# METRICAL RESULTS ON THE DISTRIBUTION OF FRACTIONAL PARTS OF POWERS OF REAL NUMBERS

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Abstract. Denote by $\{\cdot\}$ the fractional part. We establish several new metrical results on the distribution properties of the sequence $(\{x^n\})_{n\geq 1}$. Many of them are presented in a more general framework, in which the sequence of functions $(x \mapsto x^n)_{n\geq 1}$ is replaced by a sequence $(f_n)_{n\geq 1}$, under some growth and regularity conditions on the functions $f_n$.

## 1. Introduction

Let $\{\cdot\}$ denote the fractional part and $\|\cdot\|$ the distance to the nearest integer. For a given real number $x > 1$, only few results are known on the distribution of the sequence $(\{x^n\})_{n\geq 1}$. For example, we still do not know whether 0 is a limit point of $(\{e^n\})_{n\geq 1}$, nor of $(\{(\frac{3}{2})^n\})_{n\geq 1}$; see [5] for a survey of related results.

However, several metric statements have been established. The first one was obtained in 1935 by Koksma [13], who proved that for almost every $x > 1$ the sequence $(\{x^n\})_{n\geq 1}$ is uniformly distributed on the unit interval $[0, 1]$. Here and below, almost every always refers to the Lebesgue measure. In 1967, Mahler and Szekeres [15] studied the quantity

$$P(x) := \liminf \|x^n\|^{1/n} \quad (x > 1).$$

They proved that $P(x) = 0$ if $x$ is transcendental and $P(x) = 1$ for almost all $x > 1$. The function $x \mapsto P(x)$ was subsequently studied in 2008 by Bugeaud and Dubickas [6]. Among other results, it was shown in [6] that, for all $v > u > 1$ and $b > 1$, we have

$$\dim_H\{x \in (u, v) : P(x) \leq 1/b\} = \frac{\log v}{\log(bv)},$$

where $\dim_H$ denotes the Hausdorff dimension.

In a different direction, Pollington [16] showed in 1980 that there are many real numbers $x > 1$ such that $(\{x^n\})_{n\geq 1}$ is very far from being well distributed, namely he established that, for any $\varepsilon > 0$, we have

$$\dim_H \big\{x > 1 : \{x^n\} < \varepsilon \text{ for all } n\big\} = 1.$$

This result has been subsequently extended by Bugeaud and Moshchevitin [8] and, independently, by Kahane [11], who proved that for any $\varepsilon > 0$, for any sequence of real numbers $(y_n)_{n\geq 1}$, we have

$$\dim_H \big\{x > 1 : \|x^n - y_n\| < \varepsilon \text{ for all } n\big\} = 1.$$

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In the present paper, we further investigate, from a metric point of view, the Diophantine approximation properties of the sequence $(\{x^n\})_{n\ge 1}$, where $x > 1$, and extend several known results to more general families of sequences $(\{f_n(x)\})_{n\ge 1}$, under some conditions on the sequence of functions $(f_n)_{n\ge 1}$.

As a consequence of our main theorem, we obtain an inhomogeneous version of the result of Bugeaud and Dubickas [6] mentioned above.

**Theorem 1.** *Let $b > 1$ be a real number and $y = (y_n)_{n\ge 1}$ an arbitrary sequence of real numbers in $[0, 1]$. Set*

$$E(b, y) := \{x > 1 : \|x^n - y_n\| < b^{-n} \text{ for infinitely many } n\}.$$

*For every $v > 1$, we have*

$$\lim_{\varepsilon\to 0} \dim_H([v - \varepsilon, v + \varepsilon] \cap E(b, y)) = \frac{\log v}{\log(bv)}.$$

In the homogeneous case (that is, the case where $y_n = 0$ for $n \ge 1$), Theorem 1 was proved in [6] by using a classical result of Koksma [14] and the mass transference principle developed by Beresnevich and Velani [3]. The method of [6] still works when $y$ is a constant sequence, but one then needs to apply the inhomogeneous version of Koksma's theorem in [14]. Here, for an arbitrary sequence $(y_n)_{n\ge 1}$, we use a direct construction.

Letting $v$ tend to infinity in Theorem 1, we obtain the following immediate corollary.

**Corollary 2.** *For an arbitrary sequence $y$ of real numbers in $[0, 1]$ and any real number $b > 1$, the set $E(b, y)$ has full Hausdorff dimension.*

Theorem 1 gives, for every $v > 1$, the value of the localized Hausdorff dimension of $E(b, y)$ at the point $v$. We stress that, in the present context, the localized Hausdorff dimension varies with $v$, while this is not at all the case for many classical results, including the Jarník–Besicovitch Theorem and its extensions. Taking this point of view allows us also to place Theorem 1 in a more general context, where the family of functions $x \mapsto x^n$ is replaced by an arbitrary family of functions $f_n$ satisfying some regularity and growth conditions.

We consider a family of strictly positive increasing $C^1$ functions $f = (f_n)_{n\ge 1}$ defined on an open interval $I \subset \mathbb{R}$ and such that $f_n(x), f_n'(x) > 1$ for all $x \in I$. For $\tau > 1$, define

$$E(f, y, \tau) := \{x \in I : \|f_n(x) - y_n\| < f_n(x)^{-\tau} \text{ for infinitely many } n\}.$$

For $v \in I$, put

$$u(v) := \limsup_{n\to\infty} \frac{\log f_n(v)}{\log f_n'(v)}, \quad \ell(v) := \liminf_{n\to\infty} \frac{\log f_n(v)}{\log f_n'(v)}.$$

We will assume the regularity condition

$$\lim_{r\to 0} \limsup_{n\to\infty} \sup_{|x-y|<r} \frac{\log f_n'(x)}{\log f_n'(y)} = 1, \tag{1.1}$$

which guarantees the continuity of the functions $u$ and $\ell$.

For non-linear functions $f_n$, i.e., when $f_n$ is not of the form $f_n(x) = a_n \cdot x + b_n$, we also need the following condition:

$$M := \sup_{n\geq 1} \frac{\log f'_{n+1}(v)}{\log f'_n(v)} < \infty \qquad \text{for all } v \in I. \tag{1.2}$$

Theorem 1 is a particular case of the following general statement.

**Theorem 3.** *Consider a family of strictly positive increasing $C^1$ functions $f = (f_n)_{n\geq 1}$ defined on an open interval $I \subset \mathbb{R}$ and such that $f_n(x), f'_n(x) > 1$ for all $x \in I$. Assume* (1.1) *and* (1.2). *If for all $x \in I$,*

$$\forall \varepsilon > 0, \quad \sum_{n=1}^{\infty} f'_n(x)^{-\varepsilon} < \infty, \tag{1.3}$$

*then, for any $v \in I$ and any $\tau > 1$, we have*

$$\frac{1}{1+\tau u(v)} \leq \lim_{\varepsilon\to 0} \dim_H([v-\varepsilon, v+\varepsilon] \cap E(f,y,\tau)) \leq \frac{1}{1+\tau\ell(v)}.$$

*If the functions $f_n$ are linear then we do not need to assume* (1.2), *and the assertion gets strengthened to*

$$\lim_{\varepsilon\to 0} \dim_H([v-\varepsilon, v+\varepsilon] \cap E(f,y,\tau)) = \frac{1}{1+\tau\ell(v)}.$$

We remark that the condition (1.3) is satisfied if

$$\forall x \in I, \quad \lim_{n\to\infty} \frac{\log f'_n(x)}{\log n} = \infty. \tag{1.4}$$

We also observe that the condition (1.1) implies that $\ell(v) \geq 1$ for $v$ in $I$. In many cases (in particular, for $f_n(x) = x^n$), we have $u(v) = \ell(v) = 1$ for $v$ in $I$.

It follows from the formulation of Theorem 3 that the real number $\tau$ can be replaced by a continuous function $\tau : I \to (0,\infty)$, in which case the set $E(f,y,\tau)$ is defined by

$$E(f,y,\tau) := \{x \in I : \|f_n(x) - y_n\| < f_n(x)^{-\tau(x)} \text{ for infinitely many } n\}.$$

We get at once the following localized version of Theorem 3. For the classical Jarník–Besicovitch Theorem, such a localized theorem was obtained by Barral and Seuret [2], who were the first to consider localized Diophantine approximation.

**Corollary 4.** *With the above notation and under the hypotheses of Theorem 3, we have*

$$\frac{1}{1+\tau(v)u(v)} \leq \lim_{\varepsilon\to 0} \dim_H([v-\varepsilon, v+\varepsilon] \cap E(f,y,\tau)) \leq \frac{1}{1+\tau(v)\ell(v)}.$$

We illustrate Theorem 3 and Corollary 4 by some examples. If the family of functions $f = (f_n)_{n\geq 1}$ in Theorem 3 is such that, for every $x$ in $I$, the sequence $(f_n(x))_{n\geq 1}$ increases sufficiently rapidly, then

$$\lim_{\varepsilon\to 0} \dim_H([v-\varepsilon, v+\varepsilon] \cap E(f,y,\tau)) = \frac{1}{1+\tau},$$

independently of the family $f$. This applies, for example, to the families of functions $x^{n^2}, x^n, 2^n x$ and $x^{\sqrt{n}}$.

The case $f_n(x) = a_n x$, where $(a_n)_{n\geq 1}$ is an increasing sequence of positive integers, has been studied by Borosh and Fraenkel [4] (but only in the special case of a constant sequence $y$ equal to 0). Let $I$ be an open, non-empty, real interval. They proved that

$$\dim_H\{x \in I : \|a_n x\| < a_n^{-\tau}\} = \frac{1+s}{1+\tau},$$

where $s$ (usually called the convergence exponent of the sequence $(a_n)_{n\geq 1}$) is the largest real number in $[0,1]$ such that

$$\sum_{n\geq 1} a_n^{-s-\varepsilon} \quad \text{converges for any } \varepsilon > 0.$$

The case $s = 0$ of their result, which corresponds to rapidly growing sequences $(a_n)_{n\geq 1}$, follows from Theorem 3. The case $a_n = n$ for $n \geq 1$ corresponds to the Jarník–Besicovitch Theorem. We stress that the assumption (1.3) is satisfied only if $(a_n)_{n\geq 1}$ increases sufficiently rapidly.

Questions of uniform Diophantine approximation were recently studied by Bugeaud and Liao [7] for the $b$-ary and $\beta$-expansions and by Kim and Liao [12] for the irrational rotations. In this paper, we consider the uniform Diophantine approximation of the sequence $(\{x^n\})_{n\geq 1}$ with $x > 1$.

For a real number $B > 1$ and a sequence of real numbers $y = (y_n)_{n\geq 1}$ in $[0,1]$, set

$$\begin{aligned} F(B,y) := \{x > 1 : \text{ for all large integer } N,\ \|x^n - y_n\| < B^{-N} \\ \text{has a solution } 1 \leq n \leq N\}. \end{aligned}$$

Our next theorem gives a lower bound for the Hausdorff dimension of $F(B,y)$ intersected with a small interval.

**Theorem 5.** *Let $B > 1$ be a real number and $y$ an arbitrary sequence of real numbers in $[0,1]$. For any $v > 1$, we have*

$$\lim_{\varepsilon\to 0} \dim_H([v-\varepsilon, v+\varepsilon] \cap F(B,y)) \geq \left(\frac{\log v - \log B}{\log v + \log B}\right)^2.$$

Unfortunately, we are unable to decide whether the inequality in Theorem 5 is an equality. Observe that the lower bound we obtain is the same as the one established in [7] for a question of uniform Diophantine approximation related to $b$-ary and $\beta$-expansions.

Letting $v$ tend to infinity, we have the following corollary.

**Corollary 6.** *For an arbitrary sequence $y$ of real numbers in $[0,1]$ and any real number $B > 1$, the set $F(B,y)$ has full Hausdorff dimension.*

We end this paper with results on sequences $(\{x^n\})_{n\geq 1}$, with $x > 1$, which are badly distributed, in the sense that all of their points lie in a small interval. As above, we take a more general point of view. Consider a family of $C^1$ strictly positive increasing functions $f = (f_n)_{n\geq 1}$ defined on an open interval $I \subset \mathbb{R}$ and such that $f_n(x), f_n'(x) > 1$ for all $x \in I$ and for all $n \geq 1$. Let $\delta = (\delta_n)_{n\geq 1}$ be a sequence of positive real numbers such that $\delta_n < 1/4$ for $n \geq 1$. Set

$$G(f, y, \delta) := \{x \in I : \|f_n(x) - y_n\| \leq \delta_n, \ \forall n \geq 1\}.$$

We need the following hypotheses:

$$\forall \varepsilon > 0, \forall n \geq 1, \ \frac{\inf_{x\in(v-\varepsilon,v+\varepsilon)} f'_{n+1}(x)}{\sup_{x\in(v-\varepsilon,v+\varepsilon)} f'_n(x)} \cdot \delta_n \geq 2, \tag{1.5}$$

$$\forall x \in I, \quad \lim_{n\to\infty} \frac{\log f'_{n+1}(x)}{\log f'_n(x)} = \infty. \tag{1.6}$$

Our last main theorem is as follows.

**Theorem 7.** *Keep the above notation. Under the hypotheses (1.1), (1.5), and (1.6), for all $v \in I$, we have*

$$\lim_{\varepsilon\to 0} \dim_H([v-\varepsilon, v+\varepsilon] \cap G(f, y, \delta)) = \liminf_{n\to\infty} \frac{\log f'_n(v) + \sum_{j=1}^{n-1} \log \delta_j}{\log f'_n(v) - \log \delta_n}. \tag{1.7}$$

We remark that our result extends a recent result of Baker [1]. In fact, in [1], the author studied the special case $f_n(x) = x^{q_n}$ with $(q_n)_{n\geq 1}$ being a strictly increasing sequence of real numbers such that

$$\lim_{n\to\infty} (q_{n+1} - q_n) = +\infty.$$

Our result also gives the following corollary.

**Corollary 8.** *Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers such that*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = +\infty.$$

*Then, for any sequence $(y_n)_{n\geq 1}$ of real numbers, we have*

$$\dim_H\{x \in \mathbb{R} : \lim_{n\to+\infty} \|a_n x - y_n\| = 0\} = 1.$$

## 2. Basic tools

We present two lemmas which serve as important tools for estimating the Hausdorff dimension of the sets studied in this paper.

Let $[0, 1] = E_0 \supset E_1 \supset E_2 \supset \cdots$ be a decreasing sequence of sets, with each $E_k$ a finite union of disjoint closed intervals. The components of $E_k$ are called $k$-th level basic intervals. Set $F = \cap_{k=0}^{\infty} E_k$. We do not assume that each basic interval in $E_{k-1}$ contains the same number of next level basic intervals, nor that they are of the same length, nor that the gaps between two consecutive basic intervals are equal. Instead, for $x \in E_{k-1}$, we denote by $m_k(x)$ the number of $k$-th level basic intervals contained in the $(k-1)$-th level basic interval containing $x$, and by $\tilde{\varepsilon}_k(x)$ the minimal distance between two of them. Set

$$\varepsilon_k(x) = \min_{i\leq k} \tilde{\varepsilon}_i(x).$$

In the following, we generalize a lemma in Falconer's book [9, Example 4.6].

**Lemma 9.** *For any open interval $I \subset [0,1]$ intersecting $F$, we have*

$$\dim_H(I \cap F) \geq \inf_{x \in I \cap F} \liminf_{k \to \infty} \frac{\log(m_1(x) \cdots m_{k-1}(x))}{-\log(m_k(x)\varepsilon_k(x))}.$$

*Proof.* The proof is similar to that in the book of Falconer. We define a probability measure $\mu$ on $F$ by assigning the mass evenly. Precisely, for $k \geq 1$, let $I_k(x)$ be the $k$-th level interval containing $x$. For $x \in F$ and $k \geq 1$, we put a mass $(m_1(x) \cdots m_k(x))^{-1}$ to the interval $I_k(x)$. Note that any two $k$-th basic intervals contained in the same $(k-1)$-th interval have the same measure. One can check that the measure $\mu$ is well defined.

Now let us calculate the local dimension at the point $x$. Let $B(x,r)$ be the ball of radius $r$ centered at $x$. Suppose that $\varepsilon_k(x) \leq 2r < \varepsilon_{k-1}(x)$. The number of $k$-th level intervals intersecting $B(x,r)$ is at most

$$\min\left\{m_k(x),\ \frac{2r}{\varepsilon_k(x)} + 1\right\} \leq \min\left\{m_k(x),\ \frac{4r}{\varepsilon_k(x)}\right\} \leq m_k(x)^{1-s}\left(\frac{4r}{\varepsilon_k(x)}\right)^s,$$

for any $s \in [0,1]$. Thus

$$\mu(B(x,r)) \leq m_k(x)^{1-s}\left(\frac{4r}{\varepsilon_k(x)}\right)^s \cdot (m_1(x) \cdots m_k(x))^{-1}.$$

Hence

$$\frac{\log \mu(B(x,r))}{\log r} \geq \frac{s \log m_k(x)\varepsilon_k(x) - s\log(4r) + \log(m_1(x) \cdots m_{k-1}(x))}{-\log r}.$$

Let $s$ be in $(0,1)$ such that

$$s < \inf_{z \in I \cap F} \liminf_{k \to \infty} \frac{\log(m_1(z) \cdots m_{k-1}(z))}{-\log m_k(z)\varepsilon_k(z)} \leq \liminf_{k \to \infty} \frac{\log(m_1(x) \cdots m_{k-1}(x))}{-\log m_k(x)\varepsilon_k(x)}.$$

Then

$$s \log m_k(x)\varepsilon_k(x) - s\log 4 + \log(m_1(x) \cdots m_{k-1}(x)) \geq 0,$$

for $k$ large enough. Therefore

$$\liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \geq s.$$

The proof is completed by applying the mass distribution principle (see [10], Proposition 2.3). □

We also have an upper bound for the dimension of the set $I \cap F$. Denote by $|I_k(x)|$ the length of the $k$-th basic interval $I_k(x)$ containing $x$.

**Lemma 10.** *For any open interval $I \subset [0,1]$ intersecting $F$, we have*

$$\dim_H(I \cap F) \leq \sup_{x \in I \cap F} \liminf_{k \to \infty} \frac{\log(m_1(x) \cdots m_k(x))}{-\log|I_k(x)|}.$$

*Proof.* We define the same probability measure $\mu$ as in Lemma 9, i.e., the interval $I_k(x)$ has measure $(m_1(x) \cdots m_k(x))^{-1}$. Then

$$\liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \leq \liminf_{k \to \infty} \frac{\log \mu(I_k(x))}{-\log|I_k(x)|} = \liminf_{k \to \infty} \frac{\log(m_1(x) \cdots m_k(x))}{-\log|I_k(x)|}.$$

We finish the proof by applying again the mass distribution principle (see [10], Proposition 2.3). □

## 3. Asymptotic approximation

In this section, we prove Theorem 3. To see that Theorem 1 is a special case of it, take the family of functions $f$ defined by

$$f_n(x) = x^n, \quad \forall n \geq 1,$$

we have $u(v) = \ell(v) = 1$ and

$$\begin{aligned}[v-\varepsilon, v+\varepsilon] \cap E\left(f, y, \frac{\log b}{\log(v+\varepsilon)}\right) &\subset [v-\varepsilon, v+\varepsilon] \cap E(b, y) \\ &\subset [v-\varepsilon, v+\varepsilon] \cap E\left(f, y, \frac{\log b}{\log(v-\varepsilon)}\right).\end{aligned}$$

Then, Theorem 1 follows directly from Theorem 3.

Now we prove Theorem 3.

*Proof of Theorem 3.* **Lower bound**: We can assume that $u(v)$ is finite, since otherwise there is nothing to prove. Let us start by the simple observation about the condition (1.1). Given an integer $n \geq 1$, set
(3.1)

$$\eta(n) = \sup\left\{\frac{\log f_n'(w)}{\log f_n'(z)} - 1; w, z \in [v-\varepsilon, v+\varepsilon], |f_n(w) - f_n(z)| \leq 1\right\}.$$

**Lemma 11.** *If* (1.1) *and* (1.3) *hold, then*

$$\lim_{n\to\infty} \eta(n) = 0.$$

*Proof.* Assume this is not true. Then there exists a sequence of integers $(n_i)$ and a sequence of pairs of points $(w_i, z_i)$ such that $|f_{n_i}(w_i) - f_{n_i}(z_i)| \leq 1$ and

$$\frac{\log f_{n_i}'(w_i)}{\log f_{n_i}'(z_i)} > Z > 1.$$

By compactness of $[v-\varepsilon, v+\varepsilon]$, taking a subsequence if necessary, we can assume that $(w_i)_{i\geq 1}$ converges to some point $w_0$.

By (1.3), $f_n'(v) \to \infty$. Hence, (1.1) gives us

$$\lim_{n\to\infty} \inf_{x\in[v-\varepsilon,v+\varepsilon]} f_n'(x) = \infty.$$

This implies that

$$|w_i - z_i| \leq \frac{1}{\inf_{x\in[v-\varepsilon,v+\varepsilon]} f_{n_i}'(x)} \to 0$$

as $i \to \infty$, and hence any neighborhood of $w_0$ contains all except finitely many points $w_i, z_i$. Thus, in any neighbourhood $U$ of $w_0$ we have

$$\limsup_{n\to\infty} \sup_{w,z\in U} \frac{\log f_n'(w)}{\log f_n'(z)} > Z,$$

which is a contradiction with (1.1). □

Now we construct a nested Cantor set which is the intersection of unions of subintervals at level $n_i$, where $(n_i)_{i\geq 1}$ is an increasing sequence of positive integers which will be defined precisely later. Suppose we have already well chosen this subsequence. Let us describe the nested family of subintervals. For each level $i$, we need to consider the set of points $x$ such that

$$\|f_{n_i}(x) - y_{n_i}\| \leq f_{n_i}(x)^{-\tau}.$$

By the property $\|f_{n_1}(x) - y_{n_1}\| \leq f_{n_1}(x)^{-\tau}$, we take the intervals at level 1 as

$$I_1(k, v, f, y, \tau) := [f_{n_1}^{-1}(k + y_{n_1} - f_{n_1}(v+\varepsilon)^{-\tau}), f_{n_1}^{-1}(k + y_{n_1} + f_{n_1}(v+\varepsilon)^{-\tau})],$$

with $k$ being an integer in $[f_{n_1}(v - \varepsilon) + 1, \quad f_{n_1}(v + \varepsilon) - 1]$.

Suppose we have constructed the intervals at level $i-1$. Let $[c_{i-1}, d_{i-1}]$ be an interval at such level. A subinterval of $[c_{i-1}, d_{i-1}]$ at level $i$ is such that

$$[f_{n_i}^{-1}(k + y_{n_i} - f_{n_i}(d_{i-1})^{-\tau}), f_{n_i}^{-1}(k + y_{n_i} + f_{n_i}(d_{i-1})^{-\tau})],$$

with $k$ being an integer in $[f_{n_i}(c_{i-1}) + 1, \ f_{n_i}(d_{i-1}) - 1]$. By continuing this construction, we obtain intervals $I_i(\cdot)$ for all levels.

Finally, the intersection $F$ of these nested intervals is obviously a subset of $[v - \varepsilon, v + \varepsilon] \cap E(f, y, \tau)$.

Let $z \in F$ and $[c_i(z), d_i(z)]$ be the $i$-th level interval containing $z$. Then we have

$$m_{i+1}(z) \geq f'_{n_{i+1}}(w_i) \cdot (d_i - c_i) - 2 \geq f'_{n_{i+1}}(w_i) \cdot \frac{2 f_{n_i}(d_i)^{-\tau}}{f'_{n_i}(z_i)} - 2, \tag{3.2}$$

where $w_i, z_i \in [c_i(z), d_i(z)]$. Furthermore,

$$\varepsilon_{i+1}(z) \geq \frac{1 - 2 f_{n_{i+1}}(c_i(z))^{-\tau}}{f'_{n_{i+1}}(u_i)} \geq \frac{1}{2 f'_{n_{i+1}}(u_i)}, \tag{3.3}$$

where $u_i \in [c_i(z), d_i(z)]$.

Now we are going to define the subsequence $(n_i)_{i \geq 1}$.

**Lemma 12.** *Assume (1.1) and (1.2). For any $\gamma > 0$, we can find a subsequence $(n_i)_{i\geq 1}$ such that*

$$\frac{f'_{n_{i+1}}(w)}{f'_{n_{i+1}}(u)} \leq f'_{n_i}(z)^{\gamma} \quad \forall w, u \in [c_i(z), d_i(z)], \tag{3.4}$$

*and for any small $\varepsilon > 0$, we have*

$$\forall x \in (v - \varepsilon, v + \varepsilon), \quad \lim_{i\to\infty} \frac{\log f'_{n_i}(x)}{\log f'_{n_{i-1}}(x)} = \lim_{i\to\infty} \frac{\log f'_{n_i}(x)}{\log f_{n_{i-1}}(x)} = \infty, \tag{3.5}$$

*and*

$$\frac{\inf_{x\in(v-\varepsilon, v+\varepsilon)} f'_{n_{i+1}}(x)}{\sup_{x\in(v-\varepsilon,v+\varepsilon)} f'_{n_i}(x) \cdot f_{n_i}(x)^{\tau}} \geq 2. \tag{3.6}$$

*If $f_n$ are linear then we do not need to assume* (1.2)*, moreover we can choose $(n_i)$ in such a way that we have (in addition to the other parts of the assertion)*

$$\lim_{i\to\infty} \frac{\log f_{n_i}(v)}{\log f'_{n_i}(v)} = \ell(v). \tag{3.7}$$

*Proof.* In the linear case (3.4) is automatically true, and to have (3.5) and (3.6) we just need that $(n_i)_{i\geq 1}$ increases sufficiently fast (as will be clear from the proof for the general case). Hence, we will be free to choose $(n_i)$ satisfying in addition (3.7).

Let us proceed with the general case. For any $\gamma > 0$, by Lemma 11, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0, \quad \eta(n) < \frac{\gamma}{2M},$$

where $M$ is the constant in assumption (1.2).

Starting with this $n_0$, by the assumption (1.2), we can then construct a subsequence $(n_i)_{i\geq 1}$ satisfying

$$\frac{\gamma}{2\eta(n_i)\cdot M} \leq \frac{\log f'_{n_{i+1}}(v)}{\log f'_{n_i}(v)} \leq \frac{\gamma}{2\eta(n_i)}. \tag{3.8}$$

Observe that, as $\eta(n_i) \to 0$ by Lemma 11, the lefthand side of (3.8) implies the first part of (3.5). As $u < \infty$, the second part of (3.5) follows. The condition (3.6) will also follow, provided that $n_0$ was selected large enough.

We need now to prove (3.4). By (3.1), for any $w, u$ in the interval $[c_i(z), d_i(z)]$,

$$\frac{f'_{n_{i+1}}(w)}{f'_{n_{i+1}}(u)} \leq \frac{f'_{n_{i+1}}(z)^{1+\eta(n_i)}}{f'_{n_{i+1}}(z)^{1-\eta(n_i)}} = f'_{n_{i+1}}(z)^{2\eta(n_i)}. \tag{3.9}$$

Combining (3.8) and (3.9), we get (3.4).

□

We continue the proof of the lower bound of Theorem 3. By (3.2) and (3.6),

$$m_{i+1}(z) \geq f'_{n_{i+1}}(w_i)\cdot \frac{2f_{n_i}(d_i)^{-\tau}}{f'_{n_i}(z_i)} - 2 \geq 2,$$

which then implies that $F$ is non-empty. Further, by (3.4), for any $\gamma > 0$,

$$m_{i+1}(z) \geq f'_{n_{i+1}}(z)\cdot f'_{n_i}(z)^{-\gamma}\cdot \frac{f_{n_i}(d_i)^{-\tau}}{f'_{n_i}(z_i)}. \tag{3.10}$$

By (3.2), (3.3) and (3.4),

$$m_{i+1}(z)\varepsilon_{i+1}(z) \geq f'_{n_i}(z)^{-\gamma}\cdot \frac{f_{n_i}(d_i)^{-\tau}}{2f'_{n_i}(z_i)}. \tag{3.11}$$

Thus, (1.3) and (3.5) imply that $-\log m_{i+1}(z)\varepsilon_{i+1}(z)$ is unbounded. So by (3.10), (3.11) and (3.5)

$$\begin{aligned}
&\liminf_{i\to\infty} \frac{\log(m_2(z)\cdots m_i(z))}{-\log m_{i+1}(z)\varepsilon_{i+1}(z)}\\
\geq &\liminf_{i\to\infty} \frac{\sum_{j=2}^{i}(\log f'_{n_j}(z) - \gamma\log f'_{n_{j-1}}(z) - \tau\log f_{n_{j-1}}(d_j) - \log f'_{n_{j-1}}(z_j))}{\log 2 + \log f'_{n_i}(z_i) + \gamma\log f'_{n_i}(z) + \tau\log f_{n_i}(d_i)}\\
= &\liminf_{i\to\infty} \frac{\log f'_{n_i}(z)}{\log f'_{n_i}(z_i) + \gamma\log f'_{n_i}(z) + \tau\log f_{n_i}(d_i)}.
\end{aligned}$$

Hence, by the definition of $\eta(n_i)$, we have

$$\begin{aligned}&\liminf_{i\to\infty} \frac{\log(m_2(z)\cdots m_i(z))}{-\log m_{i+1}(z)\varepsilon_{i+1}(z)}\\ \geq &\frac{1}{\limsup_{i\to\infty}\left(1+\eta(n_i)+\gamma+\tau(1+\eta(n_i))\cdot\frac{\log f_{n_i}(d_i)}{\log f'_{n_i}(d_i)}\right)}.\end{aligned}$$

In the linear case, $\log f_{n_i}(d_i)/\log f'_{n_i}(d_i)$ converges to $\ell(\lim_{i\to\infty} d_i)$. In the general situation, we have

$$\limsup_{i\to\infty}\frac{\log f_{n_i}(d_i)}{\log f'_{n_i}(d_i)} \leq u(\lim_{i\to\infty} d_i).$$

As $\gamma$ can be chosen arbitrarily small, $\eta(n_i)\to 0$ by Lemma 11, and

$$\lim_{i\to\infty} d_i \in [v-\varepsilon, v+\varepsilon],$$

the lower bound is obtained by applying Lemma 9.

**Upper bound**: Since for all $x\in[v-\varepsilon,v+\varepsilon]\cap E(f,y,\tau)$, we have

$$\|f_n(x)-y_n\| < f_n(x)^{-\tau}$$

for infinitely many $n\geq 1$. Then the set $[v-\varepsilon,v+\varepsilon]\cap E(f,y,\tau)$ is covered by the union of the family of intervals

$$I_n(k) := [f_n^{-1}(k+y_n-f_n(v-\varepsilon)^{-\tau}), f_n^{-1}(k+y_n+f_n(v-\varepsilon)^{-\tau})],$$

where $k\in[f_n(v-\varepsilon),\ f_n(v+\varepsilon)]$ is an integer. Note that the length of the interval $I_n(k)$ satisfies

$$|I_n(k)| \leq \frac{2f_n(v-\varepsilon)^{-\tau}}{f'_n(z)} \quad \text{for some } z\in(v-\varepsilon,v+\varepsilon).$$

The number of the intervals at level $n$ is less than

$$f_n(v+\varepsilon)-f_n(v-\varepsilon) \leq 2\varepsilon f'_n(w) \quad \text{for some } w\in(v-\varepsilon,v+\varepsilon).$$

Thus for $s>0$

$$\sum_{n=1}^{\infty}\sum_{k\in[f_n(v-\varepsilon),f_n(v+\varepsilon)]}|I_n(k)|^s \leq \sum_{n=1}^{\infty} 2\varepsilon f'_n(w)\cdot\left(\frac{2f_n(v-\varepsilon)^{-\tau}}{f'_n(z)}\right)^s. \tag{3.12}$$

By the definition of $\ell(v)$, for any $\eta>0$, there exists $n_0=n_0(\eta)\in\mathbb{N}$ such that for any $n\geq n_0$

$$f_n(v-\varepsilon) > f'_n(v-\varepsilon)^{\ell(v-\varepsilon)-\eta}.$$

Thus by ignoring the first $n_0$ terms, we have (3.12) is bounded by

$$2^{1+s}\varepsilon\sum_{n=n_0}^{\infty} f'_n(w)\cdot f'_n(z)^{-s}\cdot\left(f'_n(v-\varepsilon)\right)^{-\tau s(\ell(v-\varepsilon)-\eta)}. \tag{3.13}$$

Hence by the assumption (1.3) if

$$s > \limsup_{n\to\infty}\frac{\log f'_n(w)}{\log f'_n(z)+\tau(\ell(v-\varepsilon)-\eta)\log f'_n(v-\varepsilon)}$$

the sum in (3.12) converges. By (1.1),

$$\lim_{n\to\infty} \frac{\log f_n'(w)}{\log f_n'(z)} = 1, \quad \lim_{n\to\infty} \frac{\log f_n'(w)}{\log f_n'(v-\varepsilon)} = 1.$$

Therefore

$$\lim_{\varepsilon\to 0} \dim_H [v-\varepsilon, v+\varepsilon] \cap E(f, y, \tau) \leq \frac{1}{1+\tau\ell(v)}.$$

□

## 4. Uniform Diophantine approximation

In this section, we study the uniform Diophantine approximation of the sequence $(\{x^n\})_{n\geq 1}$ with $x > 1$.

Recall that for any sequence of real numbers $y = (y_n)_{n\geq 1}$ in $[0,1]$, we are interested in the set

$$F(B, y) := \{x > 1 : \text{ for all large integer } N,\ \|x^n - y_n\| < B^{-N} \\ \text{has a solution } 1 \leq n \leq N\}.$$

For any $v \in F(B, y)$, for any $\varepsilon > 0$, we will give a lower bound for the Hausdorff dimension of $[v-\varepsilon, v+\varepsilon] \cap F(B, y)$. To this end, we investigate the uniform Diophantine approximation and asymptotic Diophantine approximation together. We consider the following subset of $[v-\varepsilon, v+\varepsilon] \cap F(B, y)$

$$F(v, \varepsilon, b, B, y) := \{z \in [v-\varepsilon, v+\varepsilon] : \|z^n - y_n\| < b^{-n} \text{ for infinitely many } n \\ \text{and } \forall N \gg 1, \|z^n - y_n\| < B^{-N} \text{ has a solution } 1 \leq n \leq N\}.$$

The proof of Theorem 5 will be completed by maximizing the lower bounds of $F(v, \varepsilon, b, B, y)$ with respect to $b > B$.

*Proof of Theorem 5.* We first construct a subset $F \subset F(v, \varepsilon, b, B, y)$. Suppose that $b = B^\theta$ with $\theta > 1$. Let $n_k = \lfloor \theta^k \rfloor$. Consider the points $z$ such that

$$\|z^{n_k} - y_{n_k}\| < b^{-n_k}.$$

Then one can check that

$$z \in F(v, \varepsilon, b, B, y) = F(v, \varepsilon, B^\theta, B, y) = F(v, \varepsilon, b, b^{\frac{1}{\theta}}, y).$$

We do the same construction as in Section 3. We will obtain a Cantor set $F \subset F(v, \varepsilon, b, b^{\frac{1}{\theta}}, y)$, which is the intersection of a nested family of intervals with

$$m_k(z) = \frac{2n_{k+1} c_k(z)^{n_{k+1}-1}}{n_k b^{n_k} d_k(z)^{n_k - 1}}$$

and

$$\varepsilon_k(z) = \left(1 - \frac{2}{b^{n_{k+1}}}\right) \frac{1}{n_{k+1} d_k(z)^{n_{k+1}-1}},$$

where $[c_k(z), d_k(z)]$ is the $k$-th level interval containing $z$.

By the choice of $n_k$, we will have the following estimations:

$$m_k(z) \geq \frac{2(\theta^{k+1}-1) c_k(z)^{\theta^{k+1}-1}}{\theta^k b^{\theta^k} d_k(z)^{\theta^k - 1}} \geq \theta \cdot b^{-\theta^k} \cdot \left(\frac{c_k(z)}{d_k(z)}\right)^{\theta^k} \cdot c_k(z)^{\theta^k(\theta-1)}.$$

and

$$\varepsilon_k(z) \geq \frac{1}{2\theta^{k+1}} \cdot d_k(z)^{-\theta^{k+1}}.$$

Since

$$d_k(z) - c_k(z) \leq \frac{b^{-n_k}}{n_k c_k(z)^{n_k - 1}} \leq b^{-\theta^k}$$

is much more smaller than $1/\theta^k$,

$$\left(\frac{c_k(z)}{d_k(z)}\right)^{\theta^k} = \left(1 - \frac{d_k(z) - c_k(z)}{d_k(z)}\right)^{\theta^k} \geq \frac{1}{2}.$$

Then

$$m_k(z) \geq \frac{\theta}{2} \cdot \left(\frac{c_k(z)^{\theta-1}}{b}\right)^{\theta^k} \geq \frac{\theta}{2^{\theta+1}} \left(\frac{z^{\theta-1}}{b}\right)^{\theta^k},$$

and

$$m_k(z)\varepsilon_k(z) \geq \frac{1}{4\theta^k} \cdot \left(\frac{c_k(z)^{\theta-1}}{b \cdot d_k(z)^{\theta}}\right)^{\theta^k} \geq \frac{1}{2^{\theta+3}\theta^k} \left(\frac{1}{bz}\right)^{\theta^k}.$$

Thus by Lemma 9, for any $z \in F$, we have

$$\begin{aligned}
&\liminf_{k\to\infty} \frac{\log(m_1(z)\cdots m_{k-1}(z))}{-\log m_k(z)\varepsilon_k(z)} \\
\geq &\liminf_{k\to\infty} \frac{\big((\theta-1)\log z - \log b\big)\sum_{j=1}^{k-1}\theta^j}{\theta^k \log bz} \\
= &\liminf_{k\to\infty} \frac{(\theta-1)\log z - \log b}{(\theta-1)\log bz} \cdot \frac{\theta^{k-1}-1}{\theta^{k-1}} \\
= &\frac{(\theta-1)\log z - \log b}{(\theta-1)\log bz}.
\end{aligned}$$

Hence, by the relation $b = B^\theta$, we deduce that the Hausdorff dimension of the set $F(v,\varepsilon,b,B,y) = F(v,\varepsilon,B^\theta,B,y)$ is at least equal to

$$\frac{(\theta-1)\log(v-\varepsilon) - \log b}{(\theta-1)\log(b(v-\varepsilon))} = \frac{\log(v-\varepsilon) - \frac{\theta}{\theta-1}\log B}{\log(v-\varepsilon) + \theta\log B}.$$

Taking $\theta \to \infty$ in the left side of the equality, we get the lower bound $\log(v-\varepsilon)/\log(b(v-\varepsilon))$ for the Hausdorff dimension of the set considered in Theorem 1:

$$\begin{aligned}
&[v-\varepsilon, v+\varepsilon] \cap E(b,y) \\
=&\{v-\varepsilon \leq x \leq v+\varepsilon : \|x^n - y_n\| < b^{-n} \text{ for infinitely many } n\}.
\end{aligned}$$

By maximizing the right side of the equality with respect to $\theta > 1$, we obtain the lower bound

$$\left(\frac{\log(v-\varepsilon) - \log B}{\log(v-\varepsilon) + \log B}\right)^2$$

for the Hausdorff dimension of the set

$$\begin{aligned}
&[v-\varepsilon, v+\varepsilon] \cap F(B,y) \\
=&\{v-\varepsilon \leq x \leq v+\varepsilon : \forall N \gg 1, ||x^n - y|| < B^{-N} \text{ has a solution } 1 \leq n \leq N\}.
\end{aligned}$$

By letting $\varepsilon$ tend to 0, this completes the proof of Theorem 5. □

## 5. Bad approximation

In this section, we study the bad approximation properties of the sequence $(\{x^n\})_{n\geq 1}$, where $x > 1$.

Let $q = (q_n)_{n\geq 1}$ be a sequence of positive real numbers and $y = (y_n)_{n\geq 1}$ be an arbitrary sequence of real numbers in $[0, 1]$. Define

$$G(q, y) = \{x > 1 : \lim_{n\to\infty} \|x^{q_n} - y_n\| = 0\},$$

and, for $v > 1$, define

$$G(v, q, y) = \{1 < x < v : \lim_{n\to\infty} \|x^{q_n} - y_n\| = 0\}.$$

Recently Baker [1] showed that if $q = (q_n)_{n\geq 1}$ is strictly increasing and

$$\lim_{n\to\infty} (q_{n+1} - q_n) = \infty,$$

then the set $G(q, y)$ has Hausdorff dimension 1.

We want to generalize Baker's result. Consider a family of $C^1$ functions $f = (f_n)_{n\geq 1}$ from an interval $I \subset \mathbb{R}$ to $\mathbb{R}$ such that $f_n'(x) \geq 1$ for all $x \in I$ and for all $n \geq 1$. Let $\delta = (\delta_n)_{n\geq 1}$ be a sequence of positive real numbers tending to 0. For $\varepsilon > 0$, set

$$G(\varepsilon, v, f, y, \delta) := \{v - \varepsilon < x < v + \varepsilon : \|f_n(x) - y_n\| \leq \delta_n,\ \forall n \geq 1\}.$$

To prove Theorem 7, we need to estimate $\dim_H G(\varepsilon, v, f, y, \delta)$.

*Sketch proof of Theorem 7.* **Lower bound**: We do the same construction as in the proof of the lower bound in Theorem 3. If the right-hand side inequality in (3.8) is satisfied, that is, if

$$\frac{\log f_{n+1}'(v)}{\log f_n'(v)} \leq \frac{\gamma}{2\eta(n)}, \tag{5.1}$$

for some $\gamma > 0$, for large enough $n$, and for $\eta$ defined in (3.1), then the distortion estimation (3.4) holds and we estimate the dimension in exactly the same way as in Theorem 3.

If, however, (5.1) is not satisfied, that is, at some place $f_n'$ is too sparse, with $\log f_{n+1}'(v) \gg \log f_n'(v)$ then we can apply the idea of Baker ([1], page 69): we add some new functions $\tilde{f}_m$ between $f_n$ and $f_{n+1}$, in such a way that the resulting, expanded, sequence of their logarithms of derivatives is not too sparse anymore. We also add some $\tilde{\delta}_m = 1$ for each added $\tilde{f}_m$. Observe that the right-hand side of (1.7) does not change. Naturally, the resulting set $G(\varepsilon, v, \tilde{f}, y, \tilde{\delta})$ is exactly the same as $G(\varepsilon, v, f, y, \delta)$. So, for the lower bound, we need only to estimate the lower bound of $\dim_H G(\varepsilon, v, \tilde{f}, y, \tilde{\delta})$.

This means that we can freely assume that (5.1) holds.

We will construct a subset of $G(\varepsilon, v, f, y, \delta)$ which is the intersection of a nested family of subintervals $I_n(\cdot)$.

For $n = 1$, by the property $\|f_1(x) - y_1\| \leq \delta_1$, we take the intervals at level 1 as

$$I_1(k, v, f, y, \delta) := [f_1^{-1}(k + y_1 - \delta_1), f_1^{-1}(k + y_1 + \delta_1)],$$

with $k$ being an integer in $[f_1(v - \varepsilon) + 1, \quad f_1(v + \varepsilon) - 1]$.

Suppose we have constructed the intervals at level $n-1$. Let $[c_{n-1}, d_{n-1}]$ be an interval at this level. A subinterval of $[c_{n-1}, d_{n-1}]$ at level $n$ is

$$[f_n^{-1}(k + y_n - \delta_n), f_n^{-1}(k + y_n + \delta_n)],$$

with $k$ being an integer in $[f_n(c_{n-1}) + 1,\ f_n(d_{n-1}) - 1]$. By continuing this construction, we obtain intervals $I_n(\cdot)$ for all levels. Finally, the intersection $F$ of these nested intervals is obviously a subset of $G(\varepsilon, v, f, y, \delta)$.

Let $z \in F$ and $[c_n(z), d_n(z)]$ be the $n$-th level interval containing $z$. Then by (1.5)

$$m_{n+1}(z) \geq f'_{n+1}(w_n) \cdot (d_n - c_n) - 2 \geq f'_{n+1}(w_n) \cdot \frac{2\delta_n}{f'_n(z_n)} - 2 \geq 2,$$

and

$$\varepsilon_{n+1}(z) \geq \frac{1 - 2\delta_{n+1}}{f'_{n+1}(u_n)} \geq \frac{1}{2f'_{n+1}(u_n)}.$$

with $w_n, z_n, u_n \in [c_n(z), d_n(z)]$. As we are assuming (5.1), we have (3.4) and then for any $\gamma > 0$

$$m_{n+1}(z) \geq f'_{n+1}(z) \cdot f'_n(z)^{-\gamma} \cdot \frac{\delta_n}{f'_n(z_n)}.$$

and

$$m_{n+1}(z)\varepsilon_{n+1}(z) \geq f'_n(z)^{-\gamma} \cdot \frac{\delta_n}{2f'_n(z_n)}$$

Thus,

$$\begin{aligned} &\frac{\log(m_2(z) \cdots m_n(z))}{-\log m_{n+1}(z)\varepsilon_{n+1}(z)} \\ \geq &\frac{\log f'_n(z) - \log f'_1(z) + \sum_{j=1}^{n-1} \log \delta_j + \sum_{j=1}^{n-1} \log \frac{f'_j(z)^{-\gamma}}{f'_j(z_j)}}{\log 2 + \log f'_n(z_n) + \gamma \log f'_n(z) - \log \delta_n}. \end{aligned}$$

By (1.6), we have

$$\begin{aligned} &\liminf_{n\to\infty} \frac{\log(m_2(z) \cdots m_n(z))}{-\log m_{n+1}(z)\varepsilon_{n+1}(z)} \\ \geq &\liminf_{n\to\infty} \frac{\log f'_n(z) + \sum_{j=1}^{n-1} \log \delta_j}{\log f'_n(z_n) + \gamma \log f'_n(z) - \log \delta_n}. \end{aligned}$$

Since $\gamma$ can be chosen arbitrary small and $z_n$ tends to $z$, by (1.1) we have

$$\liminf_{n\to\infty} \frac{\log(m_2(z) \cdots m_n(z))}{-\log m_{n+1}(z)\varepsilon_{n+1}(z)} \geq \liminf_{n\to\infty} \frac{\log f'_n(z) + \sum_{j=1}^{n-1} \log \delta_j}{\log f'_n(z) - \log \delta_n}.$$

Hence the lower bound of Theorem 7 is obtained by Lemma 9.

**Upper bound**: We will apply Lemma 10. For each basic interval $I_n(z)$, by (1.1), we have for any $\gamma$, for $n$ large enough

$$\frac{\delta_n}{f'_n(z) f'_{n-1}(z)^{\gamma}} \leq |I_n(z)| \leq \frac{\delta_n f'_{n-1}(z)^{\gamma}}{f'_n(z)}.$$

Thus,

$$m_n(z) \le |I_{n-1}(z)| \cdot f_n'(z) f_{n-1}'(z)^\gamma \le \frac{\delta_{n-1} f_{n-2}'(z)^\gamma}{f_{n-1}'(z)} f_n'(z) f_{n-1}'(z)^\gamma.$$

Hence,

$$\prod_{j=2}^{n} m_j(z) \le f_n'(z) \cdot \prod_{j=1}^{n-1} \delta_j \cdot \frac{\prod_{j=1}^{n-1} f_j'(z)^{2\gamma}}{f_1'(z)}.$$

Therefore, by (1.6),

$$\liminf_{n\to\infty} \frac{\log(m_1(z)\cdots m_n(z))}{-\log|I_n(z)|} \le \liminf_{n\to\infty} \frac{\log f_n'(z) + \sum_{j=1}^{n-1} \log \delta_j}{\log f_n'(z) - \log \delta_n}.$$

By Lemma 10, we conclude the proof. □

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