BIG BIRKHOFF SUMS IN d-DECAYING GAUSS LIKE ITERATED FUNCTION SYSTEMS

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Abstract. The increasing rate of the Birkhoff sums in the infinite iterated function systems with polynomial decay of the derivative (for example the Gauss map) is studied. For different unbounded potential functions, the Hausdorff dimensions of the sets of points whose Birkhoff sums share the same increasing rate are obtained.

1. Introduction

Denote by $\mathbb{N} = \{1, 2, \dots\}$ the set of positive integers. Let $d > 1$ be a real number. A family $\{f_n\}_{n\in\mathbb{N}}$ of C^1 maps from the interval $[0, 1]$ to itself is called a *d-decaying Gauss like* iterated function system if the following properties are satisfied:

(1) for any $i, j \in \mathbb{N}$ $\; f_i((0, 1)) \cap f_j((0, 1)) = \emptyset$;
(2) $\bigcup_{i=1}^{\infty} f_i([0, 1]) = [0, 1)$;
(3) if $f_i(x) < f_j(x)$ for all $x \in (0, 1)$ then $i < j$;
(4) there exists $m \in \mathbb{N}$ and $0 < A < 1$ such that for all $(a_1, ..., a_m) \in \mathbb{N}^m$ and for all $x \in [0, 1]$

$$0 < |(f_{a_1} \circ \cdots \circ f_{a_m})'(x)| \le A < 1;$$

(5) for any $\delta > 0$, we can find two constants $K_1 = K_1(\delta), K_2 = K_2(\delta) > 0$ such that for $i \in \mathbb{N}$ there exist constants ξ_i, λ_i such that

$$\forall x \in [0, 1], \quad \xi_i \le |f_i'(x)| \le \lambda_i$$

and

$$\frac{K_1}{i^{d+\delta}} \le \xi_i \le \lambda_i \le \frac{K_2}{i^{d-\delta}}.$$

We have a natural projection $\Pi : \mathbb{N}^{\mathbb{N}} \to [0, 1]$ defined by

$$\Pi(\underline{a}) = \lim_{n\to\infty} f_{a_1} \circ \cdots \circ f_{a_n}(1).$$

Its inverse gives for points $x \in [0, 1]$ their symbolic expansions in $\mathbb{N}^{\mathbb{N}}$. The symbolic expansion is unique for most points, but there can exist countably many points that have zero or two symbolic expansions. When the symbolic expansion is unique, we write $x = (a_1(x), a_2(x), \dots)$ the expansion of $x \in [0, 1]$.

For each $n \in \mathbb{N}$, and each word $a_1 \cdots a_n \in \mathbb{N}^n$, the set

$$I_n(a_1, \cdots, a_n) = f_{a_1} \circ \cdots \circ f_{a_n}([0, 1])$$

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is called an n-cylinder. Except for a countable set, the n-cylinder $I_n(a_1, \cdots, a_n)$ is identical with the set of points $x \in [0,1]$ whose symbolic expansions begin with $a_1, \cdots, a_n$. Write $I_n(x)$ the n-cylinder containing $x \in [0,1]$.

Denote by $|I|$ the diameter of the interval I.

We say the d-decaying Gauss like iterated function system $\{f_n\}_{n\in\mathbb{N}}$ satisfies the *bounded distortion property* if there exist positive constants K_3 and K_4 such that for any two finite words $a_1a_2\cdots a_n$ and $b_1b_2\cdots b_m$, we have

$$K_3 \le \frac{|I_{n+m}(a_1, \cdots, a_n, b_1, \cdots, b_m)|}{|I_n(a_1, \cdots, a_n)| \cdot |I_m(b_1, \cdots, b_m)|} \le K_4. \tag{1.1}$$

Consider a *potential function* $\varphi : [0,1] \to \mathbb{R}_+$, such that φ is a constant on the interior of $I_1(a_1)$ for all $a_1 \in \mathbb{N}$. For $n \in \mathbb{N}$, the n-th Birkhoff sum of φ at $x \in (0,1)$ is defined by

$$S_n\varphi(x) = \sum_{j=0}^{n-1} \varphi(a_j), \quad \text{if } x \in I_n(a_1, \cdots, a_n).$$

We remark that except for a countable set, the above Birkhoff sums are well defined.

For a positive *growth rate function* $\Phi : \mathbb{N} \to \mathbb{R}_+$, we are interested in the following set

$$E_\varphi(\Phi) := \left\{ x \in (0,1) : \lim_{n\to\infty} \frac{S_n\varphi(x)}{\Phi(n)} = 1 \right\}. \tag{1.2}$$

We will calculate $\dim_H E_\varphi(\Phi)$, where $\dim_H(\cdot)$ denotes the Hausdorff dimension of a set. When $\Phi(n)/n$ has a finite limit as $n \to \infty$, $E_\varphi(\Phi)$ is the classical level set of Birkhoff averages studied in [2], [4], [6],... In this paper we will consider the case when $\Phi(n)/n \to \infty$, thus necessarily the potential function φ is unbounded in $[0,1]$.

For all $j \in \mathbb{N}$, denote by $\varphi(j)$ the constant value of φ on the interior of 1-cylinder $I_1(j)$. We obtain the following multifractal analysis results on the Hausdorff dimension of $E_\varphi(\Phi)$, according to different choices of φ and Φ.

Theorem 1.1. *Suppose $\varphi(j) = j^a$ for all $j \ge 1$, with $a > 0$.*
(I) *When $\Phi(n) = e^{n^\alpha}$ with $\alpha > 0$, we have*
(I-1) $\dim_H E_\varphi(\Phi) = 1$ *if $\alpha < \frac{1}{2}$ and the distortion property (1.1) holds;*
(I-2) $\dim_H E_\varphi(\Phi) = 1/d$ *if $\alpha > \frac{1}{2}$.*
(II) *When $\Phi(n) = e^{\beta^n}$ with $\beta > 1$, we have* $\dim_H E_\varphi(\Phi) = \frac{1}{d\beta - \beta + 1}$.

Theorem 1.2. *Suppose $\varphi(j) = e^{(\log j)^b}$ for all $j \ge 1$, with $b > 1$.*
(I) *When $\Phi(n) = e^{n^\alpha}$ with $\alpha > 0$, we have*
(I-1) $\dim_H E_\varphi(\Phi) = 1$ *if $\alpha < \frac{b}{b+1}$ and the distortion property (1.1) holds;*
(I-2) $\dim_H E_\varphi(\Phi) = 1/d$ *if $\alpha > \frac{b}{b+1}$.*
(II) *When $\Phi(n) = e^{\beta^n}$ with $\beta > 1$, we have* $\dim_H E_\varphi(\Phi) = \frac{1}{d\beta^{\frac{1}{b}} - \beta^{\frac{1}{b}} + 1}$.

Theorem 1.3. *Suppose $\varphi = e^{j^c}$ for all $j \ge 1$, with $0 < c < 1$.*
(I) *When $\Phi(n) = e^{n^\alpha}$ with $\alpha > 0$, we have*
(I-1) $\dim_H E_\varphi(\Phi) = 1$ *if $\alpha < 1$ and the distortion property (1.1) holds;*
(I-2) $\dim_H E_\varphi(\Phi) = \frac{1-c}{d}$ *if $\alpha > 1$.*

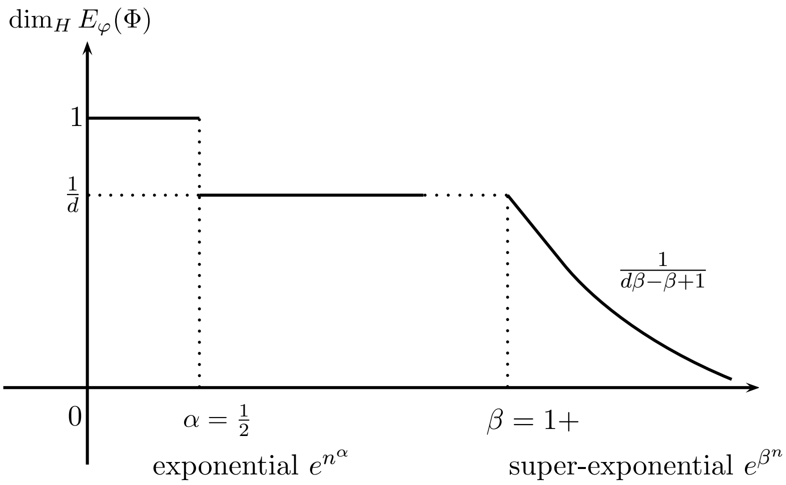

FIGURE 1. $\dim_H E_\varphi(\Phi)$ for $\varphi(j) = j^a$.

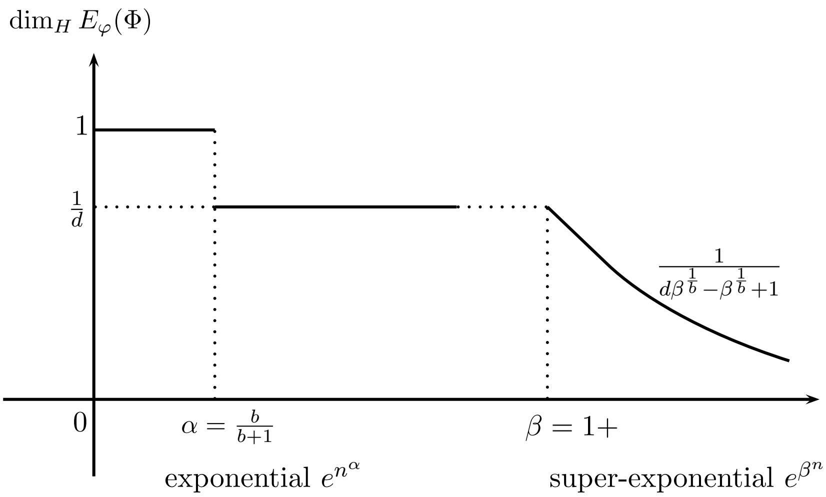

FIGURE 2. $\dim_H E_\varphi(\Phi)$ for $\varphi(j) = e^{(\log j)^b}$.

(II) *When* $\Phi(n) = e^{\beta^n}$ *with* $\beta > 1$, *we have* $\dim_H E_\varphi(\Phi) = \frac{1-c}{d}$.
(III) *When* $\Phi(n) = e^{e^{\gamma^n}}$ *with* $\gamma > 1$, *we have* $\dim_H E_\varphi(\Phi) = \frac{1-c}{d\gamma-(1-c)(\gamma-1)}$.

Theorem 1.4. *Suppose* $\varphi(j) = e^{j^c}$ *for all* $j \geq 1$, *with* $c \geq 1$. *When* $\Phi(n) = e^{n^\alpha}$, *with* $\alpha > 0$, *we have*
(I-1) $\dim_H E_\varphi(\Phi) = 1$ *if* $\alpha < 1$ *and the distortion property (1.1) holds;*
(I-2) $\dim_H E_\varphi(\Phi) = 0$ *if* $\alpha \geq 1$.

The Hausdorff dimensions in Theorems 1.1-1.4 are depicted in Figures 1-4.

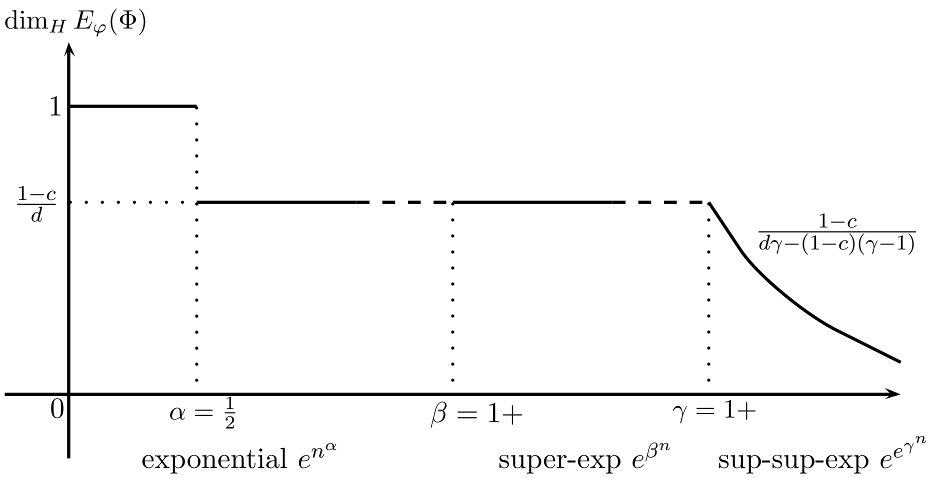

FIGURE 3. $\dim_H E_\varphi(\Phi)$ for $\varphi = e^{j^c}$ with $0 < c < 1$.

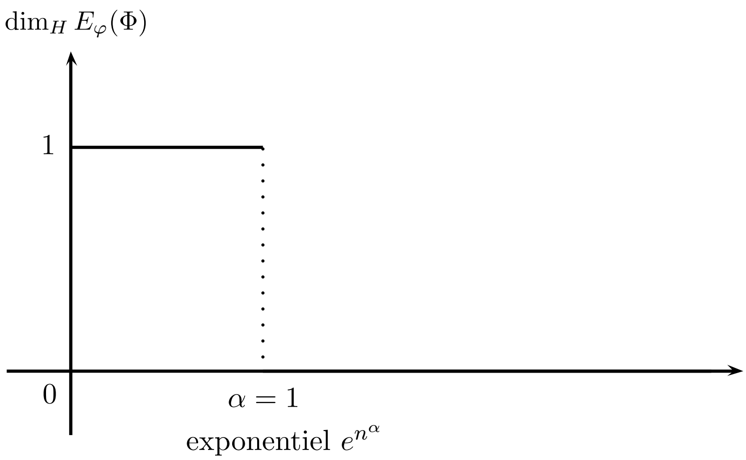

FIGURE 4. $\dim_H E_\varphi(\Phi)$ for $\varphi = e^{j^c}$ with $c \geq 1$.

Remark 1. The critical cases $\alpha = \frac{1}{2}$ in Theorems 1.1, $\alpha = \frac{b}{b+1}$ in Theorem 1.2, and $\alpha = 1$ in Theorems 1.3 and 1.4 are not investigated in this paper. However, Theorem 1.2 in [7] suggests that the Hausdorff dimension function has jumps at these points.

Remark 2. Theorem 1.1 was announced in [7, Theorem 4.1.], but with an erroneous formula in the part (iii) (now part II).

Remark 3. For simplicity, in our proofs, we assume $\delta = 0$ in the condition (5) of the d-decaying Gauss like iterated function system. For the general case, the proofs are the same. We need only to replace d by $d + \delta$ for the lower bound and by $d - \delta$ for the upper bound, then take the limit $\delta \to 0$.

2. Technical lemmas

In this section, we prove four technical lemmas. The first lemma serves for the proof of full dimension in the theorems, i.e., the proofs for (I-1) of Theorems 1.1-1.4.

Let $(n_k)_{k\geq 1}$ be a positive sequence satisfying $n_k/k \to \infty$ and $n_{k+1}/n_k \to 1$ as $k \to \infty$. Let u_k be a positive sequence such that

$$\lim_{k\to\infty} \frac{1}{n_k}\sum_{j=1}^{k} \log u_j = 0. \tag{2.1}$$

For each $M \in \mathbb{N}$, set

$$E_M := \{x \in (0,1) : a_{n_k}(x) = u_k, \text{ and } 1 \leq a_j(x) \leq M \text{ if } j \neq n_k\}.$$

Then we have the following lemma. The idea comes from the proof of Theorem 1.4 of [10].

Lemma 2.1. *Suppose the d-decaying Gauss like iterated function system $\{f_n\}_{n\in\mathbb{N}}$ satisfies the distortion property (1.1). Then we have*

$$\lim_{M\to\infty} \dim_H E_M = 1.$$

Proof. For any $k \geq 1$, let $I_{n_k}(a_1 \cdots a_{n_k})$ be an n_k-cylinder intersecting E_M. By the distorsion property (1.1), we have

$$|I_{n_k}| \geq K_3^{2k} \prod_{j=1}^{k} |I_{n_j - n_{j-1} - 1}(a_{n_{j-1}+1}, \cdots, a_{n_j - 1})| \cdot a_{n_j}^{-d},$$

where by convention $n_0 = 0$. □

Let $s(M)$ be the Hausdorff dimension of the set of points x such that all $a_j(x) \leq M$. Then $s(M)$ is increasing to 1, see for example, [9, Theorem 3.15]. Further, there exists a probability measure ν living on $\Pi(\mathbb{N}^{\mathbb{N}})$ and a positive constant C_M such that for any cylinder $I_n(a_1, \ldots, a_n)$ we have

$$\nu(I_n(a_1, \ldots, a_n)) \leq C_M |I_n(a_1, \ldots, a_n)|^{2s(M)-1}.$$

Define a probability measure μ on each cylinder I_{n_k} intersecting E_M by

$$\mu(I_{n_k}) = \prod_{j=1}^{k} \nu(I_{n_j - n_{j-1} - 1}(a_{n_{j-1}+1}, \cdots, a_{n_j - 1})).$$

By Kolmogorov Consistence Theorem, μ is well defined and is supported on E_M.

Then for each $x \in E_M$, we have

$$|I_{n_k}(x)|^{2s(M)-1} \geq C_M^{-k} \mu(I_{n_k}(x)) \prod_{j=1}^{k} a_{n_j}^{-d}.$$

Observe that (2.1) implies that $\sum_{j=1}^{k} \log a_{n_j} \ll n_k$, while the part (4) of the definition of the d-decaying Gauss like iterated function systems implies that

$$\log |I_{n_k}(x)| \leq -\frac{\log A}{m} n_k. \tag{2.2}$$

Thus,

$$\frac{\log \mu(I_{n_k}(x))}{\log |I_{n_k}(x)|} \geq 2s(M) - 1 - o(1) \tag{2.3}$$

for large k.

This allows us to estimate the local dimension $\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$ of measure μ at x. Let us first observe the following two facts.

Fact 1. For $r = |I_{n_k}(x)|$,

$$B_r(x) \cap E_M \subset I_{n_k}(x).$$

Indeed, the pair $(I_{n_k}(x), I_{n_k-1}(x))$ is an image of the pair $(I_1(a_{n_k}), [0,1])$ under the map $f_{a_1} \circ \ldots \circ f_{a_{n_k-1}}$. The cylinder $I_1(a_{n_k})$ has length $\approx a_{n_k}^{-d}$ and lies in distance $\approx a_{n_k}^{-d+1}$ from the endpoints $\{0,1\}$, and the map we apply has bounded distortion, hence it roughly preserves the proportions. Thus, $I_{n_k}(x)$ is also short and far away from the endpoints of $I_{n_k-1}(x)$.

Fact 2. When $k \to \infty$,

$$\frac{\log |I_{n_{k+1}}(x)|}{\log |I_{n_k}(x)|} \to 1.$$

Indeed, as

$$\frac{|I_{n_{k+1}}(x)|}{|I_{n_k}(x)|} \geq (K_1 M^{-d})^{n_{k+1}-n_k} \cdot K_1 a_{n_{k+1}}^{-d}$$

the statement follows from the formula (2.2) and the hypothesis $n_{k+1}/n_k \to 1$ which is equivalent to $(n_{k+1} - n_k)/n_k \to 0$.

The first fact implies that when $r = |I_{n_k}(x)|$ we can use (2.3) in the local dimension calculation. The second fact implies that we do not need to check any r not of the form $r = |I_{n_k}(x)|$. Thus, by the Mass Distribution Principle (see [1, Principle 4.2]), we have

$$\dim_H E_M \geq 2s(M) - 1.$$

Passing with M to infinity, we obtain the assertion.

The second lemma is an improved version of [3, Lemma 3.2.], [5, Proof of Theorem 1.3.], [7, Lemma 2.2.] and [8, Lemma 2.2.].

Let $(s_n)_{n\geq 1}, (t_n)_{n\geq 1}$ be two positive integer sequences. Assume that $s_n > t_n$, $s_n, t_n \to \infty$ as $n \to \infty$, and

$$\liminf_{n\to\infty} \frac{s_n - t_n}{s_n} > 0.$$

For $N \in \mathbb{N}$, let

$$B(s_n, t_n, N) := \{x \in (0,1) : s_n - t_n \leq a_n(x) \leq s_n + t_n,\ \forall n \geq N\}.$$

Lemma 2.2. *We have*

$$\dim_H B(s_n, t_n, N) = \liminf_{n\to\infty} \frac{\sum_{i=1}^{n} \log t_i}{d\sum_{i=1}^{n+1} \log s_i - \log t_{n+1}}.$$

Proof. Within this proof, we write $f(n) \sim g(n)$ if $f(n)$ and $g(n)$ differ by at most an exponential factor, that is

$$\limsup_{n\to\infty} \frac{1}{n}\left|\log \frac{f(n)}{g(n)}\right| < \infty.$$

We give the proof for the case $N = 1$. For the general case, note that

$$B(s_n, t_n, N) = \bigcup_{a_1\cdots a_{N-1}\in\mathbb{N}^{N-1}} f_{a_1}\circ\cdots\circ f_{a_{N-1}}(B(s_{n+N-1}, t_{n+N-1}, 1))$$

is a countable union of bi-Lipschitz images of $B(s_{n+N-1}, t_{n+N-1}, 1)$. Since the bi-Lipschitz maps preserve the Hausdorff dimension, we have

$$\dim_H B(s_n, t_n, N) = \dim_H(B(s_{n+N-1}, t_{n+N-1}, 1)).$$

On the other hand, notice that the dimensional formula of the lemma we will obtain does not depend on the finite number of first terms of the two sequences (s_n) and (t_n), we then have

$$\dim_H B(s_n, t_n, N) = \dim_H(B(s_n, t_n, 1)).$$

Let $n \geq 1$ and $I_n(a_1, \ldots, a_n)$ be an n-cylinder with non-empty intersection with $B(s_n, t_n, 1)$. Then for each $1 \leq k \leq n$, $a_k \in [s_k - t_k, s_k + t_k]$. Define

$$D_n(a_1, \ldots, a_n) := \big\{x \in I_n(a_1, \ldots, a_n) : a_{n+1}(x) \in [s_{n+1}-t_{n+1}, s_{n+1}+t_{n+1}]\big\}.$$

We have

$$\begin{aligned} B(s_n, t_n, 1) &= \bigcap_{n=1}^{\infty} \bigcup_{\substack{a_1,\ldots,a_n \\ a_i\in[s_i-t_i, s_i+t_i]}} I_n(a_1, \ldots, a_n) \\ &= \bigcap_{n=1}^{\infty} \bigcup_{\substack{a_1,\ldots,a_n \\ a_i\in[s_i-t_i, s_i+t_i]}} D_n(a_1, \ldots, a_n). \end{aligned}$$

At level n we have $\sim \prod_{i=1}^n t_i$ intervals $I_n(a_1, \ldots, a_n)$ and corresponding $D_n(a_1, \ldots, a_n)$. Each $I_n(a_1, \ldots, a_n)$ is of size $\sim \prod_{i=1}^n s_i^{-d}$. Moreover,

$$\frac{|D_n(a_1, \ldots, a_n)|}{|I_n(a_1, \ldots, a_n)|} \sim \sum_{i=s_{n+1}-t_{n+1}}^{s_{n+1}+t_{n+1}} i^{-d} \sim t_{n+1}s_{n+1}^{-d}.$$

Thus, using for a given n the sets $D_n(a_1, \ldots, a_n)$ as a cover for $B(s_n, t_n, 1)$, we need $\sim \prod_{i=1}^n t_i$ of them, each of size $\sim t_{n+1}\prod_{i=1}^{n+1} s_i^{-d}$. Then we obtain the upper bound

$$\dim_H B(s_n, t_n, 1) \leq \liminf_{n\to\infty} \frac{\sum_{i=1}^n \log t_i}{d\sum_{i=1}^{n+1}\log s_i - \log t_{n+1}}.$$

To get the lower bound, we consider a probability measure μ uniformly distributed on $B(s_n, t_n, 1)$, in the following sense: given $a_1, \ldots, a_{n-1}$, the probability of a_n taking any particular value between $s_n - t_n$ and $s_n + t_n$ is the same. The basic intervals $I_n(a_1, \ldots, a_n)$ and corresponding $D_n(a_1, \ldots, a_n)$ have the measure $\sim \prod_{i=1}^n t_i^{-1}$.

Our goal is to apply the Mass Distribution Principle, hence we need to calculate the local dimension of the measure μ at a μ-typical point $x \in B(s_n, t_n, 1)$. Fix any $x \in B(s_n, t_n, 1)$. Denote by r_n the diameter of the set

$D_n(a_1(x), \ldots, a_n(x))$ and by r'_n the diameter of $I_n(a_1(x), \ldots, a_n(x))$. When $r = r_n$, we have

$$\frac{\log \mu(B_r(x))}{\log r} = \frac{\log \mu(D_n(a_1(x), \ldots, a_n(x)))}{\log r_n} \approx \frac{\sum_{i=1}^n \log t_i}{d \sum_{i=1}^{n+1} \log s_i - \log t_{n+1}}.$$

For $r_n < r < r'_n$, the ball $B_r(x)$ still does not intersect any point from $B(s_n, t_n, 1) \setminus D_n(a_1(x), \ldots, a_n(x))$, hence it has the same measure as $B_{r_n}(x)$, but a larger diameter. Finally, for $r'_{n+1} < r < r_n$ we have

$$\mu(B_r(x)) \sim \frac{r}{r_n} \mu(B_{r_n}(x)),$$

since each cylinder $I_{n+1}(a_1(x), \ldots, a_n(x), j)$ contained in $D_n(a_1(x), \ldots, a_n(x))$ has the same measure and approximately the same diameter. Applying the obvious fact that

$$\frac{\log z_1 z_2}{\log z_1 z_3} > \frac{\log z_2}{\log z_3}$$

for all $z_1 < 1$ and $z_3 < z_2 < 1$, we see that for $r < r_n$

$$\frac{\log \mu(B_{r_n}(x))}{\log r_n} < \frac{\log(\mu(B_{r_n}(x)) \cdot r/r_n)}{\log r}.$$

Thus, the minimum of the function $r \to \log \mu(B_r(x))/\log r$ for $r'_{n+1} < r < r'_n$ is equal to its value at r_n, up to an error term that vanishes as $n \to \infty$. It implies

$$\liminf_{r \to 0} \frac{\log \mu(B_r(x))}{\log r} = \liminf_{n \to \infty} \frac{\log \mu(B_{r_n}(x))}{\log r_n} = \liminf_{n \to \infty} \frac{\sum_{i=1}^n \log t_i}{d \sum_{i=1}^{n+1} \log s_i - \log t_{n+1}}.$$

Applying the Mass Distribution Principle, we obtain the lower bound for $\dim_H B(s_n, t_n, 1)$ and finish the proof. □

The remaining two lemmas are generalizations of Lemma 2.1 of [7].

Let $\zeta(\cdot)$ be the Riemann zeta function. For $m, n \in \mathbb{N}$, $a > 0$ and $\varepsilon > 0$, let

$$A(m, n, a, \varepsilon) := \left\{ (i_1, \ldots, i_n) \in \mathbb{N}^n : \sum_{k=1}^n i_k^a \in [m, m + m\varepsilon)] \right\}.$$

For $s > 1/d$, write

$$G(m, n, a, \varepsilon, s) = \sum_{i_1 \cdots i_n \in A(m,n,a,\varepsilon)} \prod_{k=1}^n i_k^{-ds}.$$

Lemma 2.3. *There exist positive constants $C_1 = C_1(a, s), C_2 = C_2(s)$, and $C_3 = C_3(a)$, such that for all $C_3 \cdot (m3^{2-n})^{-1/a} < \varepsilon < 1/3$, we have*

$$G(m, n, a, \varepsilon, s) \leq C_1 C_2^{n-1} \varepsilon \cdot m^{\frac{1-ds}{a}}.$$

Proof. The proof goes by induction. First consider the case $n = 2$. Note that if $i_1^a + i_2^a \in [m, m + m\varepsilon]$ then at most one of i_1, i_2 is strictly larger than $\frac{m+m\varepsilon}{2}$. We divide the sum in the definition of $G(m, n, a, \varepsilon, s)$ into two parts, one is $i_1 > \frac{m+m\varepsilon}{2}$, the other is $i_1 \leq \frac{m+m\varepsilon}{2}$. However, by permuting i_1 and i_2, the two sums are the same. Thus

$$G(m,2,a,\varepsilon,s) \leq 2 \sum_{k=1}^{(\frac{m(1+\varepsilon)}{2})^{\frac{1}{a}}} k^{-ds}(m(1+\varepsilon)-k^a)^{-\frac{ds}{a}} \cdot N_{m,a,\varepsilon}(k),$$

with $N_{m,a,\varepsilon}(k) := \sharp\{i_2 : m-k^a \leq i_2^a \leq m-k^a+\varepsilon m\}$.

Assuming $\varepsilon < 1/3$, we can estimate for $a \geq 1$

$$N_{m,a,\varepsilon}(k) \leq \lceil a^{-1}\varepsilon m(m-k^a)^{\frac{1}{a}-1}\rceil \leq \lceil \varepsilon m^{1/a} \cdot a^{-1}3^{1-1/a}\rceil,$$

while for $a < 1$

$$N_{m,a,\varepsilon}(k) \leq \lceil a^{-1}\varepsilon m(m(1+\varepsilon)-k^a)^{\frac{1}{a}-1}\rceil \leq \lceil \varepsilon m^{1/a} \cdot a^{-1}(4/3)^{1/a-1}\rceil.$$

That is, in both cases we will get an upper estimation in the form $\lceil \varepsilon m^{1/a} \cdot C_4(a)\rceil$.

If $z > 1$, we can write $\lceil z\rceil \leq 2z$. Thus, for $\varepsilon > m^{-1/a}C_4^{-1}(a)$ we have

$$N_{m,a,\varepsilon}(k) \leq 2\varepsilon m^{1/a} \cdot C_4(a).$$

Hence

$$\begin{aligned} &G(m,2,a,\varepsilon,s) \\ \leq &2 \sum_{k=1}^{(\frac{m(1+\varepsilon)}{2})^{\frac{1}{a}}} k^{-ds}(\frac{m}{2})^{-\frac{ds}{a}} \cdot 2\varepsilon m^{\frac{1}{a}} \cdot C_4(a) \\ \leq &\zeta(ds) \cdot 2^{\frac{ds}{a}+2}C_4(a)\varepsilon m^{\frac{1-ds}{a}}. \end{aligned} \tag{2.4}$$

Assume now that the assertion is satisfied for all $n < N$ for some $N > 2$, we will prove by induction that it holds for $n = N$ as well.

As above, there is at most one i_k such that $i_k > \frac{m+m\varepsilon}{2}$. Thus the sum of $G(m,N,a,\varepsilon,s)$ can be divided into two parts, one is $i_1 \leq \frac{m+m\varepsilon}{2}$ and the other is $i_1 > \frac{m+m\varepsilon}{2}$. But the latter is the same as the first case by permuting i_1 and i_2. Further, by observing $3(m-k^a)\varepsilon > m\varepsilon$, we can deduce

$$G(m,N,a,\varepsilon,s) \leq 2 \sum_{k=1}^{(\frac{m+m\varepsilon}{2})^{\frac{1}{a}}} k^{-ds} \sum_{j=0}^{2} G((m-k^a)(1+j\varepsilon), N-1, a, \varepsilon, s).$$

Substituting the induction assumption, we get

$$\begin{aligned} G(m,N,a,\varepsilon,s) &\leq 6 \cdot C_1 C_2^{N-2}\varepsilon(\frac{m}{3})^{\frac{1-ds}{a}} \sum_{k=1}^{(\frac{m+m\varepsilon}{2})^{\frac{1}{a}}} k^{-ds} \\ &\leq 6 \cdot 3^{\frac{ds-1}{a}} C_1 C_2^{N-2}\varepsilon m^{\frac{1-ds}{a}}\zeta(ds). \end{aligned}$$

Thus, by comparing the formula (2.4), we proved the assertion for

$$C_1 = 2^{\frac{ds+a}{a}}C_4(a),\ C_2 = 6 \cdot 3^{\frac{ds-1}{a}}\zeta(ds),$$

and we needed that $\varepsilon \in ((m3^{2-n})^{-1/a}C_4^{-1}(a), 1/3)$. We can choose $C_3 = C_4^{-1}$. □

The next lemma is very similar. Let

$$\widehat{A}(m,n,b,\varepsilon) := \left\{ (i_1,\ldots,i_n) \in \mathbb{N}^n : \sum_{k=1}^{n} e^{(\log i_k)^b} \in [m, m(1+\varepsilon)] \right\}.$$

and for $s > 1/d$, write

$$\widehat{G}(m,n,b,\varepsilon,s) = \sum_{i_1\cdots i_n \in \widehat{A}(m,n,b,\varepsilon)} \prod_{k=1}^{n} i_k^{-ds}.$$

Lemma 2.4. *There exists a positive constant $\widehat{C} = \widehat{C}(s)$ such that for all $e^{-(\log(m3^{2-n}))^{1/b}} < \varepsilon < 1/3$, we have*

$$\widehat{G}(m,n,b,\varepsilon,s) \le 6 \cdot \widehat{C}^{n-1} \varepsilon \cdot e^{(1-ds)(\log m)^{1/b}}.$$

Proof. The proof goes again by induction. First consider the case $n = 2$. Similar to the proof of Lemma 2.3, we have

$$\widehat{G}(m,2,b,\varepsilon,s) \le 2 \sum_{k=1}^{e^{(\log(m(1+\varepsilon)/2)^{1/b}}} k^{-ds} e^{-ds(\log(m-e^{(\log k)^b}))^{1/b}} \cdot \widehat{N}_{m,b,\varepsilon}(k),$$

with

$$\widehat{N}_{m,b,\varepsilon}(k) := \sharp\{i_2 : m - e^{(\log k)^b} \le e^{(\log i_2)^b} \le m - e^{(\log k)^b} + \varepsilon m\}.$$

For $\varepsilon < 1/3$, short calculations give us the following estimation

$$\widehat{N}_{m,b,\varepsilon}(k) \le \lceil 3\varepsilon \cdot e^{(\log m)^{1/b}} \rceil.$$

Hence, if $\varepsilon > e^{-(\log m)^{1/b}}$,

$$\widehat{N}_{m,b,\varepsilon}(k) \le 6\varepsilon \cdot e^{(\log m)^{1/b}}.$$

Thus, by noting $e^{\log(m/3))^{1/b}} \ge \frac{1}{3} e^{(\log m)^{1/b}}$, we obtain

$$\widehat{G}(m,2,b,\varepsilon,s) \le 12 \cdot 3^{ds} \zeta(ds) e^{(1-ds)(\log m)^{1/b}}.$$

Assume now that the assertion is satisfied for all $n < N$ for some $N > 2$, we will prove by induction that it holds for $n = N$ as well. We have

$$\widehat{G}(m,N,b,\varepsilon,s) \le 2 \sum_{k=1}^{e^{(\log(m(1+\varepsilon)/2)^{1/b}}} k^{-ds} \sum_{j=0}^{2} \widehat{G}((m-e^{(\log k)^b})(1+j\varepsilon), N-1, b, \varepsilon, s).$$

Substituting the induction assumption, we get

$$\widehat{G}(m,N,b,\varepsilon,s) \le 12 \cdot 3^{ds} \widehat{C}^{N-2} \varepsilon e^{(1-ds)(\log m)^{1/b}} \zeta(ds).$$

Thus, we proved the assertion for

$$\widehat{C} = 2 \cdot 3^{ds} \zeta(ds)$$

under the assumption $\varepsilon \in (e^{-(\log(m3^{2-n}))^{1/b}}, 1/3)$.

□

3. Proofs for (I-1) of Theorems 1.1-1.4 and (I-2) of Theorem 1.4

3.1. Proofs for (I-1) of Theorems 1.1-1.4. For these parts of proofs we suppose the d-decaying Gauss like iterated function system satisfies the distortion property (1.1). We will apply Lemma 2.1.

Note that in all cases we are going to prove, the function Φ is taken as $\Phi(n) = e^{n^\alpha}$. Let $\varepsilon > 0$. Take $n_k = k^{\frac{1}{\alpha}(1-\varepsilon)}$ and $u_k = \varphi^{-1}(\Phi(n_k) - \Phi(n_{k-1}))$. Then evidently the sequence $(n_k)_{k\geq 1}$ satisfies the assumption of Lemma 2.1. We can also check that $E_M \subset E_\varphi(\Phi)$. In fact, for any $x \in E_M$ we have

$$\Phi(n_k) < S_{n_k}\varphi(x) < \Phi(n_k) + n_k\varphi(M).$$

Since $\Phi(n)/n \to \infty$, we see that

$$\frac{S_{n_k}\varphi(x)}{\Phi(n_k)} \to 1.$$

However, as $n_{k+1}/n_k \to 1$ and $S_n\varphi$ is increasing, this is enough to have

$$\lim_n \frac{S_n\varphi(x)}{\Phi(n)} = \lim_k \frac{S_{n_k}\varphi(x)}{\Phi(n_k)}$$

and we are done.

Now we need only to check for each case of φ in Theorems 1.1-1.4, the condition (2.1) is satisfied. First notice that

$$\Phi(n_k) - \Phi(n_{k-1}) = e^{k^{1-\varepsilon}} - e^{(k-1)^{1-\varepsilon}} \approx (1-\varepsilon)k^{-\varepsilon}e^{k^{1-\varepsilon}}.$$

Thus when $\varphi(j) = j^a$, we have

$$u_k \approx \left((1-\varepsilon)k^{-\varepsilon}e^{k^{1-\varepsilon}}\right)^{1/a},$$

and, if $\alpha < 1/2$ and ε is small enough,

$$\lim_{k\to\infty} \frac{1}{n_k}\sum_{j=1}^{k} \log u_j = \lim_{k\to\infty} \frac{\sum_{j=1}^k j^{1-\varepsilon}/a}{k^{\frac{1}{\alpha}(1-\varepsilon)}} = 0.$$

When $\varphi(j) = e^{(\log j)^b}$, then

$$u_k \approx e^{(\log((1-\varepsilon)k^{-\varepsilon}e^{k^{1-\varepsilon}}))^{1/b}},$$

and if $\alpha < \frac{b}{b+1}$, and ε is small enough,

$$\lim_{k\to\infty} \frac{1}{n_k}\sum_{j=1}^{k} \log u_j = \lim_{k\to\infty} \frac{\sum_{j=1}^k j^{\frac{1-\varepsilon}{b}}}{k^{\frac{1}{\alpha}(1-\varepsilon)}} = 0.$$

When $\varphi(j) = e^{j^c}$, we have

$$u_k \approx \log((1-\varepsilon)k^{-\varepsilon}e^{k^{1-\varepsilon}})^{1/c},$$

and, if $\alpha < 1$ and ε is small enough,

$$\lim_{k\to\infty} \frac{1}{n_k}\sum_{j=1}^{k} \log u_j = \lim_{k\to\infty} \frac{\sum_{j=1}^k \frac{1-\varepsilon}{c}\log j}{k^{\frac{1}{\alpha}(1-\varepsilon)}} = 0.$$

Then in all cases the condition (2.1) is satisfied.

Applying Lemma 2.1, we complete the proofs.

3.2. **Proofs for (I-2) of Theorem 1.4.** We will use a natural covering. Suppose $\Phi(n) = e^{n^\alpha}$ with $\alpha > 1$. For each $x \in E_\varphi(\Phi)$, for any small $\varepsilon > 0$, for all large enough n, we have

$$(1-\varepsilon)\Phi(n) \leq \sum_{k=1}^{n} \varphi(a_k) \leq (1+\varepsilon)\Phi(n).$$

Thus

$$(1-\varepsilon)\Phi(n) - (1+\varepsilon)\Phi(n-1) \leq \varphi(a_n) \leq (1+\varepsilon)\Phi(n) - (1-\varepsilon)\Phi(n-1).$$

Note that for $\alpha > 1$, we have

$$(1+\varepsilon)\Phi(n) - (1-\varepsilon)\Phi(n-1) = (1+\varepsilon)e^{n^\alpha} - (1-\varepsilon)e^{(n-1)^\alpha} \leq (1+\varepsilon)e^{n^\alpha},$$

and

$$(1-\varepsilon)\Phi(n) - (1+\varepsilon)\Phi(n-1) = (1-\varepsilon)e^{n^\alpha} - (1+\varepsilon)e^{(n-1)^\alpha} \geq (1-2\varepsilon)e^{n^\alpha}.$$

Hence

$$(1-2\varepsilon)e^{n^\alpha} \leq \varphi(a_n) \leq (1+\varepsilon)e^{n^\alpha}.$$

However, for $\varphi(j) = e^{j^c}$ with $c \geq 1$, there is at most one j such that

$$(1-2\varepsilon)e^{n^\alpha} \leq \varphi(j) \leq (1+\varepsilon)e^{n^\alpha},$$

Hence $E_\varphi(\Phi)$ is a countable set which has Hausdorff dimension 0.

4. Remaining proofs

We will divide the case I-2 of Theorem 1.1 into two subcases: subcase I-2a for $1/2 < \alpha < 1$, and subcase I-2b for $\alpha \geq 1$. Similarly, we will divide the case I-2 of Theorem 1.2 into subcase I-2a ($b/(b+1) < \alpha < 1$) and subcase I-2b ($\alpha \geq 1$).

Theorem 1.1, case II; Theorem 1.1, subcase I-2b; Theorem 1.2, case II; Theorem 1.2, subcase I-2b; Theorem 1.3, case I-2; Theorem 1.3, case II; Theorem 1.3, case III are all obtained by applying Lemma 2.2.

4.1. **Proof of Theorem 1.1, case II.** Let $x \in E_\varphi(\Phi)$. Fix some small $\varepsilon > 0$. For N large enough we will have $\Phi(n)(1-\varepsilon) < S_n\varphi(x) < \Phi(n)(1+\varepsilon)$ for all $n > N$. This implies

$$\begin{aligned} \varphi(a_n(x)) &= S_n\varphi(x) - S_{n-1}\varphi(x) \\ &\in \Big(\Phi(n)(1-\varepsilon) - \Phi(n-1)(1+\varepsilon), \Phi(n)(1+\varepsilon) - \Phi(n-1)(1-\varepsilon)\Big) \end{aligned} \tag{4.1}$$

for $n \geq N$. Substituting the formula for Φ, we get

$$\varphi(a_n(x)) \in \Big(e^{\beta^n}(1-2\varepsilon), e^{\beta^n}(1+2\varepsilon)\Big).$$

Hence a further substitution of the formula for φ gives us

$$e^{\beta^n/a}(1-3\varepsilon/a) < a_n(x) < e^{\beta^n/a}(1+3\varepsilon/a).$$

Thus,

$$E_\varphi(\Phi) \subset \bigcup_N B(e^{\beta^n/a}, 3\varepsilon e^{\beta^n/a}/a, N).$$

Put $s_n = e^{\beta^n/a}$ and $t_n = 3\varepsilon e^{\beta^n/a}/a$. By Lemma 2.2, we have the upper bound

$$\begin{aligned}\dim_H E_\varphi(\Phi) \leq & \liminf_{n\to\infty} \frac{\sum_{j=1}^n \log 3\varepsilon e^{\beta^j/a}/a}{d\sum_{j=1}^{n+1}\log e^{\beta^j/a} - \log 3\varepsilon e^{\beta^{n+1}/a}/a} \\ = & \liminf_{n\to\infty} \frac{\sum_{j=1}^n \beta^j/a}{d\sum_{j=1}^{n+1}\beta^j/a - \beta^{n+1}/a} \\ = & \frac{1}{d\beta - \beta + 1}.\end{aligned}$$

On the other hand, let ε_n be a sequence of positive numbers converging to 0. Let $x \in B(e^{\beta^n/a}, \varepsilon_n e^{\beta^n/a}, 1)$. For large n we have

$$e^{\beta^n}(1-\varepsilon_n)^a < S_n\varphi(x) < e^{\beta^n}(1+\varepsilon_n)^a + \sum_{i=1}^{n-1}(1+\varepsilon_i)^a \cdot e^{\beta^i} < e^{\beta^n}(1+a\varepsilon_n + o(1)).$$

Thus,

$$E_\varphi(\Phi) \supset B(e^{\beta^n/a}, \varepsilon_n e^{\beta^n/a}, 1).$$

Applying Lemma 2.2 and doing almost the same calculation as above, we obtain the lower bound.

4.2. **Theorem 1.1, case I-2b.** We can repeat the proof of Theorem 1.1, case II. From the formula (4.1), we get

$$\varphi(a_n(x)) \in \Big(e^{n^\alpha}(1-2\varepsilon), e^{n^\alpha}(1+2\varepsilon)\Big).$$

Hence,

$$E_\varphi(\Phi) \subset \bigcup_N B(e^{n^\alpha/a}, 3\varepsilon e^{n^\alpha/a}/a, N).$$

On the other hand, for a sequence of positive numbers ε_n converging to 0, we have

$$E_\varphi(\Phi) \supset B(e^{n^\alpha/a}, \varepsilon_n e^{n^\alpha/a}, 1).$$

Applying Lemma 2.2, we have

$$\dim_H E_\varphi(\Phi) = \liminf_{n\to\infty} \frac{\sum_{j=1}^n j^\alpha/a}{d\sum_{j=1}^{n+1} j^\alpha/a - (n+1)^\alpha/a} = \frac{1}{d}.$$

4.3. **Theorem 1.2, case II.** From the formula (4.1), we get

$$\varphi(a_n(x)) \in \Big(e^{\beta^n}(1-2\varepsilon), e^{\beta^n}(1+2\varepsilon)\Big).$$

Hence,

$$E_\varphi(\Phi) \subset \bigcup_N B(e^{\beta^{n/b}}, \frac{3\varepsilon}{b}\beta^{n(1/b-1)}e^{\beta^{n/b}}, N).$$

On the other hand, for a positive sequence ε_n converging to 0, we have

$$E_\varphi(\Phi) \supset B(e^{\beta^{n/b}}, \varepsilon_n \beta^{n(1/b-1)}e^{\beta^{n/b}}, 1).$$

Applying Lemma 2.2, we have

$$\dim_H E_\varphi(\Phi) = \liminf_{n\to\infty} \frac{\sum_{j=1}^n \beta^{j/b}}{d\sum_{j=1}^{n+1}\beta^{j/b} - \beta^{(n+1)/b}} = \frac{1}{d\beta^{1/b} - \beta^{1/b} + 1}.$$

4.4. **Theorem 1.2, case I-2b.** From the formula (4.1), we get

$$\varphi(a_n(x)) \in \Big(e^{n^\alpha}(1-2\varepsilon), e^{n^\alpha}(1+2\varepsilon)\Big).$$

Hence,

$$E_\varphi(\Phi) \subset \bigcup_N B(e^{n^{\alpha/b}}, \frac{3\varepsilon}{b} n^{\alpha(1/b-1)} e^{n^{\alpha/b}}, N).$$

On the other hand, for a sequence of positive numbers ε_n converging to 0, we have

$$E_\varphi(\Phi) \supset B(e^{n^{\alpha/b}}, \varepsilon_n n^{\alpha(1/b-1)} e^{n^{\alpha/b}}, 1).$$

Applying Lemma 2.2, we have

$$\dim_H E_\varphi(\Phi) = \liminf_{n\to\infty} \frac{\sum_{j=1}^n j^{\alpha/b}}{d\sum_{j=1}^{n+1} j^{\alpha/b} - (n+1)^{\alpha/b}} = \frac{1}{d}.$$

4.5. **Theorem 1.3, case I-2.** From the formula (4.1), we get

$$\varphi(a_n(x)) \in \Big(e^{n^\alpha}(1-2\varepsilon), e^{n^\alpha}(1+2\varepsilon)\Big).$$

Hence,

$$E_\varphi(\Phi) \subset \bigcup_N B(n^{\alpha/c}, \frac{3\varepsilon}{c} n^{\alpha(1/c-1)}, N).$$

On the other hand, for a sequence of positive numbers ε_n converging to 0, we have

$$E_\varphi(\Phi) \supset B(n^{\alpha/c}, \varepsilon_n n^{\alpha(1/c-1)}, 1).$$

We then apply Lemma 2.2 to obtain

$$\dim_H E_\varphi(\Phi) = \liminf_{n\to\infty} \frac{\sum_{j=1}^n \alpha(1/c-1)\log j}{d\sum_{j=1}^{n+1} \alpha/c \log j - \alpha(1/c-1)\log n} = \frac{1-c}{d}.$$

4.6. **Theorem 1.3, case II.** From the formula (4.1), we get

$$\varphi(a_n(x)) \in \Big(e^{\beta^n}(1-2\varepsilon), e^{\beta^n}(1+2\varepsilon)\Big).$$

Hence,

$$E_\varphi(\Phi) \subset \bigcup_N B(\beta^{n/c}, \frac{3\varepsilon}{c} \beta^{n(1/c-1)}, N).$$

On the other hand, for a positive sequence ε_n converging to 0, we have

$$E_\varphi(\Phi) \supset B(\beta^{n/c}, \varepsilon_n \beta^{n(1/c-1)}, 1).$$

Applying Lemma 2.2, we obtain

$$\dim_H E_\varphi(\Phi) = \liminf_{n\to\infty} \frac{\sum_{j=1}^n j(1/c-1)\log\beta}{d\sum_{j=1}^{n+1} j/c \log\beta - (n+1)(1/c-1)\log\beta} = \frac{1-c}{d}.$$

4.7. **Theorem 1.3, case III.** From the formula (4.1), we get

$$\varphi(a_n(x)) \in \Big(e^{e^{\gamma^n}}(1-2\varepsilon), e^{e^{\gamma^n}}(1+2\varepsilon)\Big).$$

Hence,

$$E_\varphi(\Phi) \subset \bigcup_N B(e^{\frac{1}{c}\gamma^n}, \frac{3\varepsilon}{c} e^{\gamma^n(1/c-1)}, N).$$

On the other hand, for a positive sequence ε_n converging to 0, we have

$$E_\varphi(\Phi) \supset B(e^{\frac{1}{c}\gamma^n}, \varepsilon_n e^{\gamma^n(1/c-1)}, 1).$$

Applying Lemma 2.2, we get

$$\dim_H E_\varphi(\Phi) = \liminf_{n\to\infty} \frac{\sum_{j=1}^{n}(1/c-1)\gamma^j}{d\sum_{j=1}^{n+1} 1/c\gamma^j - (1/c-1)\gamma^{n+1}} = \frac{1-c}{d\gamma - (1-c)(\gamma-1)}.$$

We also apply Lemma 2.2 for the lower bounds of Theorem 1.1, subcase I-2a and Theorem 1.2, subcase I-2a. But for the upper bounds we need Lemma 2.3 and Lemma 2.4 respectively.

4.8. **Proof of Theorem 1.1, case I-2a.** We first show the lower bound. Let x be points such that

$$\varphi(a_n(x)) \in \Big(\alpha n^{\alpha-1}e^{n^\alpha}(1-\varepsilon_n), \alpha n^{\alpha-1}e^{n^\alpha}(1+\varepsilon_n)\Big).$$

where ε_n is a summable positive sequence. Then

$$\sum_{j=1}^{n} \alpha j^{\alpha-1}e^{j^\alpha}(1-\varepsilon_j) \le \sum_{j=1}^{n} \varphi(a_j(x)) \le \sum_{j=1}^{n} \alpha j^{\alpha-1}e^{j^\alpha}(1+\varepsilon_j),$$

which implies

$$e^{n^\alpha} - 2\sum_{j=1}^{n} \alpha j^{\alpha-1}e^{j^\alpha}\varepsilon_j \le \sum_{j=1}^{n} \varphi(a_j(x)) \le e^{n^\alpha} - 2\sum_{j=1}^{n} \alpha j^{\alpha-1}e^{j^\alpha}\varepsilon_j.$$

Note that

$$\sum_{j=1}^{n/2} \alpha j^{\alpha-1}e^{j^\alpha}\varepsilon_j \le \sum_{j=1}^{n/2} \alpha j^{\alpha-1}e^{j^\alpha} \le e^{(n/2)^\alpha},$$

and by the summability of (ε_n),

$$\sum_{j=n/2}^{n} \alpha j^{\alpha-1}e^{j^\alpha}\varepsilon_j \le \alpha n^{\alpha-1}e^{n^\alpha}\sum_{j=1}^{n/2}\varepsilon_j = o(e^{n^\alpha}).$$

Hence, these points x are all in $E_\varphi(\Phi)$, that is

$$E_\varphi(\Phi) \supset B\Big((\alpha n^{\alpha-1}e^{n^\alpha})^{1/a}, \frac{\varepsilon_n}{a}(\alpha n^{\alpha-1}e^{n^\alpha})^{1/a}, 1\Big).$$

Applying Lemma 2.2, we obtain the lower bound.

Now we turn to the upper bound.

Take a subsequence $n_0 = 1$, and $n_k = \Phi^{-1}(e^k) = k^{1/\alpha}$ $(k \geq 1)$. If $x \in E_\varphi(\Phi)$ then for any $\varepsilon > 0$ there exists an integer $N \geq 1$ such that for all $k \geq N$,

$$(1-\varepsilon/5)\Phi(n_k) \leq S_{n_k}\varphi(x) \leq (1+\varepsilon/5)\Phi(n_k),$$

and (as $\Phi(n_k) = e^k$)

$$(1-\varepsilon/5)e^k - (1+\varepsilon/5)e^{k-1} \leq S_{n_k}(x) - S_{n_{k-1}}(x) \leq (1+\varepsilon/5)e^k - (1-\varepsilon/5)e^{k-1}.$$

Observe that

$$(1+\varepsilon/5)e^k - (1-\varepsilon/5)e^{k-1} < \Big((1-\varepsilon/5)e^k - (1+\varepsilon/5)e^{k-1}\Big) \cdot (1+\varepsilon).$$

Fix $\varepsilon = 1/3$ and denote by A_k the set of points for which the block of symbols $a_{n_{k-1}+1}(x) \cdots a_{n_k}(x)$ in the symbolic expansion of x from the position $n_{k-1}+1$ to n_k belongs to the set

$$A\left((1-\varepsilon/5)e^k - (1+\varepsilon/5)e^{k-1}, n_k - n_{k-1}, a, \varepsilon\right).$$

Then

$$E_\varphi(\Phi) \subset \bigcup_N \bigcap_{k \geq N} A_k.$$

Now, we are going to estimate the upper bound of the Hausdorff dimension of $F = \bigcap_{k\geq 1} A_k$. For $\bigcap_{k \geq N} A_k$ with $N \geq 2$ we have the same bound and the proofs are almost the same.

Let us now define $n(k) = n_k - n_{k-1}$ and $m(k) = (1-\varepsilon/5)e^k - (1+\varepsilon/5)e^{k-1}$. By the assumption $\alpha > 1/2$, we have $m(k)/3^{n(k)} \gg 1$ for k large enough. Thus we can apply Lemma 2.3 to calculate $G(m(k), n(k), a, 1/3, s)$ for all $s > 1/d$ and all k large enough. Hence

$$\begin{aligned} &\sum_{I_{k_n}(a_1,\ldots,a_{n_k}) \cap F \neq \emptyset} |I_{k_n}(a_1, \ldots, a_{n_k})|^s \\ \leq & K_2^{sn_k} \prod_{j=1}^k G(m(j), n(j), a, 1/3, s) \\ \leq & \text{const} \cdot K_2^{sn_k} C_1^k C_2^{n_k} 3^{-k} \prod_{j=1}^k m(j)^{\frac{1-ds}{a}}. \end{aligned}$$

As $ds > 1$, the right hand side is arbitrarily small for large k. This proves the s-dimensional Hausdorff measure

$$\mathcal{H}^s(F) = 0$$

for all $s > 1/d$. We thus obtain the wanted upper bounded.

4.9. **Theorem 1.2, case I-2a.** For the lower bound, we follow the proof of Theorem 1.1, case I-2a by taking those points x such that

$$\varphi(a_n(x)) \in \Big(\alpha n^{\alpha-1} e^{n^\alpha}(1-\varepsilon_n), \alpha n^{\alpha-1} e^{n^\alpha}(1+\varepsilon_n)\Big).$$

where ε_n is a summable positive sequence. Then we still have these points x are all in $E_\varphi(\Phi)$. By apply the inverse of φ, we have

$$E_\varphi(\Phi) \supset B\Big(e^{(n^\alpha + \log\alpha + (\alpha-1)\log n)^{1/b}}, \frac{2\varepsilon_n}{b} n^{\alpha(1/b-1)} e^{(n^\alpha + \log\alpha + (\alpha-1)\log n)^{1/b}}, 1\Big).$$

Applying Lemma 2.2, we obtain the lower bound.

The proof of the upper bound is also similar to that of Theorem 1.1, case I-2a. The difference is that we need to apply Lemma 2.4 in place of Lemma 2.3.

As in the proof of Theorem 1.1, case I-2a, we take a subsequence $n_0 = 1$, and $n_k = \Phi^{-1}(e^k) = k^{1/\alpha}$ $(k \geq 1)$. Denote by $\widehat{A}_k$ the set of points for which the block of symbols $a_{n_{k-1}+1}(x) \cdots a_{n_k}(x)$ in the symbolic expansion of x from the position $n_{k-1}+1$ to n_k belongs to the set

$$\widehat{A}\left(m(k), n(k), b, 1/3\right),$$

with $n(k) = n_k - n_{k-1}$ and $m(k) = \frac{14}{15}e^k - \frac{16}{15}e^{k-1}$. Then

$$E_\varphi(\Phi) \subset \bigcup_N \bigcap_{k \geq N} \widehat{A}_k.$$

We need only to estimate the upper bound of the Hausdorff dimension of $\widehat{F} = \bigcap_{k\geq 1} \widehat{A}_k$. By the assumption $\alpha > \frac{b}{b+1} > \frac{1}{2}$, we still have $m(k)/3^{n(k)} \gg 1$ for k large enough. Thus we can apply Lemma 2.4 to calculate $\widehat{G}(m(k), n(k), b, 1/3, s)$ for all $s > 1/d$ and all k large enough. Hence

$$\begin{aligned}
&\sum_{I_{k_n}(a_1,\ldots,a_{n_k}) \cap F \neq \emptyset} |I_{k_n}(a_1, \ldots, a_{n_k})|^s \\
\leq & K_2^{sn_k} \prod_{j=1}^{k} \widehat{G}(m(j), n(j), b, 1/3, s) \\
\leq & \text{const} \cdot K_2^{sn_k} \cdot 6^k \cdot \widehat{C}^{n_k} \cdot 3^{-k} \prod_{j=1}^{k} e^{(1-ds)(\log m(j))^{1/b}}.
\end{aligned}$$

Note that $\log m(j) \approx j$ and $n_k \approx k^{1/\alpha}$. Thus

$$\prod_{j=1}^{k} e^{(1-ds)(\log m(j))^{1/b}} \approx e^{(1-ds)k^{\frac{b+1}{b}}}$$

and, as $\frac{b+1}{b} > \frac{1}{\alpha}$, this is the dominating term. As $ds > 1$, this term, and the whole product, converge to 0 for $k \to \infty$. This proves the s-dimensional Hausdorff measure

$$\mathcal{H}^s(\widehat{F}) = 0$$

for all $s > 1/d$. We are done.

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