Pseudoresolvents and distribution semigroups

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Abstract. For pre-distribution semigroups in Kunstmann's sense [K] the generators are defined and their characterization is given in terms of the corresponding pseudoresolvents.

Outline of the paper

Let $\mathcal{D}([0,\infty))$ be the space of complex C^∞ functions with bounded supports, defined on the closed halfaxis $[0,\infty)$. If $\varphi \in \mathcal{D}([0,\infty))$, then the right-hand derivatives $D^k\varphi(0)$ are equal to $\lim_{t\downarrow 0} D^k\varphi(t)$ and need not vanish. $\mathcal{D}([0,\infty))$ is an LF space, and is a *convolution algebra*, the convolution being defined by the formula

$$(\varphi * \psi)(t) = \int_0^t \varphi(s)\psi(t-s)\,ds, \qquad \varphi,\psi \in \mathcal{D}([0,\infty)),\ t \in [0,\infty).$$

Let X be a complex Banach space, and $L(X)$ the Banach algebra of bounded linear operators from X into X. An $L(X)$-valued distribution S on $(-\infty,\infty)$ will be called a *distribution semigroup* if

$$S(\varphi) = S^+(\varphi|_{[0,\infty)}) \qquad \text{for every } \varphi \in \mathcal{D}((-\infty,\infty)),$$

where S^+ is a continuous representation of the convolution algebra $\mathcal{D}([0,\infty))$ on the Banach space X.

The set of $L(X)$-valued distribution semigroups will be denoted by $\mathrm{DS}(L(X))$. $S \in \mathrm{DS}(L(X))$ if and only if S is an $L(X)$-valued pre-distribution semigroup in the sense of P. C. Kunstmann [K], Definition 2.1, p. 839. We prove that a distribution $S \in \mathcal{D}'((-\infty,\infty);L(X))$ belongs to $\mathrm{DS}(L(X))$ if and only if the following two conditions are satisfied:

(a) S vanishes on $(-\infty,0)$,

(b) $\psi(0)S(\varphi) - \varphi(0)S(\psi) = S(D\varphi)S(\psi) - S(\varphi)S(D\psi)$

for every $\varphi,\psi \in \mathcal{D}((-\infty,\infty))$.

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For every $S \in \mathrm{DS}(L(X))$ we define its *null space* $\mathcal{N}$ and *range* $\mathfrak{I}$ as follows:

(c) $\quad \mathcal{N} = \{x \in X : S(\varphi)x = 0 \text{ for every } \varphi \in \mathcal{D}((-\infty, \infty))\}$,

(d) $\quad \mathfrak{I} = \Big\{ \sum_{k=1}^{n} S(\psi_k)\, x_k : n \in \mathbb{N},\ \psi_k \in \mathcal{D}((-\infty,\infty)),\ x_k \in X \text{ for } k = 1, \ldots, n \Big\}.$

$\mathcal{N}$ is a closed subspace of X, and hence the factor space $X/\mathcal{N}$ is a Banach space. By (b), there is a unique linear map $G : \mathfrak{I} \to X/\mathcal{N}$ such that

$$GS(\varphi)x = -S(D\varphi)x - \varphi(0)x \qquad \text{for every } \varphi \in \mathcal{D}((-\infty,\infty)) \text{ and every } x \in X.$$

We call this G the *generator* of S. Then G is a closed linear operator from X into $X/\mathcal{N}$. Every $S \in \mathrm{DS}(L(X))$ is uniquely determined by its generator (Theorem 4.10). For narrower classes of $L(X)$-valued distribution semigroups, considered in [L], [C] and [K], other definitions of the generator were given. Our definition is compatible with those earlier definitions (Section 4.D).

Thanks to the closedness of G, its domains $\mathfrak{I}$ equipped with the graph norm is a Banach space. The characterization (a)&(b) of the distribution semigroups implies that for any distribution $S \in \mathcal{D}'((-\infty,\infty); L(X))$, any closed subspace $\mathcal{N}$ of X, and any closed linear operator G from X into $X/\mathcal{N}$ with domain $\mathfrak{I}$ the following two conditions are equivalent:

(I) $S \in \mathrm{DS}(L(X))$, (c) and (d) hold, and $G : \mathfrak{I} \to X/\mathcal{N}$ is the generator of S,

(II) there is a unique distribution $\widetilde{S} \in \mathcal{D}'((-\infty,\infty); L(X/\mathcal{N}; \mathfrak{I}))$ having the three properties:

  (i) $\widetilde{S}$ vanishes on $(-\infty, 0)$,

  (ii) $\widetilde{S}(\varphi)E = S(\varphi)$ for every $\varphi \in \mathcal{D}((-\infty,\infty))$, where E is the canonical epimorphism of X onto $X/\mathcal{N}$,

  (iii) $\widetilde{S}$ is a fundamental solution for the first order differential operator $E|_{\mathfrak{I}}D - G$ with $E|_{\mathfrak{I}}$ and G belonging to $L(\mathfrak{I}; X/\mathcal{N})$.

Condition (II)(iii) makes it evident why, in the situation discussed here, it is reasonable to use the language of the theory of distributions, and not only to speak about representations of the convolution algebra $\mathcal{D}([0,\infty))$.

The equivalence (I)$\Leftrightarrow$(II) enables us to follow the arguments of J. Chazarain [C] and in this way obtain (Theorem 4.18) a characterization of the generator $G : \mathfrak{I} \to X/\mathcal{N}$ of a distribution semigroup in terms of the pseudoresolvent generated by G. This pseudoresolvent has null space $\mathcal{N}$, and range $\mathfrak{I}$. The fact that the generator of the pseudoresolvent coincides with the generator of a distribution semigroup is reflected in the specific properties of the pseudoresolvent: it is defined in a logarithmic region of $\mathbb{C}$, and has polynomial growth in this region.

1. Pseudoresolvents and their generators

Let $\mathcal{A}$ be an algebra over a commutative ring $\mathbb{K}$. Following [H-P], p. 126, by a *pseudoresolvent* with values in $\mathcal{A}$ defined on a non-empty subset Λ of $\mathbb{K}$ we mean a map $R_\bullet : \Lambda \to \mathcal{A}$ such that

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu \tag{1.1}$$

for every $\lambda \in \Lambda$ and $\mu \in \Lambda$. It follows that the range of a pseudoresolvent $R_\bullet : \Lambda \to \mathcal{A}$ consists of mutually commuting elements of $\mathcal{A}$.

1.1. Proposition. *Every pseudoresolvent has a unique maximal extension to a pseudoresolvent.*

Proof. Consider the binary relations $\to$ and $\sim$ on $\mathbb{K}\times\mathcal{A}$ such that if $(\lambda, A) \in \mathbb{K}\times\mathcal{A}$ and $(\mu, B) \in \mathbb{K}\times\mathcal{A}$, then $(\lambda, A) \to (\mu, B)$ means that $A - B = (\mu - \lambda)AB$, and $(\lambda, A) \sim (\mu, B)$ stands for $[(\lambda, A) \to (\mu, B)]$ & $[(\mu, B) \to (\lambda, A)]$. It follows that $(\lambda, A) \sim (\mu, B)$ is equivalent to $[(\lambda, A) \to (\mu, B)]$ & $[AB = BA]$. Suppose that $(\lambda, A) \to (\mu, B)$ and $(\mu, B) \to (\nu, C)$. Then $B = A + (\lambda - \mu)AB = C + (\nu - \mu)BC$, so that

$$\begin{aligned} A - C &= (\mu - \lambda)AB + (\nu - \mu)BC \\ &= (\mu - \lambda)A[C + (\nu - \mu)BC] + (\nu - \mu)[A + (\lambda - \mu)AB]C \\ &= (\nu - \lambda)AC, \end{aligned}$$

which means that $(\lambda, A) \to (\nu, C)$. Thus $\to$ is transitive, and $\sim$ is an equivalence relation. As a consequence, if $R_\bullet : \Lambda \to \mathcal{A}$ is a pseudoresolvent and μ is any element of Λ, then the graph of $R_\bullet$ is equal to the set $\{(\lambda, A) \in \Lambda \times \mathcal{A} : (\lambda, A) \sim (\mu, R_\mu)\}$, while the graph of the maximal extension of $R_\bullet$ to a pseudoresolvent is the whole equivalence class $\{(\lambda, A) \in \mathbb{K} \times \mathcal{A} : (\lambda, A) \sim (\mu, R_\mu)\}$. ∎

By the *maximal pseudoresolvent* we mean a pseudoresolvent which is equal to its maximal extension to a pseudoresolvent. Notice that in the case of a pseudoresolvent with values in a Banach algebra the existence of the unique maximal extension is a consequence of Theorem 5.8.6 on p. 188 of [H-P]. The following Proposition 1.2 is nothing but an equivalent formulation of Theorem 5.8.2 from p. 184 of [H-P] (see also [F-Y], p. 23, Theorem 1.6).

1.2. Proposition. *If* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{K} = \mathbb{C}$ *and* $\mathcal{A}$ *is a Banach algebra over* $\mathbb{K}$, *then every maximal pseudoresolvent with values in* $\mathcal{A}$ *is an analytic function defined on an open subset of* $\mathbb{K}$.

Proof. Let $R_\bullet : \Lambda \to \mathcal{A}$ be a maximal pseudoresolvent. If $R_\lambda = 0$ for some $\lambda \in \Lambda$, then $R_\bullet \equiv 0$ on Λ and hence $\Lambda = \mathbb{K}$, because $R_\bullet$ is maximal. If $R_\lambda \neq 0$ for every $\lambda \in \Lambda$, then take any $\mu \in \Lambda$ and let $B = \{\lambda \in \mathbb{K} : |\lambda - \mu| < \|R_\mu\|\}^{-1}$. For every $\lambda \in B$ and $k = 1, 2, \ldots$ one has $\|(\mu - \lambda)^k R_\mu^{k+1}\| \leq \|R_\mu\| \theta^k$, where $\theta = |\lambda - \mu| \|R_\mu\| \in (0, 1)$. Hence for every $\lambda \in B$ the series $R_\mu + (\mu - \lambda)R_\mu^2 + (\mu - \lambda)^2 R_\mu^3 + \ldots$ is absolutely convergent and its sum S commutes with R_μ and satisfies the equality $S - R_\mu = (\mu - \lambda)SR_\mu$, so that $(\lambda, S) \sim (\mu, R_\mu)$. Since the pseudoresolvent $R_\bullet$ is maximal, it follows that $B \subset \Lambda$ and $R_\lambda = R_\mu + (\mu - \lambda)R_\mu^2 + (\mu - \lambda)^2 R_\mu^3 + \ldots$ for every $\lambda \in B$. ∎

Suppose now that X is a Banach space over the field $\mathbb{K}$. Denote by $L(X)$ the Banach algebra of continuous linear endomorphisms of X. Let $0 \neq \Lambda \subset \mathbb{K}$ and suppose that the map $R_\bullet : \Lambda \ni \lambda \to R_\lambda \in L(X)$ is a pseudoresolvent. Then $(R_\lambda)_{\lambda\in\Lambda}$ is a commutative family of operators. Furthermore, the null space $\mathcal{N}$ and the range $\mathfrak{I}$ of R_λ are both independent of λ, so that it is legitimate to call them the *null space* and the *range of the pseudoresolvent* $R_\bullet$. Since $\mathcal{N}$ is a closed linear subspace of X, the factor space $X/\mathcal{N} = \{x+\mathcal{N} : x \in X\}$ equipped with the norm $\|x+\mathcal{N}\|_{X/\mathcal{N}} = \inf\{\|y\|_X : y \in x+\mathcal{N}\}$ is a Banach space. Let $E \in L(X; X/\mathcal{N})$ be the *canonical epimorphism* of X onto $X/\mathcal{N}$, that is, $Ex = x + \mathcal{N}$ for every $x \in X$. For every $\lambda \in \Lambda$ there is a unique invertible operator $\widehat{R}_\lambda \in L(X/\mathcal{N}; X)$ satisfying

$$\widehat{R}_\lambda E = R_\lambda. \tag{1.2}$$

The range of $\widehat{R}_\lambda$ is equal to $\mathfrak{I}$, and the inverse of $\widehat{R}_\lambda$ is the map R_λ^{-1} which to every $x \in \mathfrak{I}$ assigns the element $R_\lambda^{-1}x = \{y \in X : R_\lambda y = x\}$ of $X/\mathcal{N}$. It follows that R_λ^{-1} is a closed linear operator from X into $X/\mathcal{N}$ with domain $\mathfrak{I}$ and range $X/\mathcal{N}$. As a consequence of (1.1) and (1.2), if $\lambda \in \Lambda$ and $\mu \in \Lambda$, then

$$R_\lambda^{-1} - R_\mu^{-1} = R_\lambda^{-1}(\widehat{R}_\mu - \widehat{R}_\lambda)R_\mu^{-1} = R_\lambda^{-1}[(\lambda-\mu)\widehat{R}_\lambda E\widehat{R}_\mu]R_\mu^{-1} = (\lambda-\mu)E|_\mathfrak{I},$$

so that

$$\lambda E|_\mathfrak{I} - R_\lambda^{-1} = \mu E|_\mathfrak{I} - R_\mu^{-1}.$$

Thus there is a unique closed linear operator G from X into $X/\mathcal{N}$ with domain $\mathfrak{I}$ such that

$$G = \lambda E|_\mathfrak{I} - R_\lambda^{-1} \qquad \text{for every } \lambda \in \Lambda. \tag{1.3}$$

We will call G the *generator of the pseudoresolvent R*. This definition agrees with the one used by K. Itô and H. P. McKean [I-K], Sec. 3.7 and 7.1, in some situations when $\mathcal{N} = \{0\}$, i.e. when $R_\bullet$ is a *resolvent*. See also [Y], p. 216, Theorem 1. From (1.3) it follows that $\widehat{R}_\lambda = (\lambda E|_\mathfrak{I} - G)^{-1}$, so that

$$R_\lambda = (\lambda E|_\mathfrak{I} - G)^{-1}E \qquad \text{for every } \lambda \in \Lambda. \tag{1.4}$$

1.3. Proposition. *Let X be a Banach space over the field $\mathbb{K}$. Let $\mathfrak{I}$ be a linear subset of X, $\mathcal{N}$ a closed linear subspace of X, and G a closed linear operator from X into $X/\mathcal{N}$ with domain $\mathfrak{I}$. Put*

$$\varrho(G) = \{\lambda \in \mathbb{K} : \lambda E|_\mathfrak{I} - G \textit{ is a one-to-one map of } \mathfrak{I} \textit{ onto } X/\mathcal{N}\}. \tag{1.5}$$

If $\varrho(G) \neq \emptyset$, then the map

$$R_\bullet : \varrho(G) \ni \lambda \to R_\lambda = (\lambda E|_\mathfrak{I} - G)^{-1}E \in L(X) \tag{1.6}$$

is a maximal pseudoresolvent with null space $\mathcal{N}$ and range $\mathfrak{I}$. The generator of this pseudoresolvent is equal to G.

Proof. Suppose that $\varrho(G) \neq 0$. If $\lambda \in \varrho(G)$, then $\lambda E|_\mathfrak{I} - G$ is a closed linear operator from X into $X/\mathcal{N}$ with range $X/\mathcal{N}$, and hence, by the closed graph theorem,

$(\lambda E|_{\mathfrak{I}} - G)^{-1} \in L(X/\mathcal{N}; X)$, so that $R_\lambda \in L(X)$. If $\lambda, \mu \in \varrho(G)$, then

$$\begin{aligned} R_\lambda - R_\mu &= (\lambda E|_{\mathfrak{I}} - G)^{-1}[(\mu E|_{\mathfrak{I}} - G) - (\lambda E|_{\mathfrak{I}} - G)](\mu E|_{\mathfrak{I}} - G)^{-1}E \\ &= (\mu - \lambda)(\lambda E|_{\mathfrak{I}} - G)^{-1}E(\mu E|_{\mathfrak{I}} - G)^{-1}E = (\mu - \lambda)R_\lambda R_\mu. \end{aligned}$$

Thus $R_\bullet$ is a pseudoresolvent. It follows from (1.5) and (1.6) that the null space of $R_\bullet$ is equal to $\mathcal{N}$ and the range of $R_\bullet$ is equal to $\mathfrak{I}$. For every $\lambda \in \varrho(G)$, by (1.6) and (1.2), one has $\widehat{R}_\lambda = (\lambda E|_{\mathfrak{I}} - G)^{-1}$, whence, according to the definition of R_λ^{-1}, $\lambda E|_{\mathfrak{I}} - R_\lambda^{-1} = \lambda E|_{\mathfrak{I}} - (\lambda E|_{\mathfrak{I}} - G) = G$, so that the generator of $R_\bullet$ coincides with G.

It remains to prove that the pseudoresolvent $R_\bullet$ is maximal. To this end we have to prove that if $\widetilde{R}_\bullet : \widetilde{\Lambda} \ni \lambda \to \widetilde{R}_\lambda \in L(X)$ is a pseudoresolvent extending $R_\bullet$, then $\widetilde{\Lambda} \subset \varrho(G)$. But, by (1.3), the generators of $\widetilde{R}_\bullet$ and $R_\bullet$ coincide and hence the generator of $\widetilde{R}_\bullet$ is equal to G. Furthermore, the null space and range of $\widetilde{R}_\bullet$ coincide with those of $R_\bullet$, that is, with $\mathcal{N}$ and $\mathfrak{I}$ respectively. If $\lambda \in \widetilde{\Lambda}$, then, by (1.3), $\lambda E|_{\mathfrak{I}} - G = \widetilde{R}_\lambda^{-1}$ is a one-to-one map of $\mathfrak{I}$ onto $X/\mathcal{N}$, which means that $\lambda \in \varrho(G)$. ∎

1.4. Theorem. *Let X be a Banach space over the field $\mathbb{K}$. Let $\emptyset \neq \Lambda \subset \mathbb{K}$ and suppose that $R_\bullet : \Lambda \ni \lambda \to R_\lambda \in L(X)$ is a pseudoresolvent with null space $\mathcal{N}$, range $\mathfrak{I}$, and generator G. Put*

$$\mathcal{G} = \{(x, y) \in \mathfrak{I} \times X : Ey = Gx\}.$$

Then

$$\text{(a)} \quad \begin{aligned} &\{(x, y) \in X \times X : \exists_{\lambda \in \Lambda}\ \lambda R_\lambda x - x = R_\lambda y\} \\ &\qquad = \{(x, y) \in X \times X : \forall_{\lambda \in \Lambda}\ \lambda R_\lambda x - x = R_\lambda y\} = \mathcal{G}, \end{aligned}$$

and

$$\text{(b)} \qquad \{(R_\lambda z, \lambda R_\lambda z - z) : z \in X\} = \mathcal{G} \quad \textit{for every } \lambda \in \Lambda.$$

If $\widetilde{R}_\bullet : \widetilde{\Lambda} \ni \lambda \to \widetilde{R}_\lambda \in L(X)$ is the maximal pseudoresolvent extending $R_\bullet$, then

$$\text{(c)} \qquad \widetilde{\Lambda} = \varrho(G) \quad \textit{and} \quad \widetilde{R}_\lambda = (\lambda E|_{\mathfrak{I}} - G)^{-1}E \quad \textit{for every } \lambda \in \widetilde{\Lambda},$$

and, furthermore,

$$\text{(d)} \qquad \begin{aligned} \widetilde{\Lambda} &= \{\lambda \in \mathbb{K} : [1 + (\lambda - \mu)R_\mu]^{-1} \in L(X)\} \quad \textit{for every } \mu \in \Lambda, \\ \widetilde{R}_\lambda &= [1 + (\lambda - \mu)R_\mu]^{-1}R_\mu \quad \textit{for every } \lambda \in \widetilde{\Lambda} \textit{ and } \mu \in \Lambda. \end{aligned}$$

Proof. To prove (a), fix any $\lambda \in \Lambda$. Then, by (1.3) and (1.2), $(x, y) \in \mathcal{G} \Leftrightarrow [(x, y) \in \mathfrak{I} \times X$ and $Ey = \lambda Ex - R_\lambda^{-1}x] \Leftrightarrow [(x, y) \in \mathfrak{I} \times X$ and $\widehat{R}_\lambda Ey = \lambda \widehat{R}_\lambda Ex - x] \Leftrightarrow [(x, y) \in X \times X$ and $R_\lambda y = \lambda R_\lambda x - x]$.

For (b) notice that, by (a), $(R_\lambda z, \lambda R_\lambda z - z) \in \mathcal{G}$ for every $\lambda \in \Lambda$ and $z \in X$. On the other hand, if $\lambda \in \Lambda$, $(x, y) \in \mathcal{G}$ and $z = \lambda x - y$, then, again by (a), $R_\lambda z = R_\lambda(\lambda x - y) = x$ and $\lambda R_\lambda z - z = \lambda x - (\lambda x - y) = y$, so that $(x, y) = (R_\lambda z, \lambda R_\lambda z - z)$.

Statement (c) follows from (1.4) and Proposition 1.3.

To prove (d), for any $\mu \in \Lambda$ define

$$\Lambda_\mu = \{\lambda \in \mathbb{K} : \text{the operator } 1 + (\lambda - \mu)R_\mu \text{ has an inverse in } L(X)\}.$$

Take any $\mu \in \Lambda$, $\lambda \in \Lambda_\mu$, and put $R = [1 + (\lambda - \mu)R_\mu]^{-1}R_\mu$. Then R commutes with R_μ and $R + (\lambda - \mu)R_\mu R = [1 + (\lambda - \mu)R_\mu][1 + (\lambda - \mu)R_\mu]^{-1}R_\mu = R_\mu$, so that $(\lambda, R) \sim (\mu, R_\mu)$, and hence $\lambda \in \widetilde{\Lambda}$ and $\widetilde{R}_\lambda = R$. This shows that whenever $\mu \in \Lambda$, then $\Lambda_\lambda \subset \widehat{\Lambda}$ and $\widetilde{R}_\lambda = [1 + (\lambda - \mu)R_\mu]^{-1}R_\mu$ for every $\lambda \in \Lambda_\mu$. It remains to prove that $\widetilde{\Lambda} \subset \Lambda_\mu$ for every $\mu \in \Lambda$. By the closed graph theorem, this last will follow once it is shown that

(1.7) whenever $\mu \in \Lambda$ and $\lambda \in \widetilde{\Lambda}$, then $1 + (\lambda - \mu)R_\mu$ is a bijection of X onto X.

To prove (1.7), fix $\mu \in \Lambda$ and $\lambda \in \widetilde{\Lambda}$. Notice that $\widetilde{R}_\lambda^{-1}\widehat{R}_\mu$ is a bijection of $X/\mathcal{N}$ onto $X/\mathcal{N}$, and

$$E[1 + (\lambda - \mu)R_\mu] = \widetilde{R}_\lambda^{-1}\widetilde{R}_\lambda[1 + (\lambda - \mu)R_\mu] = \widetilde{R}_\lambda^{-1}R_\mu = \widetilde{R}_\lambda^{-1}\widehat{R}_\mu E. \tag{1.8}$$

Let $x \in X$, $y = [1 + (\lambda - \mu)R_\mu]x$. If $x \in X \setminus \mathcal{N}$, then $Ex \neq 0$ and, by (1.8), $Ey = \widetilde{R}_\lambda^{-1}\widehat{R}_\mu Ex \neq 0$, so that $y \neq 0$. If $0 \neq x \in \mathcal{N}$, then $y = x \neq 0$. Hence the operator $1 + (\lambda - \mu)R_\mu$ is invertible. Since $\widetilde{R}_\lambda^{-1}\widehat{R}_\mu E$ maps X onto $X/\mathcal{N}$, it follows form (1.8) that for every $y \in X$ there are $z \in X$ and $\nu \in \mathcal{N}$ such that $[1 + (\lambda - \mu)R_\mu]z = y + \nu$. If $x = z - \nu$, then $[1 + (\lambda - \mu)R_\mu]x = y$. Hence $1 + (\lambda - \mu)R_\mu$ maps X onto X. Statement (1.7) is proved. ∎

1.5. Remark. The set $\mathcal{G}$ occurring in Proposition 1.4 may be viewed as the graph of the multivalued operator

$$\mathfrak{I} \ni x \to E^{-1}Gx \subset X.$$

Statement (a) shows that this multivalued operator is an analogue of the *generator in the weak sense* and the *extended generator* of a markovian semigroup defined on pp. 165 and 243 of [D-M;D] by means of the corresponding markovian resolvent. Notice that, by definition, the markovian resolvent is a family of kernels in a measurable space, and hence it may determine pseudoresolvents whose operators act in various invariant spaces of measurable functions.

1.6. Remark. In the case of $\Lambda = (0, \infty)$, $\mathcal{N} = \{0\}$ and $\sup_{\lambda > 1} \lambda\|R_\lambda\| < \infty$ F. Hirsch [H], p. 98, and C. Dellacherie and P.-A. Meyer [D-M;C], p. 314, defined the generator G_0 of a resolvent $R_\bullet$ by $[x \in \mathcal{D}(G_0)$ and $y = G_0x] \Leftrightarrow [(x, y) \in X \times X$ and $\lim_{\lambda\to\infty} \|\lambda(\lambda R_\lambda x - x) - y\| = 0]$. This definition of G_0 is also applicable in the case of a pseudoresolvent, i.e. without assuming that $\mathcal{N} = \{0\}$. From Theorem 1.4(a) it may be deduced that if $R_\bullet$ is a pseudoresolvent on $\Lambda = (0, \infty)$ such that $\sup_{\lambda > 1} \lambda\|R_\lambda\| < \infty$, then

$$[x \in \mathcal{D}(G_0) \text{ and } y = G_0x] \;\Leftrightarrow\; [(x, y) \in \mathfrak{I} \times \mathcal{R} \text{ and } Ey = Gx],$$

where G is the generator of $R_\bullet$ in the sense of the present paper, and

$$\mathcal{R} = \{x \in X : \lim_{\lambda\to\infty} \|\lambda R_\lambda x - x\| = 0\}$$

is the *regularity space* of $R_\bullet$. See [D-M;C], p. 315, formula (8.1) or [D-M;D], p. 91, formula (8.1).

1.7. Remark. Let $\mathcal{A}$ be a Banach algebra over the field $\mathbb{K}$, and let $\mathcal{A}_{\mathrm{u}}$ be the unitization of $\mathcal{A}$. Denote by 1 the unit of $\mathcal{A}_{\mathrm{u}}$. If $\emptyset \neq \Lambda \subset \mathbb{K}$ and $R_\bullet : \Lambda \ni \lambda \to R_\lambda \in \mathcal{A}$ is a maximal pseudoresolvent, then

$$\Lambda = \Lambda_\mu \quad \text{for every } \mu \in \Lambda \tag{1.9}$$

where

$$\Lambda_\mu = \{\lambda \in \mathbb{K} : 1 + (\lambda - \mu)R_\mu \text{ is an invertible element of } \mathcal{A}_{\mathrm{u}}\}.$$

As a consequence of (1.9) and (1.1),

$$R_\lambda = [1 + (\lambda - \mu)R_\mu]^{-1} R_\mu \quad \text{for every } \lambda \in \Lambda \text{ and } \mu \in \Lambda.$$

The equality (1.9) follows from Lemma 5.8.1 and Theorem 5.8.6 of [H-P], p. 188, the latter being deduced from Theorem 5.8.4, p. 186, concerning the Gelfand representation of a commutative Banach subalgebra of $\mathcal{A}_{\mathrm{u}}$ containing $\{1\} \cup \{R_\lambda : \lambda \in \Lambda\}$. In order to obtain a more direct proof of (1.9), notice first that an argument similar to one used in the proof of Theorem 1.4(d) yields the inclusion $\Lambda_\mu \subset \Lambda$ for every $\mu \in \Lambda$. The opposite inclusion means that

$$\text{whenever } \lambda \in \Lambda \text{ and } \mu \in \Lambda, \text{ then } 1 + (\lambda - \mu)R_\mu \text{ is an invertible element of } \mathcal{A}_{\mathrm{u}}. \tag{1.10}$$

In order to prove (1.10), consider the pseudoresolvent $\Lambda \ni \lambda \to L_{R_\lambda} \in L(\mathcal{A}_{\mathrm{u}})$, where, for any $\lambda \in \Lambda$, L_{R_λ} is the operator of left multiplication by R_λ. Whenever $\lambda \in \Lambda$ and $\mu \in \Lambda$, then, by Theorem 1.4(d), the operator $1 + (\lambda - \mu)L_{R_\mu}$ has an inverse belonging to $L(\mathcal{A}_{\mathrm{u}})$. Now (1.10) follows from the equality $L_{1+(\lambda-\mu)R_\mu} = 1 + (\lambda - \mu)L_{R_\mu}$ and from the fact that an element $A \in \mathcal{A}_{\mathrm{u}}$ is invertible if and only if the corresponding operator $L_A \in L(\mathcal{A}_{\mathrm{u}})$ has an inverse belonging to $L(\mathcal{A}_{\mathrm{u}})$.

1.8. Remark. Pseudoresolvents appear in [F-Y] in Sections 1.1 and 1.3, pp. 21–22 and 23–24, as the maps $\varrho(A) \ni \lambda \to (\lambda - A)^{-1} \in L(X)$ where A is a *linear multivalued operator*. The symbols A, $\mathcal{D}(A)$, $A0$, $\varrho(A)$ and $(\lambda - A)^{-1}$ used in [F-Y] correspond to our $\mathcal{G}$, $\mathfrak{I}$, $\mathcal{N}$, $\varrho(G)$ and $(\lambda E|_{\mathfrak{I}} - G)^{-1}E$.

2. Distribution semigroups as representations of the convolution algebra $\mathcal{D}([0,\infty))$

Let $c \in (0, \infty]$, and let Ω be either $(-\infty, c)$ or $[0, c)$. For any closed set $F \subset \Omega$ and any $n = 0, 1, \ldots, \infty$ define

$$C_F^n(\Omega) = \{\varphi \in C^n(\Omega) : \varphi|_{\Omega \setminus F} \equiv 0\}.$$

If $n < \infty$ and F is compact, then $C_F^n(\Omega)$ is a Banach space. Otherwise $C_F^n(\Omega)$ is a Fréchet space. Each $C^n(\Omega)$ is a Fréchet space. The pseudonorms determining the topology in these spaces will be used in the form

$$|\varphi|_{a,b;n} = \sup\{|D^k\varphi(t)| : t \in [a,b],\ k = 0, 1, \ldots, n\}, \quad -\infty < a < b < c,\ n = 0, 1, \ldots$$

Let $\mathcal{D}(\Omega)$ be the inductive limit of the Fréchet spaces $C_K^\infty(\Omega)$ for K ranging over compact subsets of Ω.

Let X be a Banach space. According to [S;I–II], the space $L(\mathcal{D}(-\infty,c);X)$ of continuous linear maps of $\mathcal{D}((-\infty,c))$ into X is denoted by $\mathcal{D}'((-\infty,c));X)$, and its elements are called *distributions on* $(-\infty,c)$ *with values in* X. The subspace of $\mathcal{D}'((-\infty,c);X)$ consisting of the distributions with supports ([1]) contained in $[0,c)$ will be denoted by $\mathcal{D}'_0((-\infty,c);X)$. For any $\varphi\in\mathcal{D}((-\infty,c))$ let $\mathfrak{R}(\varphi)=\varphi|_{[0,c)}$ be the restriction of φ to $[0,c)$. Then $\mathfrak{R}\in L(\mathcal{D}((-\infty,c));\mathcal{D}([0,c)))$ and $T^+\circ\mathfrak{R}\in\mathcal{D}'_0((-\infty,c);X)$ for every $T^+\in L(\mathcal{D}([0,c));X)$.

2.1. Theorem. *$\mathfrak{R}$ is a surjection of $\mathcal{D}((-\infty,c))$ onto $\mathcal{D}([0,c))$. For any Banach space X the map*

$$\mathfrak{R}^*:L(\mathcal{D}([0,c));X)\ni T^+\to T^+\circ\mathfrak{R}\in\mathcal{D}'_0((-\infty,c);X)\tag{2.1}$$

is a linear topological isomorphism of $L_{\mathrm{b}}(\mathcal{D}([0,c));X)$ onto the closed subspace $\mathcal{D}'_0((-\infty,c);X)$ of $L_{\mathrm{b}}(\mathcal{D}((-\infty,c));X)$ ([2]).

In the proof of Theorem 2.1 we will use the following version of a theorem of Borel ([N], p. 30, Theorem 1.5.4; [T], p. 390, Theorem 38.1).

2.2. Theorem (E. Borel). *Let $c\in(0,\infty]$ and $b\in(0,c)$. If A is a set of complex sequences $a=(a_n)_{n=0,1,\dots}$ such that $\sup\{|a_n|:a\in A\}<\infty$ for every $n=0,1,\dots$, then there is a map $A\ni a\to\phi_a\in C^\infty_{[-b,b]}((-\infty,c))$ such that*

$$D^n\phi_a(0)=a_n\quad \textit{for every } a\in A \textit{ and every } n=0,1,\dots,\tag{2.2}$$

and

$$\{\phi_a:a\in A\}\ \textit{is a bounded subset of the Fréchet space}\ C^\infty_{[-b,b]}((-\infty,c)).\tag{2.3}$$

Proof of Theorem 2.2. Modifying slightly the argument from [N], we start with proving that

(2.4) for every $n=1,2,\dots$ and every $\varepsilon>0$ there is $\varphi_{n,\varepsilon}\in C^\infty_{[-b,b]}((-\infty,c))$ such that $D^m\varphi_{n,\varepsilon}(0)=\delta_{m,n}$ for $m=0,1,\dots,$ and $|\varphi_{n,\varepsilon}|_{-b,b;n-1}\le\varepsilon$.

([1]) Let $\Omega=(-\infty,c)$. An element T of $\mathcal{D}'(\Omega;X)$ is said to vanish in an open set $U\subset\Omega$ if $T(\varphi)=0$ for every $\varphi\in\mathcal{D}(\Omega)$ such that $\operatorname{supp}\varphi\subset U$. The partition of unity theorem ([B]; [S], p. 22; [T], pp. 161–165; [Y], p. 61) implies that the family of those open subsets of Ω in which T vanishes is stable under arbitrary unions ([T], p. 255, Corollary 2). Hence there exists the *support* of T, denoted by $\operatorname{supp}T$, defined as the smallest subset F of Ω with the property that F is closed and T vanishes in $\Omega\backslash F$. See [S], p. 28; [T], p. 255; [Y], p. 62.

([2]) The same is true for the pair $L_{\mathrm{s}}(\mathcal{D}([0,c));X)$, $L_{\mathrm{s}}(\mathcal{D}((-\infty,c));X)$. If $L(E;F)$ is the space of continuous linear maps of a l.c.v.s. E into a l.c.v.s. F, then $L_{\mathrm{b}}(E;F)$ denotes $L(E;F)$ equipped with the topology of uniform convergence on bounded subsets of E, and $L_{\mathrm{s}}(E;F)$ means $L(E;F)$ with the topology of pointwise convergence (see [Y], pp. 110–111, or [T], p. 337).

To prove (2.4), take any $v \in C^\infty_{[-b,b]}(\mathbb{R})$ such that $v|_{[-b/2,b/2]} \equiv 1$. For $k = 0, 1, \ldots, n-1$ put

$$C_k = |v|_{-b,b;k} \sum_{l=0}^{k} \binom{k}{l} \frac{b^{n-k+l}}{(n-k+l)!},$$

and take $\delta \in (0,1]$ so small that $\delta^{n-k} C_k \le \varepsilon$ for $k = 0, 1, \ldots, n-1$. Define

$$\varphi_{n,\varepsilon}(t) = \frac{t^n}{n!} v\left(\frac{t}{\delta}\right), \qquad t \in (-\infty, c).$$

Then $\varphi_{n,\varepsilon} \in C^\infty_{[-b,b]}((-\infty,0))$ and $D^m \varphi_{n,\varepsilon}(0) = \delta_{m,n}$ for $m = 0, 1, \ldots$, because $\varphi_{n,\varepsilon}(t) = t^n/n!$ for $t \in [-b\delta/2, b\delta/2]$, Furthermore, by the Leibniz formula, for every $k = 0, 1, \ldots, n-1$ one has

$$\sup_{-b \le t \le b} |D^k \varphi_{n,\varepsilon}(t)| = \sup_{-b\delta \le t \le b\delta} \left| \sum_{l=0}^{k} \binom{k}{l} \frac{t^{n-k+l}}{(n-k+l)!} \delta^{-l} (D^l v)\left(\frac{t}{\delta}\right) \right| \le \delta^{n-k} C_k \le \varepsilon,$$

whence $|\varphi_{n,\varepsilon}|_{-b,b;n-1} \le \varepsilon$, and (2.4) is proved.

Let $A_n = \sup\{|a_n| : a \in A\}$. Thanks to (2.4), for every $n = 0, 1, \ldots$ one can choose $\varphi_n \in C^\infty_{[-b,b]}((-\infty, c))$ so that

$$D^m \varphi_n(0) = \delta_{m,n} \qquad \text{for } m = 0, 1, \ldots \tag{2.5}$$

and

$$|\varphi_n|_{-b,b;n-1} \le 2^{-n}(A_n + 1)^{-1} \qquad \text{for } n \ge 1. \tag{2.6}$$

Now, for every sequence $a = (a_n)_{n=0,1,\ldots} \in A$ define

$$\phi_a = \sum_{n=0}^{\infty} a_n \varphi_n. \tag{2.7}$$

By (2.6), for every $a \in A$, $k = 0, 1, \ldots$ and $n = k+1, k+2, \ldots$ one has

$$|a_n \varphi_n|_{-b,b;k} \le A_n |\varphi_n|_{-b,b;n-1} < 2^{-n}. \tag{2.8}$$

This implies that for every $a \in A$ the partial sums of the series on the right side of (2.7) constitute a Cauchy sequence in the Fréchet space $C^\infty_{[-b,b]}((-\infty, c))$. From the completeness of $C^\infty_{[-b,b]}((-\infty, c))$, from the theorem about term by term differentation, and from (2.8), it follows that whenever $a \in A$, then $\phi_a \in C^\infty_{[-b,b]}((-\infty, c))$ is well defined. By (2.5) and (2.7), ϕ_a satisfies (2.2). Furthermore, for every $a \in A$ and $k = 0, 1, \ldots$, as a consequence of (2.7) and (2.8), one has

$$|\phi_a|_{-b,b;k} \le \sum_{n=0}^{k} A_n |\varphi_n|_{-b,b;k} + \sum_{n=k+1}^{\infty} A_n |\varphi_n|_{-b,b;n-1} < \sum_{n=0}^{k} A_n |\varphi_n|_{-b,b;k} + 2^{-k},$$

proving (2.3).

Proof of Theorem 2.1. Theorem 2.2 implies that

(2.9) if $b \in (0, c)$ and B^+ is a bounded subset of $C^\infty_{[0,b]}([0, c))$, then there exists a bounded subset B of $C^\infty_{[-b,b]}((-\infty, c))$ such that $\mathfrak{R}(B) = B^+$.

Indeed, let $A = \{(D^n\varphi(0))_{n=0,1,\ldots} : \varphi \in B^+\}$, where $D^n\varphi(0)$ are the right-hand derivatives. Since B^+ is bounded, one has $\sup\{|a_n| : a \in A\} < \infty$ for every $n = 0, 1, \ldots$ Let $A \ni a \to \phi_a \in C^\infty_{[-b,b]}((-\infty, c))$ be a map having the properties (2.2) and (2.3). For every $\varphi \in B^+$ define $\overline{\varphi} \in C^\infty_{[-b,b]}((-\infty, c))$ so that

$$\overline{\varphi}(t) = \begin{cases} \varphi(t) & \text{if } t \in [0, c), \\ \varphi_a(t) & \text{if } t \in (-\infty, 0), \text{ where } a = (D^n\varphi(0))_{n=0,1,\ldots} \end{cases}$$

Then $\mathfrak{R}\overline{\varphi} = \varphi$ for every $\varphi \in B^+$, and $B = \{\overline{\varphi} : \varphi \in B^+\}$ is a bounded subset of $C^\infty_{[-b,b]}((-\infty, c))$, proving (2.9).

In particular (2.9) implies that $\mathfrak{R}$ is a surjection of $\mathcal{D}((-\infty, c))$ onto $\mathcal{D}([0, c))$, whence it follows that $\mathfrak{R}^*$ is injective. Furthermore, $\mathfrak{R}^*$ is a continuous map of $L_{\mathrm{b}}(\mathcal{D}([0, c)); X)$ into $L_{\mathrm{b}}(\mathcal{D}((-\infty, c)); X)$. Indeed, the family of the sets $U_{B,\varepsilon} = \{T \in L(\mathcal{D}((-\infty, c)); X) : \|T(\varphi)\| < \varepsilon$ for every $\varphi \in B\}$, where $\varepsilon \in (0, \infty)$ and B ranges over all bounded subsets of $\mathcal{D}((-\infty, c))$, is a basis of neighbourhoods of zero in $L_{\mathrm{b}}(\mathcal{D}((-\infty, c)); X)$. If B is a bounded subset of $\mathcal{D}((-\infty, c))$, then $\mathfrak{R}(B)$ is a bounded subset of $\mathcal{D}([0, c))$, and $V_{\mathfrak{R}(B),\varepsilon} = \{T^+ \in L(\mathcal{D}([0, c)); X) : \|T^+(\psi)\| < \varepsilon$ for every $\psi \in \mathfrak{R}(B)\}$ is a neighbourhood of zero in $L_{\mathrm{b}}(\mathcal{D}([0, c)); X)$ such that $\mathfrak{R}^*(V_{\mathfrak{R}(B),\varepsilon}) \subset U_{B,\varepsilon}$.

Suppose that we already know that $\mathfrak{R}^*$ is a surjection of $L(\mathcal{D}([0, c)); X)$ onto $\mathcal{D}'_0((-\infty, c); X)$. Under this assumption we will show that $(\mathfrak{R}^*)^{-1} : \mathcal{D}'_0((-\infty, c); X) \to L_{\mathrm{b}}(\mathcal{D}([0, c)); X)$ is continuous, provided that $\mathcal{D}'_0((-\infty, c); X)$ carries the topology inherited from $L_{\mathrm{b}}(\mathcal{D}((-\infty, c)); X)$. Indeed, in this topology a basis of neighbourhoods of zero in $\mathcal{D}'_0((-\infty, c)); X)$ is constituted by the sets $W_{B,\varepsilon} = U_{B,\varepsilon} \cap \mathcal{D}'_0((-\infty, c); X)$, where, as before, $\varepsilon \in (0, \infty)$ and B ranges over all bounded subsets of $\mathcal{D}((-\infty, c))$. The sets $V_{B^+,\varepsilon} = \{T^+ \in L(\mathcal{D}([0, c)); X) : \|T^+(\psi)\| < \varepsilon$ for every $\psi \in B^+\}$, where $\varepsilon \in (0, \infty)$ and B^+ ranges over all bounded subsets of $\mathcal{D}([0, c))$, constitute a basis of neighbourhoods of zero in $L_{\mathrm{b}}(\mathcal{D}([0, c)); X)$. For every $V_{B^+,\varepsilon}$ there is a $b \in (0, c)$ such that B^+ is a bounded subset of the Fréchet space $C^\infty_{[0,b]}([0, c))$. Let $B \subset C^\infty_{[-b,b]}((-\infty, c))$ satisfy (2.9). Then $(\mathfrak{R}^*)^{-1}(W_{B,\varepsilon}) = \{T^+ \in L(\mathcal{D}([0, c)); X) : \|(T^+ \circ \mathfrak{R})(\varphi)\| < \varepsilon$ for every $\varphi \in B\} = V_{\mathfrak{R}(B),\varepsilon} = V_{B^+,\varepsilon}$, which proves the desired continuity of $(\mathfrak{R}^*)^{-1}$.

In order to complete the proof of Theorem 2.1, it remains to show that $\mathfrak{R}^*$ is a surjection of $L(\mathcal{D}([0, c)); X)$ onto $\mathcal{D}'_0((-\infty, c); X)$. To this end, let $0 < a < b < c$, and let α be any fixed C^∞ function on $(-\infty, c)$ such that $\alpha|_{(-\infty,a)} \equiv 1$, $\alpha|_{[b,c)} \equiv 0$. Given a $T \in \mathcal{D}'_0((-\infty, c); X)$, put

$$R = (1 - \alpha)T, \quad S = \alpha T,$$

so that $R, S \in \mathcal{D}'_0((-\infty, c); X)$ and $T = R + S$. Let $R^+ = T \circ F$, where, for any $\psi \in \mathcal{D}([0, c))$,

$$(F\psi)(t) = \begin{cases} (1 - \alpha(t))\psi(t) & \text{if } t \in [0, c), \\ 0 & \text{if } t \in (-\infty, 0). \end{cases}$$

Then we have $F \in L(\mathcal{D}([0, c)); \mathcal{D}((-\infty, c)))$, $R^+ \in L(\mathcal{D}([0, c)); X)$, and $(R^+ \circ \mathfrak{R})(\varphi) = T((F \circ \mathfrak{R})(\varphi)) = T((1 - \alpha)\varphi) = R(\varphi)$ for every $\varphi \in \mathcal{D}((-\infty, c))$, so that $\mathfrak{R}^* R^+ = R$.

It remains to construct an $S^+ \in L(\mathcal{D}([0,c));X)$ such that $\mathfrak{R}^*S^+ = S = \alpha T$. Since $T|_{C^\infty_{[-b,b]}((-\infty,c))}$ is continuous with respect to the topology of $C^n_{[-b,b]}((-\infty,c))$, it follows that for some $n \in \mathbb{N}_0$ there is $\overline{T} \in L(C^n_{[-b,b]}((-\infty,c));X)$ extending $T|_{C^\infty_{[-b,b]}((-\infty,c))}$. Let $\beta(t) = \alpha(|t|)$ if $t \in (-c,c)$, $\beta(t) = 0$ if $t \in (-\infty,-c]$. Then $\beta \in C^\infty_{[-b,b]}((-\infty,c))$ and $\operatorname{supp}(\alpha-\beta) \subset (-\infty,-a)$. Let $S^+ = \overline{T} \circ E$, where, for every $\psi \in \mathcal{D}([0,c))$,

$$(E\psi)(t) = \begin{cases} \beta(t)\psi(t) & \text{if } t \in [0,c), \\ \beta(t)\displaystyle\sum_{k=0}^{n} \frac{t^k}{k!} D^k\psi(0) & \text{if } t \in (-\infty,0). \end{cases}$$

Then $E \in L(\mathcal{D}([0,c)); C^n_{[-b,b]}((-\infty,c)))$ and $S^+ \in L(\mathcal{D}([0,c));X)$. Furthermore, for every $\varphi \in \mathcal{D}((-\infty,c))$ one has $(S^+ \circ \mathfrak{R})(\varphi)) = \overline{T}((E\circ\mathfrak{R})(\varphi)) = \overline{T}(\beta\varphi) = T(\beta\varphi) = T(\alpha\varphi) = S(\varphi)$, so that $\mathfrak{R}^*S^+ = S$. In the above, the equality $\overline{T}((E\circ\mathfrak{R})(\varphi)) = \overline{T}(\beta\varphi)$ follows from the fact that if $v = (E\circ\mathfrak{R})(\varphi) - \beta\varphi$, then $v \in C^n_{[-b,0]}((-\infty,c))$. Indeed, the latter implies that if $v_k \in C^\infty_{[-b,0]}((-\infty,c))$ and $\operatorname{supp} v_k \subset [-b,0)$ for every $k = 1,2,\ldots$, and $\lim_{k\to\infty} |v_k - v|_{-b,0;n} = 0$, then $\overline{T}(v_k) = T(v_k) = 0$ and $\lim_{k\to\infty} \|\overline{T}(v_k) - \overline{T}(v)\| = 0$. ■

For any $\varphi,\psi \in \mathcal{D}([0,c))$ the convolution $\varphi * \psi$ is defined on $[0,c)$ by the formula

$$[\varphi*\psi](t) = \int_0^t \varphi(s)\psi(t-s)\,ds, \qquad t \in [0,c), \tag{2.10}$$

so that $\varphi*\psi \in C^\infty([0,c))$. If $\varphi,\psi \in \mathcal{D}([0,c))$ and $\operatorname{supp}\varphi + \operatorname{supp}\psi \subset [0,c)$, then $\operatorname{supp}(\varphi*\psi) \subset \operatorname{supp}\varphi + \operatorname{supp}\psi \subset [0,c)$, whence $\varphi*\psi \in \mathcal{D}([0,c))$. If $\varphi,\psi \in \mathcal{D}([0,\infty))$, then $\varphi*\psi \in \mathcal{D}([0,\infty))$. As a consequence, $\mathcal{D}([0,\infty))$ with convolution multiplication and with the inductive limit topology discussed above is a locally convex commutative algebra. For any Banach algebra $\mathcal{A}$ we will denote by $\operatorname{Hom}(\mathcal{D}([0,\infty));\mathcal{A})$ the set of all continuous homomorphisms of $\mathcal{D}([0,\infty))$ into $\mathcal{A}$. The most important will be the case when $\mathcal{A} = L(X)$ is the Banach algebra of continuous linear endomorphisms of a Banach space X. Then $\operatorname{Hom}(\mathcal{D}([0,\infty));L(X))$ is the set of all continuous representations of the convolution algebra $\mathcal{D}([0,\infty))$ on the Banach space X ([3]).

2.3. Definition. By a *distribution semigroup* with values in a Banach algebra $\mathcal{A}$ we will mean a distribution $S \in \mathcal{D}'_0((-\infty,\infty);\mathcal{A})$ such that $S = \mathfrak{R}^*S^+$, where $S^+ \in \operatorname{Hom}(\mathcal{D}([0,\infty));\mathcal{A})$ and $\mathfrak{R}^*$ is the isomorphism (2.1). The set of $\mathcal{A}$-valued distribution semigroups will be denoted by $\mathrm{DS}(\mathcal{A})$.

In Chapter 3 we will discuss the relation between Definition 2.3 and a more restrictive definition of the *regular distribution semigroup* formulated in 1960 by J. L. Lions

([3]) A continuous representation of $\mathcal{D}([0,\infty))$ on the Banach space X is a continuous bilinear map $B : \mathcal{D}([0,\infty)) \times X \to X$ such that $B(\varphi*\psi,x) = B(\varphi,B(\psi,x))$ for every $\varphi,\psi \in \mathcal{D}([0,\infty))$ and $x \in X$. The existence of a unique $S^+ \in \operatorname{Hom}(\mathcal{D}([0,\infty));L_{\mathrm{b}}(X))$ such that $B(\varphi,x) = S^+(\varphi)x$ for every $\varphi \in \mathcal{D}([0,\infty))$ and $x \in X$ follows from the fact that $\mathcal{D}([0,\infty))$ is the inductive limit of Fréchet spaces, and from the Corollary on p. 354 of [T]. See also [F], p. 478, proof of Theorem 8.3.1.

[L]. Definition 2.3 is equivalent to Definition 2.1 on p. 839 of [K], where P. C. Kunstmann defined a *pre-distribution semigroup* with values in $L(X)$. Indeed, Definition 2.3 implies that $\mathrm{DS}(\mathcal{A})$ is equal to the set of distributions $S \in \mathcal{D}'((-\infty,\infty);\mathcal{A})$ such that

$$\operatorname{supp} S \subset [0,\infty) \tag{2.11}$$

and

(2.12) $S(\varphi)S(\psi) = S(\vartheta)$ for every $\varphi,\psi,\vartheta \in \mathcal{D}((-\infty,\infty))$ such that $(\varphi|_{[0,\infty)}) * (\psi|_{[0,\infty)}) = \vartheta|_{[0,\infty)}$.

If (2.11) is satisfied, then (2.12) is equivalent to Kunstmann's condition: $S(\varphi)S(\psi) = S(\varphi *_0 \psi)$ for every $\varphi,\psi \in \mathcal{D}((-\infty,\infty))$, where $[\varphi *_0 \psi](t) = \int_0^t \varphi(s)\psi(t-s)\,ds$ for *every* $t \in (-\infty,\infty)$.

In the following definitions $\mathcal{A}$ is a Banach algebra and $c \in (0,\infty)$ ([4]).

2.4. Definition. An operator $S^+ \in L(\mathcal{D}([0,c));\mathcal{A})$ will be called a *local homomorphism* of $\mathcal{D}([0,c))$ into $\mathcal{A}$ if $S^+(\varphi)S^+(\psi) = S^+(\varphi * \psi)$ whenever $\varphi,\psi \in \mathcal{D}([0,c))$ and $\operatorname{supp}\varphi + \operatorname{supp}\psi \subset [0,c)$. The set of local homomorphisms of $\mathcal{D}([0,c))$ into $\mathcal{A}$ will be denoted by $\mathrm{Hom}(\mathcal{D}([0,c));\mathcal{A})$.

2.5. Definition. By a *local distribution semigroup* on $(-\infty,c)$ with values in $\mathcal{A}$ we will mean a distribution $S \in \mathcal{D}'_0((-\infty,c);\mathcal{A})$ such that $S = \mathfrak{R}^*S^+$, where $S^+ \in \mathrm{Hom}(\mathcal{D}([0,c));\mathcal{A})$ and $\mathfrak{R}^*$ is the map (2.1). The set of $\mathcal{A}$-valued local distribution semigroups on $(-\infty,c)$ will be denoted by $\mathrm{DS}((-\infty,c);\mathcal{A})$.

2.6. Theorem. *Suppose that $\mathcal{A}$ is a Banach algebra, $c \in (0,\infty]$, and $S \in \mathcal{D}'_0((-\infty,c);\mathcal{A})$. Then $S \in \mathrm{DS}((-\infty,c);\mathcal{A})$ if and only if*

$$\psi(0)S(\varphi) - \varphi(0)S(\psi) = S(D\varphi)S(\psi) - S(\varphi)S(D\psi) \tag{2.13}$$

for every $\varphi,\psi \in \mathcal{D}([-\infty,c))$ such that $\operatorname{supp}\varphi + \operatorname{supp}\psi \subset (-\infty,c)$ ([5]).

Proof. Suppose that $S \in \mathrm{DS}(-\infty,c);\mathcal{A})$ and let S^+ be the unique element of $\mathrm{Hom}(\mathcal{D}([0,c));\mathcal{A})$ such that $S = \mathcal{R}^*S^+ = S^+ \circ \mathcal{R}$. Take any $\varphi,\psi \in \mathcal{D}((-\infty,c))$ such that $\operatorname{supp}\varphi + \operatorname{supp}\psi \subset (-\infty,c)$. Then

$$\psi(0)\varphi(t) - \varphi(0)\psi(t) = \int_0^t \frac{d}{ds}[\varphi(s)\psi(t-s)]\,ds = \int_0^t D\varphi(s)\,\psi(t-s)\,ds - \int_0^t \varphi(s)\,D\psi(t-s)\,ds$$

for every $t \in (-\infty,c)$, whence

$$\psi(0)\varphi|_{[0,c)} - \varphi(0)\psi|_{[0,c)} = (D\varphi)|_{[0,c)} * \psi|_{[0,c)} - \varphi|_{[0,c)} * (D\psi)|_{[0,c)}.$$

([4]) In Theorem 2.6 below, as well as in Chapters 4 and 5, there are no extra complications caused by allowing c to be finite. A profit is the proof of the unique extendability of a local distribution semigroup to a distribution semigroup.

([5]) If $c = \infty$ then the last inclusion is automatic.

Since $\operatorname{supp}\varphi|_{[0,c)} + \operatorname{supp}\psi|_{[0,c)} \subset [0,c)$ and $S^+ \in \operatorname{Hom}(\mathcal{D}([0,c));\mathcal{A})$, it follows that

$$\begin{aligned}\psi(0)S(\varphi) - \varphi(0)S(\psi) &= S^+(\psi(0)\varphi|_{[0,c)} - \varphi(0)\psi|_{[0,c)})\\ &= S^+((D\varphi)|_{[0,c)} * \psi|_{[0,c)} - \varphi|_{[0,c)} * (D\psi)|_{0,c)})\\ &= S^+((D\varphi)|_{[0,c)})S^+(\psi|_{[0,c)}) - S^+(\varphi|_{[0,c)})S^+((D\psi|_{[0,c)})\\ &= S(D\varphi)S(\psi) - S(\varphi)S(D\psi),\end{aligned}$$

so that (2.13) is satisfied.

Conversely, suppose that $S \in \mathcal{D}_0'((-\infty,c);\mathcal{A})$ satisfies (2.13). We have to prove that $S \in \mathrm{DS}((-\infty,c);\mathcal{A})$, i.e. that if S^+ is the unique operator in $L(\mathcal{D}([0,c));\mathcal{A})$ such that $\mathfrak{R}^*S^+ = S$, then $S^+ \in \operatorname{Hom}(\mathcal{D}([0,c));\mathcal{A})$. Since $\mathfrak{R}$ is a surjection of $\mathcal{D}((-\infty,c))$ onto $\mathcal{D}([0,c))$, it is sufficient to show that whenever $\varphi,\psi \in \mathcal{D}((-\infty,c))$ and $\operatorname{supp}\varphi + \operatorname{supp}\psi \subset (-\infty,c)$, then

$$S^+(\varphi|_{[0,c)})S^+(\psi|_{[0,c)}) = S^+(\varphi|_{[0,c)} * \psi|_{[0,c)}). \tag{2.14}$$

To this end, take φ, ψ as above. Then there are $a > 0$ and $b > 0$ such that $a + b < c$, $\operatorname{supp}\varphi \subset (-\infty,a)$ and $\operatorname{supp}\psi \subset (-\infty,b)$. For any $u \in [0,a]$ let φ_u be the left translate of φ by u, and let ψ_{-u} be the right translate of ψ by u. This means that whenever $u \in [0,a]$, then $\varphi_u(t) = \varphi(t+u)$ if $t \in (-\infty, c-u)$, $\varphi_u(t) = 0$ if $t \in [c-u,c)$, and $\psi_{-u}(t) = \psi(t-u)$ for every $t \in (-\infty,c)$. For every $u \in [0,a]$ one has $\varphi_u \in \mathcal{D}((-\infty,c))$, $\psi_{-u} \in \mathcal{D}((-\infty,c))$, $\operatorname{supp}\varphi_u \subset (-\infty,a-u)$, $\operatorname{supp}\psi_{-u} \subset (-\infty,b+u)$, and so $\operatorname{supp}\varphi_u + \operatorname{supp}\psi_{-u} \subset (-\infty,a+b) \subset (-\infty,c)$. Furthermore, $u \to \varphi_u$ and $u \to \psi_{-u}$ are C^∞ maps of $[0,a]$ into $\mathcal{D}((-\infty,c))$. Applying (2.13) to φ_u and ψ_{-u}, one concludes that

$$\psi(-u)S(\varphi_u) - \varphi(u)S(\psi_{-u}) = S(D\varphi_u)S(\psi_{-u}) - S(\varphi_u)S(D\psi_{-u}) = \frac{d}{du}[S(\varphi_u)S(\psi_{-u})]$$

for every $u \in [0,a]$. Since $\operatorname{supp}\varphi_a \subset (-\infty,0)$, one has $S(\varphi_a) = 0$ and hence

$$S(\varphi)S(\psi) = -\int_0^a \frac{d}{du}[S(\varphi_u)S(\psi_{-u})]\,du = S(\vartheta), \tag{2.15}$$

where

$$\vartheta = \int_0^a [\varphi(u)\psi_{-u} - \psi(-u)\varphi_u]\,du. \tag{2.16}$$

In (2.15) and (2.16) the integrals are computed in the sense of Riemann. This is legitimate because the integrands are (infinitely differentiable and a fortiori) continuous on $[0,a]$ and take values in complete l.c.v.s. $\mathcal{A}$ and $\mathcal{D}((-\infty,c))$. By (2.16), for every $t \in (-\infty,c)$ one has

$$\vartheta(t) = \int_0^a \varphi(u)\psi(t-u)\,du - \int_0^a \varphi(t+u)\psi(-u)\,du = \Big(\int_0^a - \int_t^{a+t}\Big)\varphi(u)\psi(t-u)\,du.$$

Since $\operatorname{supp}\varphi \subset (-\infty,a)$, it follows that $\vartheta(t) = \int_0^t \varphi(u)\psi(t-u)\,du$ for every $t \in [0,c)$, which implies that

$$\vartheta|_{[0,c)} = \varphi|_{[0,c)} * \psi|_{[0,c)}. \tag{2.17}$$

By (2.15) and (2.17),

$$S^+(\varphi|_{[0,c)})S^+(\psi|_{[0,c)}) = S(\varphi)S(\psi) = S(\vartheta) = S^+(\vartheta|_{[0,c)}) = S^+(\varphi|_{[0,c)} * \psi|_{[0,c)}),$$

proving (2.14). ∎

3. Regular distribution semigroups of J. L. Lions and distribution semigroups of P. C. Kunstmann

It was indicated in Chapter 2 that the notion of the distribution semigroup goes back to the pioneering work of J. L. Lions [L] (1960), and that Definition 2.3 of the distribution semigroup is adapted from P. C. Kunstmann [K] (1999). Let X be a Banach space. We will discuss the relations between the class $\mathrm{DS}(L(X))$ of distribution semigroups in the sense of Definition 2.3 (i.e. the class of pre-DSG's of Kunstmann) and the class $\mathrm{RDS}(L(X))$ of *regular distribution semigroups* of Lions ([6]). For any linear operator A denote by $\mathcal{N}(A)$ the null space of A, and by $\mathfrak{I}(A)$ the image of A. For any distribution $S \in \mathcal{D}'((-\infty,\infty); L(X))$ define ([7])

$$\mathcal{N} = \bigcap\{\mathcal{N}(S(\varphi)) : \varphi \in \mathcal{D}((-\infty,\infty))\},$$

$$\mathcal{N}_0 = \bigcap\{\mathcal{N}(S(\varphi)) : \varphi \in \mathcal{D}((-\infty,\infty)),\ \operatorname{supp}\varphi \subset [0,\infty)\},$$

$$\mathfrak{I} = \Big\{\sum_{k=1}^{n} S(\varphi_k)x_k : n \in \mathbb{N},\ \varphi_k \in \mathcal{D}((-\infty,\infty)) \text{ and } x_k \in X \text{ for } k = 1,\dots,n\Big\},$$

$$\mathfrak{I}_0 = \Big\{\sum_{k=1}^{n} S(\varphi_k)x_k : n \in \mathbb{N},\ \varphi_k \in \mathcal{D}((-\infty,\infty)),\ \operatorname{supp}\varphi_k \subset [0,\infty) \text{ and } x_k \in X \text{ for } k = 1,\dots,n\Big\}.$$

Evidently

$$\mathcal{N} \subset \mathcal{N}_0, \qquad \mathfrak{I} \supset \mathfrak{I}_0.$$

Both these inclusions may be strict: if, as in Kunstmann's Example 2.3 on p. 840 of [K], $S = \sum_{j=0}^{m} D^j\delta \otimes T^{j+1}$, where $0 \neq T \in L(X)$ and $T^{m+2} = 0$, then $S \in \mathrm{DS}(L(X))$, and $\operatorname{supp} S = \{0\}$, whence $\mathcal{N} \subsetneq X = \mathcal{N}_0$ and $\mathfrak{I} \supsetneq \{0\} = \mathfrak{I}_0$.

3.1. Definition. The class $\mathrm{RDS}(L(X))$ of *regular $L(X)$-valued distribution semigroups* consists of the elements of $\mathcal{D}'_0(-\infty,\infty); L(X))$ satisfying four conditions:

([6]) The adjective "regular" is in accord with the remarks of J. L. Lions in the footnote on p. 142 of [L].

([7]) The sets $\mathcal{N}(G)$ and $\mathfrak{I}(G)$ occurring in [K] correspond to our $\mathcal{N}_0$ and $\mathfrak{I}_0$.

(3.1) $S(\varphi)S(\psi) = S(\varphi * \psi)$ for every $\varphi, \psi \in \mathcal{D}((-\infty,\infty))$ such that $\operatorname{supp}\varphi \subset [0,\infty)$ and $\operatorname{supp}\psi \subset [0,\infty)$,

(3.2) for every $y \in \mathfrak{I}_0$ the distribution $\mathcal{D}((-\infty,\infty)) \ni \varphi \to S(\varphi)y \in X$ is represented by an X-valued function on $(-\infty,\infty)$ which vanishes on $(-\infty,0)$, is continuous on $[0,\infty)$ in the sense of the norm in X, and takes the value y at zero,

(3.3) the set $\mathfrak{I}_0$ is dense in X ([8]),

(3.4) $\mathcal{N}_0 = \{0\}$.

Kunstmann [K] proved that

(A) *every element of* $\mathrm{DS}(L(X))$ *satisfies* (3.2)

and

(B) *if* $S \in \mathcal{D}_0'((-\infty,\infty); L(X))$ *satisfies* (3.1)–(3.3), *then* $S \in \mathrm{DS}(L(X))$,

whence $\mathrm{RDS}(L(X)) \subset \mathrm{DS}(L(X))$. From Theorem 5.1 of Lions [L], p. 149, and from our Theorem 4.18 it follows that

(C) $\mathrm{RDS}(L(X)) = \{S \in \mathrm{DS}(L(X)) : \mathcal{N} = \{0\}, \mathfrak{I} \text{ is dense in } X\}$ ([9]).

Furthermore

(D) *there are distributions in* $\mathcal{D}_0'((-\infty,\infty); L(X))$ *satisfying* (3.1), (3.3) *and* (3.4), *but not belonging to* $\mathrm{DS}(L(X))$,

and

(E) *there are distributions in* $\mathcal{D}_0'((-\infty,\infty); L(X))$ *satisfying* (3.1), (3.2) *and* (3.4), *but not belonging to* $\mathrm{DS}(L(X))$.

Kunstmann's proofs of (A) and (B) make use of the fact that whenever $\varphi, \xi \in \mathcal{D}((-\infty,\infty))$, $\sup(\operatorname{supp}\varphi) = a > 0$, and $S \in \mathcal{D}'((-\infty,\infty); L(X))$, then $\varphi \mathbb{1}_{[0,\infty)} * \xi = \int_0^a \varphi(t)\xi_{-t}\,dt$ and hence

$$S(\varphi \mathbb{1}_{[0,\infty)} * \xi) = \int_0^a \varphi(t) S(\xi_{-t})\,dt. \tag{3.5}$$

Here $*$ denotes convolution on $(-\infty,\infty)$, ξ_{-t} is the right translate of ξ by t, and the integrals of C^∞ functions taking values in the complete l.c.v.s. $\mathcal{D}((-\infty,\infty))$ and $L(X)$ are computed in the sense of Riemann. Equality (3.5) implies at once

3.2. Lemma. *Let* $S \in \mathcal{D}'((-\infty,\infty); L(X))$ *and* $\xi \in \mathcal{D}((-\infty,\infty))$. *Then*

$$S(\varphi)S(\xi) = S(\varphi \mathbb{1}_{[0,\infty)} * \xi) \quad \text{for every } \varphi \in \mathcal{D}((-\infty,\infty)) \tag{3.6}$$

if and only if the distribution

$$\mathcal{D}((-\infty,\infty)) \ni \varphi \to S(\varphi)S(\xi) \in L(X) \tag{3.7}$$

([8]) In Definition 1.1 of [L], p. 142, a formally stronger condition is used: (3.3)* $\bigcup\{\mathfrak{I}(S(\varphi)) : \varphi \in \mathcal{D}((-\infty,\infty)), \operatorname{supp}\varphi \subset [0,\infty)\}$ is dense in X. However (3.3) implies (3.3)*, because (similarly to Remark 2.5 in [K], p. 841) if $y = \sum_{k=1}^n S(\varphi_k)x_k \in \mathfrak{I}_0(S)$ and $(\delta_m) \subset \mathcal{D}((-\infty,\infty))$ is a δ-like sequence such that $\operatorname{supp}\delta_m \subset [0,1/m]$, then $\lim_{m\to\infty} S(\delta_m)y = \lim_{m\to\infty}\sum_{k=1}^n S(\delta_m * \varphi_k)x_k = y$.

([9]) The conditions imposed there on $S \in \mathrm{DS}(L(X))$ are weaker than (3.3) and (3.4).

is represented by the $L(X)$-valued function F such that

$$F(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 0), \\ S(\xi_{-t}) & \text{if } t \in [0, \infty). \end{cases} \tag{3.8}$$

Proof of (A). We follow [K], lines 846^{9-12}. Suppose that $S \in \mathrm{DS}(L(X))$, $\xi_k \in \mathcal{D}((-\infty,\infty))$, $\operatorname{supp}\xi_k \subset [0,\infty)$, $x_k \in X$ for $k = 1,\dots,n$, and $y = \sum_{k=1}^{n} S(\xi_k)x_k$. Take any $\varphi \in \mathcal{D}((-\infty,\infty))$ and put $\vartheta_k = \varphi\mathbb{1}_{[0,\infty)} * \xi_k$. Then $\vartheta_k \in \mathcal{D}((-\infty,\infty))$ and since $\operatorname{supp}\xi_k \subset [0,\infty)$, one has $\operatorname{supp}\vartheta_k \subset [0,\infty)$ so that $\vartheta_k|_{[0,\infty)} = \varphi|_{[0,\infty)} * \xi_k|_{[0,\infty)}$ (in the sense of convolution on $[0,\infty)$) and, by (2.12), $S(\varphi)S(\xi_k) = S(\vartheta_k) = S(\varphi\mathbb{1}_{[0,\infty)} * \xi_k)$. Thus (3.6) is satisfied for $\xi = \xi_k$, $k = 1,\dots,n$, and, by Lemma 3.2, the distribution $\mathcal{D}((-\infty,\infty)) \ni \varphi \to S(\varphi)y \in X$ is represented by the X-valued function vanishing on $(-\infty,0)$ and equal to $\sum_{k=1}^{n} S((\xi_k)_{-t})x_k$ for every $t \in [0,\infty)$. ■

Proof of (B). We follow [K], lines 846^{14-15}. Note first that if $S \in \mathcal{D}_0'((-\infty,\infty); L(X))$ satisfies (3.1) and (3.2), then

(3.9) $S(\varphi)S(\xi) = S(\varphi\mathbb{1}_{[0,\infty)} * \xi)$ for every $\varphi \in \mathcal{D}((-\infty,\infty))$ and every $\xi \in \mathcal{D}((-\infty,\infty))$ such that $\operatorname{supp}\xi \subset [0,\infty)$.

Indeed, if $\xi \in \mathcal{D}((-\infty,\infty))$, $\operatorname{supp}\xi \subset [0,\infty)$ and $S \in \mathcal{D}_0'((-\infty,\infty); L(X))$ satisfies (3.1), then both sides of the equality in (3.9) vanish if $\operatorname{supp}\varphi \subset (-\infty,0)$, and are equal, by (3.1), if $\operatorname{supp}\varphi \subset [0,\infty)$. Hence, by (3.5), the distribution (3.7) is represented by the function (3.8) on the set $(-\infty,\infty)\setminus\{0\}$. Since the function (3.8) is continuous on $[0,\infty)$ in the sense of the $L(X)$ norm, condition (3.2) implies that the distribution (3.7) is represented by the function (3.8) on the whole $(-\infty,\infty)$. By Lemma 3.2, this last fact is equivalent to (3.6), and (3.9) is proved.

Suppose now that $S \in \mathcal{D}_0'((-\infty,\infty); L(X))$ satisfies (3.1)–(3.3). Take any $\varphi, \psi, \vartheta \in \mathcal{D}((-\infty,\infty))$ such that $\vartheta|_{[0,\infty)} = \varphi|_{[0,\infty)} * \psi|_{[0,\infty)}$ in the sense of convolution on $[0,\infty)$ or, what is equivalent, $\vartheta\mathbb{1}_{[0,\infty)} = \varphi\mathbb{1}_{[0,\infty)} * \psi\mathbb{1}_{[0,\infty)}$ in the sense of convolution on $(-\infty,\infty)$. Take any $\xi \in \mathcal{D}((-\infty,\infty))$ such that $\operatorname{supp}\xi \subset [0,\infty)$. Put $\eta = \psi\mathbb{1}_{[0,\infty)} * \xi$. Then $\eta \in \mathcal{D}((-\infty,\infty))$ and $\operatorname{supp}\eta \subset [0,\infty)$. Furthermore, $S(\psi)S(\xi) = S(\eta)$, by (3.9). Hence, again by (3.9),

$$S(\varphi)S(\psi)S(\xi) = S(\varphi)S(\eta) = S(\varphi\mathbb{1}_{[0,\infty)} * \eta).$$

By the associativity of convolution in $L^1(-\infty,\infty)$, one has

$$\varphi\mathbb{1}_{[0,\infty)} * \eta = \varphi\mathbb{1}_{[0,\infty)} * [\psi\mathbb{1}_{[0,\infty)} * \xi] = [\varphi\mathbb{1}_{[0,\infty)} * \psi\mathbb{1}_{[0,\infty)}] * \xi = \vartheta\mathbb{1}_{[0,\infty)} * \xi.$$

Hence $S(\varphi)S(\psi)S(\xi) = S(\upsilon\mathbb{1}_{[0,\infty)} * \xi)$ and, once more by (3.9),

$$S(\varphi)S(\psi)S(\xi) = S(\vartheta)S(\xi).$$

Since this holds for every $\xi \in \mathcal{D}((-\infty,\infty))$ such that $\operatorname{supp}\xi \subset [0,\infty)$, it follows that $S(\varphi)S(\psi)y = S(\vartheta)y$ for every $y \in \Im_0$. By (3.3), this implies that $S(\varphi)S(\psi) = S(\vartheta)$. Thus (2.12) is satisfied, and $S \in \mathrm{DS}(L(X))$. ■

Proof of (D). Let $S = T + \delta \otimes \mathrm{id}_X$, where $T \in \mathrm{RDS}(L(X))$. Then $S \in \mathcal{D}'_0((-\infty,\infty); L(X))$ and S satisfies (3.1), (3.3) and (3.4). However if $\varphi, \psi \in \mathcal{D}((-\infty,\infty))$, $\operatorname{supp}\psi \subset [0,\infty)$, and $\vartheta = \varphi \mathbb{1}_{[0,\infty)} * \psi$, then, by (3.9) applied to T, we have $S(\varphi)S(\psi) = [T(\varphi) + \varphi(0)\mathrm{id}_X]T(\psi) = T(\vartheta) + \varphi(0)T(\psi) = S(\vartheta) + \varphi(0)S(\psi)$, and from (3.3) it follows that S cannot satisfy (2.12). ∎

Proof of (E). Take a $T \in \mathrm{DS}(L(X))$ for which $\mathcal{N}_0 = \{0\}$ and $\mathfrak{I}$ is not dense in X ([10]). Following Kunstmann [K], lines 846^{22-25}, choose $x_0 \in X \setminus \overline{\mathfrak{I}}$ and $x_0^* \in X^*$ so that $\langle x_0^*, x_0\rangle = 1$ and $\langle x_0^*, y\rangle = 0$ for every $y \in \mathfrak{I}$. Define $P \in L(X)$ by $Px = \langle x_0^*, x\rangle x_0$. Then $Px_0 = x_0$ and $Py = 0$ whenever $y \in \mathfrak{I}$. The formula

$$S = T + \delta \otimes P$$

determines a distribution $S \in \mathcal{D}'_0((-\infty,\infty); L(X)$ which obviously satisfies (3.1) and (3.4), because T satisfies these conditions. Furthermore, if $\varphi, \psi \in \mathcal{D}((-\infty,\infty))$ and $\operatorname{supp}\psi \subset [0,\infty)$, then $S(\varphi)S(\psi) = [T(\varphi) + \varphi(0)P]T(\psi) = T(\varphi)T(\psi)$, whence it follows that S satisfies (3.2), because T satisfies (3.2), by (A).

However S does not belong to $\mathrm{DS}(L(X))$, because if $\varphi, \psi, \vartheta \in \mathcal{D}((-\infty,\infty))$ and $\vartheta|_{[0,\infty)} = \varphi|_{[0,\infty)} * \psi|_{[0,\infty)}$ (in the sense of convolution on $[0,\infty)$), then $\vartheta(0) = 0$, so that $S(\vartheta) = T(\vartheta)$, and $S(\varphi)S(\psi) = T(\varphi)T(\psi) + \psi(0)T(\varphi)P + \varphi(0)\psi(0)P$, so that $P[S(\varphi)S(\psi) - S(\vartheta)] = \varphi(0)\psi(0)P$. This implies that if $\varphi(0)\psi(0) \neq 0$, then $S(\varphi)S(\psi) \neq S(\vartheta)$. Therefore S does not satisfy (2.12). ∎

4. The generator of a distribution semigroup and the associated pseudoresolvent

4.A. *The logarithmic regions and the results of Lions, Foiaş and Chazarain.* For any $a \in [0,\infty)$ and $b \in (-\infty,\infty)$ define

$$\Lambda_{a,b} = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq a\log(1+|\lambda|) + b\},$$
$$\widetilde{\Lambda}_{a,b} = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq a\log(1+|\operatorname{Im}\lambda|) + b\}.$$

Following [C], [F] and [F-Y] we will call these sets the *logarithmic regions.*

4.1. Lemma. *Let $a \in [0,\infty)$, $b \in (0,\infty)$, $\Theta = b^{-1}\log(1+b)$. Then $\Theta \in (0,1)$ and*

$$\Lambda_{a,b} \subset \widetilde{\Lambda}_{a,b} \subset \Lambda_{a(1+\Theta a)^{-1},\, b(1+\Theta a)^{-1}}.$$

Proof. The left inclusion is obvious. For the proof of the right one, notice that the function

$$\xi \to \xi^{-1}\log(1+\xi) = \frac{1}{\xi}\int_0^{\xi} \frac{1}{1+\tau}\,d\tau$$

([10]) Such distribution semigroups T exist: it is sufficient to define $T(\varphi) = T^{+}(\varphi|_{[0,\infty)})$ for $\varphi \in \mathcal{D}((-\infty,\infty))$, where T^{+} is a continuous representation of the convolution algebra $L^1([0,\infty))$ on the Banach space X, corresponding to a *resolvent* whose image is not dense in X. See [Ki], Theorem 12.5. Example: $X = C[0,1]$, $R_\lambda = (\lambda - A)^{-1}$ for $\lambda > 0$, where $\mathcal{D}(A) = \{f \in C^2[0,1] : f(0) = f(1) = 0\}$, $Af = f''$ for $f \in \mathcal{D}(A)$.

decreases on $(0,\infty)$ from 1 to 0. It follows that $\Theta \in (0,1)$ and $\log(1+\xi) \le \Theta\xi$ for $\xi \in [b,\infty)$, so that

$$\begin{aligned}\widetilde{\Lambda}_{a,b} &\subset \{\xi + i\eta \in \mathbb{C} : \xi \ge a\log(1+|\eta|) + a\log(1+\xi) - a\Theta\xi + b\}\\ &= \{\xi + i\eta \in \mathbb{C} : (1+\Theta a)\xi \ge a\log((1+\xi)(1+|\eta|)) + b\}\\ &\subset \Lambda_{a(1+\Theta a)^{-1},b(1+\Theta a)^{-1}}. \ \blacksquare\end{aligned}$$

The sets $\widetilde{\Lambda}_{a,b}$ appeared in connection with distribution semigroups in 1960 in the paper of C. Foiaş [Fo], where a set $\Lambda \subset \mathbb{C}$ is said to be *logarithmic of degree* 1 if $\Lambda \subset \overline{\mathbb{C}\backslash\widetilde{\Lambda}_{a,b}}$ for some a,b. Foiaş proved that, in the Hilbert space setting, there is a one-to-one correspondence between the regular distribution semigroups S of normal operators, and the spectral measures $\mathcal{E}$ with supports logarithmic of degree 1, such that the infinitesimal generator A of S has the spectral representation

$$A = \int \lambda \, d\mathcal{E}(\lambda),$$

and

$$S(\varphi) = \int \widehat{\varphi}(i\lambda)\, d\mathcal{E}(\lambda) \qquad \text{for every } \varphi \in \mathcal{D}((-\infty,\infty)),$$

where $\widehat{\varphi}$ is the Fourier transform of φ.

From the results of J. L. Lions [L], 1960, Theorem 4.1, p. 147, and J. Chazarain [C], 1971, Theorem 1.6, p. 394, and Sec. I.3, pp. 397–400, it follows that if X is a complex Banach space then there is a one-to-one correspondence between the *regular* distribution semigroups S with values in $L(X)$ and the resolvents $R_\bullet$ with values in $L(X)$, defined on some logarithmic regions $\Lambda_{a,b}$ and having polynomial growth. This correspondence is given by

$$R_\lambda = (\lambda - A)^{-1},$$

where A is the infinitesimal generator of S, and

$$S(\varphi) = \frac{1}{2\pi i}\int_{\mathcal{C}} \widehat{\varphi}(i\lambda) R_\lambda \, d\lambda \qquad \text{for every } \varphi \in \mathcal{D}((-\infty,\infty)),$$

where $\mathcal{C}$ is the oriented boundary of $\Lambda_{a,b}$.

Our purpose is to extend these last formulas to arbitrary distribution semigroups S in the sense of Definition 2.3. In this general situation the definition of the *infinitesimal* generator of S, due to Lions [L], does not work, and the generator (no longer infinitesimal) must be defined in another way.

4.B. *Notation.* In what follows, X is a Banach space, and $c \in (0,\infty]$ is arbitrarily fixed. $\mathrm{DS}((-\infty,c);L(X))$ denotes the set of $L(X)$-valued (local) distribution semigroups in the sense of Definition 2.3 or 2.5, according as $c = \infty$ or $c \in (0,\infty)$. For any $S \in \mathrm{DS}((-\infty,c);L(X))$ we define its range $\mathfrak{I}$ and null space $\mathcal{N}$ by

$$\begin{aligned}\mathfrak{I} &= \Big\{\sum_{k=1}^{n} S(\psi_k)x_k : n \in \mathbb{N},\ \psi_k \in \mathcal{D}((-\infty,c)),\ x_k \in X\Big\},\\ \mathcal{N} &= \bigcap\{\mathcal{N}(S(\varphi)) : \varphi \in \mathcal{D}((-\infty,c))\},\end{aligned}$$

where $\mathcal{N}(S(\varphi))$ is the null space of $S(\varphi)$. The canonical epimorphism of X onto the factor space $X/\mathcal{N}$ is denoted by E. Since $\mathcal{N}$ is a closed subspace of X, it follows that $X/\mathcal{N}$ is a Banach space, $E \in L(X; X/\mathcal{N})$ and $\|E\| = 1$.

4.C. *The generator of a distribution semigroup*

4.2. Propostion. *For every $S \in \mathrm{DS}(-\infty, c); L(X))$ there is a unique linear map G from X into $X/\mathcal{N}$ with domain $\mathcal{D}(G)$ such that*

$$\mathcal{D}(G) = \mathfrak{I}, \tag{4.1}$$

and

$$GS(\psi)x = -E[S(D\psi)x + \psi(0)x] \quad \textit{for every } \psi \in \mathcal{D}((-\infty, c)) \textit{ and } x \in X. \tag{4.2}$$

Proof. Whenever $\sum_{k=1}^n S(\psi_k)x_k = 0$ for some $n \in \mathbb{N}$, $\psi_k \in \mathcal{D}((-\infty, c))$ and $x_k \in X$, then $E\sum_{k=1}^n [S(D\psi_k)x_k + \psi_k(0)x_k] = 0$ because, by (2.13), for every $\varphi \in \mathcal{D}((-\infty, c))$,

$$S(\varphi)\sum_{k=1}^n [S(D\psi_k)x_k + \psi_k(0)x_k] = [S(D\varphi) + \varphi(0)]\sum_{k=1}^n S(\psi_k)x_k. \ \blacksquare$$

4.3. Definition. The linear map G determined by (4.1)–(4.2) will be called the *generator* of the (local) distribution semigroup S.

4.4. Proposition. *If $S \in \mathrm{DS}((-\infty, c); L(X))$ and G is the generator of S, then for every $(x, y) \in X \times X$ the following two conditions are equivalent*:

$$x \in \mathfrak{I} \quad \textit{and} \quad Ey = Gx, \tag{4.3}$$

$$S(\varphi)y = -S(D\varphi)x - \varphi(0)x \quad \textit{for every } \varphi \in \mathcal{D}((-\infty, c)). \tag{4.4}$$

Proof. If $(x, y) \in \mathfrak{I} \times X$ and $Ey = Gx$, then $x = \sum_{k=1}^n S(\psi_k)z_k$ and $Ey = -E\sum_{k=1}^n [S(D\psi_k)z_k + \psi_k(0)z_k]$ for some $n \in \mathbb{N}$, $\psi_k \in \mathcal{D}((-\infty, c))$ and $z_k \in X$, whence, by (2.13),

$$\begin{aligned} S(\varphi)y &= -S(\varphi)\sum_{k=1}^n [S(D\psi_k)z_k + \psi_k(0)z_k] = -[S(D\varphi) + \varphi(0)]\sum_{k=1}^n S(\psi_k)z_k \\ &= -S(D\varphi)x - \varphi(0)x \end{aligned}$$

for every $\varphi \in \mathcal{D}((-\infty, \infty))$. Hence (4.3) implies (4.4).

Suppose now that (4.4) holds for some $(x, y) \in X \times X$. Then $x \in \mathfrak{I}$, because $x = -S(D\varphi)x - S(\varphi)y$ whenever $\varphi \in \mathcal{D}((-\infty, c))$ and $\varphi(0) = 1$. Furthermore, if $(x, y) \in \mathfrak{I} \times X$ satisfies (4.4) and $x = \sum_{k=1}^n S(\psi_k)z_k$, where $\psi_k \in \mathcal{D}((-\infty, c))$ and $z_k \in X$, then, by (2.13), for every $\varphi \in \mathcal{D}((-\infty, c))$ one has

$$S(\varphi)y = -S(D\varphi)x - \varphi(0)x = -S(\varphi)\sum_{k=1}^n [S(D\psi_k)z_k + \psi_k(0)z_k] = -S(\varphi)Gx.$$

It follows that $y + Gx \in \mathcal{N}$, i.e. $Ey = Gx$. Hence (4.4) implies (4.3). $\blacksquare$

4.5. Corollary. *For every $S \in \mathrm{DS}((-\infty, c); L(X))$ its generator is a closed linear operator from X into $X/\mathcal{N}$.*

Proof. By Proposition 4.4, the graph of the generator is equal to

$$\{(x, Ey) : x \in X,\ y \in X,\ S(\varphi)y = -S(D\varphi)x - \varphi(0)x \text{ for every } \varphi \in \mathcal{D}((-\infty, c))\},$$

which is a closed subset of $X \times X/\mathcal{N}$. ■

4.D. *Some characterizations of the generator under additional assumptions about the distribution semigroup*

4.6. *A characterization of the generator of a distribution semigroup satisfying* (3.4). Assume that $S \in \mathrm{DS}(L(X))$ satisfies (3.4). Then $\mathcal{N} = \mathcal{N}_0 = \{0\}$, so that $X/\mathcal{N} = X$ and $E = \mathrm{id}_X$. Let G be the generator of S in the sense of Definition 4.3. Then $\mathcal{D}(G) = \mathfrak{I}$. Furthermore, for every $(x, y) \in X \times X$ the following two conditions are equivalent:

(4.5) $$x \in \mathcal{D}(G) \quad \text{and} \quad y = Gx,$$

(4.6) $S(\psi)y = -S(D\psi)x$ for every $\psi \in \mathcal{D}((-\infty, \infty))$ such that $\operatorname{supp} \psi \subset [0, \infty)$.

Indeed, (4.5) implies (4.6), by Proposition 4.4. On the other hand, if (4.6) is satisfied for some $(x, y) \in X \times X$, then, by (2.13), for every $\varphi \in \mathcal{D}((-\infty, \infty))$ and every $\psi \in \mathcal{D}((-\infty, \infty))$ such that $\operatorname{supp} \psi \subset [0, \infty)$ one has

$$S(\psi)S(\varphi)y = S(\varphi)S(\psi)y = -S(\varphi)S(D\psi)x = -S(\psi)[S(D\varphi)x + \varphi(0)x].$$

Hence $S(\varphi)y + S(D\varphi)x + \varphi(0)x \in \mathcal{N}_0(S)$ for every $\varphi \in \mathcal{D}((-\infty, \infty))$. Since $\mathcal{N}_0(S) = \{0\}$, it follows that (x, y) satisfies (4.4), so that, by Proposition 4.4, (x, y) satisfies (4.5). ■

The equivalence (4.5)$\Leftrightarrow$(4.6) was used by R. Shiraishi and Y. Hirata [S-H] as the definition of the generator. The same definition was adopted by P. C. Kunstmann [K]. In both these papers the distribution semigroups satisfy (3.4).

4.7. *The infinitesimal generator of a regular distribution semigroup.* If $S \in \mathrm{RDS}(L(X))$ and Φ is a scalar distribution on $(-\infty, \infty)$ with compact support contained in $[0, \infty)$, then, according to J. L. Lions [L], pp. 143–145, there is a unique linear operator $S(\Phi)$ from X into X with domain $\mathcal{D}(S(\Phi))$ such that for every $(x, y) \in X \times X$ the following two conditions are equivalent:

(4.7) $x \in \mathcal{D}(S(\Phi))$ and $y = S(\Phi)x$,

(4.8) there is a regularizing sequence δ_n, $n = 1, 2, \ldots$, of functions in $\mathcal{D}((-\infty, \infty))$ with supports contained in $[0, \infty)$ such that

$$\lim_{n\to\infty} \|S(\delta_n)x - x\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|S(\delta_n * \Phi)x - y\| = 0.$$

Lemma 2.3 on p. 145 of [L] says that the operator $S(\Phi)$ is closable. According to Definition 2.3 on p. 145 of [L], the *infinitesimal generator* of an $S \in \mathrm{RDS}(L(X))$ is equal to the smallest closed extension $\overline{S(-\delta')}$ of the operator $S(-\delta')$, where δ' is the derivative of the Dirac distribution.

4.8. PROPOSITION. *Suppose that* $S \in \mathrm{RDS}(L(X))$. *Let* G *be the generator of* S *in the sense of Definition* 4.3. *Then* $G = \overline{S(-\delta')}$.

Proof of $\overline{S(-\delta')} \subset G$. Since, by Corollary 4.5, the operator G is closed, it is sufficient to show that $S(-\delta') \subset G$. To this end, suppose that $x \in \mathcal{D}(S(-\delta'))$ and $y = S(-\delta')x$. Take a regularizing sequence δ_n, $n = 1, 2, \ldots$, satisfying (4.8) for $\Phi = -\delta'$. Let $x_n = S(\delta_n)x$, $y_n = S(-\delta_n * \delta')x$. Then $y_n = -S(D\delta_n)x = Gx_n$, because $\delta_n(0) = 0$. Since $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Gx_n = \lim_{n\to\infty} y_n = y$, and since G is closed, it follows that $x \in \mathcal{D}(G)$ and $y = Gx$. ■

Proof of $G \subset \overline{S(-\delta')}$. We have to show that $\mathfrak{I} = \mathcal{D}(G) \subset \mathcal{D}(\overline{S(-\delta')})$ and $\overline{S(-\delta')}x = Gx$ for every $x \in \mathfrak{I}$. These conditions mean that

$$S(\varphi)z \in \mathcal{D}(\overline{S(-\delta')}) \quad \text{and} \quad \overline{S(-\delta')}S(\varphi)z = -S(D\varphi)z - \varphi(0)z \tag{4.9}$$
$$\text{for every } \varphi \in \mathcal{D}((-\infty,\infty)) \text{ and every } z \in X,$$

which corresponds to (4.10) and (4.11) on p. 148 of the paper of J. L. Lions [L] (where these last formulas constitute an essential step in the proof of Theorem 4.1). For completeness, we will prove (4.9) following [L].

LEMMA. *If* $x = S(\varphi)S(\psi)z$, *where* $\varphi \in \mathcal{D}((-\infty,\infty))$, $\psi \in \mathcal{D}((-\infty,\infty))$, $z \in X$ *and* $\operatorname{supp}\psi \subset [0,\infty)$, *then* $x \in \mathcal{D}(S(-\delta'))$ *and* $S(-\delta')x = Gx$.

Proof of the Lemma. Put $\vartheta = \varphi \mathbb{1}_{[0,\infty)} * \psi$. Then $\vartheta \in \mathcal{D}((-\infty,\infty))$, $\operatorname{supp}\vartheta \subset [0,\infty)$, and $x = S(\vartheta)z$, by (3.9). It follows that $Gx = -S(D\vartheta)z$. Let δ_n, $n = 1, 2, \ldots$, be any regularizing sequence of functions in $\mathcal{D}((-\infty,\infty))$ with supports in $[0,\infty)$. Then $S(\delta_n)x = S(\delta_n)S(\vartheta)z = S(\delta_n * \vartheta)z$, so that $\lim_{n\to\infty} S(\delta_n)x = S(\vartheta)z = x$, and

$$S(-\delta_n * \delta')x = -S(D\delta_n)x = -S(D\delta_n)S(\vartheta)z = -S(D\delta_n * \vartheta)z = -S(\delta_n * D\vartheta)x,$$

so that $\lim_{n\to\infty} S(-\delta_n * \delta')x = -S(D\vartheta)z = Gx$. This means that $x \in \mathcal{D}(S(-\delta'))$ and $S(-\delta')x = Gx$. ■

Proof of (4.9). Take any $\varphi \in \mathcal{D}((-\infty,\infty))$ and $z \in X$. By (3.3), there is a sequence ψ_n, $n = 1, 2, \ldots$, of functions in $\mathcal{D}((-\infty,\infty))$ with supports in $[0,\infty)$, and a sequence z_n, $n = 1, 2, \ldots$, of elements of X such that $\lim_{n\to\infty} S(\psi_n)z_n = z$. By the Lemma, for every $n = 1, 2, \ldots$ one has $x_n = S(\varphi)S(\psi_n)z_n \in \mathcal{D}(S(-\delta'))$ and

$$S(-\delta')x_n = Gx_n = -S(D\varphi)S(\psi_n)z_n - \varphi(0)S(\psi_n)z_n.$$

It follows that $\lim_{n\to\infty} x_n = S(\varphi)z$ and $\lim_{n\to\infty} S(-\delta')x_n = -S(D\varphi)z - \varphi(0)z$, which implies (4.9). ■

4.E. *Uniqueness of the distribution semigroup with given generator.* Let $S \in \mathrm{DS}((-\infty,c); L(X))$ and let $G : \mathfrak{I} \to X/\mathcal{N}$ be the generator of S. According to Corollary 4.5, G is a closed linear operator from X into $X/\mathcal{N}$, so that its domain $\mathfrak{I}$ equipped with the norm

$$\|x\|_{\mathfrak{I}} = \|x\|_X + \|G\|_{X/\mathcal{N}}, \qquad x \in \mathfrak{I}, \tag{4.10}$$

is a Banach space.

4.9. Proposition. (A) *For every* $S \in \mathrm{DS}((-\infty, c); L(X))$ *there is a unique* $\widetilde{S} \in \mathcal{D}'_0((-\infty, c); L(X/\mathcal{N}; \mathfrak{I}))$ *such that*

$$\widetilde{S}(\varphi)E = S(\varphi) \quad \text{for every } \varphi \in \mathcal{D}((-\infty, c)). \tag{4.11}$$

This $\widetilde{S}$ *satisfies for every* $\varphi \in \mathcal{D}((-\infty, c))$ *the equations*

$$-E|_{\mathfrak{I}}\widetilde{S}(D\varphi) - G\widetilde{S}(\varphi) = \varphi(0)\,\mathrm{id}_{X/\mathcal{N}}, \tag{4.12}$$

$$-\widetilde{S}(D\varphi)E|_{\mathfrak{I}} - \widetilde{S}(\varphi)G = \varphi(0)\,\mathrm{id}_{\mathfrak{I}}\,. \tag{4.13}$$

(B) *Suppose that* $\mathcal{N}$ *is a closed subspace of the Banach space* X, *and* G *is a closed linear operator from* X *into* $X/\mathcal{N}$ *with domain* $\mathfrak{I}$. *Then* $\mathfrak{I}$ *equipped with the norm* (4.10) *is a Banach space. Suppose that* $S \in \mathcal{D}'_0((-\infty, c); L(X))$ *and* $\widetilde{S} \in \mathcal{D}'_0((-\infty, c); L(X/\mathcal{N}; \mathfrak{I}))$ *satisfy* (4.11)–(4.13). *Then* $S \in \mathrm{DS}((-\infty, c); L(X))$ *and* G *coincides with the generator of* S.

Proof of (A). Let $\varphi \in \mathcal{D}((-\infty, c))$. Then $\mathcal{N}(S(\varphi)) \supset \mathcal{N}$, and hence there is a unique operator $\widetilde{S}(\varphi) \in L(X/\mathcal{N}; X)$ satisfying (4.11). The definition of the norm in $X/\mathcal{N}$ implies that

$$\|\widetilde{S}(\varphi)\|_{L(X/\mathcal{N};X)} = \|S(\varphi)\|_{L(X)}. \tag{4.14}$$

One has $\mathfrak{I}(\widetilde{S}(\varphi)) = \mathfrak{I}(S(\varphi)) \subset \mathfrak{I}$ and, by (4.2) and (4.11), $-E|_{\mathfrak{I}}\widetilde{S}(D\varphi)E - G\widetilde{S}(\varphi)E = \varphi(0)E$, whence (4.12) follows. Since $\widetilde{S}(D\varphi) \in L(X/\mathcal{N}; X)$, $E \in L(X; X/\mathcal{N})$ and, by (4.12), $G\widetilde{S}(\varphi) = -\varphi(0)\,\mathrm{id}_{X/\mathcal{N}} - E\widetilde{S}(D\varphi)$, it follows that $G\widetilde{S}(\varphi) \in L(X/\mathcal{N})$. Thus, by (4.10), $\widetilde{S}(\varphi) \in L(X/\mathcal{N}; \mathfrak{I})$ and $\|\widetilde{S}(\varphi)\|_{L(X/\mathcal{N};\mathfrak{I})} \le \|\widetilde{S}(\varphi)\|_{L(X/\mathcal{N};X)} + \|G\widetilde{S}(\varphi)\|_{L(X/\mathcal{N})}$. By (4.11) and (4.14), it follows that

$$\begin{aligned}
\|\widetilde{S}(\varphi)\|_{L(X/\mathcal{N};\mathfrak{I})} &\le \|S(\varphi)\|_{L(X)} + \|E\widetilde{S}(D\varphi) + \varphi(0)\,\mathrm{id}_{X/\mathcal{N}}\|_{L(X/\mathcal{N})} \\
&\le \|S(\varphi)\|_{L(X)} + |\varphi(0)| + \|\widetilde{S}(D\varphi)\|_{L(X/\mathcal{N};X)} \\
&= \|S(\varphi)\|_{L(X)} + |\varphi(0)| + \|S(D\varphi)\|_{L(X)}.
\end{aligned}$$

This estimate shows that the map $\mathcal{D}((-\infty, c)) \ni \varphi \to \widetilde{S}(\varphi) \in L(X/\mathcal{N}; \mathfrak{I})$ is continuous, so that $\widetilde{S} \in \mathcal{D}'((-\infty, c); L(X/\mathcal{N}, \mathfrak{I}))$. From (4.11) it is evident that $\operatorname{supp} \widetilde{S} = \operatorname{supp} S \subset [0, c)$, and hence $\widetilde{S} \in \mathcal{D}'_0((-\infty, c); L(X/\mathcal{N}; \mathfrak{I}))$. Finally, take any $x \in \mathfrak{I}$ and choose $y \in X$ so that $Ey = Gx$. By Proposition 4.4 one then has

$$\widetilde{S}(\varphi)Gx = \widetilde{S}(\varphi)Ey = S(\varphi)y = -S(D\varphi)x - \varphi(0)x = -\widetilde{S}(D\varphi)Ex - \varphi(0)x,$$

which shows that (4.13) is satisfied. ∎

Proof of (B). By (4.11)–(4.13) for every φ and ψ in $\mathcal{D}((-\infty,c))$ one has

$$\begin{aligned}S(\varphi)[S(D\psi)+\psi(0)\,\mathrm{id}_X] &= \widetilde{S}(\varphi)[E\widetilde{S}(D\psi)+\psi(0)\,\mathrm{id}_{X/\mathcal{N}}]E = -\widetilde{S}(\varphi)[G\widetilde{S}(\psi)]E\\ &= -[\widetilde{S}(\varphi)G]S(\psi) = [\widetilde{S}(D\varphi)E|_{\mathfrak{I}}+\varphi(0)\,\mathrm{id}_{\mathfrak{I}}]S(\psi)\\ &= [S(D\varphi)+\varphi(0)\,\mathrm{id}_X]S(\psi).\end{aligned}$$

Thus S satisfies (2.13), whence $S\in \mathrm{DS}((-\infty,c);L(X))$, by Theorem 2.6. ∎

4.10. Theorem. *Let X be a Banach space, $\mathcal{N}$ a closed subspace of X, and G a closed linear operator from X into the factor space $X/\mathcal{N}$. Then for every $c\in(0,\infty]$ there is at most one $S\in \mathrm{DS}((-\infty,c);L(X))$ whose generator is G.*

Proof. Denote by $\mathfrak{I}$ the domain of G and suppose that $S_i\in \mathrm{DS}((-\infty,c);L(X))$, $i=1,2$, both have the same generator $G:\mathfrak{I}\to X/\mathcal{N}$. This implies that $\bigcap_{\varphi\in\mathcal{D}((-\infty,c))}\mathcal{N}(S_i(\varphi))=\mathcal{N}$ and $\{\sum_{k=1}^n S_i(\psi_k)x_k : n\in\mathbb{N},\ \psi_k\in\mathcal{D}((-\infty,c)),\ x_k\in X\}=\mathfrak{I}$ for $i=1,2$. Let $\widetilde{S}_i\in\mathcal{D}_0'((-\infty,c);L(X/\mathcal{N};\mathfrak{I}))$ be the distributions such that $\widetilde{S}_i(\varphi)E=S_i(\varphi)$ for every $\varphi\in\mathcal{D}((-\infty,c))$. The partition of unity theorem implies that every $\varphi\in\mathcal{D}((-\infty,c))$ may be represented as a finite sum $\varphi=\sum_k(\varphi_k*\delta_{t_k})|_{(-\infty,c)}$ where $*$ denotes convolution on $(-\infty,\infty)$ and, for every k, $\varphi_k\in\mathcal{D}((-\infty,\infty))$, $\operatorname{supp}\varphi_k\subset(-c/2,c/2)$, $t_k\in(-\infty,c/2)$ and δ_{t_k} is the unit mass concentrated at t_k. Since $\operatorname{supp}\widetilde{S}_i\subset[0,c)$, one has

$$\widetilde{S}_i(\varphi)=\sum_{t_k\in(-c/2,c/2)}\widetilde{S}_i((\varphi_k*\delta_{t_k})|_{(-\infty,c)}).$$

By regularization of the δ_{t_k}'s, it follows that for the proof of $S_1=S_2$ it is sufficient to show that

$$\widetilde{S}_1((\varphi*\psi)|_{(-\infty,c)})=\widetilde{S}_2((\varphi*\psi)|_{(-\infty,c)}) \tag{4.15}$$

for every φ and ψ in $\mathcal{D}((-\infty,\infty))$ such that $\operatorname{supp}\varphi\subset(-c/2,c/2)$ and $\operatorname{supp}\psi\subset(-c/2,c/2)$. So, take such φ and ψ, and, for $t\in(-\infty,\infty)$, denote by φ_t and ψ_t their left translates by t ($\varphi_t(u)=\varphi(u+t)$, $\psi_t(u)=\psi(u+t)$). Then

$$(\varphi*\psi)|_{(-\infty,c)}=\int_{-c/2}^{c/2}\psi(-t)\varphi_t|_{(-\infty,c)}\,dt=\int_{-c/2}^{c/2}\varphi(t)\psi_{-t}|_{(-\infty,c)}\,dt,$$

where the Riemann integrals of C^∞ functions with values in $\mathcal{D}((-\infty,c))$ make sense, because $\mathcal{D}((-\infty,c))$ is a complete l.c.v.s. ([Y], pp. 105 and 237; [S], p. 66, Theorem 1; [T], p. 129, Theorem 13.1). Since $\operatorname{supp}\varphi_{c/2}\subset(-c,0)$, $\operatorname{supp}\psi_{c/2}\subset(-c,0)$ and $\operatorname{supp}\widetilde{S}_i\subset[0,c)$, one has $\widetilde{S}_i(\varphi_{c/2})=\widetilde{S}_i(\psi_{c/2})=0$.

In the calculations which follow, let us write $\widetilde{S}_i(\vartheta)$ instead of $\widetilde{S}_i(\vartheta)|_{(-\infty,c)})$, provided that $\vartheta\in\mathcal{D}((-\infty,\infty))$ and $\operatorname{supp}\vartheta\subset(-\infty,c)$. This abbreviation is applied to φ_t and ψ_{-t} whenever $t\in[-c/2,c/2]$. Since, for $t\in[-c/2,c/2]$,

$$\frac{d}{dt}[\widetilde{S}_1(\varphi_t)E|_{\mathfrak{I}}\widetilde{S}_2(\psi_{-t})]=\widetilde{S}_1(D\varphi_t)E|_{\mathfrak{I}}\widetilde{S}_2(\psi_{-t})-\widetilde{S}_1(\varphi_t)E|_{\mathfrak{I}}\widetilde{S}_2(D\psi_t),$$

it follows that

$$(4.16)\qquad \int_{-c/2}^{c/2} \widetilde{S}_1(D\varphi_t)E|_{\mathfrak{I}}\widetilde{S}_2(\psi_{-t})\,dt = \int_{-c/2}^{c/2} \widetilde{S}_1(\varphi_t)E|_{\mathfrak{I}}\widetilde{S}_2(D\psi_{-t})\,dt,$$

where the integrals of C^∞ functions with values in $L(X/\mathcal{N};\mathfrak{I})$ are computed in the sense of Riemann. From (4.12) applied to $\widetilde{S}_2$, (4.16), and (4.13) applied to $\widetilde{S}_1$, one obtains

$$\begin{aligned}
\widetilde{S}_1((\varphi*\psi)_{(-\infty,c)}) &= \widetilde{S}_1\Big(\int_{-c/2}^{c/2}\psi(-t)\varphi_t|_{(-\infty,c)}\,dt\Big)\\
&= \int_{-c/2}^{c/2}\widetilde{S}_1(\varphi_t)[\psi_{-t}(0)\,\mathrm{id}_{X/\mathcal{N}}]\,dt\\
&= -\int_{-c/2}^{c/2}\widetilde{S}_1(\varphi_t)[E|_{\mathfrak{I}}\widetilde{S}_2(D\psi_{-t}) + G\widetilde{S}_2(\psi_{-t})]\,dt\\
&= -\int_{-c/2}^{c/2}[\widetilde{S}_1(D\varphi_t)E|_{\mathfrak{I}} + \widetilde{S}_1(\varphi_t)G]\widetilde{S}_2(\psi_{-t})\,dt\\
&= \int_{-c/2}^{c/2}[\varphi_t(0)\,\mathrm{id}_{\mathfrak{I}}]\widetilde{S}_2(\psi_{-t})\,dt = \widetilde{S}_2\Big(\int_{-c/2}^{c/2}\varphi(t)\psi_{-t}|_{(-\infty,c)}\,dt\Big)\\
&= \widetilde{S}_2((\varphi*\psi)|_{(-\infty,c)}),
\end{aligned}$$

which is (4.15). ∎

4.11. Remarks. The equalities (4.12) and (4.13) mean that

$$E|_{\mathfrak{I}}(D\widetilde{S}) - G\widetilde{S} = \delta\otimes\mathrm{id}_{X/\mathcal{N}},\qquad (D\widetilde{S})E|_{\mathfrak{I}} - \widetilde{S}G = \delta\otimes\mathrm{id}_{\mathfrak{I}},$$

where $D\widetilde{S}$ is the derivative of $\widetilde{S}$ in the sense of the theory of distributions. If $c=\infty$, then rewriting the above in the form

$$(D\delta\otimes E|_{\mathfrak{I}} - \delta\otimes G)*\widetilde{S} = \delta\otimes\mathrm{id}_{X/\mathcal{N}},\qquad \widetilde{S}*(D\delta\otimes E|_{\mathfrak{I}} - \delta\otimes G) = \delta\otimes\mathrm{id}_{\mathfrak{I}},$$

one concludes that the distributions $\widetilde{S}\in\mathcal{D}_0'((-\infty,\infty);L(X/\mathcal{N};\mathfrak{I}))$ and $D\delta\otimes E|_{\mathfrak{I}} - \delta\otimes G\in\mathcal{D}_0'((-\infty,\infty);L(\mathfrak{I};X/\mathcal{N}))$ are mutually inverse with respect to convolution on $(-\infty,\infty)$. See [L], p. 147, Theorem 4.1; [F], p. 480 and p. 483, Theorem 8.4.1; [K], p. 844, Definition 3.9. In other words, the $L(X/\mathcal{N};\mathfrak{I})$-valued distribution $\widetilde{S}$ with support contained in $[0,\infty)$ is a *fundamental solution* (or "elementary solution") for the differential operator $E|_{\mathfrak{I}}D - G$ whose coefficients are the operators $E|_{\mathfrak{I}}$ and G belonging to $L(\mathfrak{I};X/\mathcal{N})$. See [L], p. 149, Corollary 4.1; [C], p. 392, Definition 1.1 and Proposition 1.2; [F], pp. 483–484; [K], p. 844, Theorem 3.10.

If $c=\infty$, then the proof of Theorem 4.10 becomes algebraically more transparent when (as in the case of Corollary 4.1 on p. 149 of [L] or Lemma 8.4.2 on p. 484 of [F]) a more advanced argument is used, based on the associativity of convolution of vector-valued distributions on $(-\infty,\infty)$ with supports bounded from the left. The required associativity follows from Proposition 39 on pp. 167–168 of [S;II] or formula (8.2.7) on p. 470 of [F]. The scheme of the proof of Theorem 4.10 (in the case of $c=\infty$) based on the associativity of convolution is as follows. Since $S_i \in \mathrm{DS}(L(X))$, $i=1,2$, both have the same generator $G:\mathfrak{I}\to X/\mathcal{N}$, it follows that the distribution $T = D\delta\otimes E|_{\mathfrak{I}} - \delta\otimes G \in \mathcal{D}'((-\infty,\infty);L(\mathfrak{I};X/\mathcal{N}))$ is the common convolution inverse for both the distributions $\widetilde{S}_i \in \mathcal{D}'_0((-\infty,\infty);L(X/\mathcal{N};\mathfrak{I}))$, $i=1,2$. Thus $\widetilde{S}_1 * T = \delta\otimes \mathrm{id}_{\mathfrak{I}}$ and $T*\widetilde{S}_2 = \delta\otimes\mathrm{id}_{X/\mathcal{N}}$, whence $\widetilde{S}_1 = \widetilde{S}_1 * (\delta\otimes\mathrm{id}_{X/\mathcal{N}}) = \widetilde{S}_1 * (T*\widetilde{S}_2) = (\widetilde{S}_1 * T)*\widetilde{S}_2 = (\delta\otimes\mathrm{id}_{\mathfrak{I}})*\widetilde{S}_2 = \widetilde{S}_2$, by associativity.

If $c=\infty$, then (similarly to part 3 of the proof of Theorem 5.1 in [L], pp. 150–152, and the proof of Theorem 3.10 in [K], p. 844) Proposition 4.9(B) may be deduced from the uniqueness of the solution of the equation

$$(D\delta\otimes E|_{\mathfrak{I}} - \delta\otimes G)*u = f, \tag{4.17}$$

in which $f\in\mathcal{D}'_0((-\infty,\infty);X/\mathcal{N})$ is given, and $u\in\mathcal{D}'_0((-\infty,\infty);\mathfrak{I})$ is unknown. This uniqueness follows from the fact that if $u\in\mathcal{D}'_0((-\infty,\infty);\mathfrak{I})$ is a solution of (4.17), and $\widetilde{S}\in\mathcal{D}'_0((-\infty,\infty);L(X/\mathcal{N};\mathfrak{I}))$ satisfies (4.13), then $u=\widetilde{S}*f$. This last fact follows by an argument based on the associativity of convolution, similar to the one discussed above.

4.F. *The pseudoresolvent associated with a distribution semigroup.* Let X be a complex Banach space. Suppose that $S\in\mathrm{DS}((-\infty,c);L(X))$ and G is the generator of S. By Corollary 4.5, $G:\mathfrak{I}\to X/\mathcal{N}$ is a closed linear operator from X into $X/\mathcal{N}$, so that $\mathfrak{I}$ equipped with the norm (4.10) is a Banach space. Let $\widetilde{S}\in\mathcal{D}'_0((-\infty,c);L(X/\mathcal{N};\mathfrak{I}))$ be determined by (4.11). For every $\lambda\in\mathbb{C}$ and $t\in(-\infty,c)$ let $e_\lambda(t)=\exp(\lambda t)$. Suppose that $\vartheta\in\mathcal{D}((-\infty,c))$, $\vartheta(0)=1$ and $\lambda\in\mathbb{C}$. Then, by (2.13),

$$[S(e_{-\lambda}D\vartheta)-\lambda S(e_{-\lambda}\vartheta)+\mathbb{1}]S(\varphi) = S(e_{-\lambda}\vartheta)[S(D\varphi)+\varphi(0)\mathbb{1}] \quad \text{for every } \varphi\in\mathcal{D}((-\infty,c)) \tag{4.18}$$

or, what is the same,

$$\widetilde{S}(e_{-\lambda}\vartheta)[\lambda E|_{\mathfrak{I}} - G]y = [\mathbb{1}+S(e_{-\lambda}D\vartheta)]y \qquad \text{for every } y\in\mathfrak{I}. \tag{4.19}$$

Furthermore, since the operators in (4.18) commute, from Proposition 4.4 and from (4.18) it follows that

$$[\lambda E|_{\mathfrak{I}} - G]S(e_{-\lambda}\vartheta)x = E[\mathbb{1}+S(e_{-\lambda}D\vartheta)]x \qquad \text{for every } x\in X. \tag{4.20}$$

Put

$$\Lambda(\vartheta) = \{\lambda\in\mathbb{C} : \text{the operator } \mathbb{1}+S(e_{-\lambda}D\vartheta) \text{ has an inverse in } L(X)\}. \tag{4.21}$$

Suppose that $\Lambda(\vartheta)\neq\emptyset$ for some $\vartheta\in\mathcal{D}((-\infty,c))$ such that $\vartheta(0)=1$, and let us discuss the consequences. If $\lambda\in\Lambda(\vartheta)$, then

$$S(e_{-\lambda}\vartheta)[\mathbb{1}+S(e_{-\lambda}D\vartheta)]^{-1} = [\mathbb{1}+S(e_{-\lambda}D\vartheta)]^{-1}S(e_{-\lambda}\vartheta), \tag{4.22}$$

as $S(e_{-\lambda}\vartheta)$ commutes with $\mathbb{1}+S(e_{-\lambda}D\vartheta)$. By (4.19), (4.20), (4.22) and (4.10), if $\lambda \in \Lambda(\vartheta)$ then $\mathfrak{I}([\mathbb{1}+S(e_{-\lambda}D\vartheta)]^{-1}\widetilde{S}(e_{-\lambda}\vartheta)) \subset \mathfrak{I}$, $[\mathbb{1}+S(e_{-\lambda}D\vartheta)]^{-1}\widetilde{S}(e_{-\lambda}\vartheta)[\lambda E|_{\mathfrak{I}}-G]y = y$ for every $y \in \mathfrak{I}$, $[\lambda E|_{\mathfrak{I}} - G][\mathbb{1}+S(e_{-\lambda}D\vartheta)]^{-1}\widetilde{S}(e_{-\lambda}\vartheta)z = z$ for every $z \in X/\mathcal{N}$, and $[\mathbb{1}+S(e_{-\lambda}D\vartheta)]^{-1}\widetilde{S}(e_{-\lambda}\vartheta) \in L(X/\mathcal{N};\mathfrak{I})$. This proves that

(4.23) if $\lambda \in \Lambda(\vartheta)$, then the operator $[1+S(e_{-\lambda}D\vartheta)]^{-1}\widetilde{S}(e_{-\lambda}\vartheta) \in L(X/\mathcal{N};\mathfrak{I})$ is inverse to the operator $E|_{\mathfrak{I}} - G \in L(\mathfrak{I};X/\mathcal{N})$.

Put

(4.24) $\varrho(G) = \{\lambda \in \mathbb{C} : \text{the operator } \lambda E|_{\mathfrak{I}} - G \text{ is an isomorphism of } \mathfrak{I} \text{ onto } X/\mathcal{N}\}$.

Proposition 1.3, (4.22) and (4.23) imply at once

4.12. Proposition. *If $\Lambda(\vartheta) \neq \emptyset$ for some $\vartheta \in \mathcal{D}((-\infty,c))$ such that $\vartheta(0)=1$, then*

(i) $\Lambda(\vartheta) \subset \varrho(G)$,

(ii) *the map $\varrho(G) \ni \lambda \to (\lambda E|_{\mathfrak{I}} - G)^{-1}E \in L(X)$ is a maximal pseudoresolvent with null space $\mathcal{N}$, range $\mathfrak{I}$, and generator G (in the sense of (1.3)),*

(iii) $(\lambda E|_{\mathfrak{I}} - G)^{-1}E = S(e_{-\lambda}\vartheta)[\mathbb{1}+S(e_{-\lambda}D\vartheta)]^{-1} = [\mathbb{1}+S(e_{-\lambda}D\vartheta)]^{-1}S(e_{-\lambda}\vartheta)$ ([11]) *for every $\lambda \in \Lambda(\vartheta)$.*

Now, repeating the arguments of J. Chazarain [C], Sec. I.1.2, pp. 394–397, we will show that if $\vartheta|_{[0,\alpha]} \equiv 1$ for some $\alpha \in (0,c)$, then $\Lambda(\vartheta)$ contains a logarithmic region $\Lambda_{a,b}$ such that

$$\sup_{\lambda \in \Lambda_{a,b}} (1+|\lambda|)^{-n}\|S(e_{-\lambda}\vartheta)[\mathbb{1}+S(e_{-\lambda}D\vartheta)]^{-1}\|_{L(X)} < \infty \qquad \text{for some } n \in \mathbb{N}_0.$$

4.13. Lemma. *Suppose that $S \in \mathcal{D}'_0((-\infty,c);L(X))$, $\psi \in \mathcal{D}((-\infty,c))$ and $\psi|_{[0,\alpha]} \equiv 0$ for some $\alpha \in (0,c)$. Then there are finite positive a and b such that $\|S(e_{-\lambda}\psi)\| \le 1/2$ for every $\lambda \in \Lambda_{a,b}$.*

Proof. We follow Chazarain (*loc. cit.*). Let $\beta = \sup(\operatorname{supp}\psi)$. We may assume that $\beta \in (\alpha,c)$, because otherwise $\psi|_{[0,c)} \equiv 0$ and hence $S(e_{-\lambda}\psi)=0$ for every $\lambda \in \mathbb{C}$. So, let $\beta \in (\alpha,c)$. By Theorem 2.1 there are $n \in \mathbb{N}_0$ and $C \in [0,\infty)$ such that

(4.25) $$\|S(\varphi)\| = \|S^+(\varphi|_{[0,c)})\| \le C|\varphi|_{0,\beta;n} \quad \text{for every } \varphi \in \mathcal{D}((-\infty,c)) \text{ such that } \operatorname{supp}\varphi \subset (-\infty,\beta].$$

Applying (4.25) to $\varphi = e_{-\lambda}\psi$ one obtains

$$\|S(e_{-\lambda}\psi)\| \le C|e_{-\lambda}\psi|_{\alpha,\beta;n}.$$

([11]) These equalities imply that if $\lambda \in \Lambda(\vartheta)$, then $\mathcal{N}(S(e_{-\lambda}\vartheta)) = \mathcal{N}$ and $\mathfrak{I}(S(e_{-\lambda}\vartheta)) = \mathfrak{I}$, which may also be deduced directly from (4.18). Since $e_{-\lambda}\vartheta\mathbb{1}_{[0,\infty)} = e_{-\lambda}\mathbb{1}_{[0,\infty)} * (\delta + e_{-\lambda}D\vartheta\mathbb{1}_{[0,\infty)})$, it follows that if $S \in \mathrm{DS}(L(X))$ has exponential growth (see [L], p. 154, Definition 6.1) and $\operatorname{Re}\lambda$ is sufficiently large, then $S(e_{-\lambda}\vartheta)[\mathbb{1}+S(e_{-\lambda}D\vartheta)]^{-1} = S(e_{-\lambda})$ is the value at λ of the Laplace transform of S.

Since $|e_{-\lambda}\psi|_{\alpha,\beta;n} = \sup\{|D^k(e_{-\lambda}\psi)(t)| : k = 0,\dots,n;\ t \in [\alpha,\beta]\}$ and, by the Leibniz formula, $D^k(e_{-\lambda}\psi)(t) = \sum_{l=0}^{k}\binom{k}{l}(-\lambda)^l e^{-\lambda t}D^{k-l}\psi(t)$, it follows that there is a finite constant D_n depending only on n such that

$$|e_{-\lambda}\psi|_{\alpha,\beta;n} \le D_n(1+|\lambda|)^n e^{-\alpha \operatorname{Re}\lambda}|\psi|_{\alpha,\beta;n}.$$

If $\lambda \in \Lambda_{a,b}$, then

$$(1+|\lambda|)^n e^{-\alpha \operatorname{Re}\lambda} \le (1+|\lambda|)^n e^{-\alpha a \log(1+|\lambda|)-\alpha b} = (1+|\lambda|)^{n-\alpha a}e^{-\alpha b},$$

so that

$$\|S(e_{-\lambda}\psi)\| \le CD_n|\psi|_{\alpha,\beta;n}(1+|\lambda|)^{n-\alpha a}e^{-\alpha b}.$$

From this Lemma 4.13 follows by choosing $a \ge \alpha^{-1}n$ and $b \ge \alpha^{-1}\log(2CD_n|\psi|_{\alpha,\beta;n})$. ∎

4.14. Lemma. *Suppose that $S \in \mathcal{D}_0'((-\infty,c);L(X))$ and $\vartheta \in \mathcal{D}((-\infty,c))$. Then there are $n \in \mathbb{N}_0$ and $K \in [0,\infty)$ such that $\|S(e_{-\lambda}\vartheta)\| \le K(1+|\lambda|)^n$ whenever $\operatorname{Re}\lambda \ge 0$.*

Proof. Let $\beta = \sup(\operatorname{supp}\vartheta)$. By (4.25) one has

$$\|S(e_{-\lambda}\vartheta)\| \le C|e_{-\lambda}\vartheta|_{0,\beta;n}$$

for every $\lambda \in \mathbb{C}$. By the Leibniz formula,

$$|e_{-\lambda}\vartheta|_{0,\beta;n} \le D_n(1+|\lambda|)^n \sup_{0\le t\le\beta} e^{-t\operatorname{Re}\lambda}$$

for every $\lambda \in \mathbb{C}$, where D_n is a finite constant depending only on n. If $\operatorname{Re}\lambda \ge 0$, then $\sup_{0\le t\le\beta} e^{-t\operatorname{Re}\lambda} = 1$. ∎

4.15. Lemma. *Suppose that $S \in \mathcal{D}_0'((-\infty,c);L(X))$, $\vartheta \in \mathcal{D}((-\infty,c))$ and $\vartheta|_{[0,\alpha)} \equiv 1$ for some $\alpha \in (0,c)$. Then there are finite positive a and b such that for every $\lambda \in \Lambda_{a,b}$ the operator $\mathbb{1} + S(e_{-\lambda}D\vartheta)$ has an inverse belonging to $L(X)$, and*

$$\sup_{\lambda\in\Lambda_{a,b}}(1+|\lambda|)^{-n}\|S(e_{-\lambda}\vartheta)[\mathbb{1}+S(e_{-\lambda}D\vartheta)]^{-1}\| < \infty \qquad \textit{for some } n \in \mathbb{N}_0. \ ∎$$

Proof. By Lemma 4.13 applied to $\psi = D\vartheta$ there are finite positive a and b such that $\|S(e_{-\lambda}D\vartheta)\| \le 1/2$ for every $\lambda \in \Lambda_{a,b}$. It follows that $[\mathbb{1}+S(e_{-\lambda}D\vartheta)]^{-1} \in L(X)$ and $\|[\mathbb{1}+S(e_{-\lambda}D\vartheta)]^{-1}\| \le 2$ for every $\lambda \in \Lambda_{a,b}$. By Lemma 4.14, there are $n \in \mathbb{N}_0$ and $K \in [0,\infty)$ such that $\|S(e_{-\lambda}\vartheta)\| \le K(1+|\lambda|)^n$ whenever $\operatorname{Re}\lambda \ge 0$, and hence for every $\lambda \in \Lambda_{a,b}$. As a consequence,

$$\|S(e_{-\lambda}\vartheta)[\mathbb{1}+S(e_{-\lambda}D\vartheta)]^{-1}\| \le 2K(1+|\lambda|)^n \qquad \text{for every } \lambda \in \Lambda_{a,b}. \ ∎$$

The following theorem is an immediate consequence of Proposition 4.12 and Lemma 4.15.

4.16. Theorem. *Suppose that $S \in \mathrm{DS}((-\infty,c);L(X))$ and let $G : \mathfrak{I} \to X/\mathcal{N}$ be the generator of S. Let E denote the canonical epimorphism of X onto $X/\mathcal{N}$, and let*

$\Lambda(\vartheta)$ and $\varrho(G)$ be defined by (4.21) *and* (4.24). *Then $\varrho(G) \neq \emptyset$ and the map*

$$\varrho(G) \ni \lambda \to (\lambda E|_{\mathfrak{I}} - G)^{-1} E \in L(X) \tag{4.26}$$

is a maximal pseudoresolvent. The null space, range, and generator (*in the sense of* (1.3)) *of this pseudoresolvent are $\mathcal{N}$, $\mathfrak{I}$ and G respectively. Furthermore, whenever $\vartheta \in \mathcal{D}((-\infty, c))$ and $\vartheta|_{[0,\alpha]} \equiv 1$ for some $\alpha \in (0, c)$, then $\Lambda(\vartheta) \subset \varrho(G)$,*

$$(\lambda E|_{\mathfrak{I}} - G)^{-1} E = S(e_{-\lambda}\vartheta)[\mathbb{1} + S(e_{-\lambda}D\vartheta)]^{-1} = [\mathbb{1} + S(e_{-\lambda}D\vartheta)]^{-1} S(e_{-\lambda}\vartheta) \quad \textit{for every } \lambda \in \Lambda(\vartheta), \tag{4.27}$$

and there are finite positive a and b such that $\Lambda_{a,b} \subset \Lambda(\vartheta)$ and

$$\sup_{\lambda \in \Lambda_{a,b}} (1 + |\lambda|)^{-n} \|(\lambda E|_{\mathfrak{I}} - G)^{-1} E\|_{L(X)} < \infty \quad \textit{for some } n \in \mathbb{N}_0. \tag{4.28}$$

4.17. Definition. We will call (4.26) the *pseudoresolvent associated with the (local) distribution semigroup S.*

4.G. *The generation theorem of the Hille–Yosida type for distribution semigroups.* Suppose that $\mathcal{N}$ is a closed subspace of a complex Banach space X. Let $X/\mathcal{N}$ be the corresponding *factor space* whose elements are the equivalence classes $Ex = \{y \in X : y - x \in \mathcal{N}\}$, $x \in X$. Equipped with the norm $\|Ex\|_{X/\mathcal{N}} = \inf\{\|y\|_X : y \in Ex\}$, $X/\mathcal{N}$ is a Banach space. The map $E : x \to Ex$ is an epimorphism of X onto $X/\mathcal{N}$. The definition of $\| \ \|_{X/\mathcal{N}}$ implies that $\|E\|_{L(X;X/\mathcal{N})} = 1$. The case of $\mathcal{N} = \{0\}$ is not excluded. If $\mathcal{N} = \{0\}$, then $X/\mathcal{N} = X$ and $E = \mathrm{id}_X$.

Suppose that G is a closed linear operator from X into $X/\mathcal{N}$ with domain $\mathfrak{I}$. The closedness of G is equivalent to the completeness of $\mathfrak{I}$ with respect to the graph norm $\|x\|_{\mathfrak{I}} = \|x\|_X + \|Gx\|_{X/\mathcal{N}}$. Hence $\mathfrak{I}$ equipped with this norm is a Banach space. Put

$$\varrho(G) = \{\lambda \in \mathbb{C} : \lambda E|_{\mathfrak{I}} - G \text{ is an isomorphism of } \mathfrak{I} \text{ onto } X/\mathcal{N}\}.$$

According to Propositions 1.2 and 1.3, if $\varrho(G) \neq \emptyset$, then $\varrho(G)$ is an open subset of $\mathbb{C}$, and the map

$$\varrho(G) \ni \lambda \to (E|_{\mathfrak{I}} - G)^{-1} E \in L(X)$$

is a *maximal pseudoresolvent* whose *null space* is equal to $\mathcal{N}$, *range* is equal to $\mathfrak{I}$, and the *generator* is G. (See Sec. 1 for the definitions related to a pseudoresolvent.).

Let us remind the reader that, according to Sec. 4.11, a distribution $\widetilde{S} \in \mathcal{D}((-\infty, \infty); L(X/\mathcal{N}; \mathfrak{I}))$ is called a *fundamental solution* for the first order differential operator $E|_{\mathfrak{I}} D - G$ with coefficients in $L(\mathfrak{I}; X/\mathcal{N})$ if

$$E|_{\mathfrak{I}} D\widetilde{S} - G\widetilde{S} = \delta \otimes \mathrm{id}_{X/\mathcal{N}} \quad \text{and} \quad D\widetilde{S} E|_{\mathfrak{I}} - \widetilde{S} G = \delta \otimes \mathrm{id}_{\mathfrak{I}}$$

(where $D\widetilde{S}$ is the derivative of $\widetilde{S}$ in the sense of the theory of distributions). Thus $\widetilde{S} \in \mathcal{D}((-\infty, \infty); L(X/\mathcal{N}, \mathfrak{I}))$ is a fundamental solution for $E|_{\mathfrak{I}} - G$ if and only if the equalities (4.12) and (4.13) are satisfied, that is,

$$-E|_{\mathfrak{I}} \widetilde{S}(D\varphi) - G\widetilde{S}(\varphi) = \varphi(0)\, \mathrm{id}_{X/\mathcal{N}} \quad \text{and} \quad -\widetilde{S}(D\varphi) E|_{\mathfrak{I}} - \widetilde{S}(\varphi) G = \varphi(0)\, \mathrm{id}_{\mathfrak{I}}$$

for every $\varphi \in \mathcal{D}((-\infty,\infty))$. According to Proposition 4.9 and Theorem 4.10, if G is a closed linear operator from X into $X/\mathcal{N}$ with domain $\mathfrak{I}$, then

1° there is at most one fundamental solution $\widetilde{S}$ for $E|_{\mathfrak{I}}-G$ belonging to $\mathcal{D}'_0((-\infty,\infty); L(X/\mathcal{N};\mathfrak{I}))$, that is, such that $\operatorname{supp}\widetilde{S} \subset [0,\infty)$,

2° there is at most one $L(X)$-valued *distribution semigroup* (in the sense of Definition 2.3) whose *generator* (in the sense of Definition 4.3) is G,

3° there exist a (unique) $L(X)$-valued distribution semigroup with generator G if and only if there exists a (unique) fundamental solution $\widetilde{S} \in \mathcal{D}'_0((-\infty,\infty); L(X/\mathcal{N};\mathfrak{I}))$ for the differential operator $E|_{\mathfrak{I}}D - G$, and if it is the case, then $\widetilde{S}(\varphi)E = S(\varphi)$ for every $\varphi \in \mathcal{D}((-\infty,\infty))$.

4.18. Theorem. *Suppose that $\mathcal{N}$ is a closed subspace of a complex Banach space X, the case of $\mathcal{N} = \{0\}$ not being excluded. Let G be a closed linear operator from X into $X/\mathcal{N}$ with domain $\mathfrak{I}$. Then the following three assertions are equivalent*:

(i) *$\varrho(G)$ contains a logarithmic region $\widetilde{\Lambda}_{a,b}$ such that $a \in [0,\infty)$, $b \in [0,\infty)$ and, for some $n \in \mathbb{N}_0$,*

$$\sup_{\lambda \in \widetilde{\Lambda}_{a,b}} (1+|\lambda|)^{-n} \|(\lambda E|_{\mathfrak{I}} - G)^{-1}E\|_{L(X)} < \infty,$$

(ii) *there is a unique distribution $\widetilde{S} \in \mathcal{D}'_0((-\infty,\infty); L(X/\mathcal{N};\mathfrak{I}))$ which is a fundamental solution for the differential operator $E|_{\mathfrak{I}} - G$,*

(iii) *there is a unique distribution semigroup $S \in \mathrm{DS}(L(X))$ with generator G.*

If these equivalent assertions are valid, then

(a) *$S(\varphi) = \widetilde{S}(\varphi)E$ for every $\varphi \in \mathcal{D}((-\infty,\infty))$,*

(b) *for every $\vartheta \in \mathcal{D}((-\infty,\infty))$ such that $\vartheta|_{[0,\alpha]} \equiv 1$ for some $\alpha > 0$ there is a logarithmic region $\widetilde{\Lambda}_{a,b}$ for which* (i) *is valid and, moreover, $[\mathbb{1} + S(e_{-\lambda}D\vartheta)]^{-1} \in L(X)$ and $(\lambda E|_{\mathfrak{I}} - G)^{-1}E = S(e_{-\lambda}\vartheta)[\mathbb{1} + S(e_{-\lambda}D\vartheta)]^{-1} = [\mathbb{1} + S(e_{-\lambda}D\vartheta)]^{-1}S(e_{-\lambda}\vartheta)$ for every $\lambda \in \widetilde{\Lambda}_{a,b}$,*

(c) *$\widetilde{S}(\varphi) = (2\pi i)^{-1}\int_{\mathcal{C}_{a,b}} \widehat{\varphi}(i\lambda)(\lambda E|_{\mathfrak{I}} - G)^{-1}\,d\lambda$ for every $\varphi \in \mathcal{D}((-\infty,\infty))$, where $\widehat{\varphi}$ the Fourier transform of φ, and the curve $\mathcal{C}_{a,b} : (-\infty,\infty) \ni \eta \to a\log(1+|\eta|)+b+i\eta \in \mathbb{C}$ is a parametrization of the boundary of any logarithmic region $\widetilde{\Lambda}_{a,b}$ for which* (i) *holds.*

Theorem 4.18, concerning the distribution semigroups in the wide sense of Definition 2.3, is an analogue of the Hille–Yosida generation theorem for one-parameter strongly continuous semigroups of operators. It is interesting to compare the condition (i) from Theorem 4.18, going back to J. Chazarain [C], with the Hadamard–Petrovsky condition for forward evolution, known from the theory of the initial value problem for PDE with constant coefficients. See the book of J. Rauch [R], Sec. 3.10 and 3.11, pp. 126–136. In Theorem 4.18, (ii)⇔(iii) and (a) follow from Proposition 4.9 and Theorem 4.10, while (iii)⇒(i) and (b) are consequences of Theorem 4.16 and Lemma 4.1. Thus it remains to prove (i)⇒(ii) and (c), for which it is sufficient to repeat almost verbatim the arguments of J. Chazarain from [C], Sec. I.1.3, pp. 397–400.

4.19. Lemma. *Let* $0 \le a < \infty$, $0 < b < c < \infty$, $-\infty < \alpha < \beta < \infty$, $\beta^+ = 0 \vee \beta$, $\varphi \in \mathcal{D}((-\infty,\infty))$, $\operatorname{supp}\varphi \subset [\alpha,\beta]$, $n \in \mathbb{N}_0$ *and* $k \in \mathbb{N}$. *Then*

$$\text{(A)} \quad \widehat{\varphi}(i\lambda)|(1+|\lambda|)^n \le e^{\beta^+ c}\left(1+\frac{1}{b}\right)^k \frac{\beta-\alpha}{(1+|\operatorname{Im}\lambda|)^{k-n-\beta^+ a}} \sup_{\alpha\le t\le\beta}|D^k\varphi(t)|$$

whenever $k > n$ *and* $b \le \operatorname{Re}\lambda - a\log(1+|\operatorname{Im}\lambda|) \le c$,

and

$$\text{(B)} \quad \frac{1}{2\pi}\int_{\mathcal{C}_{a,b}} |\widehat{\varphi}(i\lambda)|(1+|\lambda|)^n\,|d\lambda| \le e^{\beta b}\left(1+\frac{1}{b}\right)^k \sqrt{1+a^2}\,\frac{\beta-\alpha}{\pi(\gamma-1)} \sup_{\alpha\le t\le\beta}|D^k\varphi(t)|$$

whenever $k > n$ *and* $k-n-\beta a = \gamma > 1$.

Proof. One has

$$\widehat{\varphi}(i\lambda) = \int_\alpha^\beta e^{\lambda t}\varphi(t)\,dt = \frac{1}{(-\lambda)^k}\int_\alpha^\beta e^{\lambda t}D^k\varphi(t)\,dt,$$

so that

$$|\widehat{\varphi}(i\lambda)| \le e^{\beta\operatorname{Re}\lambda}\frac{\beta-\alpha}{|\lambda|^k}\sup_{\alpha\le t\le\beta}|D^k\varphi(t)| \quad \text{whenever } \operatorname{Re}\lambda > 0.$$

If $k > n$ and $b \le \operatorname{Re}\lambda - a\log(1+|\operatorname{Im}\lambda|) \le c$, then $|\lambda| \ge \operatorname{Re}\lambda \ge b$, so that $(1+|\lambda|)^k/|\lambda|^k \le (1+1/b)^k$, and

$$\begin{aligned}|\widehat{\varphi}(i\lambda)|(1+|\lambda|)^n &\le e^{\beta^+(c+a\log(1+|\operatorname{Im}\lambda|))}\left(1+\frac{1}{b}\right)^k \frac{\beta-\alpha}{(1+|\lambda|)^{k-n}}\sup_{\alpha\le t\le\beta}|D^k\varphi(t)| \\ &\le e^{\beta^+ c}\left(1+\frac{1}{b}\right)^k\frac{\beta-\alpha}{(1+|\operatorname{Im}\lambda|)^{k-n-\beta^+ a}}\sup_{\alpha\le t\le\beta}|D^k\varphi(t)|,\end{aligned}$$

proving (A). If $k > n$, $k-n-\beta a = \gamma > 1$ and $\operatorname{Re}\lambda = a\log(1+|\operatorname{Im}\lambda|)+b$ (which means that λ lies on the curve $\mathcal{C}_{a,b}$), then

$$\begin{aligned}|\widehat{\varphi}(i\lambda)|(1+|\lambda|)^n &\le e^{\beta(b+a\log(1+|\operatorname{Im}\lambda|))}\left(1+\frac{1}{b}\right)^k\frac{\beta-\alpha}{(1+|\lambda|)^{k-n}}\sup_{\alpha\le t\le\beta}|D^k\varphi(t)| \\ &\le e^{\beta b}\left(1+\frac{1}{b}\right)^k\frac{\beta-\alpha}{(1+|\operatorname{Im}\lambda|)^{\gamma}}\sup_{\alpha\le t\le\beta}|D^k\varphi(t)|.\end{aligned}$$

Since on $\mathcal{C}_{a,b}$ one has

$$\frac{d\lambda}{d\eta} = i + \frac{a}{1+|\eta|}\operatorname{sign}\eta$$

and hence $|d\lambda/d\eta| \le \sqrt{1+a^2}$, it follows that

$$\begin{aligned}&\frac{1}{2\pi}\int_{\mathcal{C}_{a,b}}|\widehat{\varphi}(i\lambda)|(1+|\lambda|)^n\,|d\lambda| \\ &\qquad \le e^{\beta b}\left(1+\frac{1}{b}\right)^k\sqrt{1+a^2}\,\frac{\beta-\alpha}{2\pi}\sup_{\alpha\le t\le\beta}|D^k\varphi(t)|\int_{-\infty}^{\infty}\frac{d\eta}{(1+|\eta|)^\gamma}.\end{aligned}$$

This proves (B), because the integral on the right is equal to $2/(\gamma-1)$. ∎

Completion of the proof of Theorem 4.18. Suppose that (i) holds. Fix a, b, n and C such that $a \in [0,\infty)$, $b \in (0,\infty)$, $n \in \mathbb{N}_0$, $C \in (0,\infty)$, $\widetilde{\Lambda}_{a,b} \subset \varrho(G)$ and $\|(\lambda E|_\mathfrak{I} - G)^{-1}E\|_{L(X)} \le C(1+|\lambda|)^n$ for every $\lambda \in \widetilde{\Lambda}_{a,b}$. Then $\|(\lambda E|_\mathfrak{I} - G)^{-1}\|_{L(X/\mathcal{N};X)} = \|(\lambda E|_\mathfrak{I} - G)^{-1}E\|_{L(X)}$ and

$$\begin{aligned}\|G(\lambda E|_\mathfrak{I} - G)^{-1}\|_{L(X/\mathcal{N})} &= \|\lambda E(\lambda E|_\mathfrak{I} - G)^{-1} - \mathrm{id}_{X/\mathcal{N}}\|_{L(X/\mathcal{N})} \\ &\le 1 + |\lambda|\|(\lambda E|_\mathfrak{I} - G)^{-1}\|_{L(X/\mathcal{N};X)}\end{aligned}$$

for every $\lambda \in \widetilde{\Lambda}_{a,b}$, whence

$$\begin{aligned}\|(\lambda E|_\mathfrak{I} - G)^{-1}\|_{L(X/\mathcal{N};\mathfrak{I})} &\le \|(\lambda E|_\mathfrak{I} - G)^{-1}\|_{L(X/\mathcal{N};X)} + \|G(\lambda E|_\mathfrak{I} - G)^{-1}\|_{L(X/\mathcal{N})} \\ &\le 1 + (1+|\lambda|)\|(\lambda E|_\mathfrak{I} - G)^{-1}E\|_{L(X)} \le 1 + C(1+|\lambda|)^{n+1}.\end{aligned}$$

Thus

$$\|(\lambda E|_\mathfrak{I} - G)^{-1}\|_{L(X/\mathcal{N};\mathfrak{I})} \le \widetilde{C}(1+|\lambda|)^{n+1} \quad \text{for every } \lambda \in \widetilde{\Lambda}_{a,b}, \tag{4.29}$$

where $\widetilde{C} = 1 + C$. From (4.29) and Lemma 4.19(B) it follows that

(4.30) the integral $(2\pi i)^{-1}\int_{\mathcal{C}_{a,c}} \widehat{\varphi}(i\lambda)(\lambda E|_\mathfrak{I} - G)^{-1}d\lambda$ is absolutely convergent in the $L(X/\mathcal{N};\mathfrak{I})$ norm for every $c \in [b,\infty)$ and every $\varphi \in \mathcal{D}((-\infty,\infty))$, and

$$\begin{aligned}&\bigg\|\frac{1}{2\pi i}\int_{\mathcal{C}_{a,c}} \widehat{\varphi}(i\lambda)(\lambda E|_\mathfrak{I} - G)^{-1}\,d\lambda\bigg\|_{L(X/\mathcal{N};\mathfrak{I})} \\ &\qquad \le \widetilde{C}\sqrt{1+a^2}\,e^{\beta c}\Big(1+\frac{1}{c}\Big)^k \frac{\beta-\alpha}{\pi(k-n-2-\beta a)} \sup_{\alpha\le t\le\beta}|D^k\varphi(t)|\end{aligned}$$

whenever $\operatorname{supp}\varphi \subset [\alpha,\beta]$, $k \in \mathbb{N}$, $k > n$ and $k-n-2-\beta a > 0$.

For any $c \in [b,\infty)$ and $r \in (0,\infty)$ let $\mathcal{C}_{a,c;r}$ denote the parametrized curve $[-r,r] \ni \eta \to a\log(1+|\eta|) + c + i\eta \in \mathbb{C}$. By Lemma 4.19(B) for every $c \in [b,\infty)$ and every $\varphi \in \mathcal{D}((-\infty,\infty))$ one has

$$\lim_{r\to\infty}\bigg\|\frac{1}{2\pi i}\int_{\mathcal{C}_{a,c;r}} \widehat{\varphi}(i\lambda)(\lambda E|_\mathfrak{I} - G)^{-1}\,d\lambda - \frac{1}{2\pi i}\int_{\mathcal{C}_{a,c}} \widehat{\varphi}(i\lambda)(\lambda E|_\mathfrak{I} - G)^{-1}\,d\lambda\bigg\|_{L(X/\mathcal{N};\mathfrak{I})} = 0. \tag{4.31}$$

Furthermore, since the map $U \ni \lambda \to \widehat{\varphi}(i\lambda)(\lambda E|_\mathfrak{I} - G)^{-1} \in L(X/\mathcal{N};\mathfrak{I})$ is holomorphic in some neighbourhood U of $\widetilde{\Lambda}_{a,b}$, from Cauchy's integral theorem and from Lemma 4.19(A) one obtains the estimate

$$\begin{aligned}&\bigg\|\frac{1}{2\pi i}\int_{\mathcal{C}_{a,c;r}} \widehat{\varphi}(i\lambda)(\lambda E|_\mathfrak{I} - G)^{-1}\,d\lambda - \frac{1}{2\pi i}\int_{\mathcal{C}_{a,b;r}} \widehat{\varphi}(i\lambda)(\lambda E|_\mathfrak{I} - G)^{-1}\,d\lambda\bigg\|_{L(X/\mathcal{N};\mathfrak{I})} \\ &\qquad = \bigg\|\frac{1}{2\pi i}\int_b^c \widehat{\varphi}(i(\xi+ir))((\xi+ir)E|_\mathfrak{I} - G)^{-1}\,d\xi \\ &\qquad\qquad - \frac{1}{2\pi i}\int_b^c \widehat{\varphi}(i(\xi-ir))((\xi-ir)E|_\mathfrak{I} - G)^{-1}\,d\xi\bigg\|_{L(X/\mathcal{N};\mathfrak{I})} \\ &\qquad \le \frac{c-b}{\pi}e^{\beta+c}\Big(1+\frac{1}{b}\Big)^k \frac{\beta-\alpha}{(1+r)^{k-n-1-\beta+a}} \sup_{\alpha\le t\le\beta}|D^k\varphi(t)|,\end{aligned}$$

in which $[\alpha,\beta]$ is an interval containing $\operatorname{supp}\varphi$, and $k\in\mathbb{N}$ is greater than $n+1+\beta^+a$. As a consequence,

$$\text{(4.32)}\qquad \lim_{r\to\infty}\bigg\|\frac{1}{2\pi i}\int\limits_{\mathcal{C}_{a,c;r}}\widehat{\varphi}(i\lambda)(\lambda E|_{\Im}-G)^{-1}\,d\lambda - \frac{1}{2\pi i}\int\limits_{\mathcal{C}_{a,b;r}}\widehat{\varphi}(i\lambda)(\lambda E|_{\Im}-G)^{-1}\,d\lambda\bigg\|_{L(X/\mathcal{N};\Im)}=0.$$

From (4.31) and (4.32) it follows that

(4.33) for every $\varphi\in\mathcal{D}((-\infty,\infty))$ the integral $(2\pi i)^{-1}\int_{\mathcal{C}_{a,c}}\widehat{\varphi}(i\lambda)(\lambda E|_{\Im}-G)^{-1}\,d\lambda$ is independent of the choice of $c\in[b,\infty)$.

From (4.30) it follows that the formula

$$\text{(4.34)}\qquad \widetilde{S}(\varphi)=\frac{1}{2\pi i}\int\limits_{\mathcal{C}_{a,b}}\widehat{\varphi}(i\lambda)(\lambda E|_{\Im}-G)^{-1}\,d\lambda,\qquad \varphi\in\mathcal{D}((-\infty,\infty)),$$

defines a distribution $\widetilde{S}\in\mathcal{D}'((-\infty,\infty);L(X/\mathcal{N};\Im))$. If $\operatorname{supp}\varphi\subset[\alpha,\beta]\subset(-\infty,0)$, then from (4.33) and (4.30), by letting $c\to\infty$, one concludes that $\widetilde{S}(\varphi)=0$. Thus the distribution $\widetilde{S}$ vanishes on $(-\infty,0)$, and so $\widetilde{S}\in\mathcal{D}_0'((-\infty,\infty);L(X/\mathcal{N};\Im))$. We will now prove that this distribution is a fundamental solution for the differential operator $E|_{\Im}D-G$. To this end notice that if $\varphi\in\mathcal{D}((-\infty,\infty))$, then $\widehat{D\varphi}(i\lambda)=-\lambda\widehat{\varphi}(i\lambda)$ and

$$\widetilde{S}(D\varphi)=\frac{1}{2\pi i}\int\limits_{\mathcal{C}_{a,b}}\widehat{D\varphi}(i\lambda)(\lambda E|_{\Im}-G)^{-1}\,d\lambda=-\frac{1}{2\pi i}\int\limits_{\mathcal{C}_{a,b}}\widehat{\varphi}(i\lambda)\lambda(\lambda E|_{\Im}-G)^{-1}\,d\lambda,$$

where, by (4.30) applied to $D\varphi$, the integral is absolutely convergent in the $L(X/\mathcal{N};\Im)$ norm. Since $E|_{\Im}\in L(\Im;X/\mathcal{N})$ and $G\in L(\Im;X/\mathcal{N})$ it follows that, for every $\varphi\in\mathcal{D}((-\infty,\infty))$,

$$\begin{aligned}-E|_{\Im}\widetilde{S}(D\varphi)-G\widetilde{S}(\varphi)&=\frac{1}{2\pi i}\int\limits_{\mathcal{C}_{a,b}}\widehat{D\varphi}(i\lambda)(\lambda E|_{\Im}-G)(\lambda E|_{\Im}-G)^{-1}\,d\lambda\\&=\frac{1}{2\pi i}\int\limits_{\mathcal{C}_{a,b}}\widehat{\varphi}(i\lambda)\,d\lambda\cdot\mathrm{id}_{X/\mathcal{N}}\end{aligned}$$

and

$$\begin{aligned}-\widetilde{S}(D\varphi)E|_{\Im}-\widetilde{S}(\varphi)G&=\frac{1}{2\pi i}\int\limits_{\mathcal{C}_{a,b}}\widehat{\varphi}(i\lambda)(\lambda E|_{\Im}-G)^{-1}(\lambda E|_{\Im}-G)\,d\lambda\\&=\frac{1}{2\pi i}\int\limits_{\mathcal{C}_{a,b}}\widehat{\varphi}(i\lambda)\,d\lambda\cdot\mathrm{id}_{\Im}\,.\end{aligned}$$

By Lemma 4.19(A) with $n=0$, and by Cauchy's integral theorem,

$$\frac{1}{2\pi i}\int\limits_{\mathcal{C}_{a,b}}\widehat{\varphi}(i\lambda)\,d\lambda=\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\widehat{\varphi}(i(i\eta))i\,d\eta=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\widehat{\varphi}(-x)\,dx.$$

By the Fourier inversion formula, this last integral is equal to $\varphi(0)$. Hence $\widetilde{S}$ is a fundamental solution for $E|_{\Im}D-G$. As proved earlier, $\widetilde{S}\in\mathcal{D}_0'((-\infty,\infty);L(X/\mathcal{N};\Im))$.

By Proposition 4.9 and Theorem 4.10, these two properties determine $\widetilde{S}$ uniquely. Thus (ii) is proved, and (c) is a consequence of the definition (4.34). ∎

Theorems 4.10, 4.16 and 4.18 have the following consequence.

4.20. Corollary. *Every local distribution semigroup* $S_c \in \mathrm{DS}((-\infty, c); L(X))$, $c \in (0, \infty)$, *has a unique extension to a distribution semigroup* $S_\infty \in \mathrm{DS}(L(X))$. *The semigroups* S_c *and* S_∞ *have the same generator, and hence the same associated pseudoresolvent.*

Notice that if $\mathcal{N}_0 = \{0\}$, then Corollary 4.20 may be deduced from Theorem 4.11 of [K], p. 851. If $\mathcal{N}_0 = \{0\}$ and $\mathfrak{I}_0$ is dense in X, then Corollary 4.20 follows from Theorem 4.1 in [A-El-M-K], p. 581. This last result gives additional information about the order of the distribution S_∞ on compact subintervals of $[0, \infty)$.

4.21. Example. Consider again Kunstmann's Example 2.3 from [K], p. 480, mentioned earlier in our Section 3. Let $0 \neq T \in L(X)$, $T^{m+2} = 0$. Then $S = \sum_{j=0}^{m} D^j \delta \otimes T^{j+1} \in \mathrm{DS}(L(X))$, $\operatorname{supp} S = \{0\}$, $\mathcal{N} = \mathcal{N}(T)$, $\mathfrak{I} = \mathfrak{I}(T)$. The generator of S is the operator $G : \mathfrak{I} \ni y \to T^{-1}(y) \in X/\mathcal{N}$, and the corresponding pseudoresolvent is the map $\mathbb{C} \ni \lambda \to \sum_{j=0}^{m} (-\lambda)^j T^{j+1} \in L(X)$.

4.22. Example. For any logarithmic region $\widetilde{\Lambda}_{a,b}$, $a, b \in [0, \infty)$, and any $n = 2, 3, \ldots$ we will construct a closed linear operator G from a Hilbert space X into itself, with domain dense in X, such that

$$\varrho(G) \text{ is equal to the interior of } \widetilde{\Lambda}_{a,b} \tag{4.35}$$

and for every $c \in (b, \infty)$ there is $K \in (0, \infty)$ such that

$$|\lambda|^{n-2} \le \|(\lambda - G)^{-1}\|_{L(X)} \le K(1 + |\lambda|)^{2n-2} \tag{4.36}$$

for every $\lambda \in \widetilde{\Lambda}_{a,c}$ ([12]). By Theorem 4.18 and by equality (C) from Section 3, this operator G is the infinitesimal generator (in the sense of the definition due to J. L. Lions [L], discussed in Section 4.7) of an $L(X)$-valued regular distribution semigroup.

Suppose that $\widetilde{\Lambda}_{a,b}$, $a, b \in [0, \infty)$, and $n \in \mathbb{N}$ are given. Let $X = L^2(\mathbb{C} \setminus \widetilde{\Lambda}_{a,b}; \mathbb{C}^n)$ be the space of $\mathbb{C}^n$-valued functions square integrable on $\mathbb{C} \setminus \widetilde{\Lambda}_{a,b}$ with respect to the two-dimensional Lebesgue measure. For every $z \in \mathbb{C}$ let $T(z)$ be the triangular $n \times n$ matrix of the form

$$T(z) = \begin{bmatrix} z & z^2 & \cdots & \cdots & z^n \\ & z & \ddots & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & z & z^2 \\ & & & & z \end{bmatrix}$$

([12]) The existence of such a G does not follow from Theorem 4.18.

and let G be the operator from X into X with domain $\mathcal{D}(G)$ such that

$$\mathcal{D}(G) = \{x \in X : \text{the function } \mathbb{C} \setminus \widetilde{\Lambda}_{a,b} \ni z \to T(z)x(z) \in \mathbb{C}^n \text{ belongs to } X\},$$
$$(Gx)(z) = T(z)x(z) \text{ a.e. in } \mathbb{C} \setminus \widetilde{\Lambda}_{a,b} \text{ for every } x \in \mathcal{D}(G).$$

It is evident that $\mathcal{D}(G)$ is dense in X, and G is a closed linear operator from X into X.

We will prove that G satisfies (4.35) and (4.36). For every $z \in \mathbb{C}$ let $U(z)$ be the $n \times n$ matrix

$$U(z) = \begin{bmatrix} z & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & z \end{bmatrix}.$$

Then $[T(z)]^{-1} = U(z^{-1})$ for every $z \in \mathbb{C} \setminus \{0\}$. Furthermore if $\lambda, z \in \mathbb{C} \setminus \{0\}$ and $\lambda \neq z$, then $U(z^{-1}) - U(z^{-1} - \lambda^{-1}) = \lambda^{-1}$, whence

$$[U(z^{-1} - \lambda^{-1})]^{-1} - [U(z^{-1})]^{-1} = \lambda^{-1}[U(z^{-1})]^{-1}[U(z^{-1} - \lambda^{-1})]^{-1},$$

so that $(\lambda - [U(z^{-1})]^{-1})(\lambda^{-1} + \lambda^{-2}[U(z^{-1} - \lambda^{-1})]^{-1}) = 1$, and

$$(\lambda - T(z))^{-1} = \lambda^{-1} + \lambda^{-2} T\left(\frac{\lambda z}{\lambda - z}\right). \tag{4.37}$$

If $0 \neq \lambda \in \mathbb{C}$ is fixed, then, by (4.37), the matrix elements on and over the diagonal of the triangular matrix $(\lambda - T(z))^{-1}$ are functions of z essentially unbounded in any neighbourhood of λ. This implies that $\overline{\mathbb{C} \setminus \widetilde{\Lambda}_{a,b}}$ is contained in the spectrum of G, and hence $\varrho(G)$ is contained in the interior of $\widetilde{\Lambda}_{a,b}$. It remains to show that whenever $c \in (b, \infty)$, then $\widetilde{\Lambda}_{a,c} \subset \varrho(G)$ and (4.36) holds. The equality (4.35) will then follow, because $\bigcup_{c \in (b,\infty)} \widetilde{\Lambda}_{a,c}$ is equal to the interior of $\widetilde{\Lambda}_{a,b}$.

Fix any $c \in (b, \infty)$. Notice first that

$$\inf\{|z - \lambda| : z \in \mathbb{C} \setminus \widetilde{\Lambda}_{a,b},\ \lambda \in \widetilde{\Lambda}_{a,c}\} \geq \frac{c - b}{a + 1}. \tag{4.38}$$

Indeed, if m denotes the infimum on the left side of (4.38), then

$$m = \inf_{-\infty < \eta_1, \eta_2 < 0} |f(\eta_1) - f(\eta_2) + c - b|, \quad \text{where} \quad f(\eta) = a \log(1 + |\eta|) + i\eta.$$

For every $\delta > 0$ one has

$$\begin{aligned} m &= \min\{\inf_{|\eta_1 - \eta_2| \geq \delta} |f(\eta_1) - f(\eta_2) + c - b|, \inf_{|\eta_1 - \eta_2| \leq \delta} |f(\eta_1) - f(\eta_2) + c - b|\} \\ &\geq \min\left\{\delta, \inf_{|\eta_1 - \eta_2| \leq \delta} \left| c - b + a \log \frac{1 + |\eta_1|}{1 + |\eta_2|} \right| \right\}. \end{aligned}$$

Since

$$\left| \log \frac{1 + |\eta_1|}{1 + |\eta_2|} \right| \leq ||\eta_1| - |\eta_2|| \leq |\eta_1 - \eta_2|$$

it follows that $m \geq \min\{\delta, c - b - a\delta\}$. For $\delta = (c - b)/(a + 1)$ this last minimum is equal to $(c - b)/(a + 1)$, which proves (4.38). From (4.37) it follows that if $\lambda, z \in \mathbb{C}$

and $0 \neq \lambda \neq z$, then

$$|\lambda|^{-2}\left|\frac{\lambda z}{\lambda - z}\right|^n \le \|(\lambda - T(z))^{-1}\|_{L(\mathbb{C}^n)} \le |\lambda|^{-1} + |\lambda|^{-2}\left\|T\left(\frac{\lambda z}{\lambda - z}\right)\right\|_{L(\mathbb{C}^n)},$$

because $|\lambda|^{-2}\left|\frac{\lambda z}{\lambda - z}\right|^n$ is the modulus of the matrix element at the right upper corner of the triangular matrix $(\lambda - T(z))^{-1}$. Since

$$\left\|T\left(\frac{\lambda z}{\lambda - z}\right)\right\|^2_{L(\mathbb{C}^n)} \le \sum_{k=1}^{n}(n-k+1)\left|\frac{\lambda z}{\lambda - z}\right|^{2k} \le \frac{n(n+1)}{2}\left(1 + \left|\frac{\lambda z}{\lambda - z}\right|^n\right)^2,$$

it follows that if $\lambda \in \widetilde{\Lambda}_{a,c}$ and $z \in \mathbb{C} \setminus \widetilde{\Lambda}_{a,b}$, then

$$\begin{aligned}(4.39)\qquad |\lambda|^{-2}\left|\frac{\lambda z}{\lambda - z}\right|^n &\le \|(\lambda - T(z))^{-1}\|_{L(\mathbb{C}^n)} \\ &\le c^{-1} + c^{-2}\left(\frac{n(n+1)}{2}\right)^{1/2} + \left(\frac{n(n+1)}{2}\right)^{1/2}|\lambda|^{-2}\left|\frac{\lambda z}{\lambda - z}\right|^n.\end{aligned}$$

By (4.38), for $\lambda \in \widetilde{\Lambda}_{a,c}$ and $z \in \mathbb{C} \setminus \widetilde{\Lambda}_{a,b}$ one has

$$\left|\frac{\lambda z}{\lambda - z}\right| \le |\lambda|\frac{|\lambda| + |\lambda - z|}{|\lambda - z|} \le |\lambda|\left(\frac{a+1}{c-b}|\lambda| + 1\right) \le \left(\frac{a+1}{c-b} + \frac{1}{c}\right)|\lambda|^2,$$

and so, by (4.39),

$$\begin{aligned}|\lambda|^{n-2} &= \lim_{z \in \mathbb{C}\setminus\widetilde{\Lambda}_{a,b},\, z\to\infty} |\lambda|^{-2}\left|\frac{\lambda z}{\lambda - z}\right|^n \\ &\le \sup_{\lambda \in \mathbb{C}\setminus\widetilde{\Lambda}_{a,b}} \|(\lambda - T(z))^{-1}\|_{L(\mathbb{C}^n)} = \|(\lambda - G)^{-1}\|_{L(X)} \\ &\le c^{-1} + c^{-2}\left(\frac{n(n+1)}{2}\right)^{1/2} + \left(\frac{n(n+1)}{2}\right)^{1/2}\left(\frac{a+1}{c-b} + \frac{1}{c}\right)^n |\lambda|^{2n-2}.\end{aligned}$$

This proves that $\widetilde{\Lambda}_{a,c} \subset \varrho(G)$, and that (4.36) holds.

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