Global asymptotics of Jack–deformed random Young diagrams via Łukasiewicz paths

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joint work with

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β-ensemble

β-ensemble (a.k.a. log-gas system): $(\ell_1 > \cdots > \ell_N) \in \mathbb{R}^N$ random particles:

$$\mathbb{P}(\ell_1 > \cdots > \ell_N) \sim \prod_{1\leq i<j\leq N} |\ell_i - \ell_j|^\beta \prod_{i=1}^N \exp(-V(\ell_i))$$

Example: $\beta = 2, V(x) = \frac{x^2}{2}$ **G**aussian **U**nitary **E**nsembles

$H = (H_{i,j})_{1\leq i,j\leq N}$ - random Hermitian matrix, $H_N \sim \exp\left(-\frac{N}{2}\operatorname{Tr}(H)\right)$.
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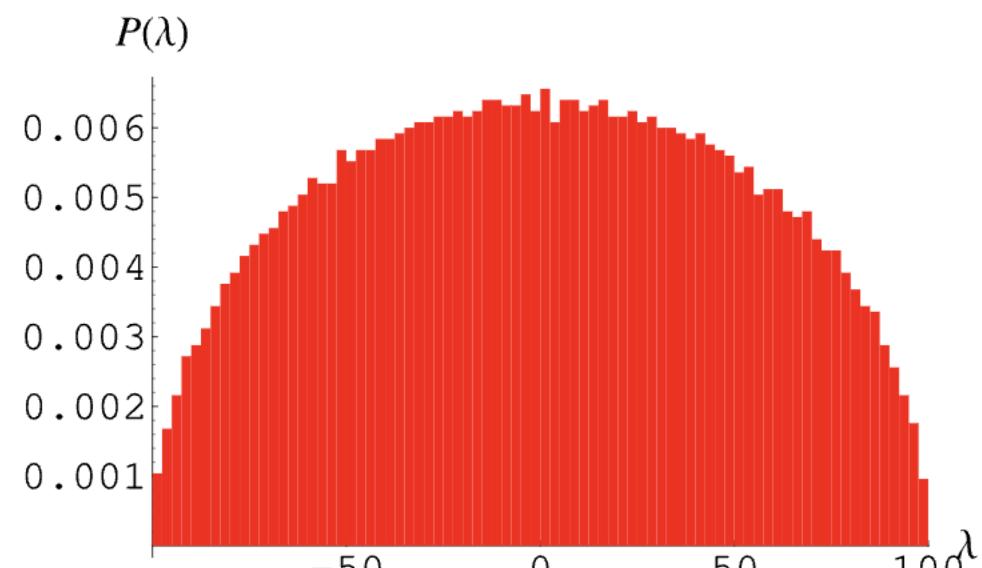
Theorem: $\beta = 2, V(x) = \frac{x^2}{2}$ **[Wigner '55]**

(LLN) $\mu_V^{(\beta)} := \frac{1}{N} \sum_{i=1}^N \delta_{\frac{\sqrt{2}\ell_i}{\sqrt{N\beta}}} \to \mu_V$



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Theorem: **[Johansson '98]**

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Theorem: $\beta = 2, V(x) = \frac{x^2}{2}$ **[Tracy–Widom '93]**,
$\beta > 0, V(x) = \frac{x^2}{2}$ **[Ramírez–Rider–Virág '11]**
$\beta > 0$ **[Bourgade–Erdős–Yau '14]**

$$\mathbb{P}\left(\frac{\sqrt{2}\ell_1 - 2\sqrt{N\beta}}{(N\beta)^{-1/6}}\right) \le x \longrightarrow TW_\beta(x)$$

High temperature regime

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Theorem: $V(x) = \frac{x^2}{2}$ **[Allez–Bouchaud–Guionnet '12]**,

$$\text{(LLN)} \quad \mu_V^{(\beta)} \to \mu_\gamma,$$

which interpolates between the **semi-circle** law ($\gamma = \infty$) and the **normal law** ($\gamma = 0$)

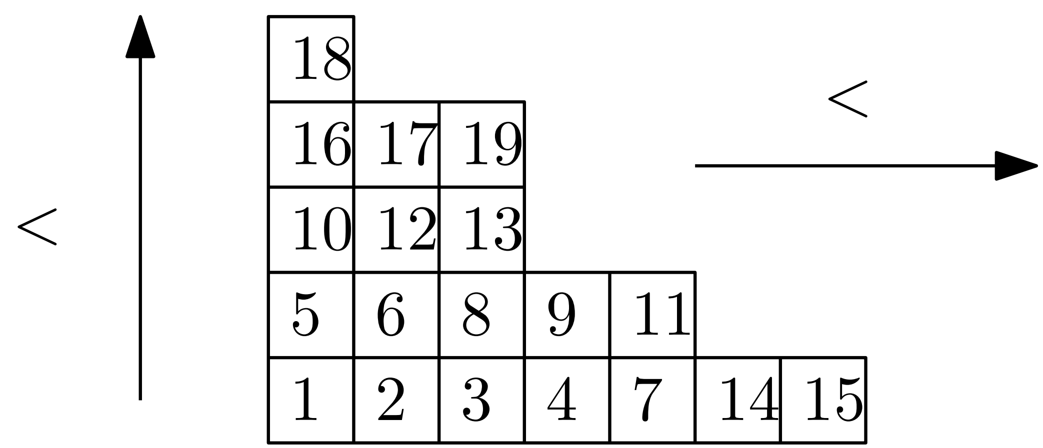
Theorem: $V(x) = \frac{x^2}{2}$ **[Nakano–Trinkh '18]**,

$$\text{(CLT)} \quad \sqrt{N}\left(\int_{\mathbb{R}} x^k \mu_V^{(\beta)} - \int_{\mathbb{R}} x^k \mu_V^{(\beta)}\right) \to \mathcal{N}(0, \sigma^2)$$

Random Young diagrams

Definition: A partition λ of n is a finite, non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ such that $\sum_i \lambda_i = d$ (denote $|\lambda| = d$).

Example: $\lambda = (7, 5, 3, 3, 1), |\lambda| = 19, \ell(\lambda) = 5$.



Standard Young tableau $T \in SYT(\lambda)$

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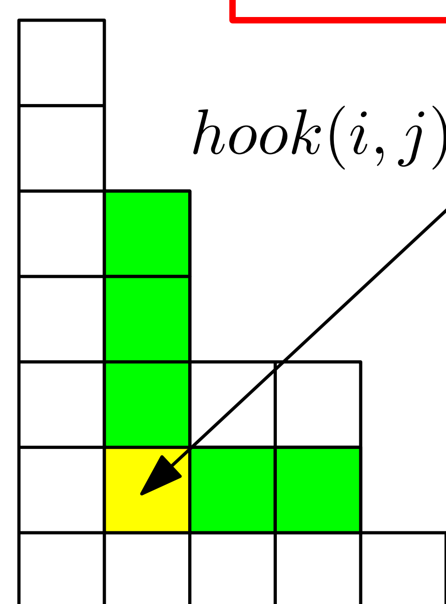
Curious identities:

[Rep. Theo.]

$$\sum_{\lambda \vdash d} |SYT(\lambda)|^2 = d!$$

[Frame–Robinson–Thrall '53]

$$|SYT(\lambda)| = \frac{d!}{\prod_{\square \in \lambda} hook(\square)}$$



$$hook(i,j) := (\lambda_i - j) + (\lambda'_j - i) + 1 = 6$$

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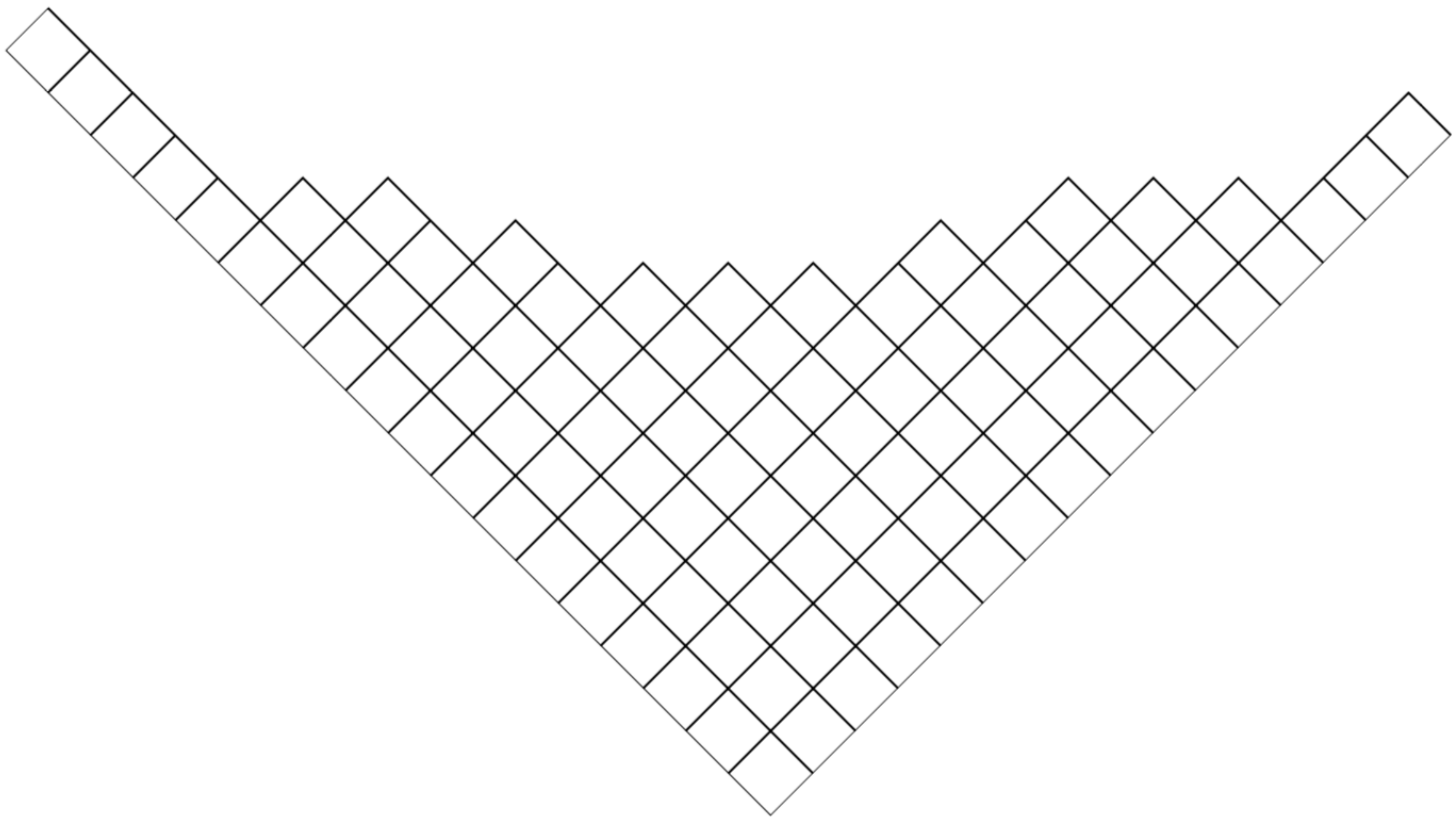
Plancherel measure:

$$\mathbb{P}_d^{(1)}(\lambda) := \frac{d!}{\prod_{(i,j)\in\lambda} \Big((\lambda_i - j) + (\lambda_j' - i) + 1\Big)\Big((\lambda_i - j) + (\lambda_j' - i) + 1\Big)}$$

Logan–Schepp Vershik–Kerov phenomenon



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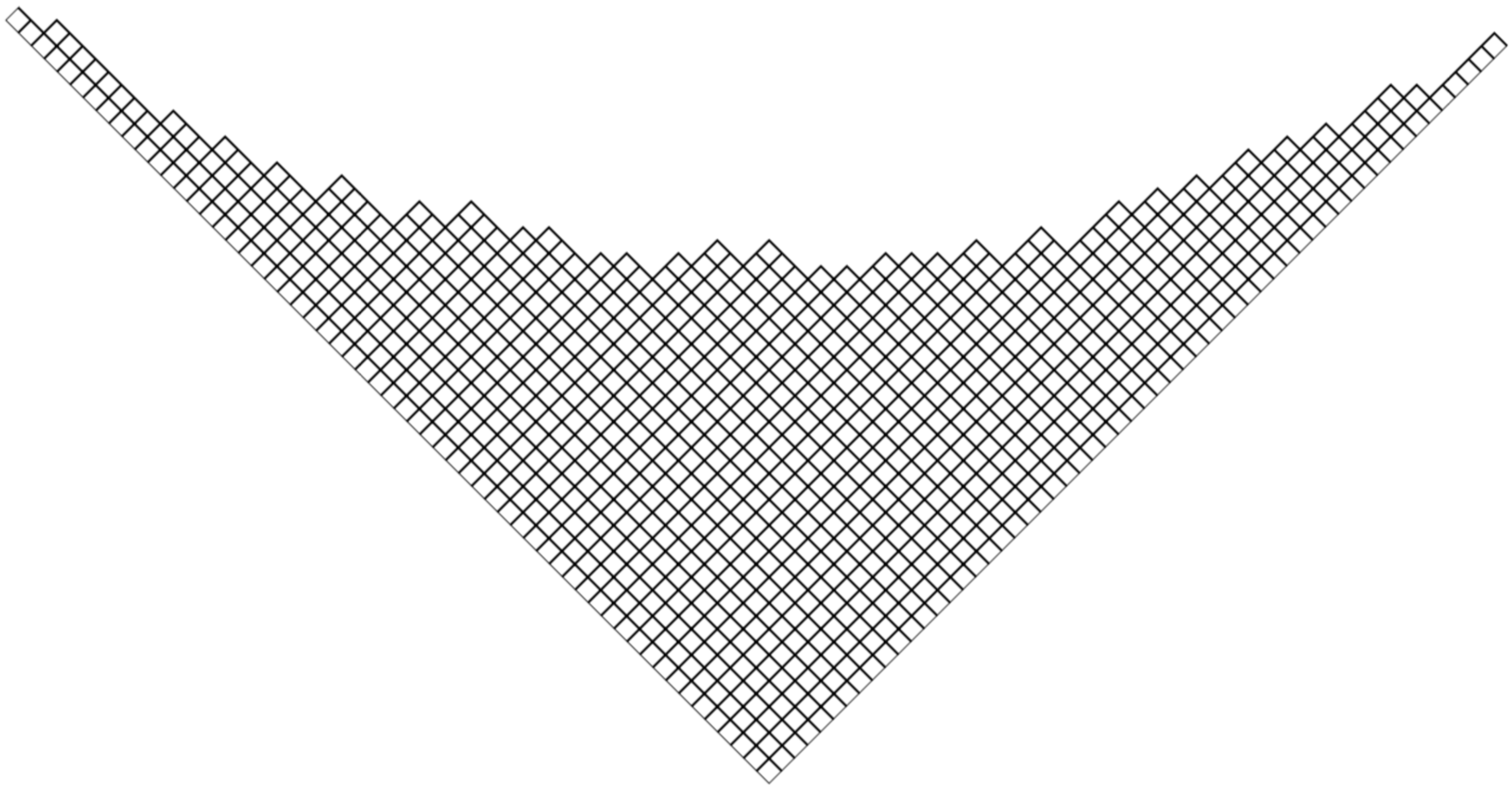

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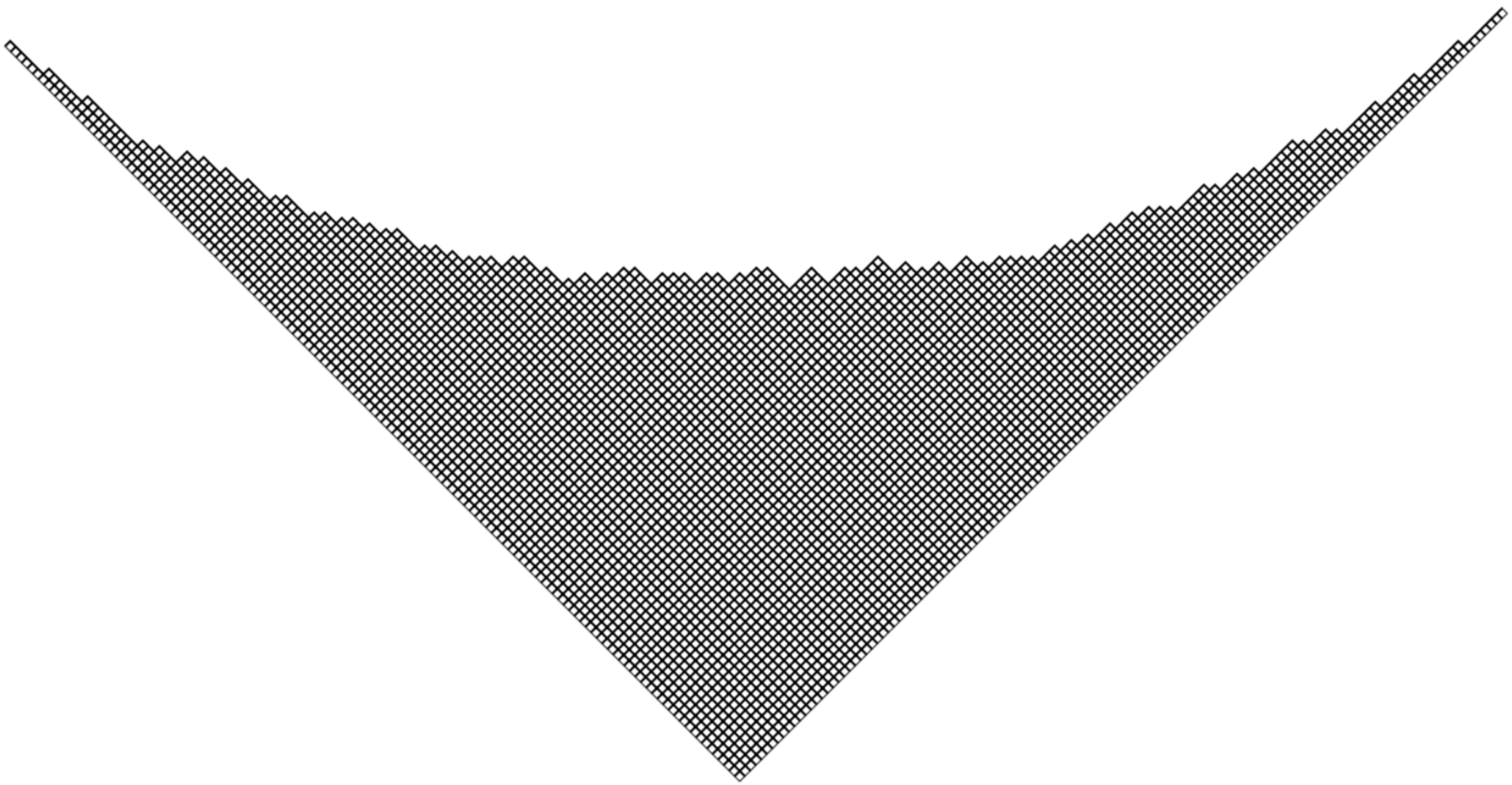

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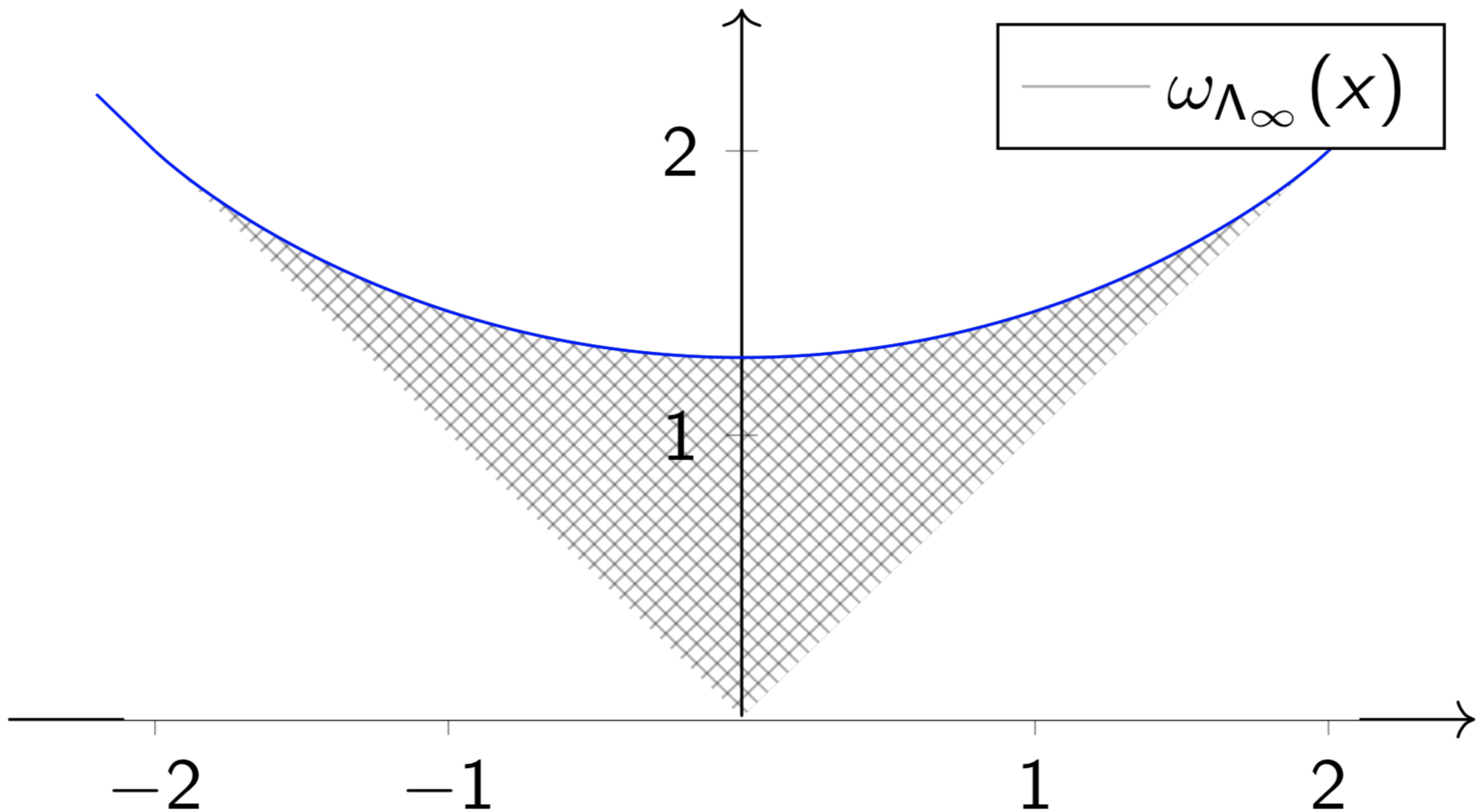

$$\omega_{\Lambda_\infty}(x) = \begin{cases} |x| & \text{if } |x| \geq 2; \\ \frac{2}{\pi}\left(x \cdot \arcsin\frac{x}{2} + \sqrt{4 - x^2}\right) & \text{otherwise.} \end{cases}$$

Logan–Schepp Vershik–Kerov phenomenon

Theorem: [Vershik–Kerov, Logan–Schepp '77]

$\lambda^{(d)} \vdash d$ - random (Plancherel) Young diagram. Then

$$\text{(LLN)} \quad \omega_{\Lambda_d} \to \omega_{\Lambda_\infty},$$

Theorem: [Kerov '93]

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Theorem [Baik–Deift–Johansson '98]

- $\lambda^{(d)}$ - Plancherel-sampled Young diagram of size d
- $\ell_1 > \cdots > \ell_N$ eigenvalues of a random $N \times N$ Hermitian matrix (GUE)

$$\left(\frac{\lambda_1^{(d)} - 2\sqrt{d}}{d^{1/6}}\right) \sim \left(\frac{\ell_1 - 2\sqrt{N}}{N^{-1/6}}\right) \text{ as } d, N \to \infty$$

Conjecture [Baik–Deift–Johansson '98]

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Proved in [Okounkov '00], [Borodin–Olshanski–Okounkov '01], [Johansson '01]

Discrete model of GβE by the Jack–Plancherel measure

Sugestion: **[Kerov'00, Okounkov '03]** Jack–Plancherel measure.

$$\mathbb{P}_d^{(1)}(\lambda) := \frac{d!}{\prod_{(i,j)\in\lambda}\Big((\lambda_i - j) + (\lambda'_j - i) + 1\Big)\Big((\lambda_i - j) + (\lambda'_j - i) + 1\Big)} = \frac{\rho(s_\lambda)\cdot\rho(s_\lambda)}{d!}\Bigg|_{\rho(p_i)=\delta_{i,1}}$$

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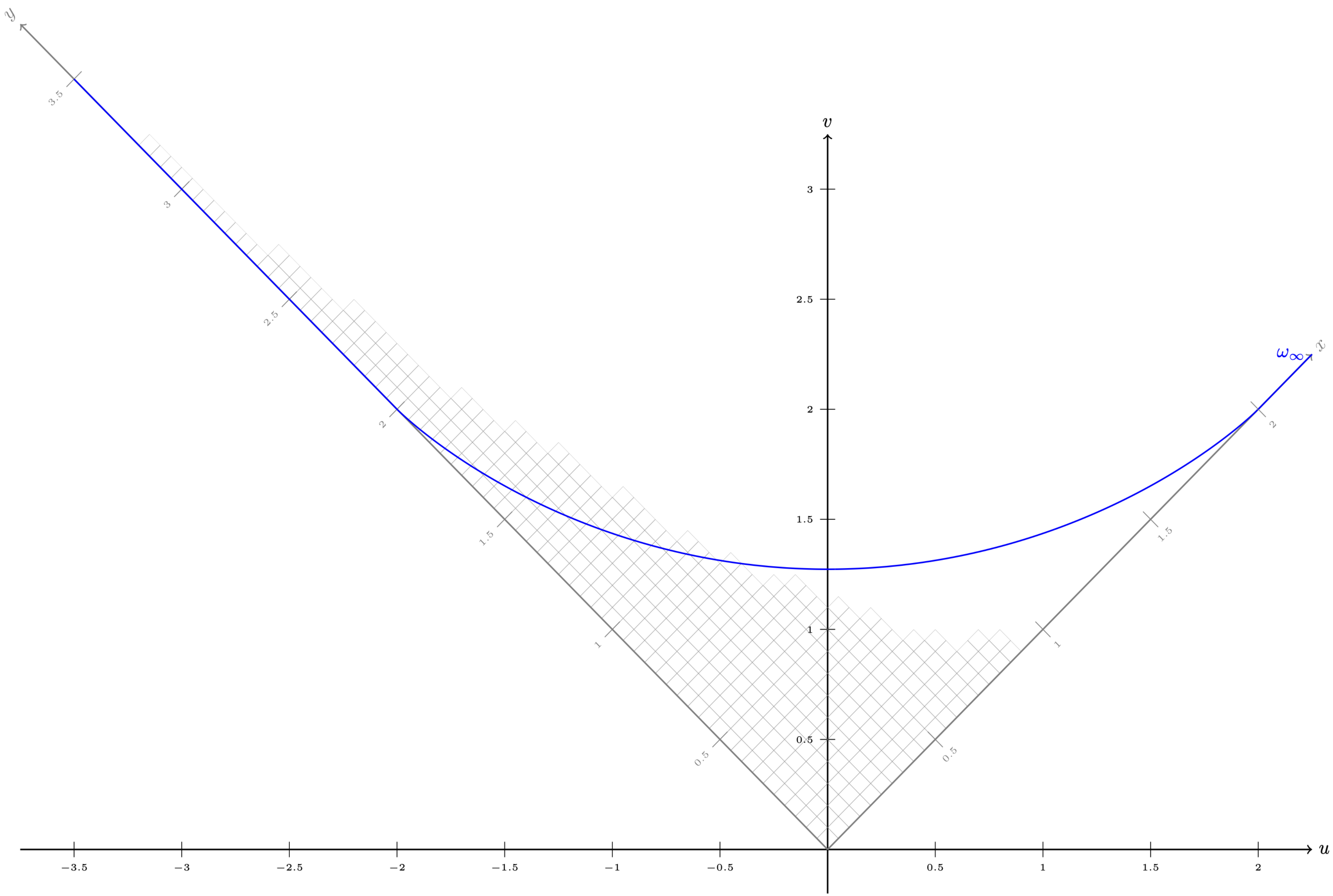

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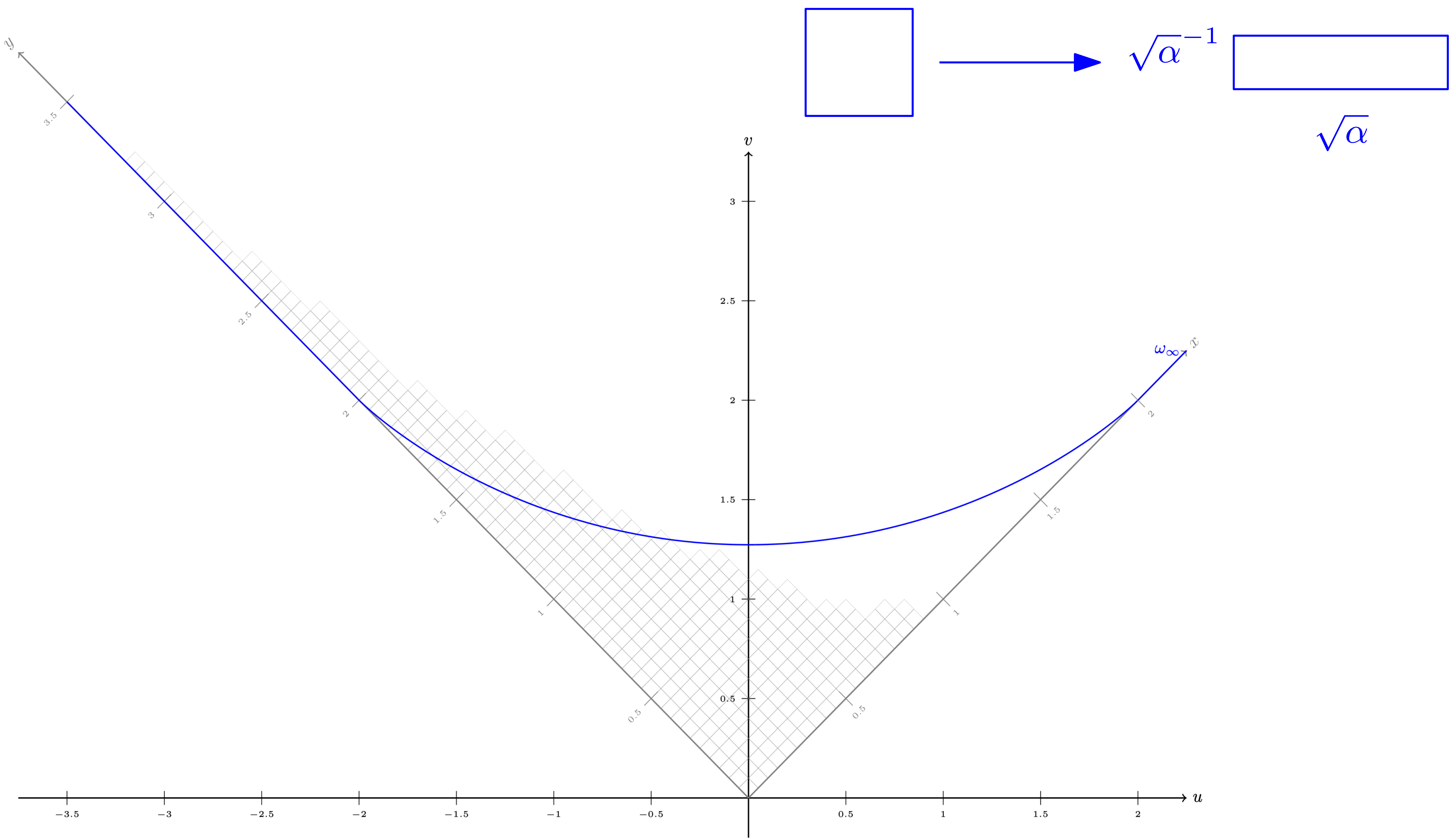

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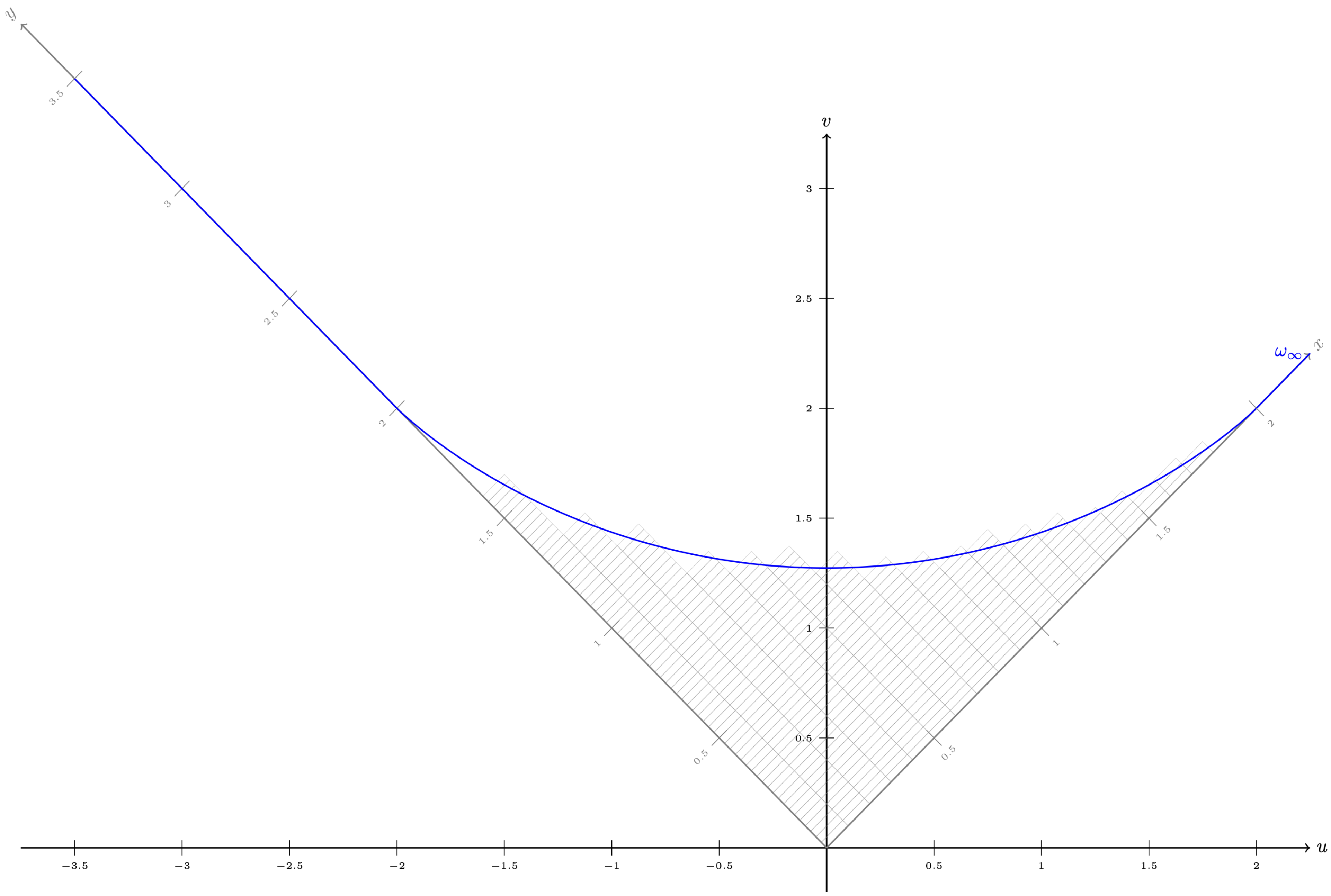

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Question: What is the right discrete analog of β-ensambles with general potential?

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- $\chi_d \colon \mathbb{Y}_d \to \mathbb{C}$
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- $\chi^{(\alpha)}_\lambda(\mu) = C(\mu,\alpha)\,[p_\mu]J^{(\alpha)}_\lambda$

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- $\sum_{k \geq 1} \frac{\rho_1(p_k)\overline{\rho_2(p_k)}}{k} < \infty,$
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Jack measures of Plancherel type

Theorem: [Kerov–Okounkov–Olshanski '98]

ρ: $\mathrm{Sym} \to \mathbb{C}$: α-Jack positive specialization ⟷ Thoma simplex:

$$\Omega = \{(a,b,c) \in \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R} \mid a = (a_1, a_2, \cdots), \quad b = (b_1, b_2, \cdots), \\ a_1 \geq a_2 \geq \cdots \geq 0, \quad b_1 \geq b_2 \geq \cdots \geq 0, \quad \textstyle\sum_{i=1}^\infty (a_i + b_i) \leq c\}.$$

$\omega \in \Omega$ ⟷ $\rho_\omega(p_1) = c,\ \rho_\omega(p_k) = \sum_{i=1}^\infty a_i^k + (-\alpha)^{1-k} \sum_{i=1}^\infty b_i^k, k \geq 2$

Jack measures of Plancherel type

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Jack measure of Plancherel type:

$$\rho_1(p_k) = u \cdot v_k;\ \rho_2(p_k) = u \cdot \delta_{k,1}$$

$$\boxed{M_{u;\mathbf{v}}^{(\alpha)}(\lambda) \sim \frac{J_\lambda^{(\alpha)}(u\cdot\mathbf{v})\cdot u^{|\lambda|}}{j_\lambda^{(\alpha)}}} \quad \mathbf{v} = (v_1, v_2, \ldots) \in \mathbb{R}^\infty,\ u \in \mathbb{R}_{>0}.$$

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Examples:

- $\mathbf{v} = (1, 0, 0, \dots)$; $\quad M^{(\alpha)}_{u;\mathbf{v}}(\lambda) = \exp\left(-\frac{u^2}{\alpha}\right) \frac{u^{2|\lambda|}}{j^{(\alpha)}_\lambda}$; Poissonized Jack–Plancherel measure
- $\mathbf{v} = (1, c, c^2, \dots)$; $\quad M^{(\alpha)}_{u;\mathbf{v}}(\lambda) = \exp\left(-\frac{u^2}{\alpha}\right) \cdot \frac{(uc)^{|\lambda|}}{j^{(\alpha)}_\lambda} \cdot \prod_{(i,j)\in\lambda} \left(\frac{u}{c} + \alpha(j-1) - (i-1)\right)$
  $\frac{u}{c} \in \mathbb{Z}_{\geq 1}$ or $-\frac{u}{c\cdot\alpha} \in \mathbb{Z}_{\geq 1}$

  Poissonized Jack–Schur–Weyl measure

Main results

Assumptions: $\lim_{d\to\infty} \frac{\alpha}{u^2} = 0, \qquad \lim_{d\to\infty} \frac{\alpha-1}{u} = g \in \mathbb{R}, \qquad M^{\alpha}_{u;\mathbf{v}}(\lambda) \geq 0$

Theorem: [Cuenca–D.–Moll '23]

(LLN) $$\omega_{\Lambda_{\alpha;u}} \to \omega_{\Lambda_{g;\mathbf{v}}},$$

(CLT) $$\frac{u}{\sqrt{\alpha}} \left(\int_{\mathbb{R}} x^k \omega_{\Lambda_{\alpha;u}}(x)dx - \int_{\mathbb{R}} x^k \omega_{\Lambda_{g;\mathbf{v}}}(x)dx \right) \to \mathcal{N}(0, \sigma^2)$$

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Assumptions: $\frac{\sqrt{\alpha}-\frac{1}{\sqrt{\alpha}}}{\sqrt{d}}=g+\frac{g'}{\sqrt{d}}+o\left(d^{-\frac{1}{2}}\right),$ as $d\to\infty,$

Theorem: [Cuenca–D.–Moll '23]

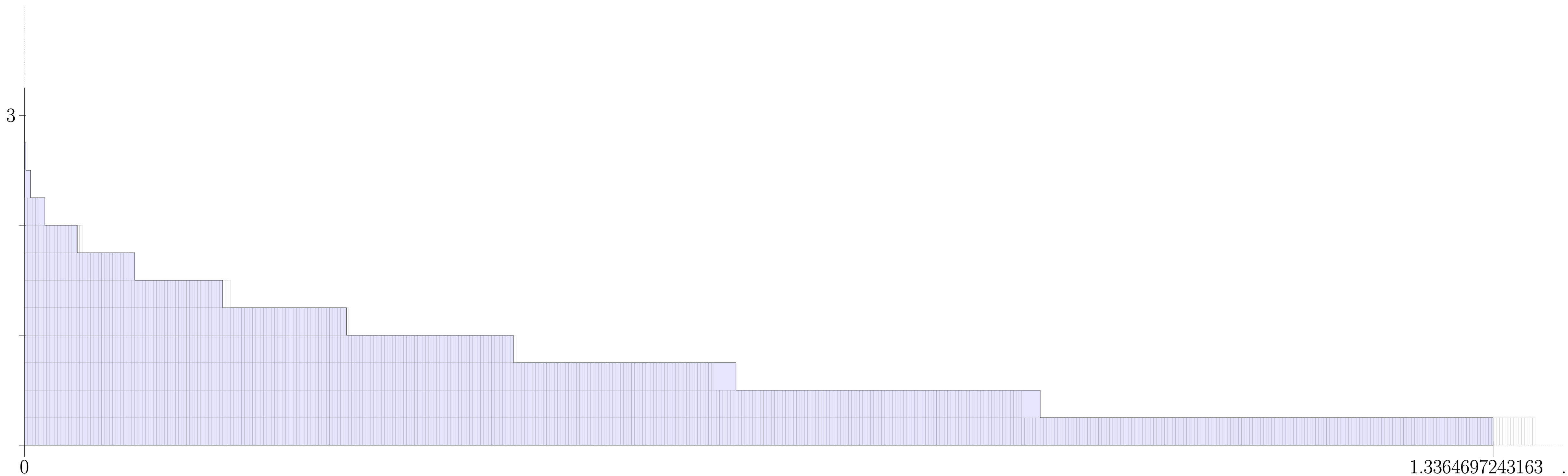
$\chi_d\colon\mathbb{Y}_d\to\mathbb{C}$ - sequence of Jack characters with the (AFP)

(LLN) $$\omega_{\Lambda_{\alpha;u}}\to\omega_{\Lambda_{g;\mathbf{v}}},$$ Universal object

(CLT) $$\sqrt{d}\left(\int_{\mathbb{R}}x^k\omega_{\Lambda_{\alpha;u}}(x)dx-\int_{\mathbb{R}}x^k\omega_{\Lambda_{g;\mathbf{v}}}(x)dx\right)\to\mathcal{N}(0,\sigma^2)$$

Low/High Temperature Limit

Example: Low-temperature limit ($g = -\frac{1}{4}$) of the Jack–Plancherel measure



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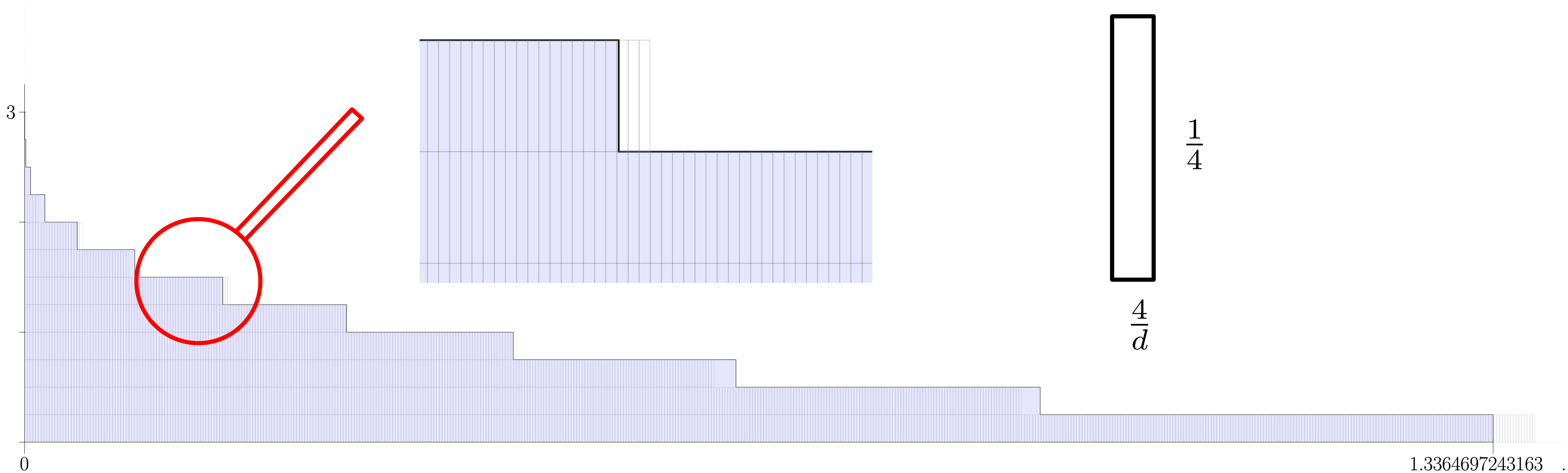

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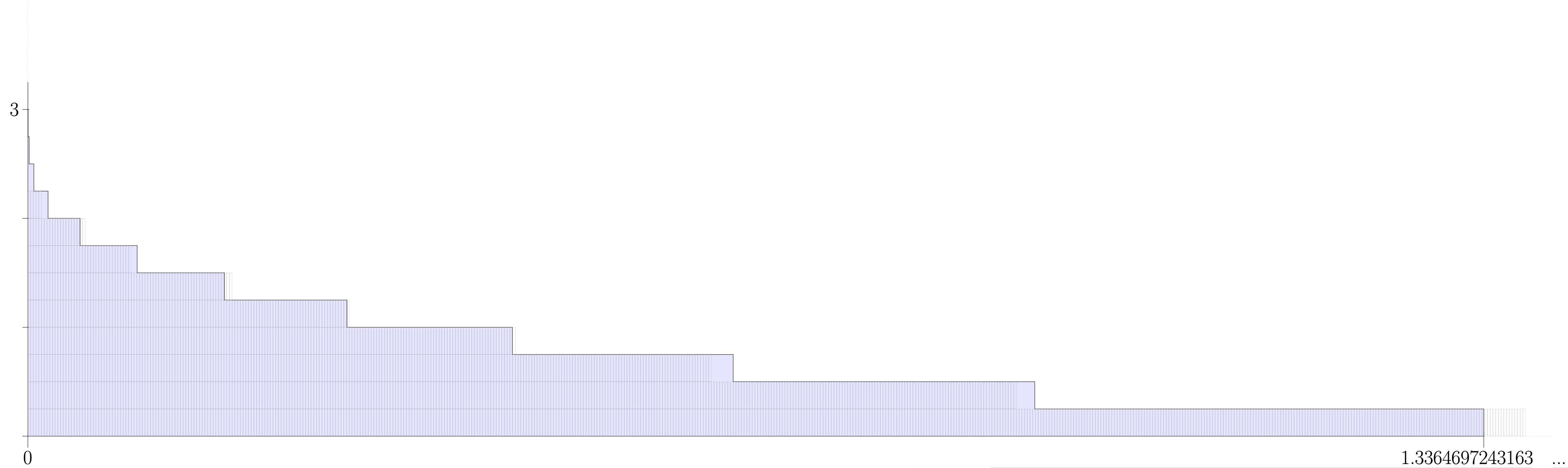


Theorem: **[Cuenca–D.–Moll '23]**

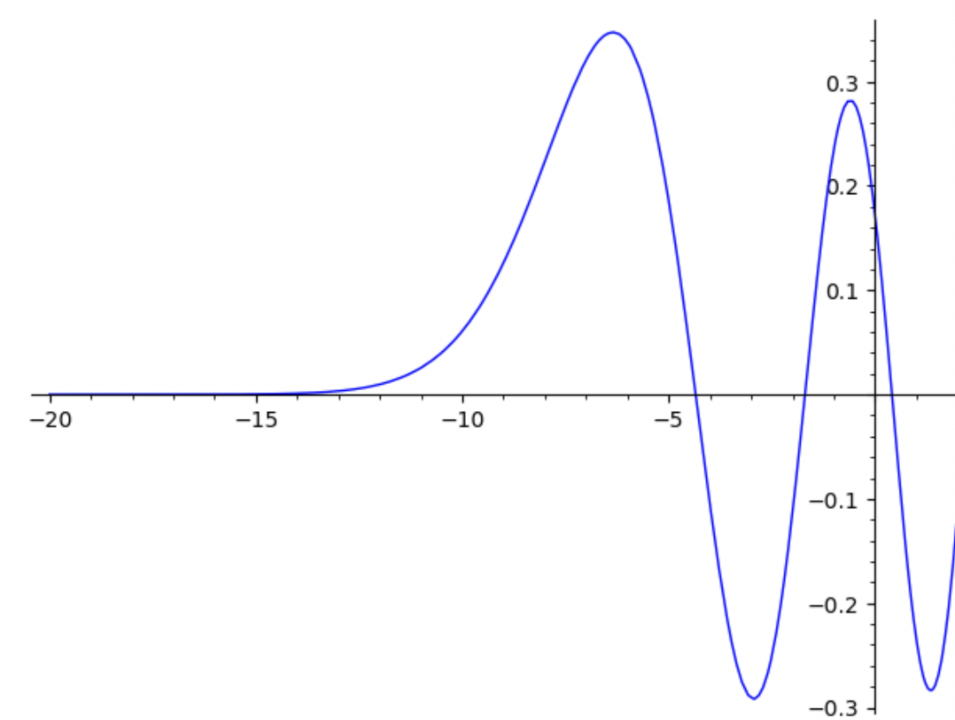
- $\lambda^{(d)} \vdash d$ - Jack–Plancherel distributed
- $J_x(y)$ - Bessel function
- $g < 0, \quad \eta_1^{(g)} < \eta_2^{(g)} < \cdots$ - zeros of $J_{-z}(-\frac{2}{g})$

Then

$$\frac{\lambda_i^{(d)}}{d} \longrightarrow g^2 \cdot (i - \eta_i^{(g)})$$

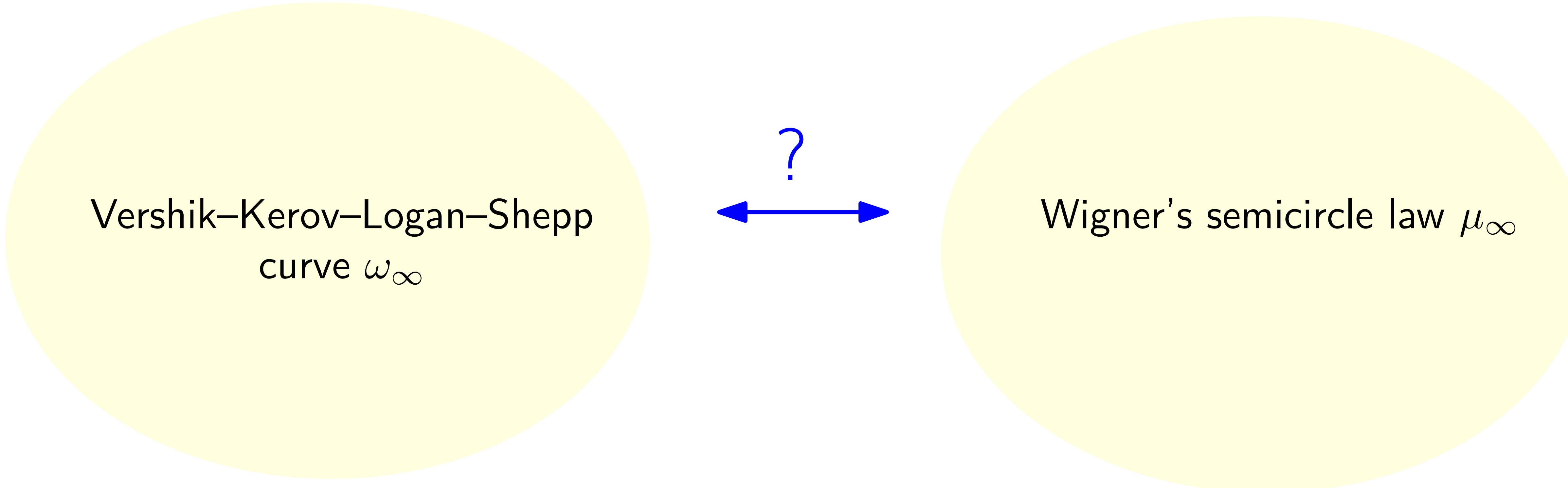
in probability as $d \to \infty, \sqrt{\alpha d} \to -g^{-1}$.

Out[86]:




```
In [84]: 1/4*(1-find_root(f(x),-5,-4))
Out[84]: 1.3364697243163097
```

Markov–Krein correspondence



Markov–Krein correspondence

Continuous Young diagrams:

ω

$\frac{1}{z} \exp \int_{\mathbb{R}} \frac{1}{x-z} \left(\frac{\omega_\Lambda(x) - |x|}{2} \right)' dx$

Markov–Krein

↔

$=$

Probability measures (on $\mathbb{R}$)

μ_ω

$\int_{\mathbb{R}} \frac{d\mu_\omega(x)}{z-x}$

Markov–Krein correspondence

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Markov–Krein ⟷

$=$

Probability measures (on $\mathbb{R}$) μ_ω

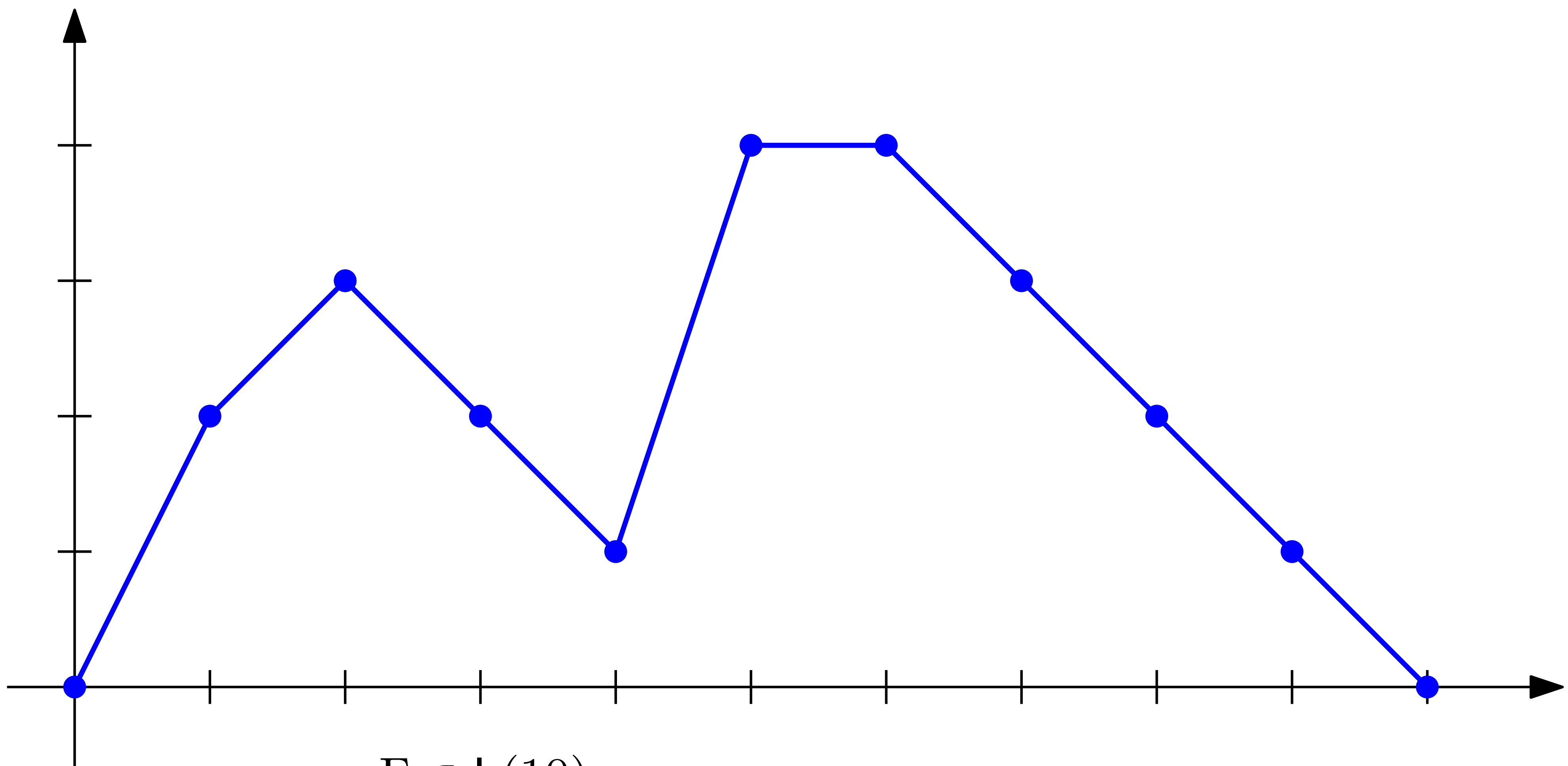
$\int_{\mathbb{R}}\frac{d\mu_\omega(x)}{z-x}$

$\omega_{g;\Lambda}$ uniquely characterized by the moments:

$$M_\ell := \int_{\mathbb{R}} x^\ell d\mu_{\omega_{g;\Lambda}}(x)$$

Moments via Łukasiewicz paths

Steps of the form: $(1, i)$, $(1, 0)$, $(1, -1)$

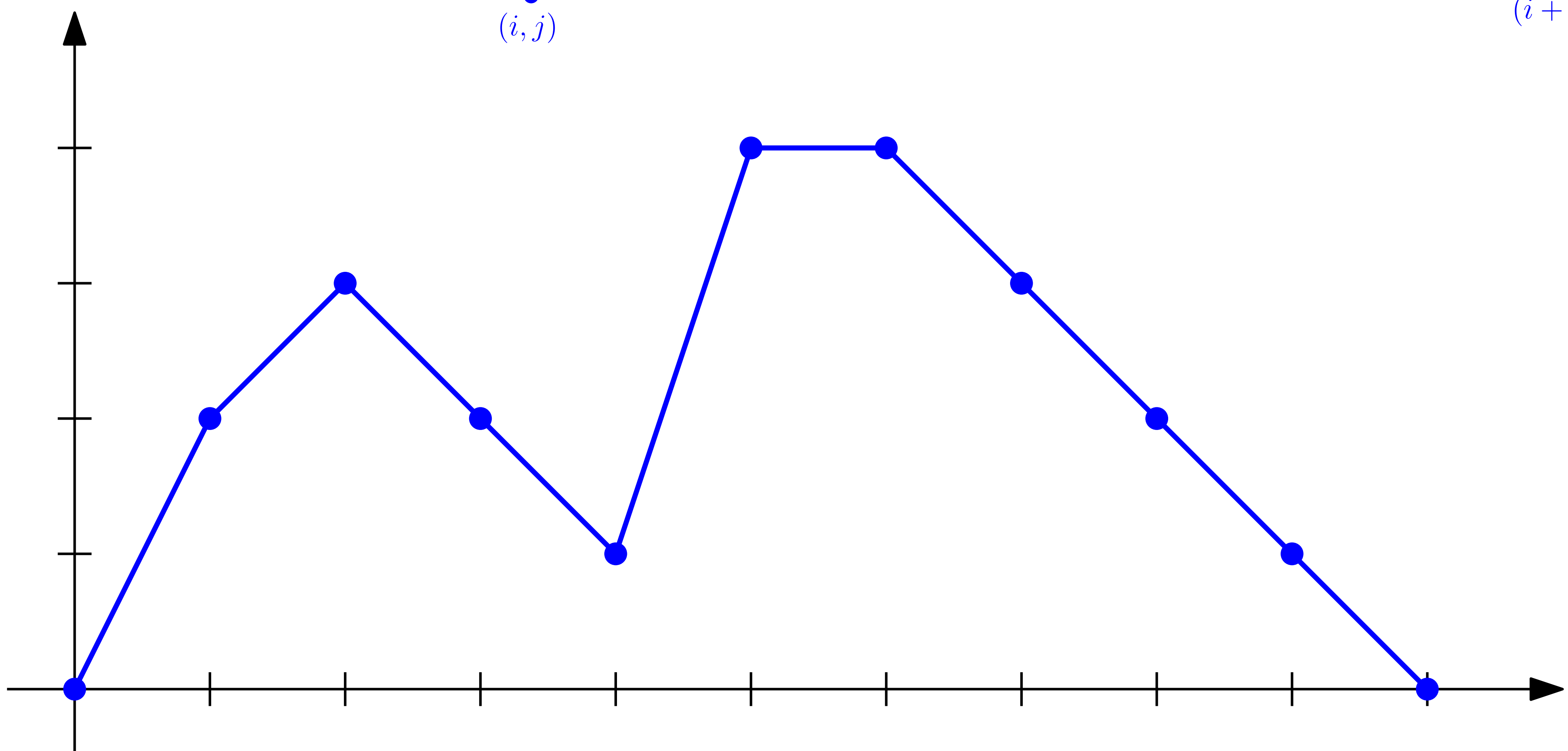


$\Gamma \in \mathsf{L}(10)$

Moments via Łukasiewicz paths

Steps of the form: $(1,i)$, $(1,0)$, $(1,-1)$

Counting with weights: wt [step from (i,j) to $(i+1,j+k)$] $= v_k$ wt [step from (i,j) to $(i+1,j)$] $= j \cdot g$ wt [step from $(i,j+1)$ to $(i+1,j)$] $= 1$



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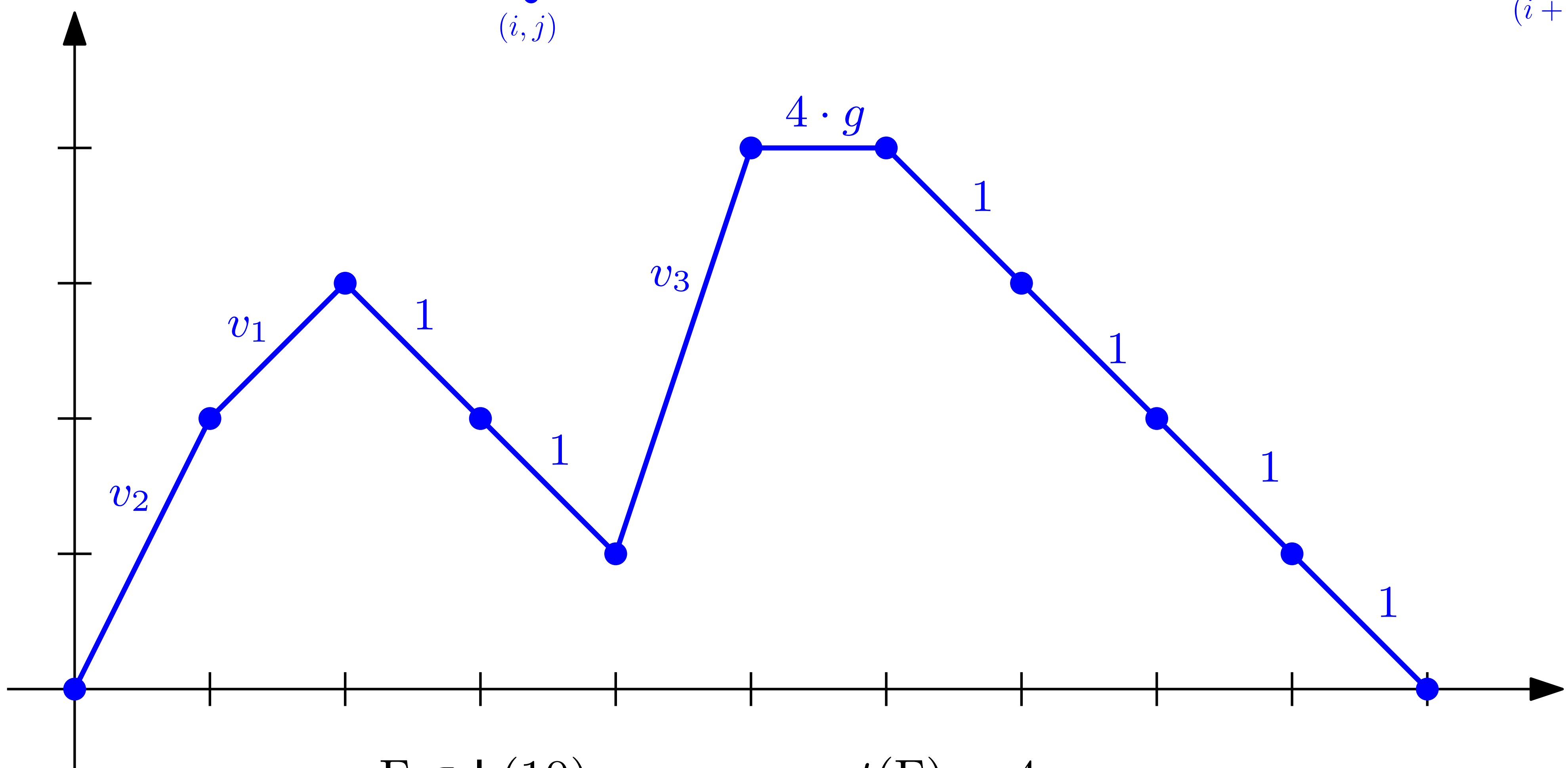

$\Gamma \in \mathsf{L}(10)$ $wt(\Gamma) = 4 \cdot g \cdot v_1 v_2 v_3$

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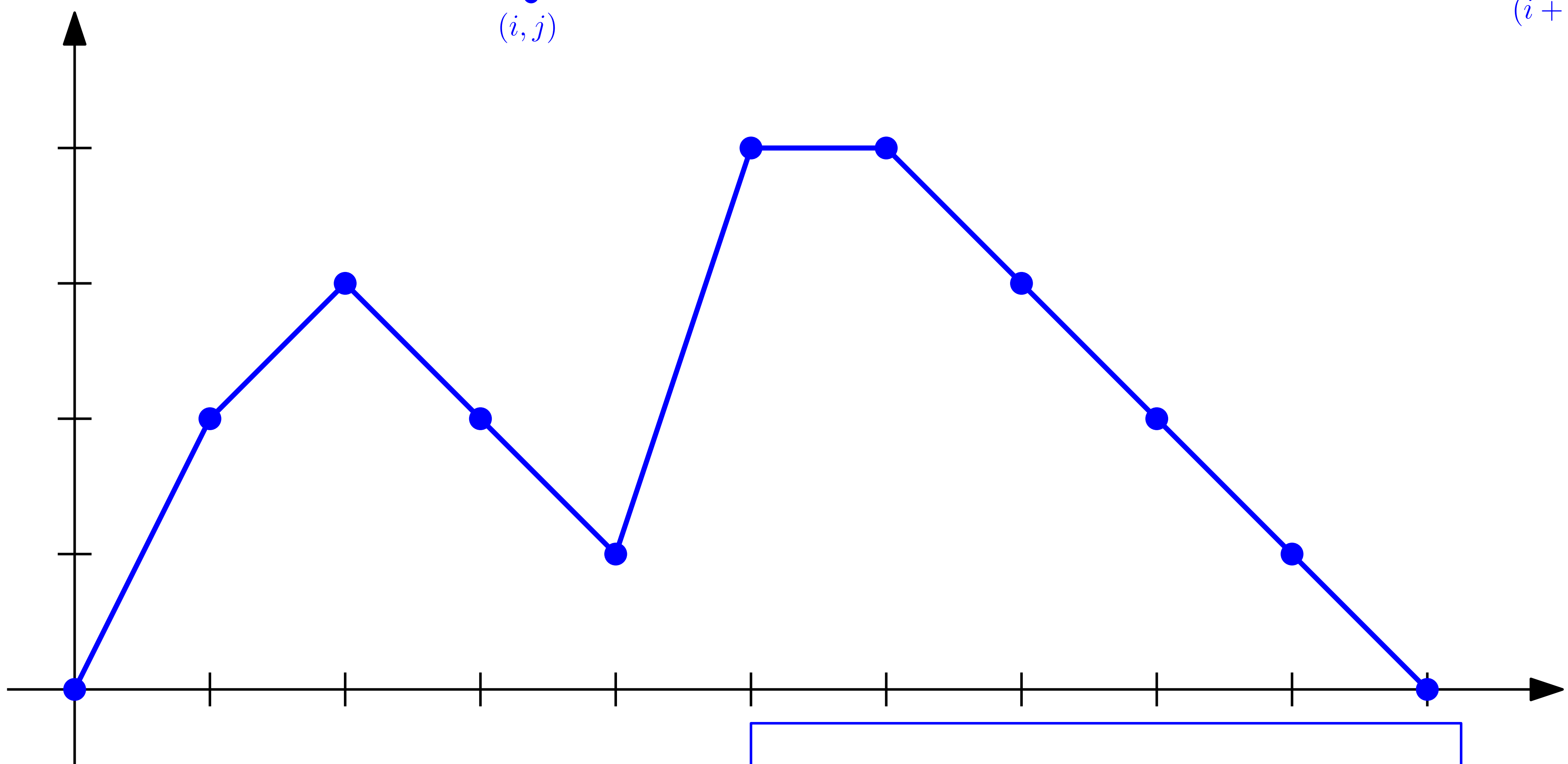

Theorem: [Cuenca–D.–Moll ’23]

$$M_\ell(\mu_{\omega_{g;\Lambda}}) = \sum_{\Gamma \in \mathsf{L}(\ell)} wt(\Gamma)$$

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Examples:

- $g = 0, \mathbf{v} = (1, 0, 0, \dots)$

$$M_\ell = \sum_{\Gamma \in Dyck(\ell)} = \begin{cases} \frac{1}{k+1}\binom{2k}{k} & \text{if } = \ell = 2k, \\ 0 & \text{otherwise.} \end{cases}$$

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Theorem: [Biane '98] $\alpha = 0$

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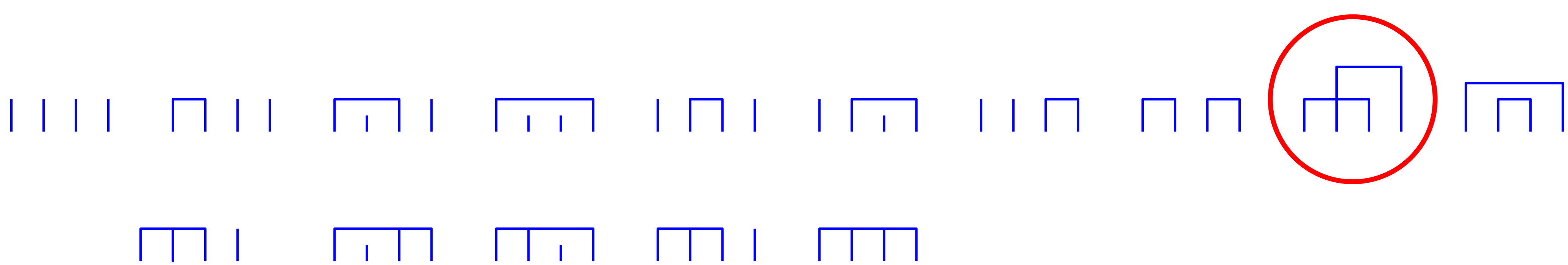
Example: $\ell = 4$



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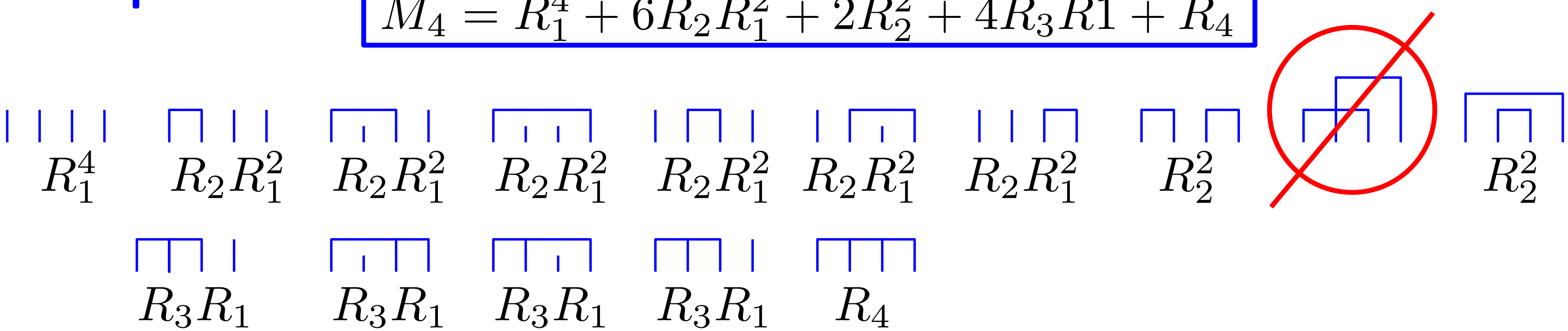
Example: $\ell = 4$ $\boxed{M_4 = R_1^4 + 6R_2R_1^2 + 2R_2^2 + 4R_3R1 + R_4}$



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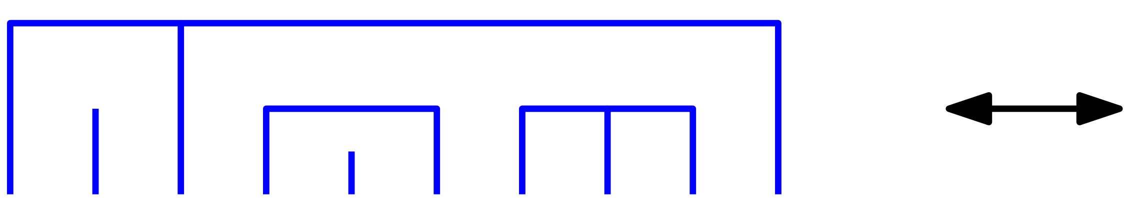

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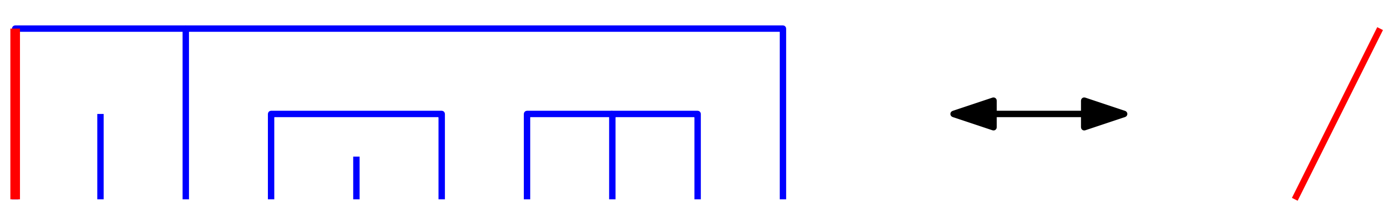

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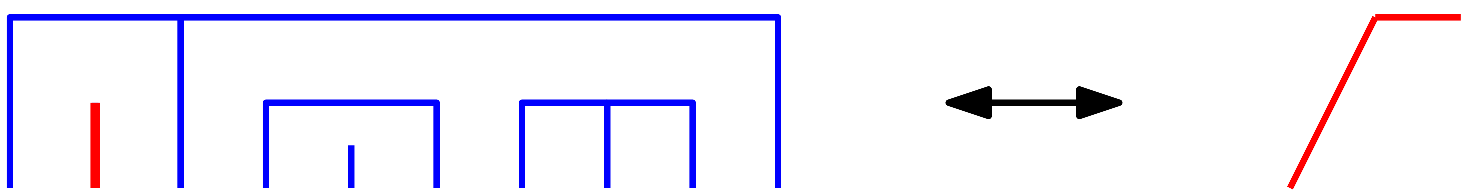

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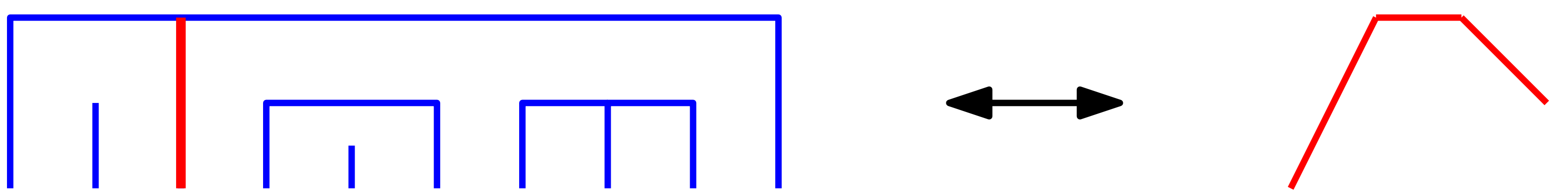

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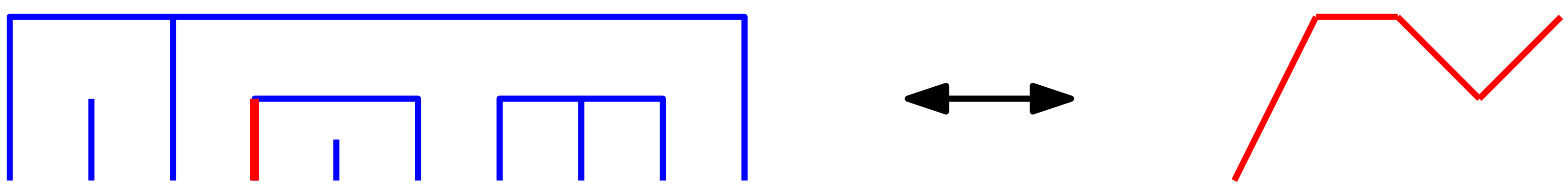

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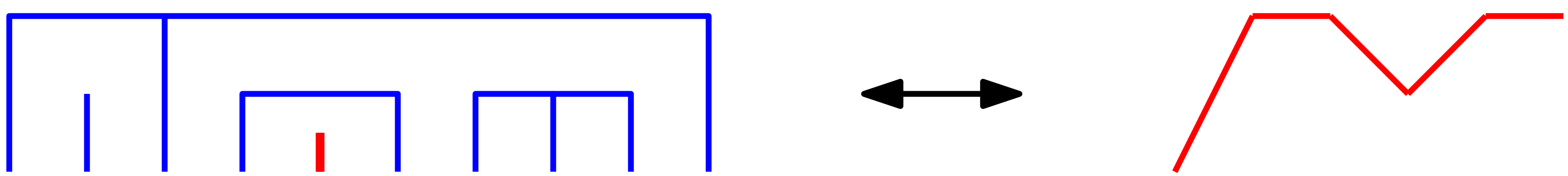

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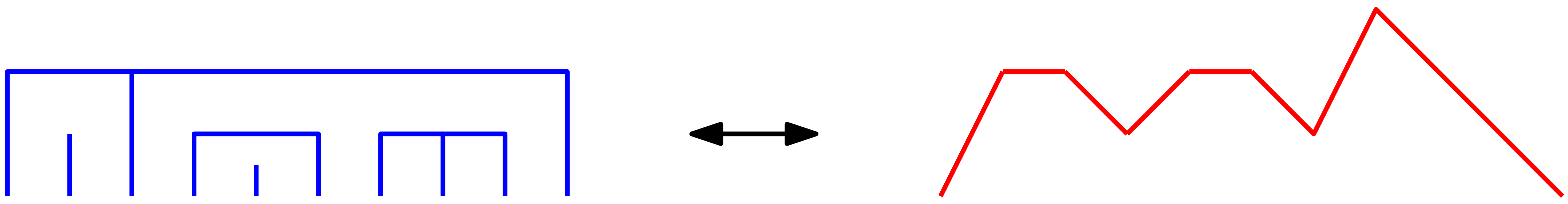

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- $g \neq 0, \mathbf{v} = (v_1, v_2, \dots)$

v_ℓ - one-parameter deformation of the free cumulant $R_{\ell+1}$

Curious combinatorial identity

$$M_\ell = \sum_{\Gamma \in \mathbf{L}(\ell)} wt(\Gamma)$$

Theorem: [Śniady '19]

$$v_\ell = \sum_M \sum_{f:\, V_\bullet(M) \to \mathbb{Z}_{\geq 2}} (-g)^{\ell+1-|V(M)|} \prod_{v \in V_\bullet(M)} R_{f(v)}$$

- M - rooted bipartite oriented maps with ℓ edges
- $\sum_{v \in V_\bullet(M)} f(v) = |V(M)|$
- $\forall \emptyset \subsetneq A \subsetneq V_\bullet(M) \quad \sum_{v \in A} f(v) > |N(A)|$

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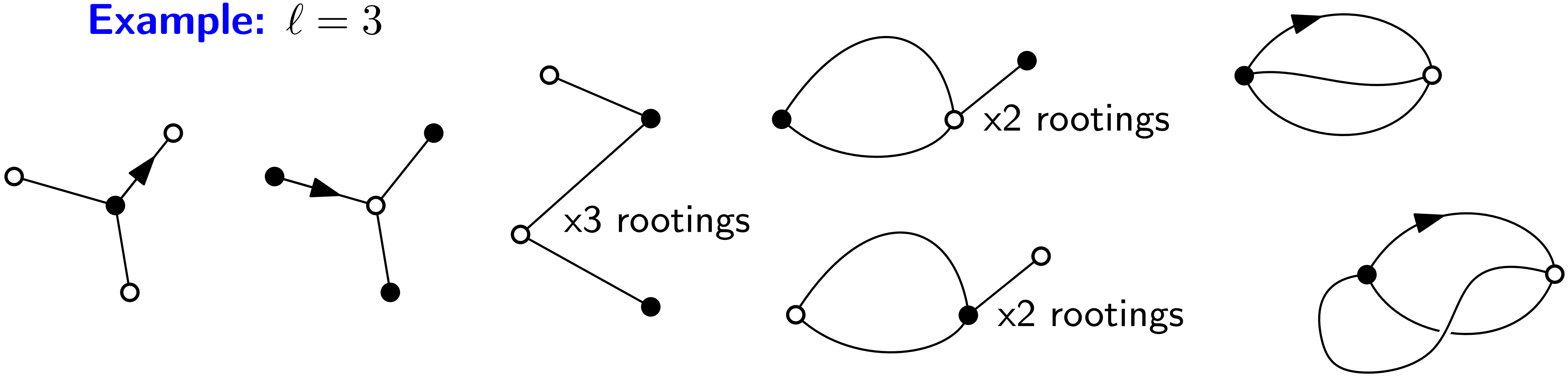
- $\sum_{v \in V_\bullet(M)} f(v) = |V(M)|$
- $\forall \emptyset \subsetneq A \subsetneq V_\bullet(M) \quad \sum_{v \in A} f(v) > |N(A)|$

Example: $\ell = 3$



Curious combinatorial identity

$$M_\ell = \sum_{\Gamma \in \mathbf{L}(\ell)} wt(\Gamma)$$

Theorem: [Śniady '19]

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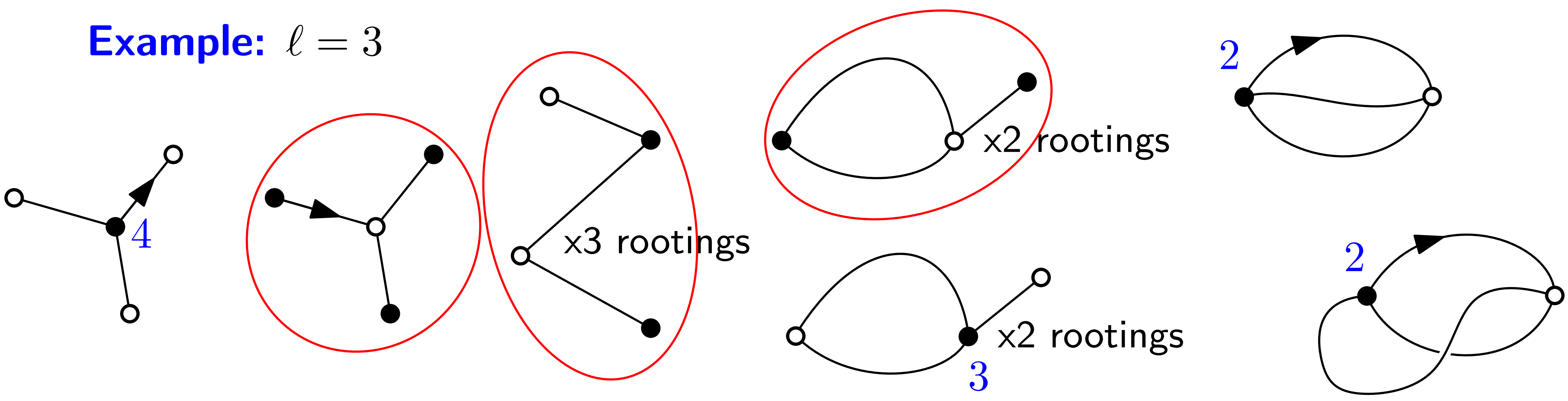

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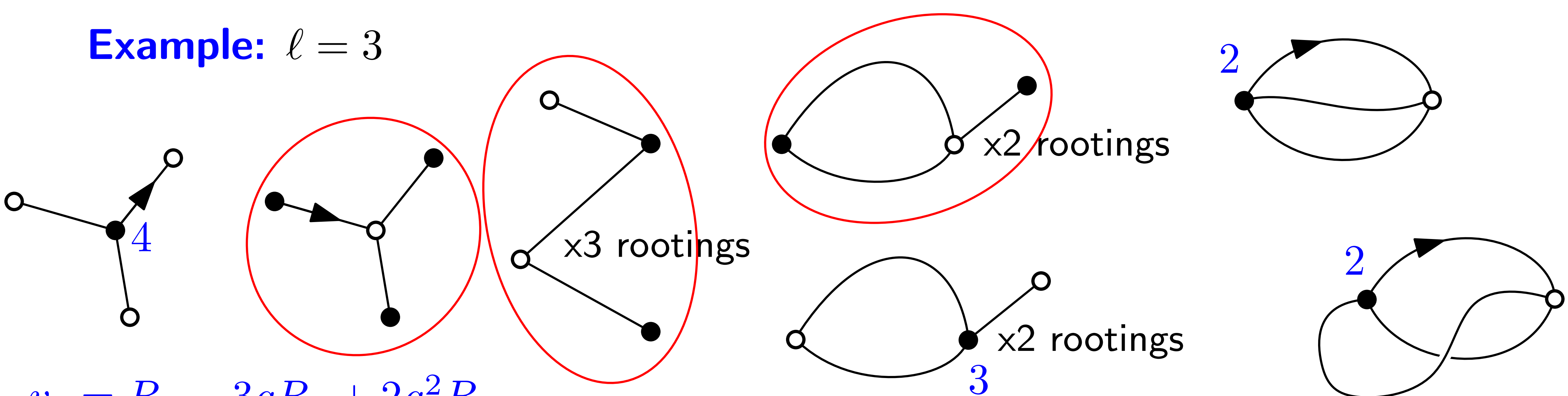

$v_3 = R_4 - 3gR_3 + 2g^2 R_2$

Curious combinatorial identity

$$M_\ell = \sum_{\Gamma \in \mathbf{L}(\ell)} wt(\Gamma)$$

Theorem: **[Śniady '19]**

Problem: Deduce one from another.

$$v_\ell = \sum_M \sum_{f\colon V_\bullet(M) \to \mathbb{Z}_{\geq 2}} (-g)^{\ell+1-|V(M)|} \prod_{v \in V_\bullet(M)} R_{f(v)}$$

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Thank You!