.

Examples.

1. $\operatorname{supp}\delta = \{0\}$.
2. $\operatorname{supp}\frac{1}{(\cdot)} = \mathbb{R}$.

We will now characterize distributions with compact support (see also [3], p. 27; [12] p. 320).

First we will introduce one more space of functions.

Definition 4.3.

A sequence (φ_n), $\varphi_n \in C^\infty$, is said to be convergent in $C^\infty(\Omega)$ if the sequences $(\varphi_n^{(m)})$ converge uniformly on every compact set $K \subset \Omega$ for each $m \in \mathbb{N}_0$. The space C^∞ with this convergence will be denoted by $\mathcal{E}(\Omega)$.

The dual $\mathcal{E}'(\Omega)$ of $\mathcal{E}(\Omega)$ is a space of distributions. The next theorem characterizes the distributions belonging to $\mathcal{E}'(\Omega)$.

Theorem 4.3 (see [11], p. 44 and [12], p. 320). *Let Ω be an open subset of $\mathbb{R}$. A distribution $\Lambda \in \mathfrak{D}'(\Omega)$ belongs to $\mathcal{E}'(\Omega)$ if and only if* $\operatorname{supp} \Lambda$ *is a compact subset of* Ω.

The elements of $\mathcal{E}'(\Omega)$ are called *distributions with compact support in* Ω.

Example.

$\Lambda = \sum_{n=0}^{m} \delta^{(n)}$ is a distribution with a point support $\{0\}$.

Remark 4.1 (see [12], p. 347).

Every distribution with a compact support is integrable.

H. J. Bremermann and L. Durand have shown in [2] that the Cauchy transform of any distribution Λ with a compact support is a function holomorphic in the whole complex plane minus the support of Λ and tending to zero at infinity. More precisely, $\mathrm{C}\Lambda(\mathrm{z})$ vanishes at least as $A|z|^{-1}$ for some $A > 0$.

In accordance with the above statement the integrals

$$\int_{-x-i\eta}^{-x+i\eta} C\Lambda(z)dz \quad \text{and} \quad \int_{x-i\eta}^{x+i\eta} C\Lambda(z)dz$$

tend to zero as x tends to infinity. Since the function $\mathrm{C}\Lambda$ is holomorphic outside the support of Λ, we have

$$\int_{-\infty}^{\infty} [C\Lambda(x+i\eta) - C\Lambda(x-i\eta)]dx = \int_{\Gamma} C\Lambda(z)dz.$$

By Theorem 4.1 (see p. 43) and these conditions we get the following

Theorem 4.4 (see [2]). *If Λ is a distribution with a compact support, then*

$$\int \Lambda = \int_{\Gamma} C\Lambda(z)dz,$$

where Γ is any closed curve surrounding the support of Λ clockwise.

3. Application of the Cauchy Transform to Determining the Fourier Transform of some Distributions.

We will use the Cauchy transform to obtain in an easy way the Fourier transforms of finite parts $Fp\frac{1}{(\cdot)^k}$, $k \in \mathbb{N}$, treated as distributions belonging to $\mathfrak{D}'_{L^p}$.

For brevity we will write $\frac{1}{(\cdot)^k}$ for $Fp\frac{1}{(\cdot)^k}$. We define the Cauchy finite part of the integral in the following way:

Let $\varphi \in \mathfrak{D}_{L^\infty}$. Then we define (see also [7], p. 77)

$$\begin{aligned}\frac{1}{(\cdot)^{2m}}(\varphi) :&= \int_{\mathbb{R}} x^{-2m}\Big[\frac{1}{2}(\varphi(x)+\varphi(-x)) - \Big(\varphi(0)+\frac{1}{2!}\varphi^{(2)}(0)x^2 + \\ &\quad ... + \frac{1}{(2m-2)!}\varphi^{(2m-2)}(0)x^{2m-2}\Big)\Big]dx\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\frac{1}{(\cdot)^{2m+1}}(\varphi) :&= \int_{\mathbb{R}} x^{-(2m+1)}\Big[\frac{1}{2}(\varphi(x)+\varphi(-x)) - \Big(\frac{1}{1!}\varphi'(0)x+\frac{1}{3!}\varphi^{(3)}(0)x^3 + \\ &\quad ... + \frac{1}{(2m-1)!}\varphi^{(2m-1)}(0)x^{2m-1}\Big)\Big]dx,\end{aligned} \tag{4.14}$$

for $m = 1, 2, ...$

For $m = 0$ we take

$$\frac{1}{(\cdot)}(\varphi) := \int_{\mathbb{R}} \frac{1}{2x}\Big[\varphi(x) - \varphi(-x)\Big]dx. \tag{4.15}$$

This definition coincides with the definition on the page 19.

Now, we will show that the above integrals exist for $\varphi \in \mathfrak{D}_{L^\infty}$ and determine distributions.

We will restrict ourselves to estimating (4.13). We have

$$\begin{aligned}\frac{1}{(\cdot)^{2m}}(\varphi) :=& \\ &\int_{|x|<1} x^{-2m}\Big[\frac{1}{2}(\varphi(x)+\varphi(-x)) - \Big(\varphi(0)+\frac{1}{2!}\varphi^{(2)}(0)x^2 \\ &+ ... + \frac{1}{(2m-2)!}\varphi^{(2m-2)}(0)x^{2m-2}\Big)\Big]dx \\ &+ \int_{|x|\geq 1} x^{-2m}\frac{1}{2}\Big[\varphi(x)+\varphi(-x)\Big]dx \\ &- \int_{|x|\geq 1} \sum_{\nu=0}^{m-1} \frac{\varphi^{(2\nu)}(0)}{(2\nu)!}x^{-2(m-\nu)}dx.\end{aligned}$$

It is easy to verify that the above integrals exist for $\varphi \in \mathfrak{D}_{L^\infty}$. Moreover, these integrals determine linear continuous forms on $\mathfrak{D}_{L^\infty}$ with respect to β-convergence.

Therefore $\frac{1}{(\cdot)^{2m}}$ is in $\mathfrak{D}'_{L^1}$.

By Remark 3.1 (see p. 36), $\frac{1}{(\cdot)^k}$ is in $\mathfrak{D}'_{L^p}$ for $p \geq 1$ and $k > 1$. In Chapter II we have shown that $\frac{1}{(\cdot)}$ belongs to $\mathfrak{D}'_{L^p}$, for $1 < p < \infty$. In fact using a similar method one can show that $\frac{1}{(\cdot)^k}$ is in $\mathfrak{D}'_{L^p}$ for $k \geq 1$.

Now, we determine the Cauchy transform of $\frac{1}{(\cdot)}$. Putting $\varphi(x) = \frac{1}{2\pi i}\frac{1}{x-z}$, $\Im z \neq 0$ in (4.15) we obtain

$$C\frac{1}{(\cdot)}(x) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{1}{x^2 - z^2}dx.$$

The computation of residues gives us

$$\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{1}{x^2 - z^2}dx = \begin{cases} \frac{1}{2z}, & \text{if } \Im z > 0, \\ -\frac{1}{2z}, & \text{if } \Im z < 0. \end{cases} \tag{4.16}$$

Now we will prove that

$$C\frac{1}{(\cdot)^{k+1}}(z) = \frac{1}{z}C\frac{1}{(\cdot)^k}(z) \tag{4.17}$$

for $k = 1, 2, ...$ and $\Im z \neq 0$.

The integrals (4.13) and (4.14) for $\varphi(x) = \frac{1}{2\pi i}\frac{1}{x-z}$ can be written as follows

$$C\frac{1}{(\cdot)^{2m}}(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} x^{-2m}\left(\frac{z}{x^2 - z^2} + \frac{1}{z} + \frac{x^2}{z^3} + ... + \frac{x^{2(m-1)}}{z^{2m-1}}\right)dx$$

and

$$C\frac{1}{(\cdot)^{2m+1}}(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} x^{-(2m+1)}\left(\frac{x}{x^2 - z^2} + \frac{x}{z^2} + \frac{x^3}{z^4} + ... + \frac{x^{2m-1}}{z^{2m}}\right)dx.$$

Let us notice that

$$\begin{aligned} C\frac{1}{(\cdot)^{2m}}(z) &= \frac{1}{2\pi i}\frac{1}{z}\int_{\mathbb{R}} x^{-2m}\left(\frac{z^2}{x^2 - z^2} + 1 + \frac{x^2}{z^2} + ... + \frac{x^{2(m-1)}}{z^{2(m-1)}}\right)dx \\ &= \frac{1}{2\pi i}\frac{1}{z}\int_{\mathbb{R}} x^{-2m+1}\left(\frac{x}{x^2 - z^2} + \frac{x}{z^2} + .. + \frac{x^{2(m-1)-1}}{z^{2(m-1)}}\right)dx \\ &= \frac{1}{z}C\frac{1}{(\cdot)^{2m-1}}(z) \end{aligned}$$

for $m = 1, 2, ...$ In the same way we obtain that

$$C\frac{1}{(\cdot)^{2m+1}}(z) = \frac{1}{z}C\frac{1}{(\cdot)^{2m}}(z)$$

for $m = 1, 2, ...$

Applying (4.16) to (4.17) we get

$$C\frac{1}{(\cdot)^k}(z) = \begin{cases} \frac{1}{2z^k}, & \text{if} \quad \Im z > 0; \\ -\frac{1}{2z^k}, & \text{if} \quad \Im z < 0, \end{cases} \tag{4.18}$$

for $k = 1, 2, ...$

Let us notice that by Remark 3.1 (see p. 36) the distributions $\frac{1}{(\cdot)^k}$ for $k > 1$ belong to $\mathfrak{D}'_{L^p}$ with $p \geq 1$. Hence we can estimate the Cauchy transforms of $\frac{1}{(\cdot)^k}$ in the L^p norm for $p > 1$. In order to do this we have to recall an important fact that has been used in the proofs of Theorem 1.3 (see p. 10) and Theorem 2.2 (see p. 20). Namely,

$$\int_{\mathbb{R}} \left(\frac{1}{x^2+1}\right)^p dx < \infty \quad \text{if} \quad p > \frac{1}{2}.$$

We have

$$\int_{\mathbb{R}} \left|\frac{1}{2(x+iy)^k}\right|^p dx = \frac{1}{2^p}\int_{\mathbb{R}} \left(\frac{1}{(x^2+y^2)}\right)^{\frac{pk}{2}} dx$$

$$= \frac{1}{2^p}|y|^{-kp}\int_{\mathbb{R}} \left(\left(\frac{x}{y}\right)^2+1\right)^{-\frac{pk}{2}} dx$$

$$= \frac{1}{2^p}|y|^{-pk+1}\int_{\mathbb{R}} \left((s^2+1)\right)^{-\frac{pk}{2}} ds \leq M|y|^{-kp+1}$$

if $pk > 1$. But for $k = 1$ the distribution $\frac{1}{(\cdot)}$ is in $\mathfrak{D}'_{L^p}$ with $p > 1$. Hence $pk > 1$.

Let us notice that if $0 < |y| < 1$ then

$$|y|^{-k+\frac{1}{p}} < \frac{1}{|y|^k}.$$

Therefore, we obtain

$$\| \, C\frac{1}{(\cdot)}(\cdot + iy) \, \|_{L^p} \leq M\frac{1}{|y|^m},$$

where

$$m = \begin{cases} k, & \text{for} \quad 0 < |y| < 1, \\ 0, & \text{for} \quad |y| \geq 1. \end{cases}$$

In the same way we can estimate the Cauchy transforms of integrable distributions $\frac{1}{(\cdot)^k}$ for $k > 1$ in the L^∞ norm. We have

$$\sup_{x\in\mathbb{R}} |C\frac{1}{(\cdot)^k}(x+iy)| \leq M\frac{1}{|y|^m},$$

where

$$m = \begin{cases} k, & \text{for} \quad 0 < |y| < 1, \\ 0, & \text{for} \quad |y| \geq 1. \end{cases}$$

4. The Fourier Transform of Finite Parts $Fp\frac{1}{(\cdot)^k}$.

The Cauchy transform allows us to find the Fourier transform of the finite parts $\frac{1}{(\cdot)^k}$.

Since $\mathcal{S} \subset \mathfrak{D}_{L^p}$ (see [22], p. 66), Theorem 2.4 (see p. 26) will be valid for any function $\varphi \in S$. The following convergence takes place

$$\frac{1}{2}\int_{\mathbb{R}}\left(\frac{1}{(x+i\varepsilon)^k}+\frac{1}{(x-i\varepsilon)^k}\right)\varphi(x)dx \to \left(\frac{1}{(\cdot)^k},\varphi\right)$$

as $\varepsilon \to 0$, for each function $\varphi \in S$.

Applying the theory of residues we obtain the following results

$$\mathcal{F}\left(\frac{1}{(x+i\varepsilon)^k}\right)(t) = -2\pi i\frac{(it)^{k-1}}{(k-1)!}\mathrm{e}^{\varepsilon t}H(-t),$$

$$\mathcal{F}\left(\frac{1}{(x-i\varepsilon)^k}\right)(t) = 2\pi i\frac{(it)^{k-1}}{(k-1)!}\mathrm{e}^{\varepsilon t}H(t),$$

where H is the Heaviside function. Hence we obtain

$$\mathcal{F}\left[C\frac{1}{(\cdot)^k}(x+i\varepsilon) - C\frac{1}{(\cdot)^k}(x-i\varepsilon)\right](t)$$

$$= \frac{1}{2}2\pi i\frac{(it)^{k-1}}{(k-1)!}\mathrm{e}^{\varepsilon t}\operatorname{sgn} t \to \pi i\frac{(it)^{k-1}}{(k-1)!}\operatorname{sgn} t, \quad as \quad \varepsilon \to 0.$$

Therefore for each $\varphi \in \mathcal{S}$ we have

$$\int_{\mathbb{R}}\mathcal{F}\left[C\frac{1}{(\cdot)^k}(x+i\varepsilon) - C\frac{1}{(\cdot)^k}(x-i\varepsilon)\right](t)\varphi(t)dt$$

$$\to \pi i\int_{\mathbb{R}}\frac{(it)^{k-1}}{(k-1)!}\operatorname{sgn} t\varphi(t)dt, \quad as \quad \varepsilon \to 0.$$

Hence by continuity of $\mathcal{F}$ in $\mathcal{S}'$ the following formula holds

$$\mathcal{F}\left(\frac{1}{(\cdot)^k}\right)(t) = \pi i \frac{(it)^{k-1}}{(k-1)!}\operatorname{sgn} t, \tag{4.19}$$

for $k \geq 1$.

In particular, for $k = 1$ we have $\mathcal{F}\frac{1}{(\cdot)}(t) = i\pi \operatorname{sgn} t$.

5. Holomorphic representation and its application to the Hilbert problem.

We will describe the Hilbert problem as in [24].

Let $g(x)$ be a function defined on the real line. We wish to find a holomorphic function $G(z)$ analytic on the complex plane cut along the real line, such that

$$\lim_{y\to 0^+} G(x+iy) + \lim_{y\to 0^-} G(x+iy) = g(x).$$

Denoting the two limit functions by γ_+ and γ_-, respectively, we have

$$\gamma_+ + \gamma_- = g(x).$$

In many problems of mathematical physics more difficult problems appear. Given functions $g(x)$ and $k(x)$ defined on the real line, we wish to find a function $G(z)$ analytic on the complex plane cut along the real line such that

$$\gamma_+ + k(x)\gamma_- = g(x),$$

where γ_+ and γ_- are the limits of $G(z)$ as described above. The latter problem can be reduced to the first, simpler form by using the factorization technique. We are however interested in its $\mathfrak{D}'_{L^p}$ counterpart.

Let $\Lambda \in \mathfrak{D}'_{L^p}$, $1 < p < \infty$. We want to find an analytic function $G(z)$ in the region $\Im z \neq 0$ such that

$$\gamma_+ + \gamma_- = \Lambda,$$

where

$$\gamma_+ = \lim_{y\to 0^+} G(x+iy) \quad in \quad D'_{L^p} \quad weakly,$$

$$\gamma_- = \lim_{y\to 0^-} G(x+iy) \quad in \quad D'_{L^p} \quad weakly.$$

In the thesis we have shown that $G(z) = C\Lambda(z)$ is the solution to that problem.

It is worth mentioning that the Hilbert problem as described above appears in the so called Griffith crack problem, in aerofoil theories and has also some other applications.

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List of Symbols

$\mathbb{N}$ denotes the set of positive integers.

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Ω denotes an open set $\Omega \subset \mathbb{R}$.

$C^1(\Omega)$ denotes the space of continuously differentiable complex valued functions in Ω.

C^∞ denotes the space of infinitely differentiable (smooth) functions.

supp$f$ denotes the support of f.

$\mathfrak{D}(\Omega)$ is the space of smooth functions with compact supports contained in Ω (see p. 12).

$\mathfrak{D}'(\Omega)$ is the space of Schwartz distributions in Ω (see p. 12).

$\mathcal{S}$ is the space of rapidly decreasing functions (see p. 15).

$\mathcal{S}'$ is the space of tempered distributions (see p. 16).

$\mathcal{E}$ is the space of smooth functions with convergence defined on p. 45.

$\mathcal{E}'$ is the space of distributions with compact supports (see p. 46).

$\mathfrak{D}_{L^q}(\mathbb{R})$ (see p. 17).

$\mathfrak{D}'_{L^p}(\mathbb{R})$ (see p. 18).

$\mathfrak{D}_{L^\infty}(\mathbb{R})$ is the space of smooth functions with bounded derivatives (see p. 35).

$\mathfrak{D}'_{L^1}(\mathbb{R})$ is the space of integrable distributions (see p. 35).

H_{L^p} (see p. 29)

Π^+, Π^- (see p. 32)

$H(\Pi^+ \cup \Pi^-)$ denotes the set of all holomorphic functions in $\Pi^+ \cup \Pi^-$ (see p. 32).

$\mathcal{F}f$, $\hat{f}$ denote the Fourier transform of f (see p. 6).

$\mathcal{H}f$ denotes the Hilbert transform of f (see p. 8).

Cf denotes the Cauchy transform of f (see p. 9).

$\wedge$ denotes exterior multiplication (see [10], pp. 2, 24).

Pv, $Fp(\frac{1}{x})$, $\frac{1}{(\cdot)}$ denote the Cauchy finite part of an integral (see p. 14).

Inclusion relations among some function spaces introduced in the thesis.

$$\mathfrak{D} \subset \mathcal{S} \subset \mathfrak{D}_{L^1} \subset \mathfrak{D}_{L^p} \subset \mathcal{E}, \quad \text{where} \quad p \geq 1.$$