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Stability and limit behavior of a distributed replicator system

The replicator equation is known to provide a general modeling framework for several distinct areas in mathematical biology. In particular, it arises as a selection equation in population genetics, as a dynamic description of the evolutionary game theory, and as a model for putative chemical reactions describing prebiotic evolution. In its simplest form, when the fitness of the species is a linear function of the relative abundances of other species, the replicator equation takes the form

$$\dot{v}_i = v_i \left[(\mathbf{A}\mathbf{v})_i - f^{loc}(t) \right], \quad i = 1, \dots, n, \tag{1}$$

where $\mathbf{v} = \mathbf{v}(t) = (v_1, \dots, v_n)$, $\mathbf{A}$ is an $n \times n$ matrix with elements a_{ij} describing the contribution of the j-th species to the fitness of the i-th species, $(\mathbf{A}\mathbf{v})_i = \sum_{j=1}^{n} a_{ij} v_j$, and the mean fitness $f^{loc}(t) = \langle \mathbf{A}\mathbf{v}, \mathbf{v} \rangle = \sum_{i=1}^{n} (\mathbf{A}\mathbf{v})_i v_i$ is chosen such that $\mathbf{v} \in S_n = \{\mathbf{v}\colon \sum_{i=1}^{n} v_i = 1\}$.

There are several different approaches to add space to (1). We suggest that the global regulation represents a natural and convenient approach to consider the replicator equation with an explicit spatial structure. To be exact, as a counterpart of the local model (1) we consider the model

$$\frac{\partial u_i}{\partial t} = u_i \left[(\mathbf{A}\mathbf{u})_i - f^{sp}(t) \right] + d_i \Delta u_i, \quad i = 1, \dots, n, \tag{2}$$

where now $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, $\mathbf{x} \in \Omega \subset \mathbb{R}^k$, $k = 1, 2, 3$, $d_i > 0$ are diffusion coefficients, and the mean integral fitness is given, assuming Niemann's boundary conditions, by $f^{sp}(t) = \int_\Omega \langle \mathbf{A}\mathbf{u}, \mathbf{u} \rangle \, d\mathbf{x}$. This approach allows analytical investigation of (2): the tool which was mainly missing in the analysis of replicator equations with explicit space. In particular, it is possible to find the conditions for asymptotically stable rest points of (1) to be asymptotically stable homogeneous equilibria of (2). In our work, we show that for some values of the diffusion coefficients spatially heterogeneous solutions appear. Using a definition for the stability in the mean integral sense we prove that these heterogeneous solutions can be attracting; in particular this is the case for Eigen's hypercycle. Defining in some natural way evolutionary stable states for the distributed system (2), we provide the conditions for this distributed state to be an asymptotically stable stationary solution to (2).

References

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