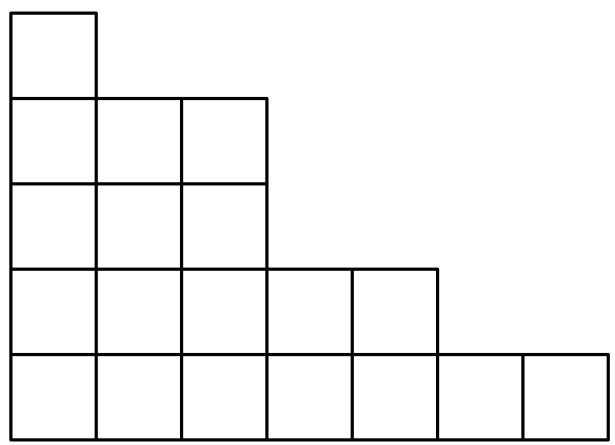
Young diagrams and tableaux - a curious identity...

Definition: A partition λ of n is a finite, non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $\sum_i \lambda_i = n$ (denote $|\lambda| = n$).

Example: $\lambda = (7, 5, 3, 3, 1), |\lambda| = 19, \ell(\lambda) = 5$.



Young diagram λ

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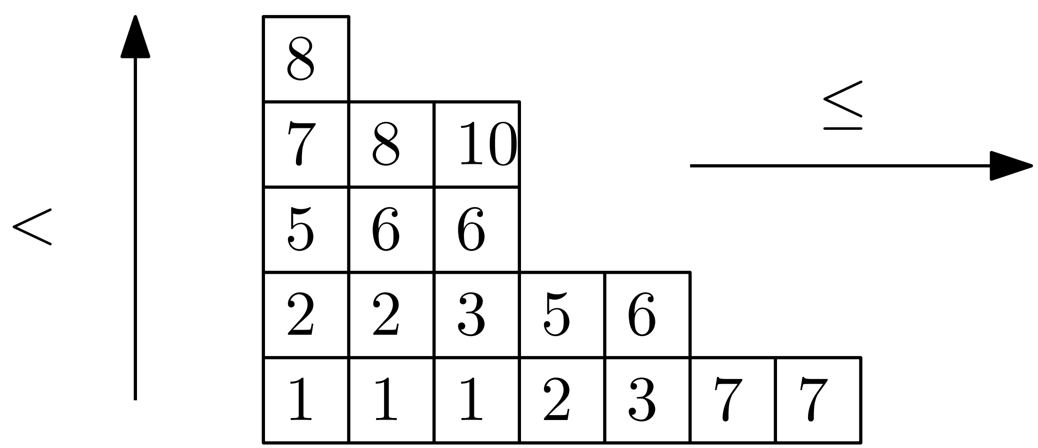

Semistandard Young tableaux $T \in SSYT(\lambda)$

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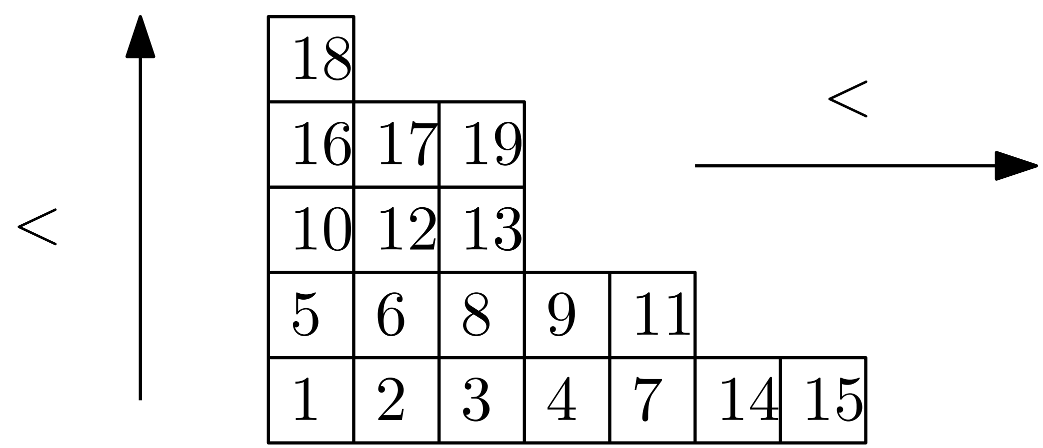

Standard Young tableau $T \in SYT(\lambda)$

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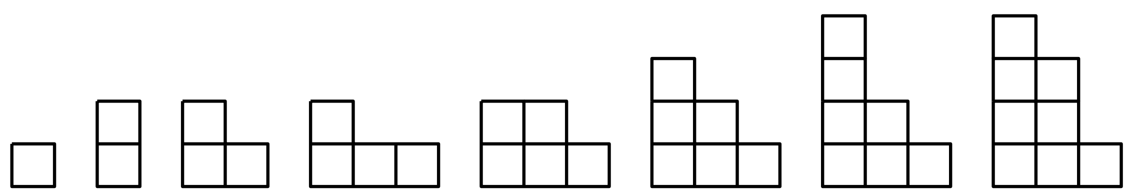
Remark Standard Young tableaux of size $n \leftrightarrow$ seuqnces of Young dirgrams $\lambda^1 \subset \lambda^2 \subset \cdots \subset \lambda^n$ s.t. $|\lambda^i| = i$.

Example:



7		
6	8	
2	5	
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Young diagrams and tableaux - a curious identity...

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Curious identity: (consequence of the representation theory of symmetric groups)

$$\sum_{\lambda \vdash n} |SYT(\lambda)|^2 = n!$$

Problem: Find a combinatorial explanation of this identity $\equiv$ find a bijection:

$$F: S_n \to \bigcup_{|\lambda|=n} SYT(\lambda) \times SYT(\lambda).$$

RSK algorithm

Input:

- a word $w = (w_1, \cdots, w_n) \in \mathbb{N}_{\geq 0}^n$

Output:

- a semistandard tableau $P \in SSYT(\lambda)$
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Example:

$w = (23, 53, 74, 16, 99, 70, 82, 37, 41)$

74	99		
23	53	70	
16	37	41	82

insertion tableau $P(w)$

8	9		
4	6	7	
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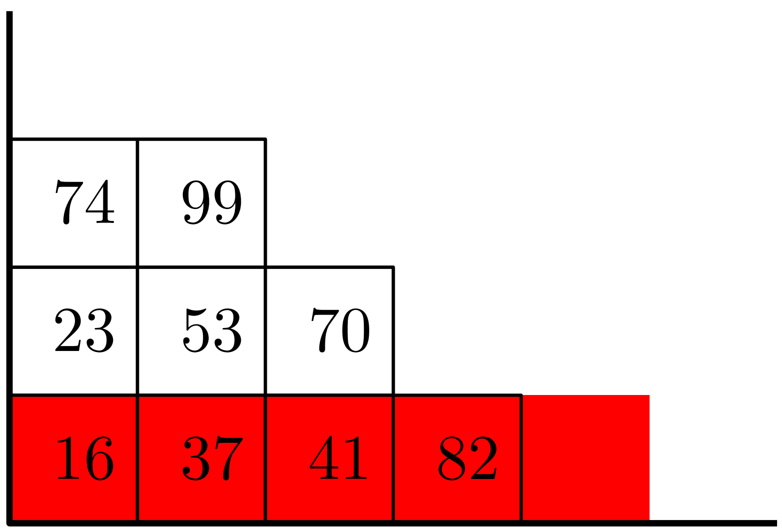

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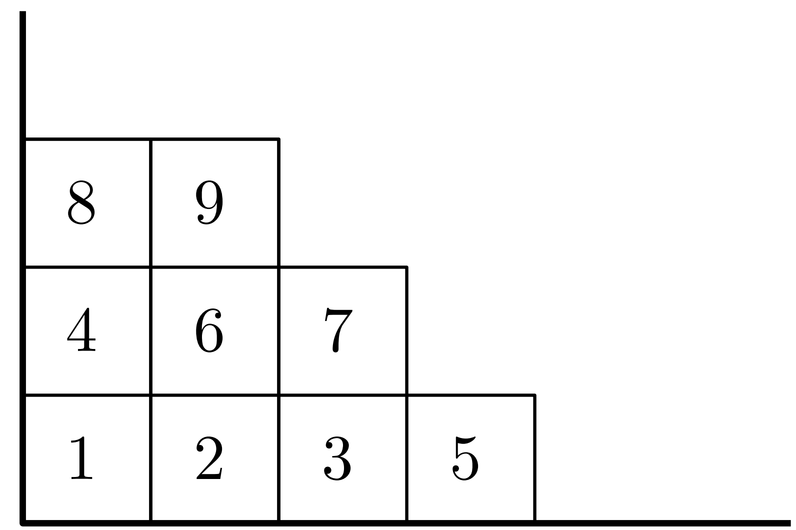

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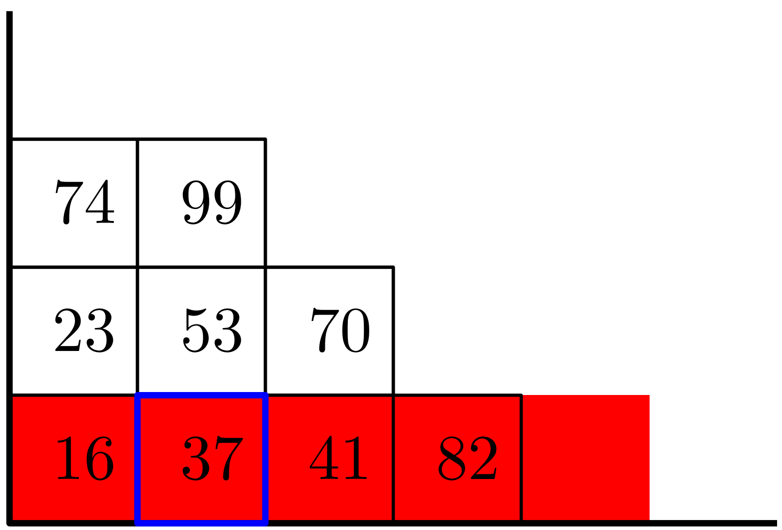

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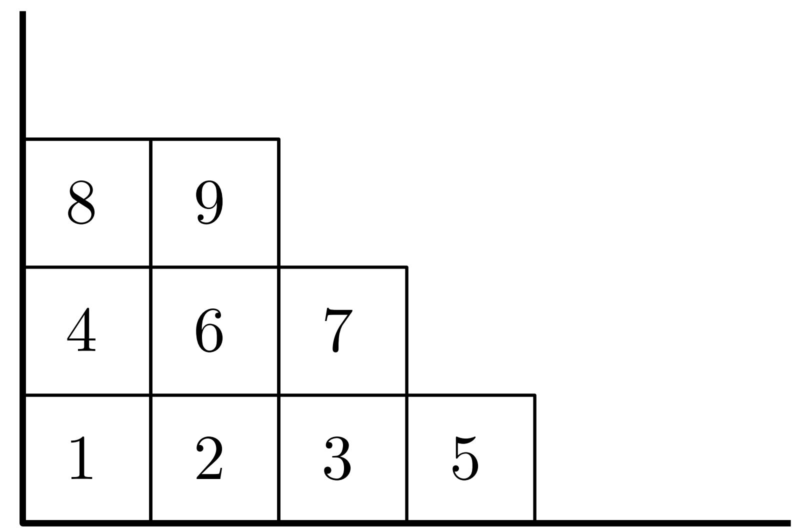

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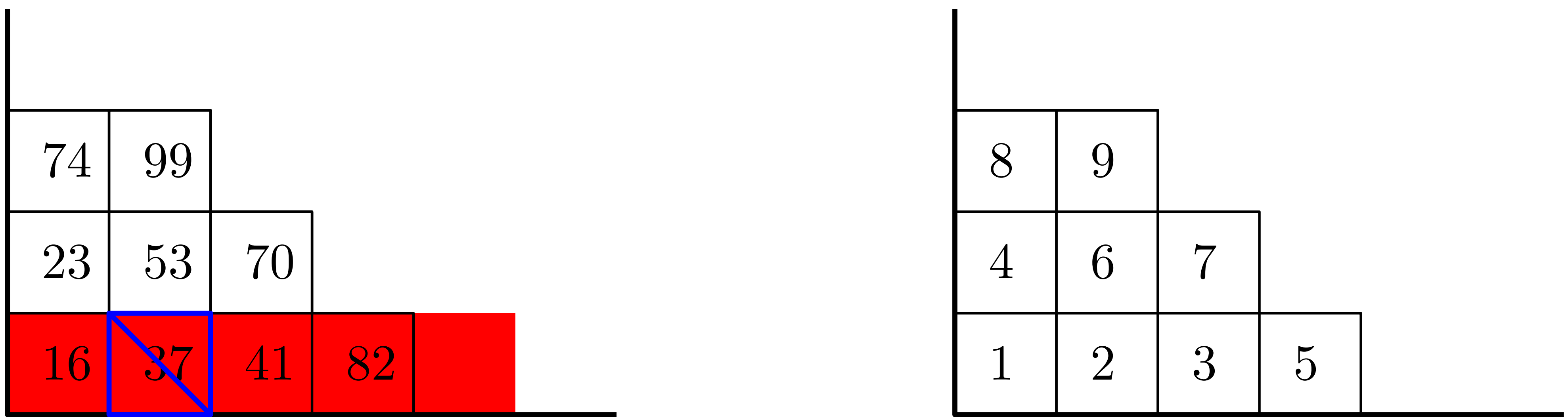

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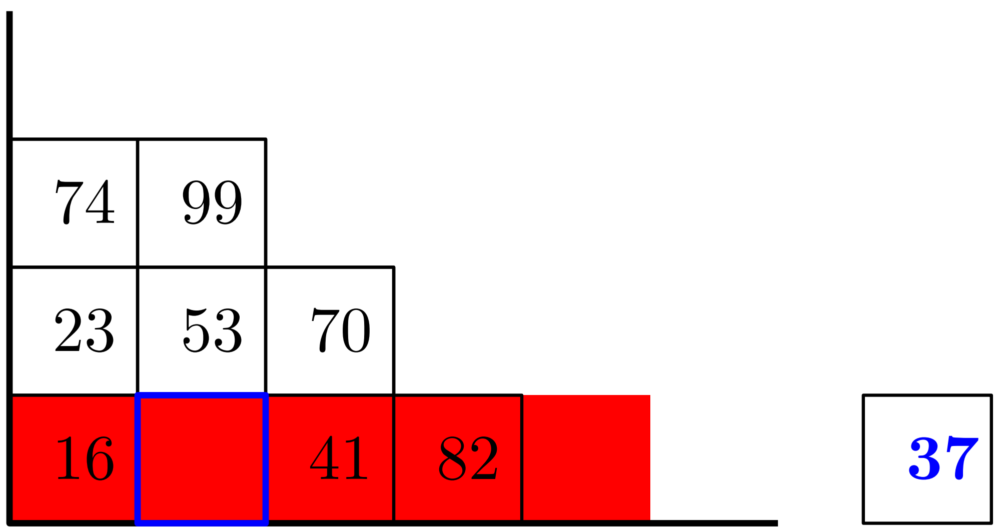

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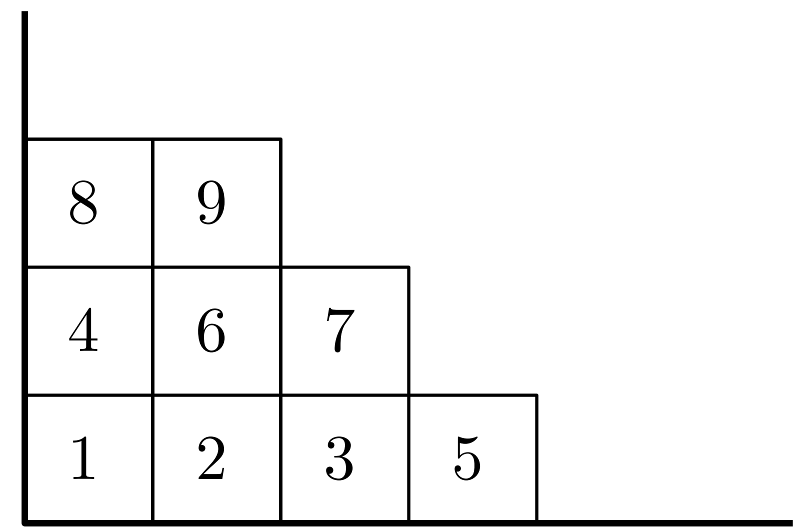

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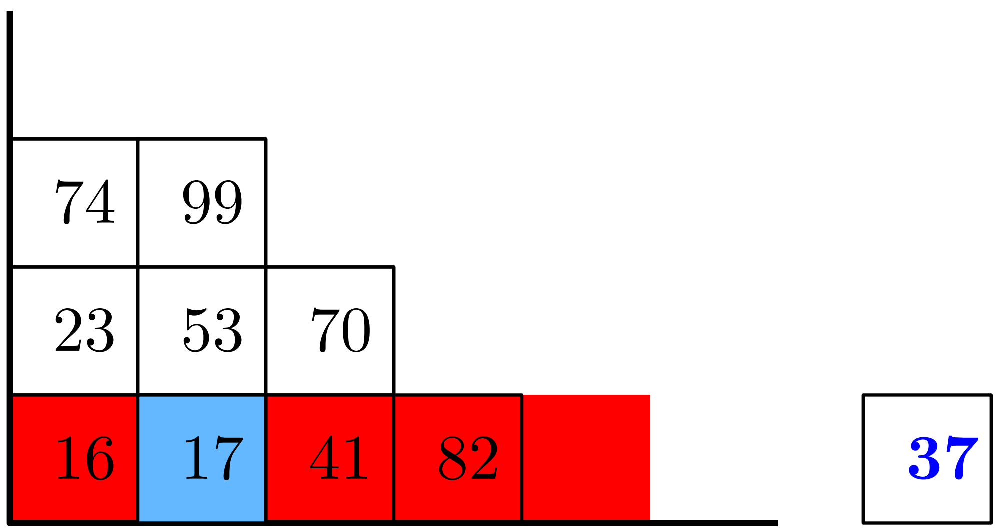

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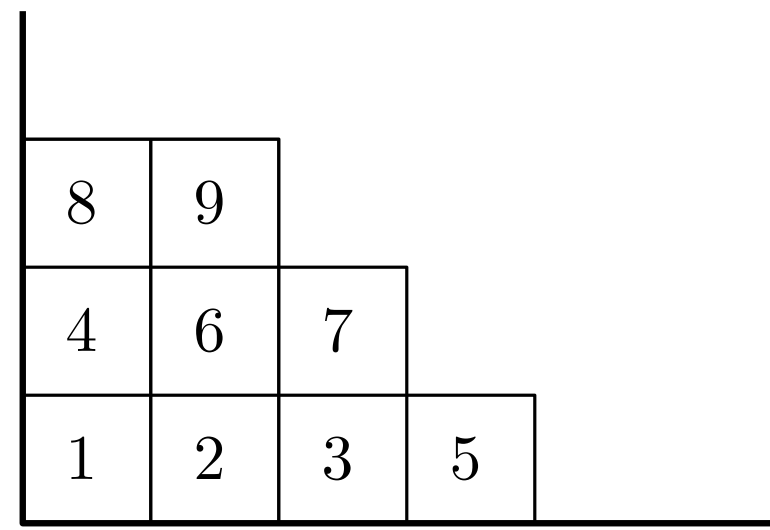

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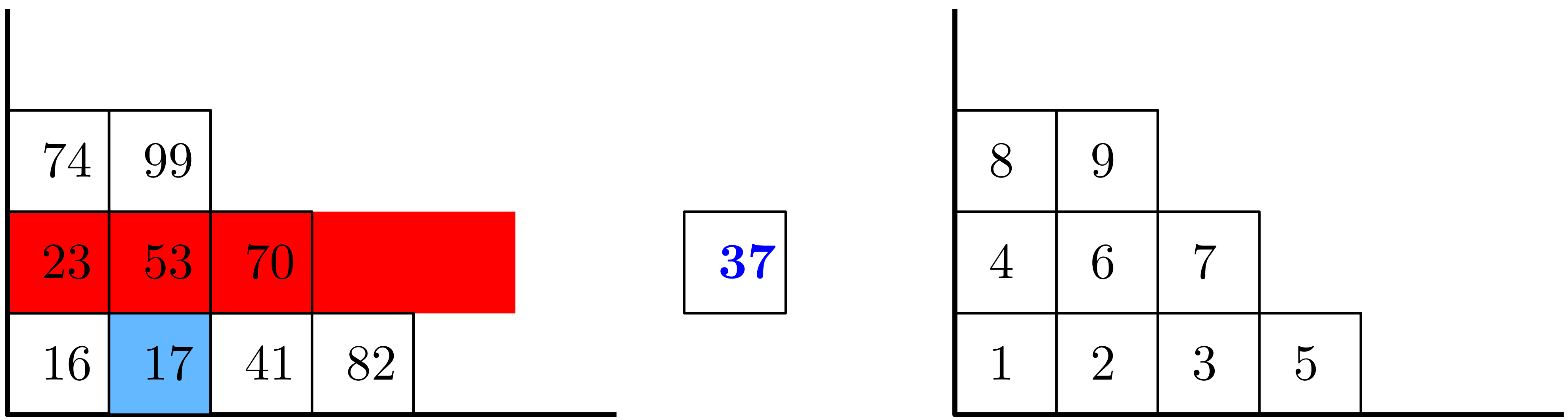

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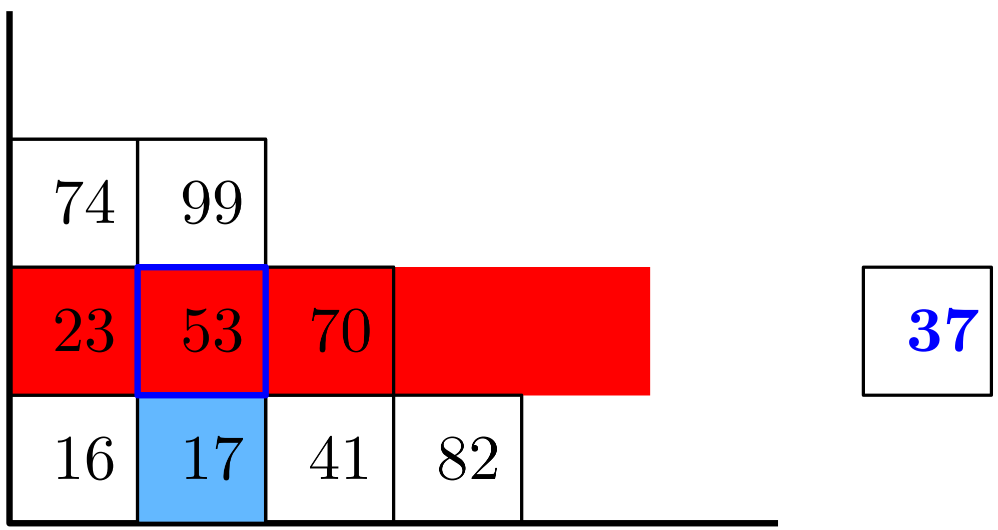

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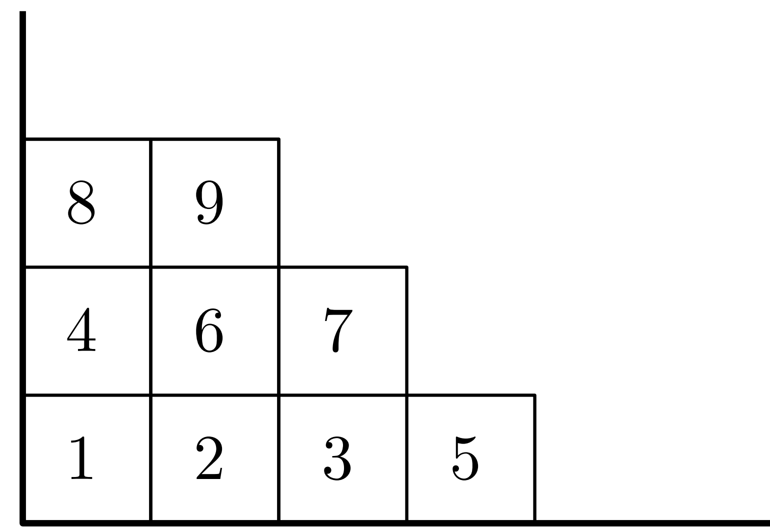

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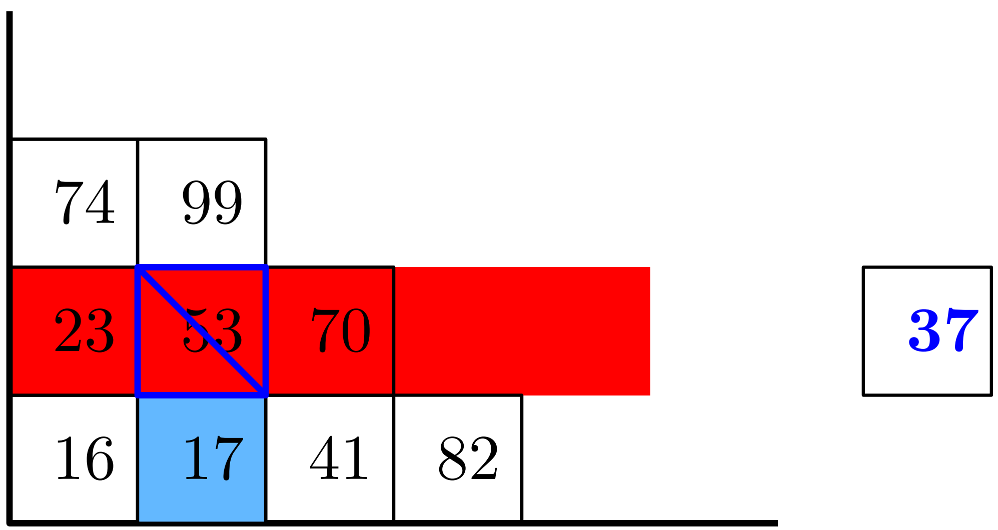

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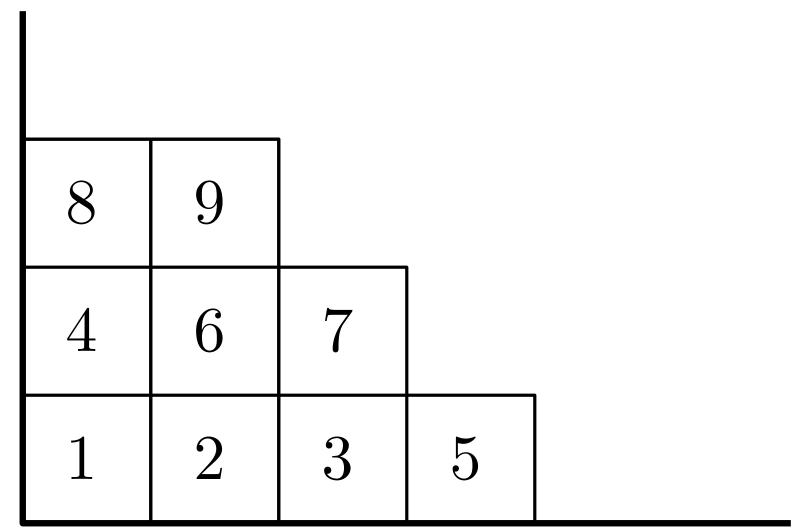

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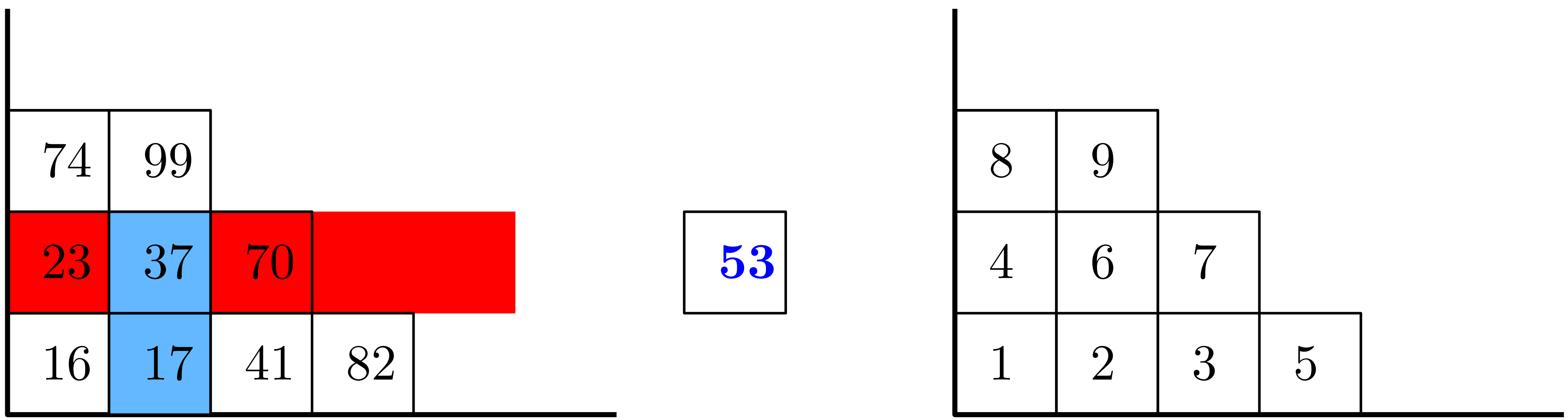

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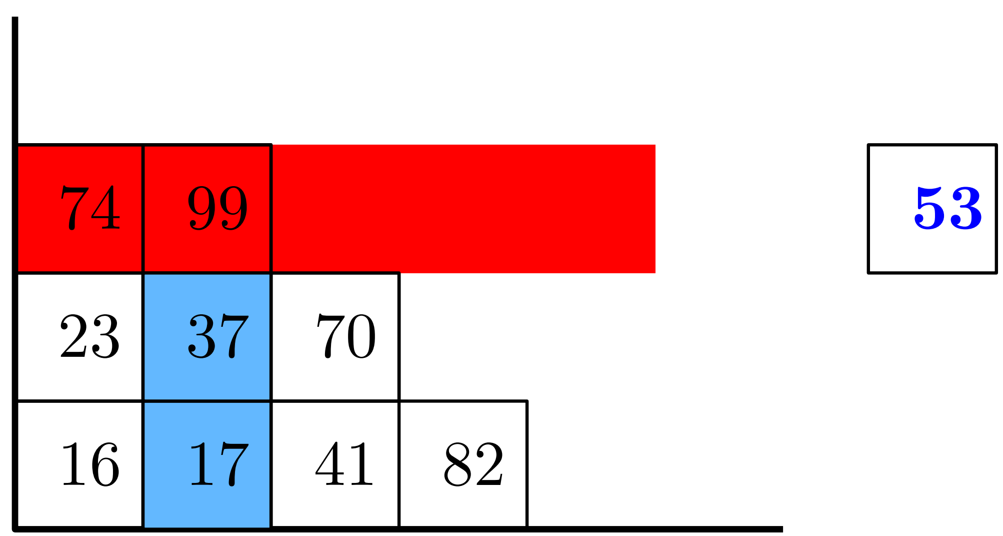

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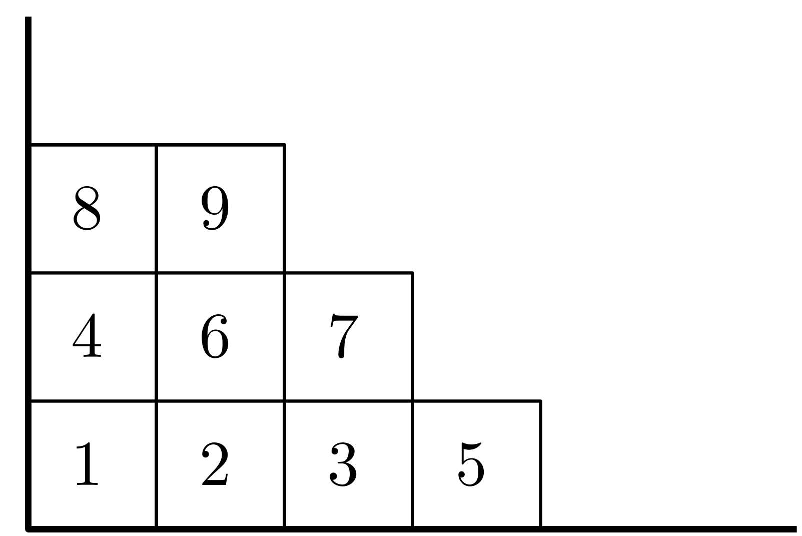

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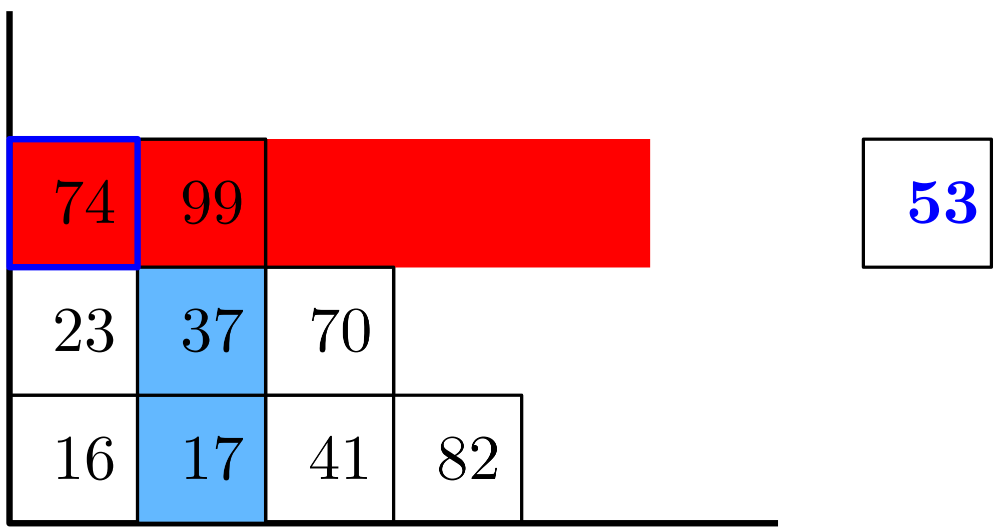

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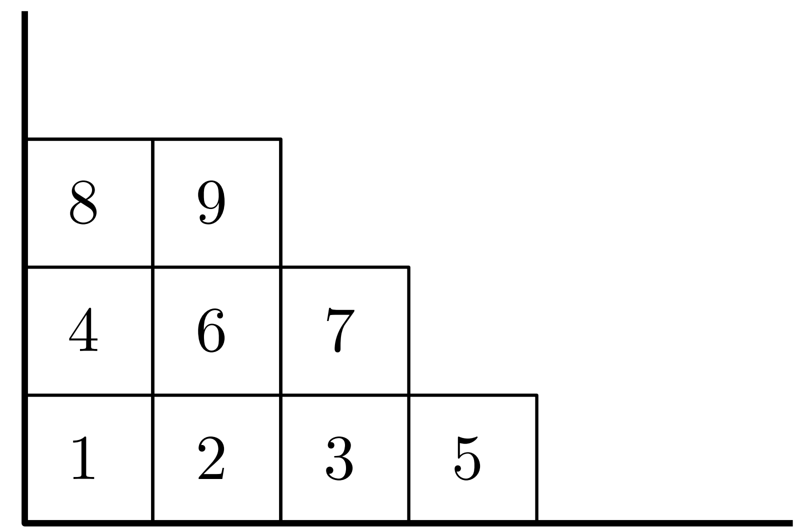

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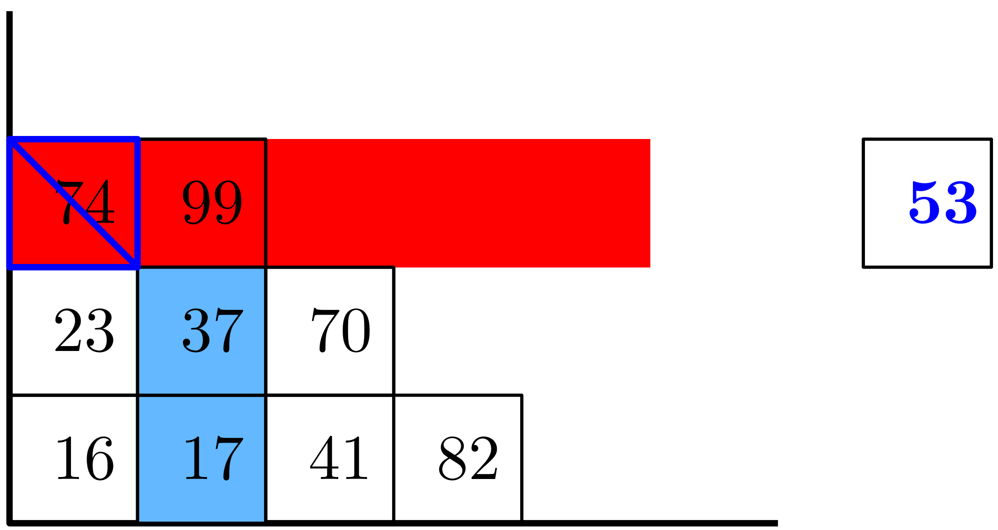

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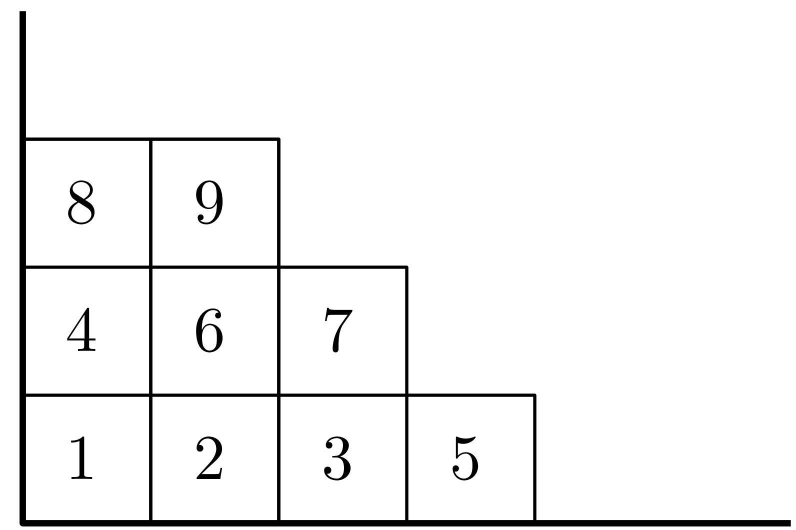

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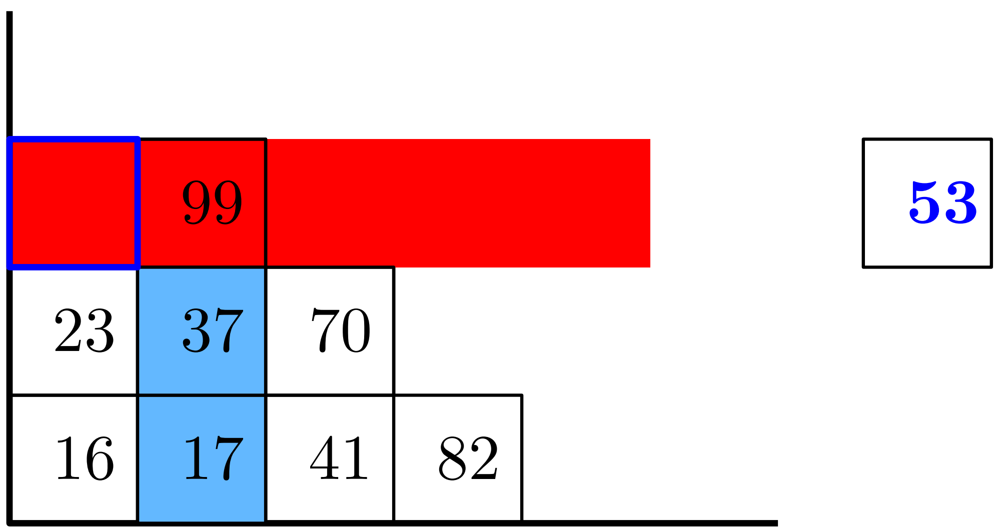

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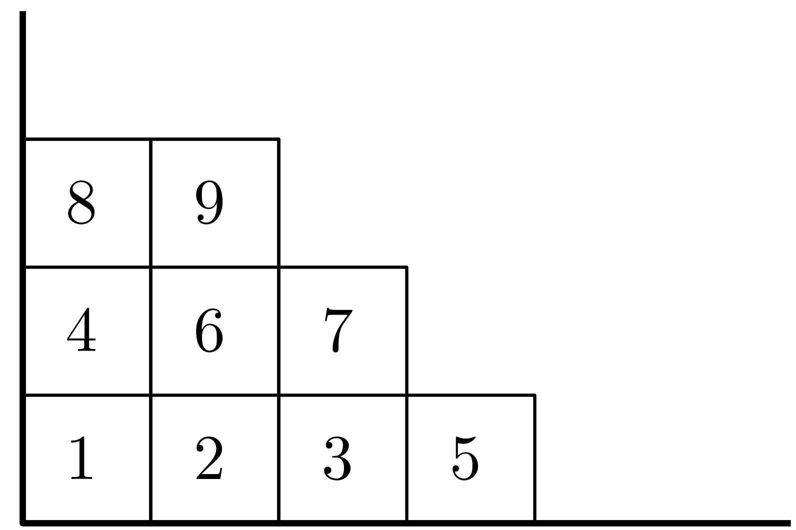

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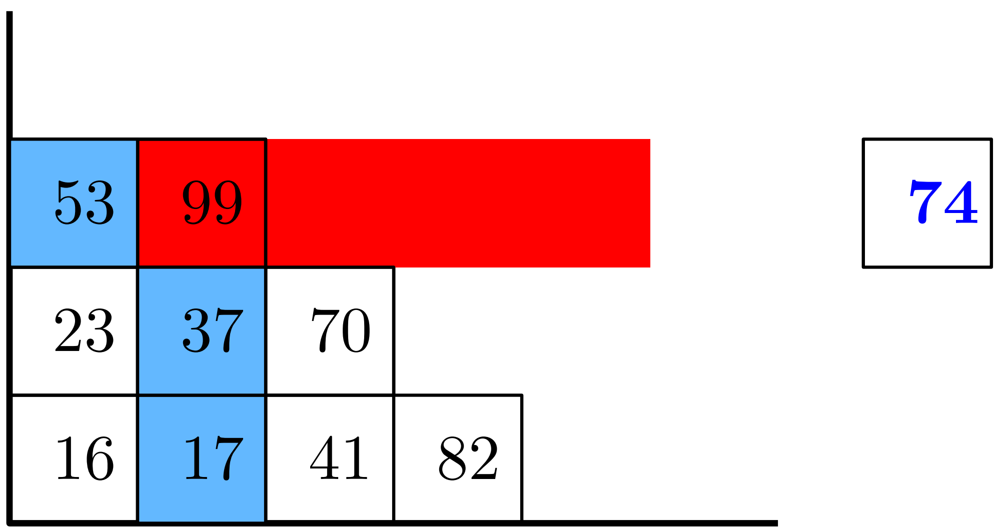

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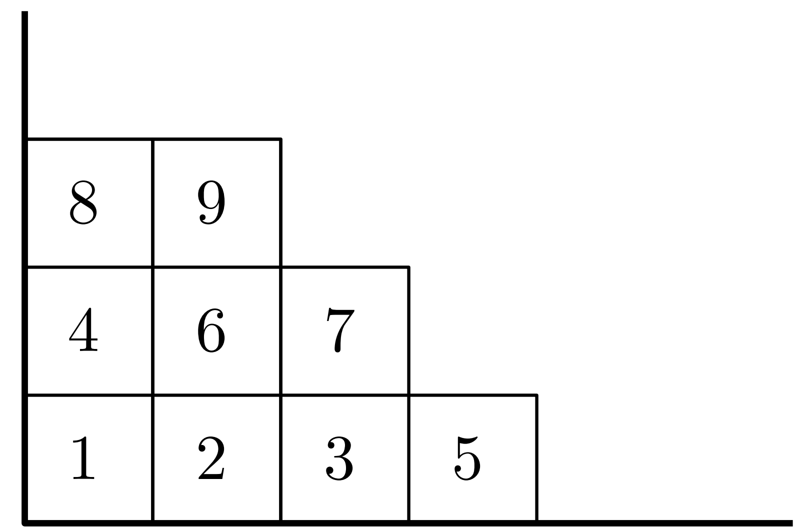

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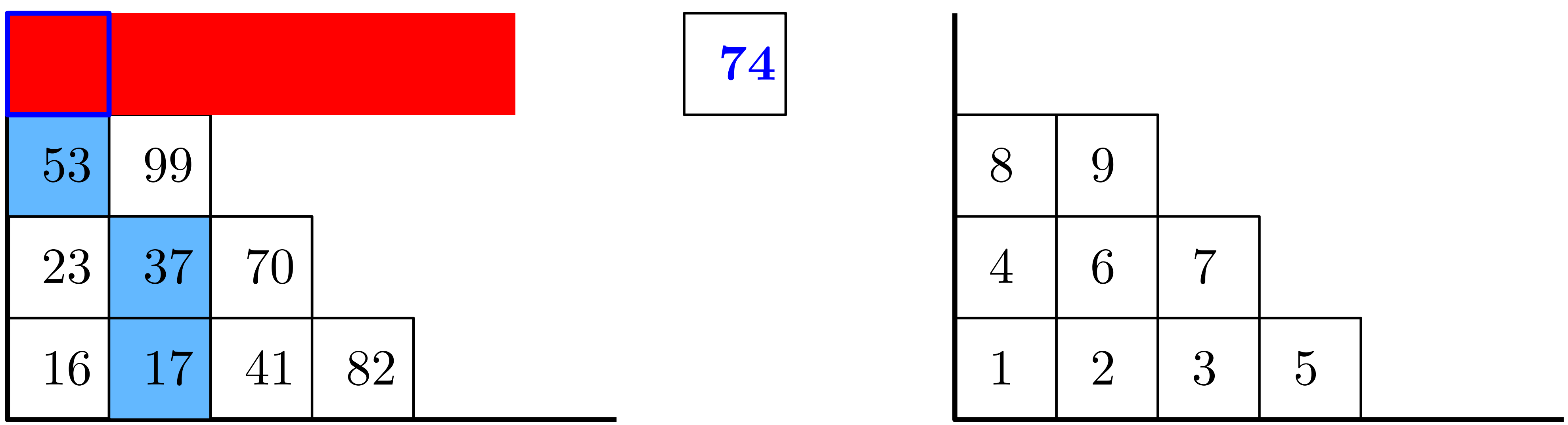

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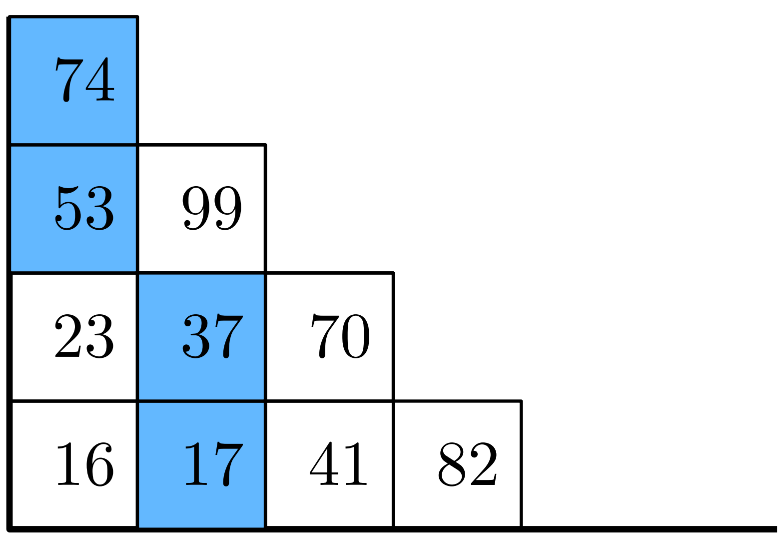

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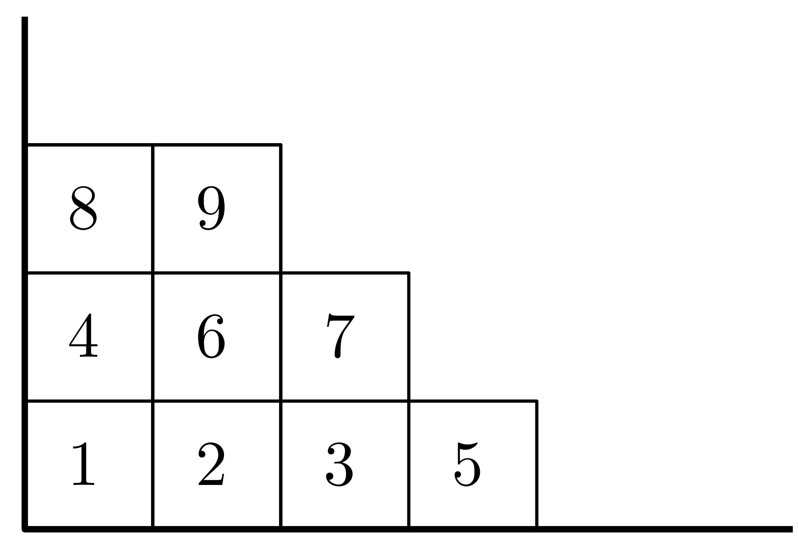

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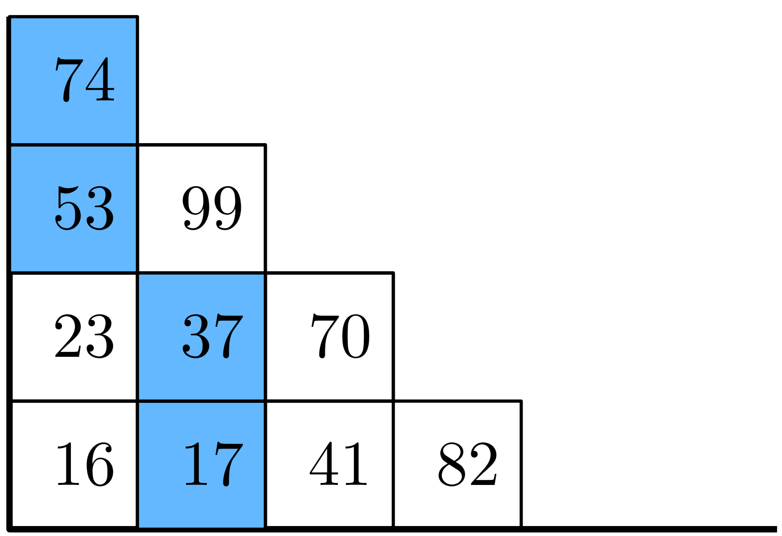

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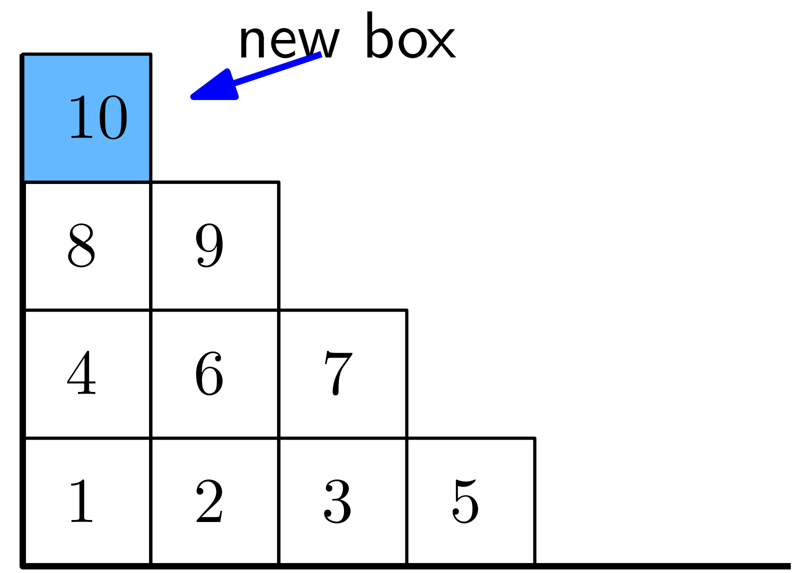

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Theorem: [Robinson–Schensted–Knuth '38 + '61+'70]

- RSK: $\mathbb{N}^n_+ \to \bigcup_{|\lambda|=n} SSYT(\lambda) \times SYT(\lambda)$ is a bijection
- $\ell(\sigma) = \lambda_1$ for a permutation σ.

Ulam's problem revisited

Corollary:

The distribution of $\ell(\sigma_n)$ when $\sigma_n \in S_n$ ⟷ The distribution of λ_1 when λ is a random Young diagram of size n sampled proportionally to $|SYT(\lambda)|^2$.

Ulam's problem revisited

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Plancherel measure

Problem: How to count $|SYT(\lambda)|$?

Ulam's problem revisited

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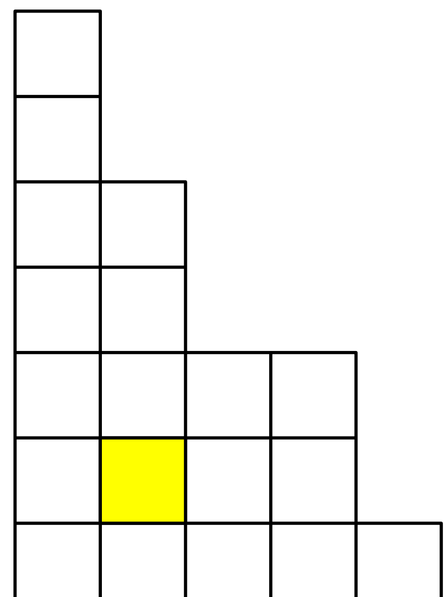

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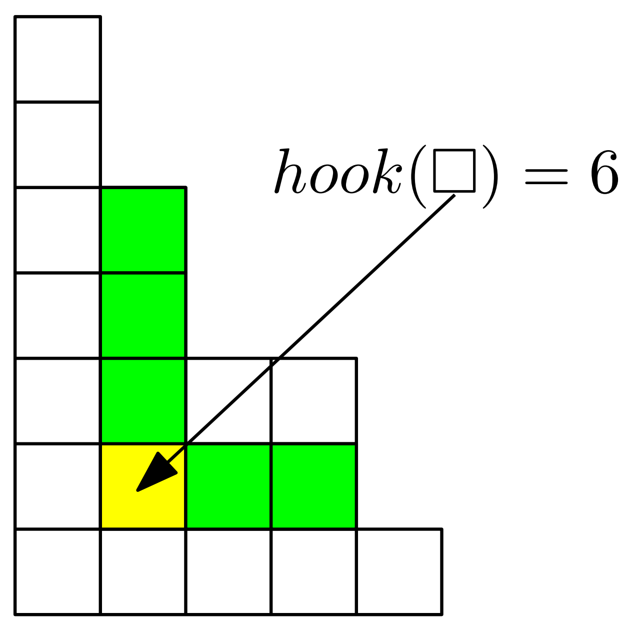

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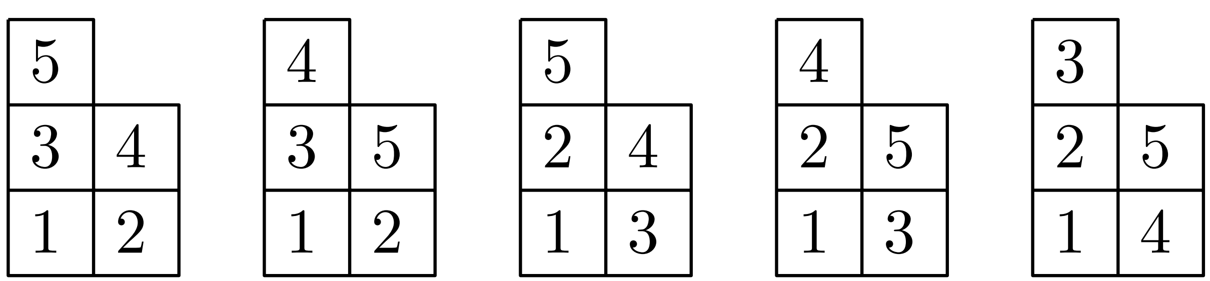

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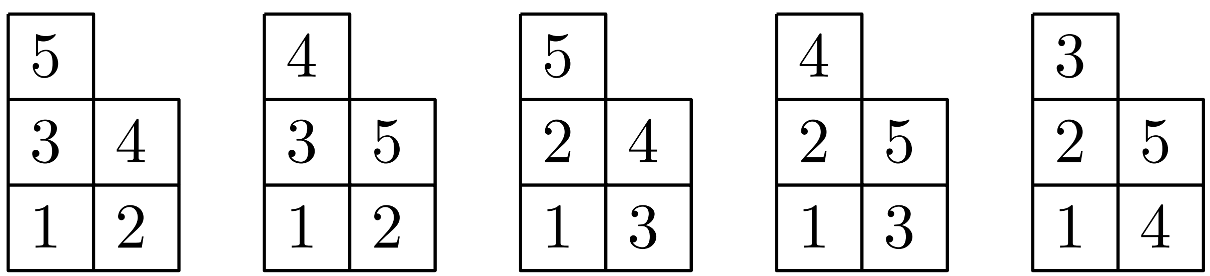

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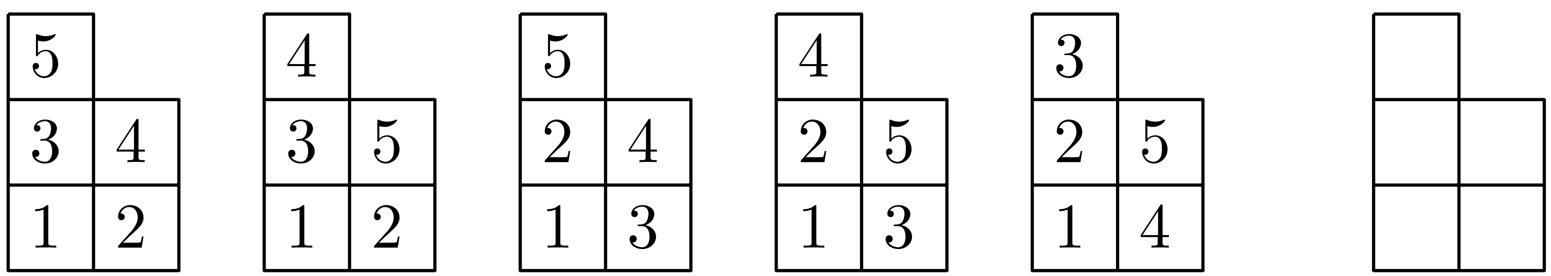

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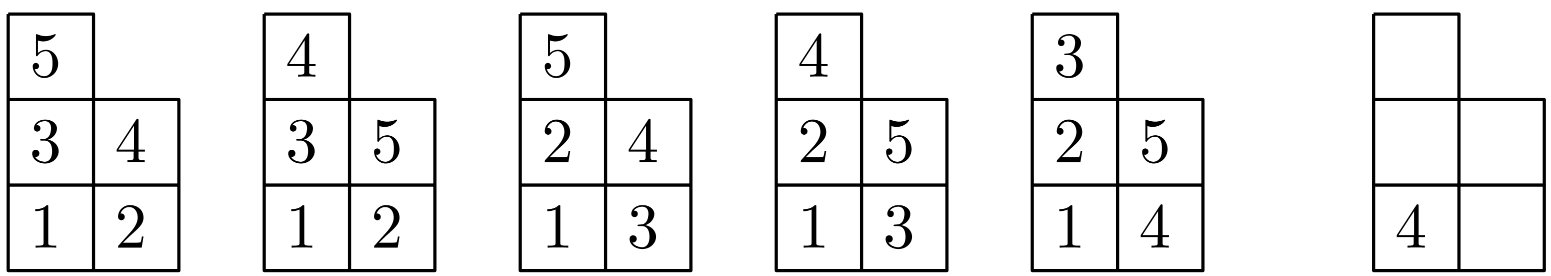

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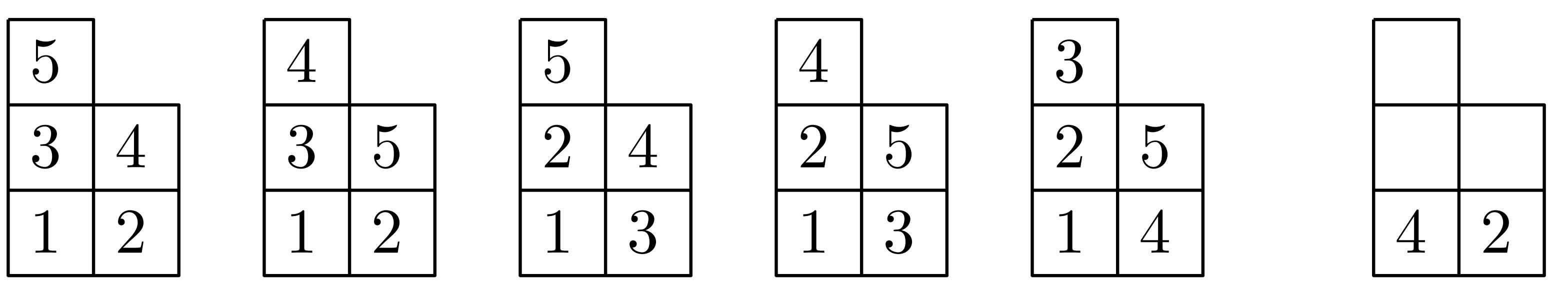

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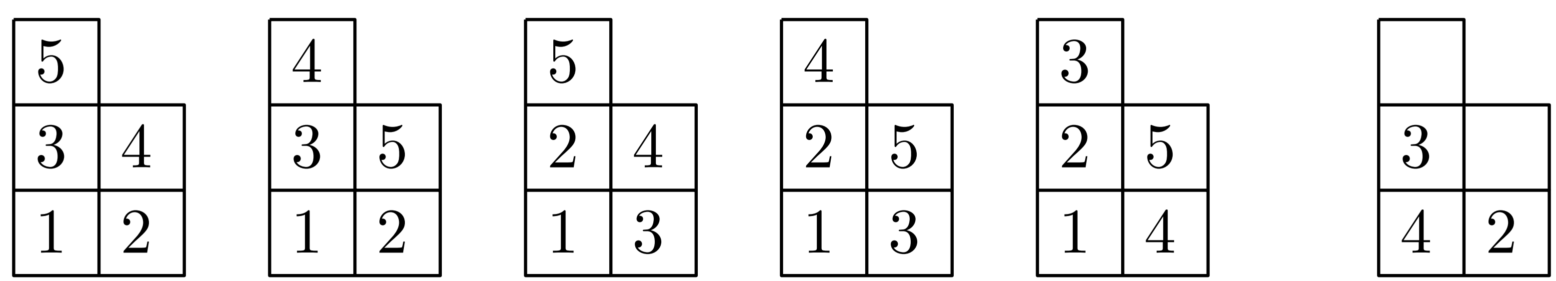

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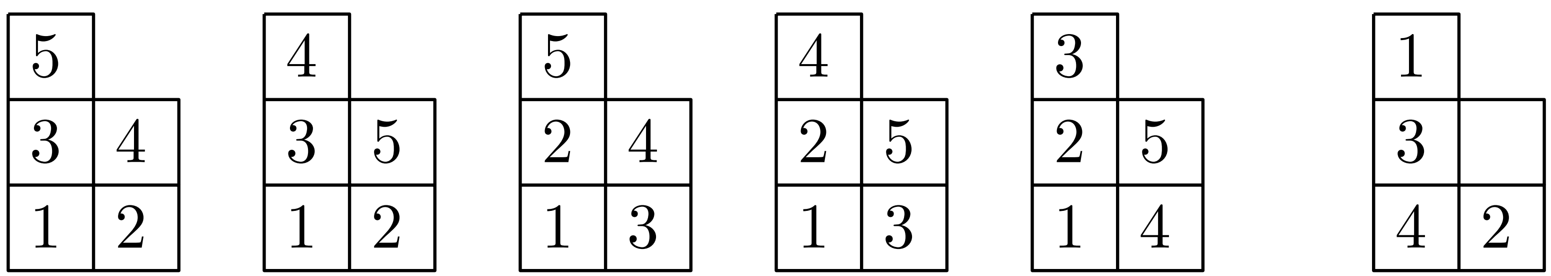

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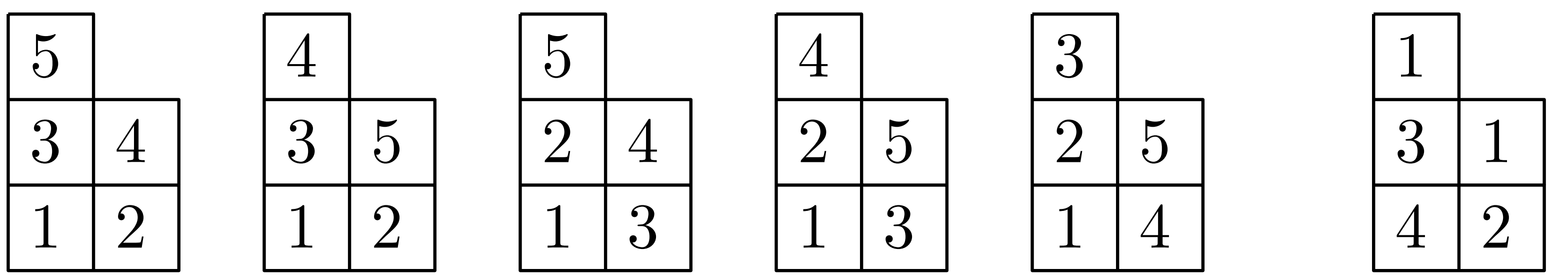

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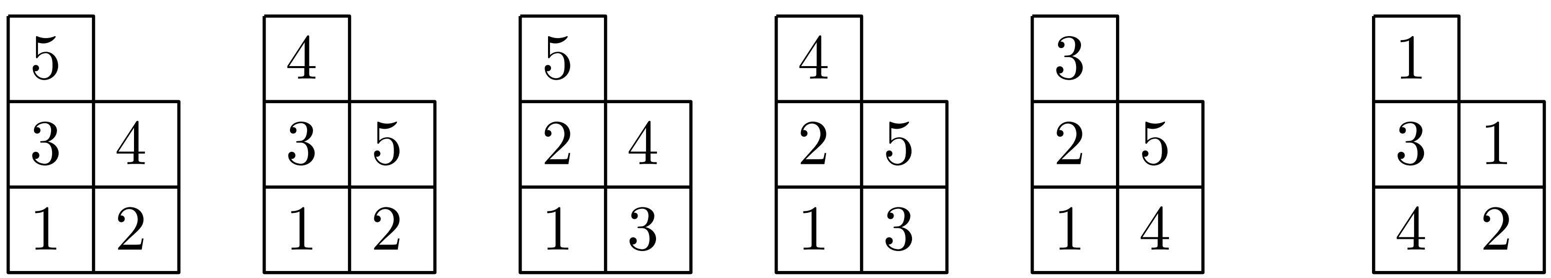

$$|SYT((2,2,1))| = \frac{5!}{1\cdot 1\cdot 2\cdot 3\cdot 4} = 5.$$

Logan–Schepp Vershik–Kerov phenomenon

Idea: Instead of studying λ_1 try to look at the global shape of λ.

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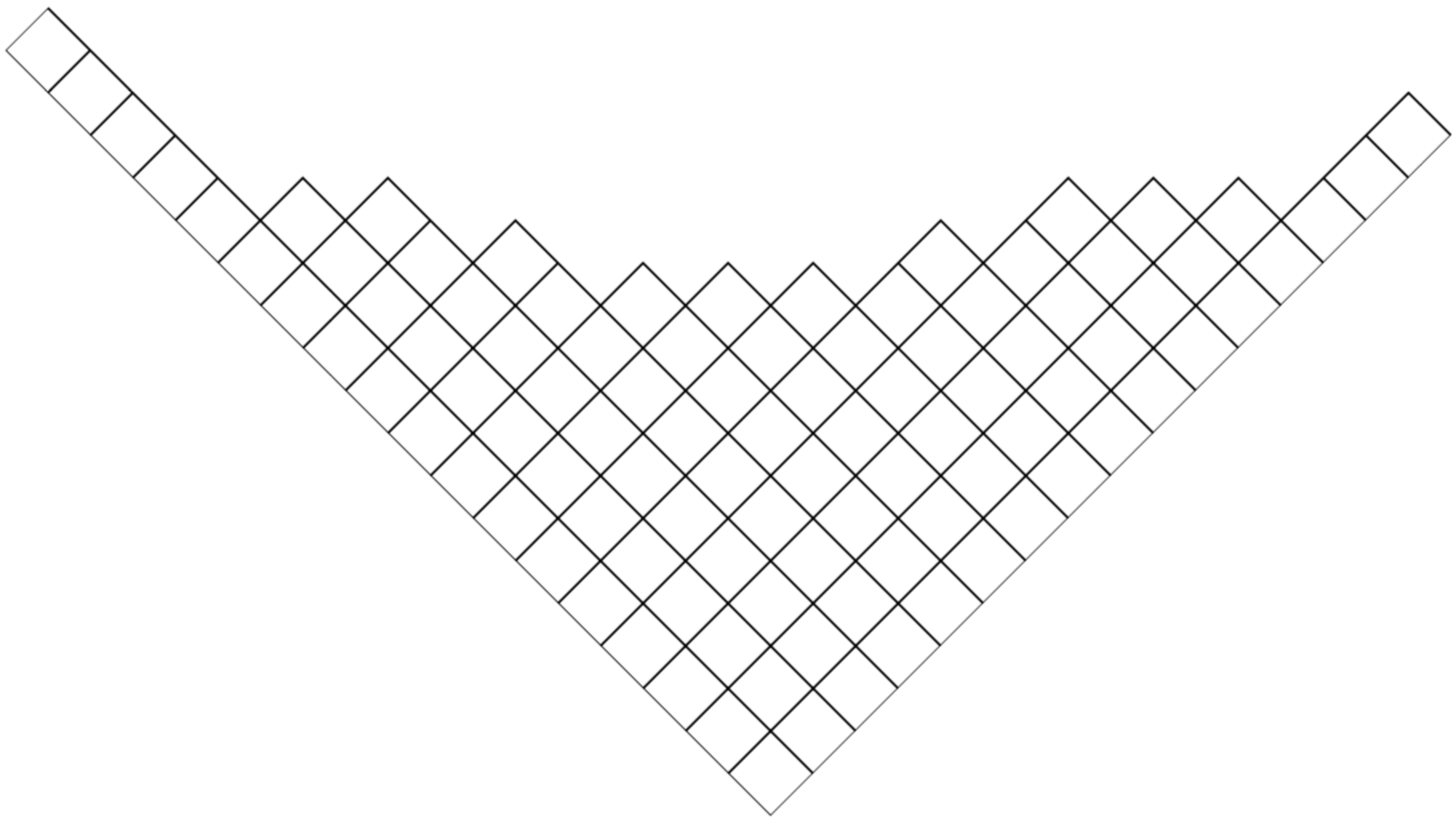

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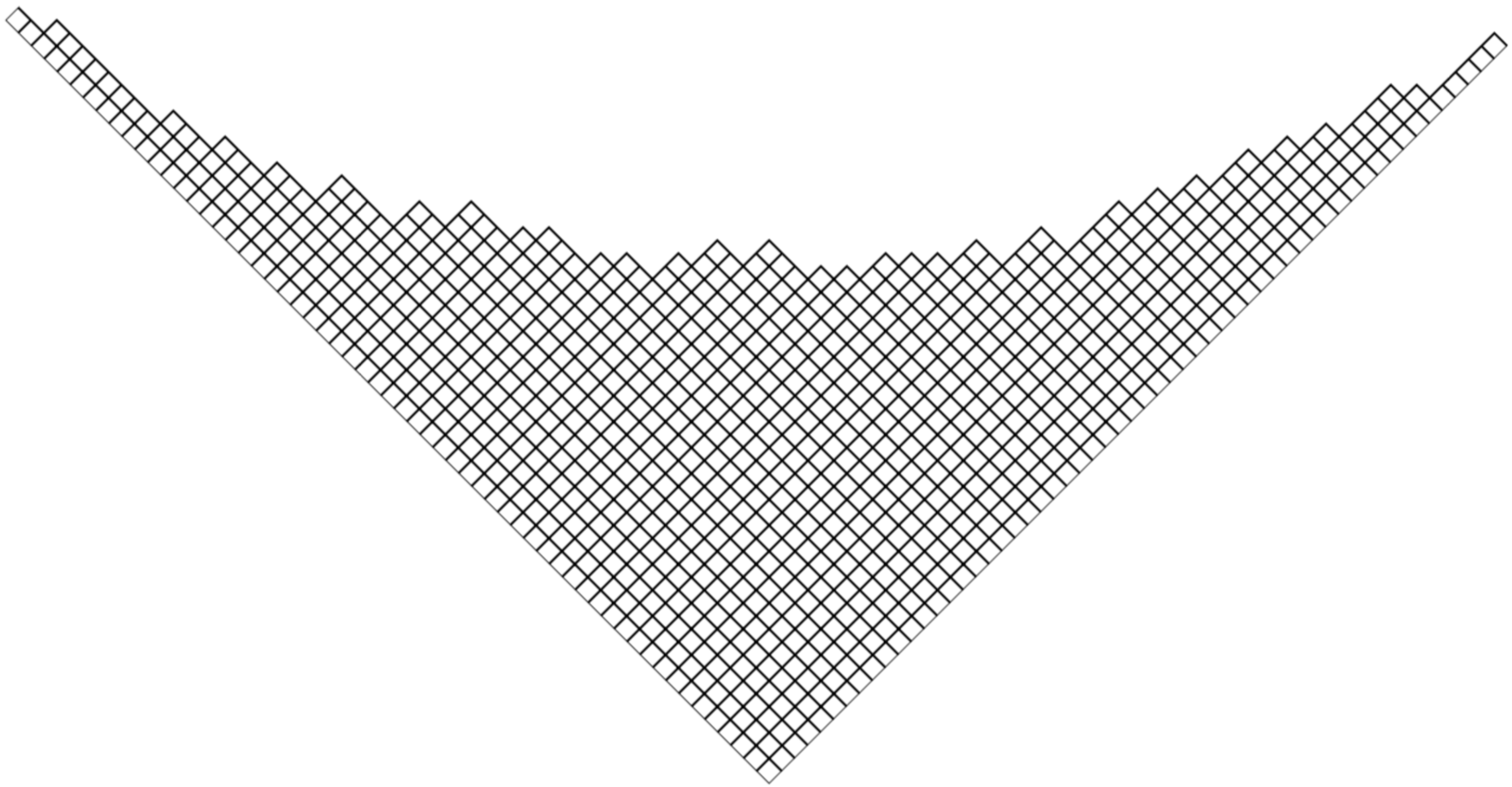

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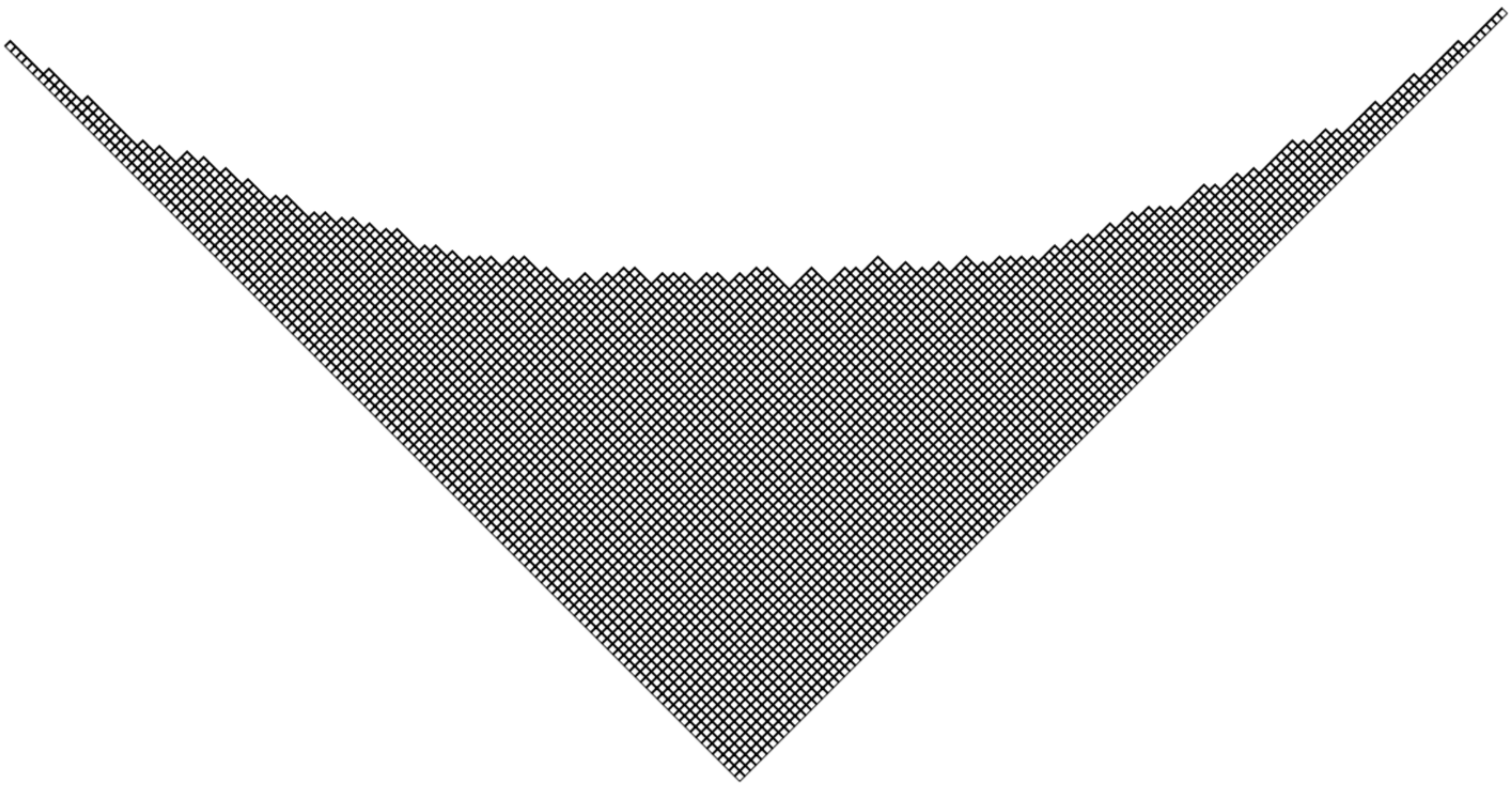

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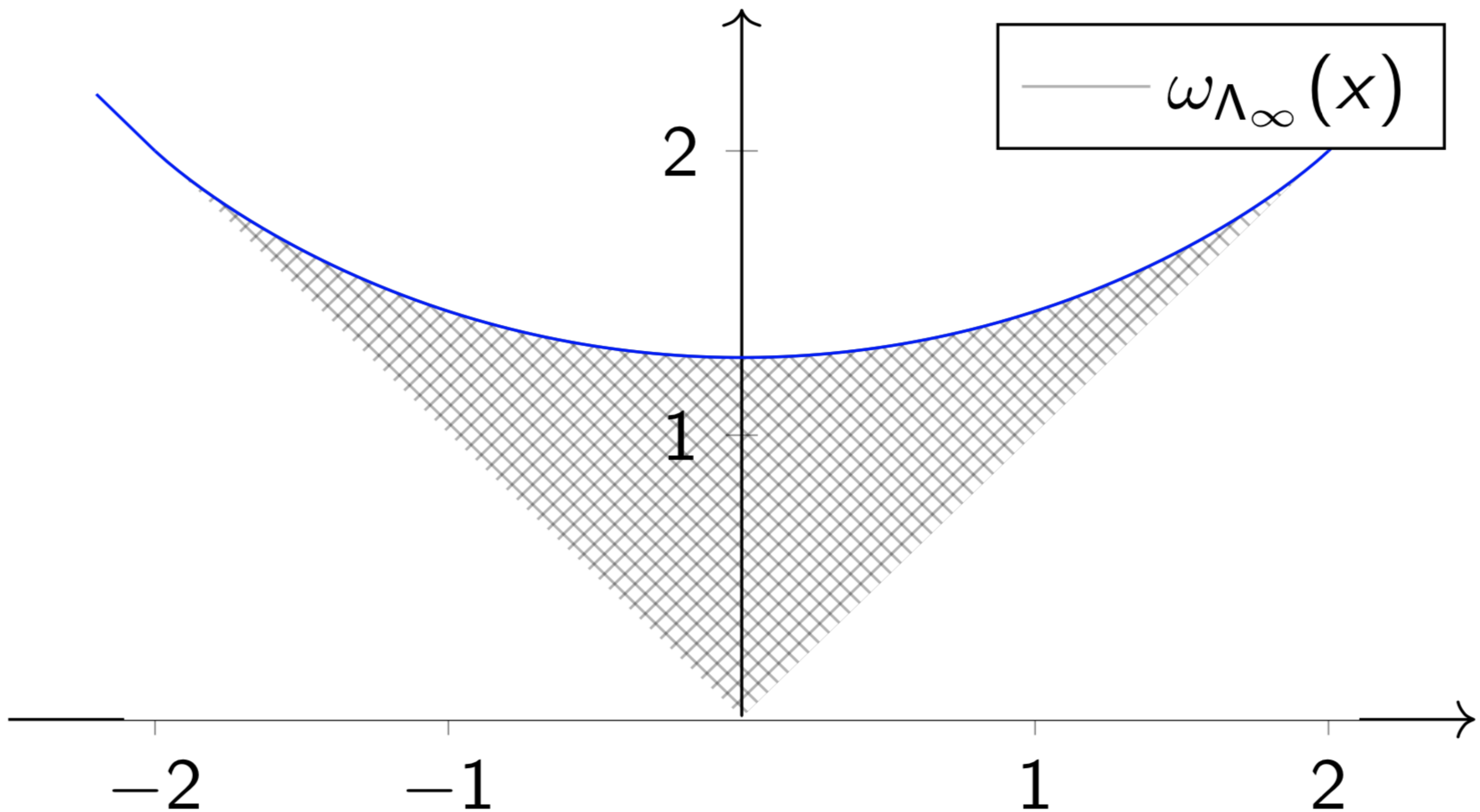

$$\omega_{\Lambda_\infty}(x) = \begin{cases} |x| & \text{if } |x| \geq 2; \\ \frac{2}{\pi}\left(x \cdot \arcsin\frac{x}{2} + \sqrt{4-x^2}\right) & \text{otherwise.} \end{cases}$$

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Theorem: [Vershik–Kerov, Logan–Schepp '77]

Suppose that $\lambda^{(n)} \vdash n$ is random Young diagram sampled w.r.t the Plancherel measure. Then its global shape concentrates around ω_{Λ_∞} when $n \to \infty$.

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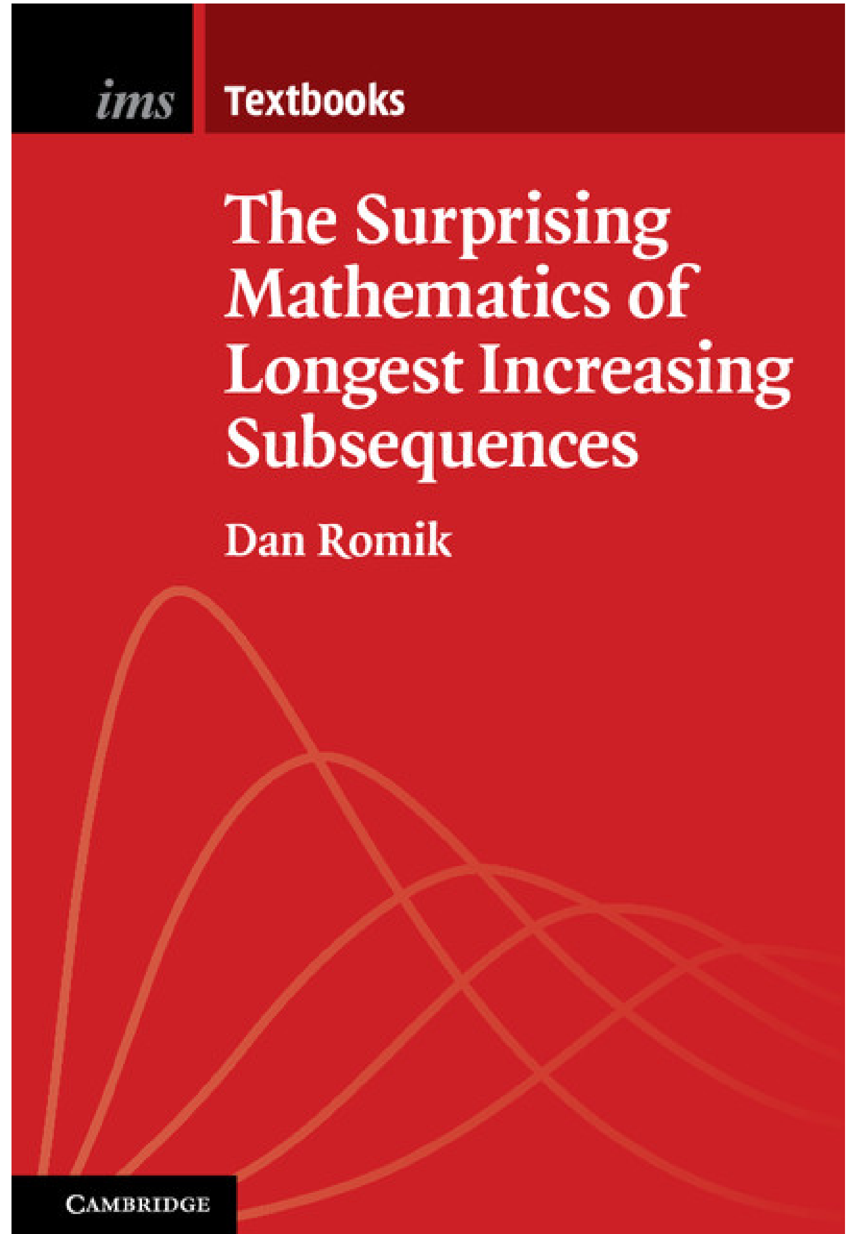
Corollary: $\frac{\ell(\sigma_n)}{\sqrt{n}} \to 2$ with high probability as $n \to \infty$.

more to come...



https://www.math.ucdavis.edu/ romik/book/