1. Asymptotic representation theory of S_n
   (discrete counterpart of random matrices)

2. β-extension
   (discrete counterpart of β-ensembles)

What we will learn?

(A) A bit of rep. th. $\rightsquigarrow$ symmetric group

(B) A bit of prob. theory $\rightsquigarrow$ method of moments

(C) A bit of symm. function theory

(D) Related combinatorics

Let $\sigma \in S_n$. $\ell(\sigma)$ = length of a maximal increasing subsequence

__Ex.__ $n = 8$ $\quad \sigma = 15326478$

$= 15326478$ $\quad \ell(\sigma) = 5$

$= 15326478$

15326478

Problem: (Ulam '60s)

Let σ_n – uniform random permutation in S_n

What can we say about $\ell(\sigma_n)$ as $n \to \infty$?

random variable

Def: $(X_n)_{n \geq 1}$ - seq. of random variables (r.v.)

X - r.v.

(1) $X_n \xrightarrow{d} X$ converges weakly if $\forall h \in C_b(\mathbb{R})$

$$Eh(X_n) \longrightarrow Eh(X) \quad n \to \infty.$$

(2) $X_n \xrightarrow{P} X$ converges in probability to X

if $\forall \varepsilon > 0$ $\lim\limits_{n\to\infty} P(|X_n - X| > \varepsilon) \to 0$.

(3) $X_n \xrightarrow{a.s.} X$ converges almost surely to X

if $P(\omega : \lim\limits_{n\to\infty} X_n(\omega) = X(\omega)) = 1$.

$(3) \Rightarrow (2) \Rightarrow (1)$

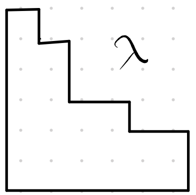
Q1: $E\ell(\sigma_n)$ — how quickly does it grow?

A1: $\sim \sqrt{n}$

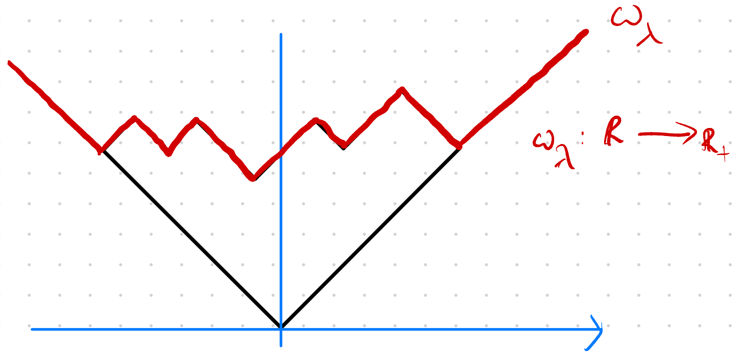
Q2: does the limit $\dfrac{E\ell(\sigma_n)}{\sqrt{n}}$ exist? A2: Yes $= 2$

Q3: How close (and in which sense) $\dfrac{\ell(\sigma_n)}{\sqrt{n}}$ is to 2?

A3: $\lim\limits_{n} P\left(\left|\dfrac{\ell(\sigma_n)}{\sqrt{n}} - 2\right| > \varepsilon\right) = 0 \quad \forall \varepsilon > 0$



French convention



Russian convention

For $\lambda \vdash n$ $\quad \tilde{\omega}_\lambda(x) := \dfrac{\omega_\lambda(x \cdot \sqrt{n})}{\sqrt{n}}$ $\quad$ the "shape" of $\omega_\lambda \equiv \tilde{\omega}_\lambda$

$$\underset{\displaystyle 1}{\overset{\|}{\operatorname{Area}(\tilde{\omega}_\lambda)}} := \int_{\mathbb{R}} \frac{\tilde{\omega}_\lambda(x) - |x|}{2}\, dx$$

<u>Thm:</u> (Vershik-Kerov, Logan-Shepp '77)

$\lambda^{(n)} \in \mathbb{Y}_n$ Plancherel distributed Young diagrams

Then $\forall \varepsilon > 0 \quad \lim_{n \to \infty} \mathbb{P}_n\left(\left\| \tilde{\omega}_{\lambda^{(n)}} - \omega_\infty \right\|_\infty > \varepsilon \right) = 0$

<u>Corollary:</u> (1) $\dfrac{\mathbb{E}\, \ell(\sigma_n)}{\sqrt{n}} \longrightarrow \underset{\text{if exists}}{\Lambda} \Rightarrow \dfrac{\ell(\sigma_n)}{\sqrt{n}} \xrightarrow{\mathbb{P}} \Lambda$

(2) $\Lambda \geq 2$

(3) $\Lambda \leq 2$

P-f: $\mathbb{P}_n\left(\lambda_1^{(n)} \geq (2-o(1))\sqrt{n}\right) \to 1$

Suppose not $\exists\, \varepsilon$ s.t.

$$\mathbb{P}_n\left(\left(\tilde{\omega}_{\lambda^{(n)}}(x) - |x| = 0\right) \text{ on } [2-\varepsilon, 2]\right) > 0$$

for ∞-many n. $\lightning$

Modern approach to VK, L-S theorem

Rep. theory of finite groups

G - finite group

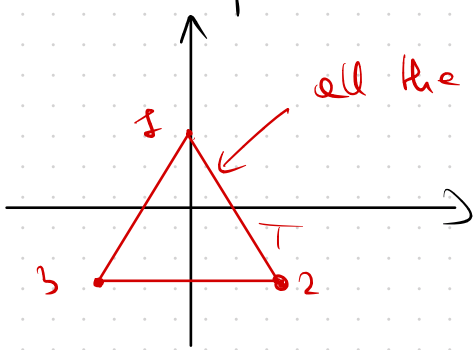
Idea: G is a group of symmetries of a geometric object

$\rho : G \longrightarrow GL(V)$

representation — group homomorphism — vector space $/K$ (Thinks $K = \mathbb{C}$)

Examples:

(1) S_3 - permutation group



all the symmetries of T

$\rho : S_3 \longrightarrow GL(\mathbb{C}^2)$

② $S_n \curvearrowright \mathbb{C}^n$ by permuting coordinates

$$\rho(\sigma)(x_1, \dots, x_n) := (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}).$$

This is a rep. of dimension n.

$$\mathbb{C}^n := \mathbb{C}\cdot(1,1,\dots,1) \oplus \mathbb{C}(1,\dots,1)^{\perp}$$

Another language (R-modules).

R - ring. M is an (left) R-module if $(M,+)$ is an abelian group equiped with $\cdot : R\times M \to M$ s.t.

① $r\cdot(x+y) = r\cdot x + r\cdot y$

② $(r_1+r_2)\cdot x = r_1\cdot x + r_2\cdot x$

③ $r_1\cdot(r_2\cdot x) = (r_1\cdot r_2)\cdot x$

④ $1\cdot x = x$. $\quad \forall x,y\in M,\ r_1,r_2,r\in R.$

The group algebra $K[G] = \mathrm{span}_K(g : g\in G)$ with the multiplication $\left(\sum a_g \cdot g\right)\cdot\left(\sum b_g \cdot g\right) =$

$= \sum_{g,h} a_g\cdot b_h\,(g\cdot h).$

Representations of G over K $\equiv K[G]$ - modules (G-modules in short)

$\rho : G \to GL(V) \longleftrightarrow V$ with the structure $\sum_g a_g\cdot g(v) := \sum_g a_g\cdot\rho(g)v$

<u>Def</u>: V - an R-module $W \leq V$ is a submodule if it is an R-module (with an induced structure) and $W\subseteq V$. Every module has two trivial submodules:

V is called an <u>reducible</u> module if it contains a non-trivial submodule. Otherwise it is called irreducible.

$\rho_1 : G \longrightarrow GL(V)$ is isomorphic to $\rho_2 : G \to GL(W)$ if $\exists\ \phi : V \longrightarrow W$ s.t.

$$\phi \circ \rho_1 = \rho_2 \circ \phi$$

$V \simeq W$ if $\phi : V \longrightarrow W$ which is $K[G]$-isomorphism

G_{Irr} – set of irreducible rep of G up to $\simeq$ (over $\mathbb{C}$)

GOAL: understand G_{Irr}

CHARACTERS

Let $\rho : G \longrightarrow GL(V)$ repr.

We define a character of ρ as

$$\chi_\rho(g) := \mathrm{Tr}(\rho(g))$$

trace

$$\chi : G \longrightarrow \mathbb{C}$$

<u>Idea</u> Classification of Irr. rep. $\Longleftrightarrow$ classification of characters

Sometimes it is convenient to think that

$$\mathbb{C}[G] \equiv \{f : G \to \mathbb{C}\}$$

$$\sum a_g \cdot g \longmapsto f(g) = a_g$$

$\langle \cdot , \cdot \rangle$ on $\mathbb{C}[G]$ by

$$\langle \psi, \chi \rangle := \frac{1}{|G|} \sum_{g} \psi(g) \chi(g^{-1})$$

<u>Lemma:</u> ρ_1, ρ_2 - irr. rep of G

$$\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = \begin{cases} 1 & \text{if } \rho_1 \cong \rho_2 \\ 0 & \neg \end{cases}$$

CLASSIFICATION THEOREM:

Let $C(G) \equiv$ conjugacy classes of G

★ G_{irr} is indexed by $C(G)$

★ $\mathbb{C}[G] = \bigoplus_{\mu \in C(G)} \dim(V_\mu) V_\mu$

★ $|G| = \sum_{\mu \in C(G)} \dim(V_\mu)^2$

<u>Ex:</u> We saw an irr. rep. of S_3 of dimension 2. $\forall G$ $\rho(g) = 1$; $\rho(g) = \mathrm{sgn}(g)$.

$1^2 + 1^2 + 2^2 = 6$

$G = S_n$

$C(S_n) \simeq \mathcal{Y}_n \to (S_n)_{Irr} \simeq \mathcal{Y}_n$

What are Irr. characters of S_n?

$\rightsquigarrow \chi_\lambda$

Def: Symmetric polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ s.t. $f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ $\forall \sigma \in S_n$

Let Λ_n^k - set of homogenous symm. polynomials in n variables of degree k

$$\Lambda_n^k \longleftarrow \Lambda_{n+1}^k$$

$$f(x_1, \dots, x_n, 0) \longleftarrow\!\!\shortmid f(x_1, \dots, x_n, x_{n+1})$$

Then $\Lambda^k := \varprojlim_n \Lambda_n^k \qquad \Lambda = \bigoplus_{k \geq 0} \Lambda^k$

Ex: $p_k(x_1, \dots, x_n) := x_1^k + x_2^k + \dots + x_n^k$

$$p_k := \sum_{i \geq 1} x_i^k$$

Fact: $\Lambda = \mathbb{C}[p_1, p_2, \dots]$

Schur polynomial: Fix λ, and let $n \geq \ell(\lambda)$

$$s_\lambda(x_1,\dots,x_n) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{1\le i,j\le n}}{\det\left(x_j^{n-j}\right)_{1\le i,j\le n}}$$

Jacobi's bialternant formula

Let $s_\lambda = \sum_{T\in SSYT(\lambda)} x^T$ (check: $s_\lambda(x_1,\dots,x_n,0,0,\dots) = s_\lambda(x_1,\dots,x_n)$)

<u>Ex:</u> $\lambda = (2,1)$

$i<j<k$

$$\det\begin{pmatrix} x_1^4 & x_1^2 & 1 \\ x_2^4 & x_2^2 & 1 \\ x_3^4 & x_3^2 & 1 \end{pmatrix} \Big/ \prod_{i>j}(x_i - x_j)$$

$$\begin{array}{|c|c}\hline k & \\ \hline i & \multicolumn{1}{c|}{j} \\ \hline\end{array} + \begin{array}{|c|c}\hline j & \\ \hline i & \multicolumn{1}{c|}{k} \\ \hline\end{array} + \begin{array}{|c|c}\hline j & \\ \hline i & \multicolumn{1}{c|}{i} \\ \hline\end{array} \quad \begin{array}{|c|c}\hline j & \\ \hline i & \multicolumn{1}{c|}{j} \\ \hline\end{array}$$

$$\Rightarrow s_{(2,1)} = 2\sum_{i<j<k} x_i x_j x_k + \sum_{i<j}\left(x_i^2 x_j + x_i x_j^2\right)$$

$$\mathrm{ch} : C(S_n) \longrightarrow \Lambda_n$$

$$\mathrm{ch}(f) := \frac{1}{n!}\sum_{\sigma\in S_n} f(\sigma)\, p_{ct(\sigma)} = \sum_{\mu\vdash n} z_\mu^{-1} f(\mu) p_\mu$$

Frobenius characteristic

<u>Theorem:</u> $\mathrm{ch}(\chi_\lambda) = s_\lambda$

irreducible character of V_λ.

<u>Corollary:</u> $s_\lambda = \sum_{\mu\vdash n} \frac{\chi_\lambda(\mu)}{z_\mu}\cdot p_\mu$ $\qquad \chi_\lambda(\sigma)$, $\sigma\in C_\mu$.

Asymptotic representation theory

Problem: What can we say about a "typical" irr. rep. of S_n when $n \to \infty$.

"Typical" $\equiv$ sampled w.r.t. a reasonable probabilistic model

Plancherel model

G - finite $\quad$ $\mathbb{P}$ - prob. measure on G_{irr}

$$\mathbb{P}(V_\lambda) := \frac{(\dim V_\lambda)^2}{|G|}$$

V-K, L-S $\quad V_\lambda \longrightarrow V_\Omega$ as $n \to \infty$

How to study random Young diagrams?

$\rightsquigarrow$ Convergence of the associated shape?

① Analogy with random matrix theory:

Let H_N - Hermitian matrix

$H_N \in M_{N \times N}(\mathbb{C})$, $\quad H_N^T = \overline{H_N}$

Then $\mathrm{Spec}(H_N) \subseteq \mathbb{R}$

Define $\mu_{H_N} := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$ — probability measure on $\mathbb{R}$

$\mathrm{Spec}(H_N) = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N\}$

Gaussian Unitary Ensemble — $H_N = (h_{ij})_{1 \leq i,j \leq N}$

s.t. $h_{ii} \sim \mathcal{N}(0,1)$

for $k<l$: $h_{kl} = u_{kl} + i\, v_{kl}$, $u_{kl} \sim \mathcal{N}(0,\frac{1}{2})$, $v_{kl} \sim \mathcal{N}(0,\frac{1}{2})$

$h_{lk} := \overline{h_{kl}} = u_{kl} - i v_{kl}$

Let $\mu_{(H_N/\sqrt{N})}$ — spectral measure of GUE (random probability measure)

Theorem: (Wigner '55)

$$\mu_{(H_N/\sqrt{N})} \xrightarrow[N\to\infty]{d} \mu_{S-C} := \underbrace{\frac{1}{2\pi}\sqrt{4-x^2}\,\chi_{[-2,2]}(x)}_{\text{UNIVERSAL OBJECT}}$$

PLAYS THE SAME ROLE IN FREE PROBABILITY AS THE NORMAL DISTR. IN CLASSICAL PROBABILITY

METHOD OF MOMENTS:

Compactly supported prob. measures are characterized by moments.

Cauchy transform:

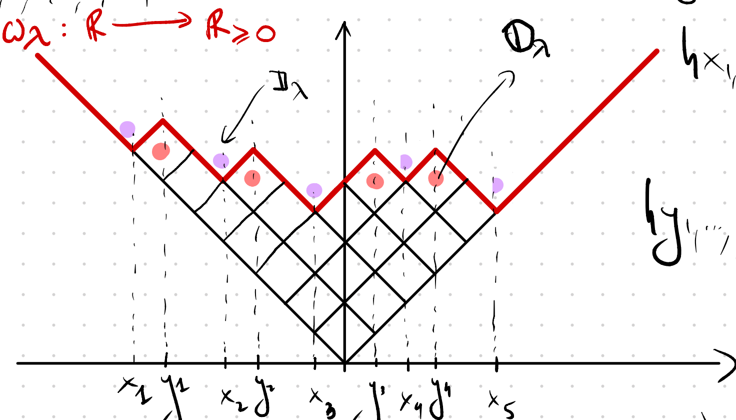
- $G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-x} d\mu(x) := z^{-1} + \sum_{i=1}^{\infty} M_i(\mu) z^{-i-1}$

$$M_i(\mu) := \int_{\mathbb{R}} x^i d\mu(x)$$

Idea of Kerov: Young diagrams $\overset{1-1}{\longleftrightarrow}$ interlacing seq.

$x_1 < y_1 < \dots < y_{k-1} < x_k$

$\lambda = (5,5,4,2,2,1,1)$



$\{x_1, \dots, x_k\} = \{c(\square) : \square \in \mathbb{I}_\lambda\}$

$\{y_1, \dots, y_{k-1}\} = \{c(\square) : \square \in \mathbb{O}_\lambda\}$

$c(i,j) := j - i$

Young diagram $\sim$ 1-Lipschitz function

$\omega(x) = |x|$ for $|x|$ large

$\mu_\lambda \in \mathcal{P}_c(\mathbb{R})$ s.t. $G_{\mu_\lambda}(z) = \dfrac{\prod_{i=1}^{k-1}(z - y_i)}{\prod_{i=1}^{k}(z - x_i)}$

check: supported on $\{x_1, \dots, x_k\}$

Idea 2: Continuous Young diagrams $\equiv$ Probability measures

Def: $\mathcal{Y}_{[a,b]} = \{ \omega : \mathbb{R} \to \mathbb{R}_+ \mid \omega$ - 1-Lipschitz, $\operatorname{supp}(\omega(x) - |x|) \subseteq [a,b] \}$

Markov-Krein correspondence:

Theorem: $$\int_{\mathbb{R}} \frac{1}{z-x} \, d\mu(x) = z^{-1} \exp\left(- \int_{\mathbb{R}} \frac{\omega(x) - |x|}{2(z-x)} dx \right)$$

defines a homeomorphism between

$$\left\{ \begin{array}{c} \mu \in \mathcal{P}_{[a,b]}(\mathbb{R}) \\ \text{weak convergence} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \omega \in \mathcal{Y}_{[a,b]} \\ \text{with the } \|\cdot\|_\infty \end{array} \right\}$$

Conclusion: $\mu_{\lambda^n} \xrightarrow{d} \mu_\Omega \Rightarrow \omega_{\lambda^n} \xrightarrow{\|\cdot\|_\infty} \Omega$

$M_k(\mu_{\lambda^n}) \xrightarrow[n\to\infty]{} m_k \ \forall k$ $\quad m_k$ - seq. of moments then $\mu_{\lambda^n} \to \mu$ s.t. $\int x^k d\mu = m_k$

We want to study $\mathbb{E} M_k(\mu_{\lambda^n})$.

Fact: The continuous Young diagram associated with μ_{S-C} is precisely $\Omega_{V-K, L-S}$.

Plancherel measure via characters

(1) $\mathbb{C}[S_n] = \bigoplus_\lambda V_\lambda^{\dim(V_\lambda)}$

$$\Rightarrow \frac{\chi_{reg}}{n!} = \sum_\lambda \frac{\dim \lambda}{n!} \cdot \chi_\lambda$$

$$\mathbb{P}_{Planch}(\lambda) := \langle \chi_{reg}, \chi_\lambda \rangle \cdot \frac{\chi_\lambda(\mathrm{id})}{\underbrace{\chi_{reg}(\mathrm{id})}_{= n!}}$$

More generally

$$\rho : S_n \longrightarrow GL(V) \qquad \mathbb{P}_\rho(\lambda) := \langle \chi_\rho, \chi_\lambda \rangle \frac{\chi_\lambda(\mathrm{id})}{\chi_\rho(\mathrm{id})}$$

(2) $\mathbb{P}_\rho(\lambda)$ is uniquely determined by χ_ρ

Let $\tilde{\chi}_\rho(\sigma) := \dfrac{\chi_\rho(\sigma)}{\chi_\rho(\mathrm{id})}$

$$\tilde{\chi}_\rho(\sigma) = \mathbb{E}_\rho\left(\tilde{\chi}_{(\cdot)}(\sigma) \right)$$

random variable

$\chi_{(\cdot)}(\sigma) : \mathbb{Y}_n \longrightarrow \mathbb{C}$

<u>Ex:</u> $\mathbb{E}_{Planch_n}\left(\tilde{\chi}_{(\cdot)}(\sigma)\right) = \begin{cases} 1 & \text{if } \sigma = \mathrm{id} \\ 0 & \text{otherwise} \end{cases}$

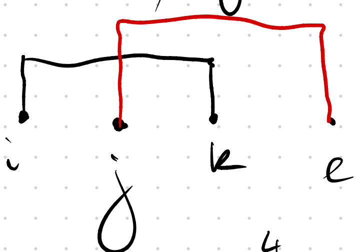
③ Behaviour of moments $\equiv$ behaviour of characters

$$G_\mu(z) = z^{-1} + \sum_{i \geq 1} M_i(\mu) z^{-i-1}$$

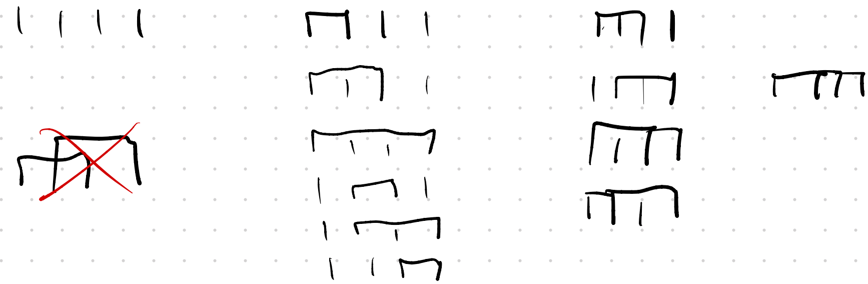
$$R_\mu(z) = G_\mu^{(-1)}(z) - z^{-1} = \sum_{i \geq 1} R_i(\mu) z^{i-1}$$

<u>Ex:</u> $$M_i = \sum_{\pi \in NP([i])} \prod_{B \in \pi} R_{|B|}$$

$\pi \in NP([n])$ if there are no $i<j<k<l$, $i,k \in B_1$, $j,l \in B_2$, $B_1 \neq B_2$, $B_1, B_2 \in \pi$.



<u>Ex:</u> $$M_4 = R_1^4 + \binom{4}{2} R_2 R_1^2 + 4 R_3 R_1 + R_4$$



$+ \, 2R_2^2$

μ_{s-c} has free cumulants equal to $R_i = \delta_{i,2}$

Def: μ - fixed partition

$$Ch_\mu : \mathbb{Y} \longrightarrow \mathbb{C}$$

$$Ch_\mu(\lambda) = \begin{cases} 0 & \text{if } |\lambda| < |\mu| \\ n(n-1)\cdots(n-|\mu|+1)\, \tilde{\chi}_\lambda(\mu\, 1^{n-|\mu|}) \end{cases}$$

where $|\lambda| = n$

Theorem: (Kerov - Olshanski '94)

$\mathrm{Span}_{\mathbb{Q}} \{ Ch_\mu(\cdot) \mid \mu \in \mathbb{Y} \}$ - algebra $\mathcal{A}$

$$\mathcal{A} = \mathbb{Q}[R_2(\cdot), R_3(\cdot), \ldots] = \mathbb{Q}[M_2(\cdot), M_3(\cdot), \ldots]$$
$$= \mathbb{Q}[Ch_1(\cdot), Ch_2(\cdot), \ldots]$$

where $M_i(\lambda) := M_i(\mu_\lambda)$

$R_i(\lambda) := R_i(\mu_\lambda)$

Corollary: For each μ there exists a polynomial $P_\mu(R_2, R_3, \ldots)$ s.t.

$$Ch_\mu = P_\mu(R_2, R_3, \ldots)$$

Ex:

$$Ch_{(1)} = R_2$$
$$Ch_{(2)} = R_3$$
$$Ch_{(3)} = R_4 + R_2$$
$$Ch_{(4)} = R_5 + 5R_3$$
$$Ch_{(5)} = R_6 + 15R_4 + 5(R_2)^2 + 8R_2$$

Theorem 1: (Biane '98)

$Ch_{(k)} = R_{k+1}$ + smaller degree terms (with r.t. $\deg(R_i) = i$)

Theorem 2: (Biane '98)

Let $\rho^{(n)}$ be a seq. of rep. of S_n s.t.

- $\lim_n \tilde{\chi}_{\rho^{(n)}}(k, 1^{n-k}) \sqrt{n}^{\,k-1} = r_{k+1}$ exists

- $\forall\, \sigma_1, \sigma_2$ with disjoint support

$$\tilde{\chi}_{\rho^{(n)}}(\sigma_1 \cdot \sigma_2) - \tilde{\chi}_{\rho^{(n)}}(\sigma_1) \cdot \tilde{\chi}_{\rho^{(n)}}(\sigma_2) =$$

$$o\left(\sqrt{n}^{\,(\ell(\mu_1) - |\mu_1| + \ell(\mu_2) - |\mu_2|)}\right) \text{ where } ct(\sigma_i) = \mu_i$$

Then $\exists\, \omega_\infty : \mathbb{R} \longrightarrow \mathbb{R}_+$ s.t. $\mathbb{P}_n(\|\omega_{\lambda^{(n)}} - \omega_\infty\|_\infty > \varepsilon) \longrightarrow 0$

$$R_k(\mu_{\omega_\infty}) = r_k$$

Ex: (1) $\rho^{(n)}$ – reg. rep. of S_n

$$P_{\rho^{(n)}} = P_{Plench}$$

★ $\chi_{\rho^{(n)}}(k, 1^{n-k}) = \begin{cases} 1 & k=1 \\ 0 & k>1 \end{cases}$

$$\Rightarrow \quad r_k = \begin{cases} 1 & k=2 \\ 0 & k>2 \end{cases}$$

$$\Rightarrow r_k = R_k(\mu_{S-C})$$

$$\chi_{\rho^{(n)}}(\sigma_1 \cdot \sigma_2) - \chi_{\rho^{(n)}}(\sigma_1)\chi_{\rho^{(n)}}(\sigma_2) = 0$$

★ $V = \mathbb{C}^N$ Then

$$S_n \curvearrowright \underbrace{V \otimes \cdots \otimes V}_{n}$$

$$e_1 \otimes \cdots \otimes e_n \longmapsto e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)}$$

$\rho_N^{(n)}$ – Schur–Weyl rep.

Let $\text{ct}(\sigma) = \mu$

Then $\chi_{\rho_N^{(n)}}(\mu) = N^{\ell(\mu)}$

$$\Rightarrow \lim_{n\to\infty} \frac{\chi_{\rho_N^{(n)}}(k, 1^{n-k})}{\chi_{\rho_N^{(n)}}(1^n)} \cdot \sqrt{n}^{k-1} = \lim \left(\frac{\sqrt{n}}{N}\right)^{k-1} = c^{k-1}$$

for $\sqrt{n} \sim c \cdot N$

Therefore μ_{ω_∞} is characterized

$$R_k(\mu_{\omega_\infty}) = c^{k-2} \quad (\text{Free Poisson}) \quad k \geq 2$$

Sketch of the proof of Th. 1

Step 1: Let $\mu \vdash n$ $\quad p_\mu = \sum_{\lambda \vdash n} \chi_\lambda(\mu) \cdot s_\lambda$

Therefore $\chi_\lambda(1^n) = [s_\lambda] \, p_{1^n}$

$$\chi_\lambda(k, 1^{n-k}) = [s_\lambda] \, p_{1^{n-k}} \cdot p_k$$

Fix λ, $N \geq \ell(\lambda)$ and let $\ell_i := \lambda_i + N - i$

$$[s_\lambda] \, p_{1^n} = \frac{n!}{\prod \ell_i!} \operatorname{Vand}(\ell_1, \dots, \ell_N)$$

$$[s_\lambda] \, p_{1^{n-k}} \cdot p_k = (n-k)! \sum_{j=1}^{N} \frac{\operatorname{Vand}(\ell_1, \dots, \ell_j - k, \dots, \ell_N)}{\prod_{i \neq j} \ell_i! \, (\ell_j - k)!}$$

$$\Rightarrow \operatorname{Ch}_k(\lambda) = \sum_{j=1}^{N} \frac{\ell_j!}{(\ell_j - k)!} \frac{\prod_{i \neq j} (\ell_i - \ell_j + k)}{\prod_{i \neq j} (\ell_i - \ell_j)}$$

Step 2: Let $\phi_\lambda(x) = \prod_{i=1}^{N} (x - \ell_i)$

Then the previous formula is equivalent to

$$\operatorname{Ch}_k(\lambda) = -\frac{1}{k} [x^{-1}] \, x(x-1) \cdots (x-k+1) \frac{\phi_\lambda(x-k)}{\phi_\lambda(x)}$$

Step 3:

Recognize that $$\frac{z\,\phi_\lambda(z-1)}{\phi_\lambda(z)} = \frac{1}{G_\mu(z+N-1)}$$

$$\Longrightarrow \quad Ch_{(k)}(\lambda) = \frac{-1}{k}[x^{-1}]\, \frac{1}{G_\mu(x+N-1)\cdots G_\mu(x+N-k)}$$

$$\approx -\frac{1}{k}[x^{-1}]\, G_{\mu_\lambda}(x)^{-k} = [z^k]\, R_{\mu_\lambda}(z)$$

Lagrange inversion formule

$$R_{\mu_\lambda}(z) = R_{k+1}(\lambda).$$

□