QUANTUM LÉVY PROCESSES ON TOPOLOGICAL QUANTUM GROUPS

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This series of talks has been delivered during the 3rd ILJU School of Mathematics in Gyeongju in Korea, during the week 17-21 January 2011. The School's website can be found at `http://math.postech.ac.kr/home/special/2011/3rd_ilju_homepage/index.htm`. In these lectures we will motivate and present the definition of quantum Lévy processes on (locally) compact quantum groups and show that such processes can be realised in a concrete way as quantum stochastic processes on a symmetric Fock space. The plan of the talks is as follows.

Lecture 1

Definition of classical Lévy processes on topological groups. Introduction to the concept of noncommutative mathematics (C^*-algebras as 'quantum' topological spaces). Definition and basic properties of (locally) compact quantum semigroups and groups.

Lecture 2

Convolution of states on a quantum group; convolution semigroups of states and their generators. Basic concepts of noncommutative probability. Definition of (weak) quantum Lévy processes on a locally compact quantum group and their first properties.

Lecture 3

Quantum stochastic calculus for processes on a symmetric Fock space. Quantum stochastic differential equations and their solutions.

Lecture 4

A topological version of the Schürmann Reconstruction Theorem for quantum Lévy processes with norm-continuous Markov semigroups. Some comments on convolution semigroups of states with unbounded generators. Further directions of research and related open problems.

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1. Lecture 1

1.1. **Lévy processes in classical probability.** Classical Lévy processes on topological groups are stochastic processes, indexed by continuous 'time' variable and having independent and identically distributed increments. They can often be used to describe natural physical processes involving random phenomena. Their mathematical theory turns out to be at the same time very rich and quite rigid.

Let G be a topological group (a group, which is also a topological space and such that the multiplication and the operation of taking the inverse element are continuous). We will usually assume that G is a *locally compact* group, i.e. that every point of G has a compact neighbourhood. Any group equipped with a discrete topology is locally compact; other examples are given by $\mathbb{T}^n$, $\mathbb{R}^n$ or classical Lie groups. By a *G-valued random variable* we understand a Borel measurable function from a probability space (Ω, μ) to G. By the *distribution* of such a random variable $X : \Omega \to G$ we understand the measure $\mu \circ X^{-1}$ on G; sometimes we just write dX, so that for a Borel set $S \subset G$ we have $dX(S) = \mu(X^{-1}(S))$.

Definition 1.1. A Lévy process on G is a family $\{X_t : t \geq 0\}$ of G-valued random variables (defined on the same probability space (Ω, μ)) such that

(i) $X_0 = e$ (almost surely);
(ii) the increments of the process are independent: i.e. if $0 \leq s_1 < t_1 \leq \cdots \leq s_n < t_n$ then the random variables $X_{s_1}^{-1}X_{t_1}, \ldots, X_{s_n}^{-1}X_{t_n}$ are independent;
(iii) the increments are identically distributed: for each $s, t > 0$ the distribution of X_s and $X_t^{-1}X_{t+s}$ is the same;
(iv) the distribution of X_t tends weakly to that of X_0 as $t \to 0^+$ – for each $f \in C_0(G)$ (a complex-valued continuous function on G vanishing at infinity)

$$\int_G f(g) d_{X_t}(g) \overset{t \to 0^+}{\longrightarrow} f(e).$$

We can reformulate the definition above in terms of the increments. Writing $X_{s,t}$ for $X_s^{-1}X_t$ $(0 \leq s \leq t)$ we obtain the following list of conditions:

(i) $X_{s,s} = e$;
(ii) $X_{r,t} = X_{r,s}X_{s,t}$ $(0 \leq r \leq s \leq t)$;
(iii) the increments $X_{s,t}$ are independent: if $0 \leq s_1 < t_1 \leq \cdots \leq s_n < t_n$ then the random variables $X_{s_1,t_1}, \ldots, X_{s_n,t_n}$ are independent;

(iv) the increments are identically distributed – the distribution of $X_{0,s}$ is identical to that of $X_{t,t+s}$;

(v) for each $f \in C_0(G)$

$$\int_G f(g) d_{X_{s,t}}(g) \stackrel{t \to s^+}{\longrightarrow} f(e).$$

Note that once we pass to the two-parameter setup thinking about the increments, we can just assume that G is a locally compact semigroup.

All the probabilistic information about the process $\{X_t : t \geq 0\}$ is contained in the collection $\{\mu_t = dX_t : t \geq 0\}$ of probability measures on G. It is easy to check that $\{\mu_t : t \geq 0\}$ forms a convolution semigroup: for all $s, t \geq 0$ we have

$$\mu_s \star \mu_t = \mu_{s+t}.$$

(the definition of the convolution will be recalled in Lecture 2). We call it the *Markov convolution semigroup* of the process $\{X_t : t \geq 0\}$; the condition (v) implies that it satisfies a natural continuity condition at $t = 0$.

Theorem 1.2 (Lévy-Khintchin)**.** *Each (continuous at* 0*) convolution semigroup of probability measures* $(\mu_t)_{t \geq 0}$ *on* $\mathbb{R}^n$ *is given by the following equation describing the characteristic functions of* μ_t *(*$t \geq 0, \overrightarrow{u} \in \mathbb{R}^n$*):*

$$\begin{aligned}\phi_t(\overrightarrow{u}) &:= \int_{\mathbb{R}^n} \exp(i \overrightarrow{u} \cdot \overrightarrow{x}) d\mu_t(\overrightarrow{x}) \\ &= \exp\left(t\left(i \overrightarrow{b} \cdot \overrightarrow{u} - \frac{1}{2}\overrightarrow{u} \cdot A\overrightarrow{u} + \int_{\mathbb{R}^n \setminus \{0\}} [\exp(i\overrightarrow{u} \cdot \overrightarrow{y}) - 1 - i\overrightarrow{u} \cdot \overrightarrow{y} \chi_{B_n}(\overrightarrow{y})]\, d\nu(\overrightarrow{y})\right)\right),\end{aligned}$$

where $\overrightarrow{b} \in \mathbb{R}^n$*,* $A \in M_n(\mathbb{R})$ *is a symmetric positive-definite matrix,* ν *is a* Lévy measure *on* $\mathbb{R}^n \setminus \{0\}$ *(that is* $\int_{\mathbb{R}^n} (\|\overrightarrow{y}\|^2 \wedge 1) d\nu(\overrightarrow{y}) < \infty$*), and* B_n *denotes the ball* $\{\overrightarrow{x} \in \mathbb{R}^n : \|\overrightarrow{x}\| \leq 1\}$*.*

For more information on classical Lévy processes we refer for example to the book [App].

1.2. **Noncommutative mathematics.** The concept of 'noncommutative mathematics', initially motivated by the quantum theory and popularised by the famous book [Con] of Alain Connes has now become an inspiration for a very important and active area of research. Its starting point is the idea of replacing the study of a given space X (a topological space, a measure space, a manifold) by the study of a suitable algebra of complex functions on it, and then following the general scheme:

(i) identify these commutative algebras which arise as the algebras of functions in our class and show that their study is equivalent to the study of spaces that we are interested in;
(ii) drop the commutativity assumption and consider the resulting class of algebras as the algebras of functions on a 'quantum space'.

The fundamental example is given by the study of compact spaces. For a given compact space X it is natural to consider all continuous functions on it; equipped with the natural algebraic operations and the usual supremum norm they form a commutative unital C^*-algebra (a Banach algebra with involution which satisfies the C^*-condition – $\|a^*a\| = \|a\|^2$).

Theorem 1.3 (Gelfand-Najmark)**.** *Any commutative unital C^*-algebra A is (isomorphic to) the algebra of continuous functions on a compact space X. Moreover if Y is another compact space such that $C(X) \approx C(Y)$ as a C^*-algebra, then X is homeomorphic to Y.*

To conclude that the category of compact spaces is equivalent to the category of commutative unital C^*-algebras it suffices to note two more facts: if X, Y are compact spaces and $T : X \to Y$ is continuous, then the map $\alpha_T : C(Y) \to C(X)$ given by

$$\alpha_T(f) = f \circ T, \quad f \in C(Y),$$

is a unital *-homomorphism; moreover for any unital *-homomorphism $\alpha : C(Y) \to C(X)$ there exists a continuous map $T : X \to Y$ such that $\alpha = \alpha_T$.

Note the inversion of arrows!

Following the ideas described above, we can then say:

Arbitrary unital C^*-algebras — algebras of functions on 'compact quantum spaces'

Similar results hold for *locally compact spaces* – we only need to replace the commutative unital C^*-algebras by arbitrary commutative C^*-algebras.

Theorem 1.4 (Gelfand-Najmark)**.** *Any commutative C^*-algebra A is (isomorphic to) the algebra of continuous functions vanishing at infinity on a locally compact space X. Moreover if Y is another locally compact space such that $C_0(X) \approx C_0(Y)$ as a C^*-algebra, then X is homeomorphic to Y.*

To describe the appropriate morphisms we will need some more definitions.

Definition 1.5. Let A be a C^*-algebra. The multiplier C^*-algebra $M(\mathsf{A})$ is the largest C^*-algebra containing A as an essential ideal (i.e. a closed two-sided ideal such that if J is another ideal in $M(\mathsf{A})$ and $\mathsf{A} \cap J = \{0\}$, then $J = \{0\}$).

The multiplier algebra $M(\mathsf{A})$ is always unital. If X is a locally compact space and $\mathsf{A} = C_0(X)$ (the algebra of continuous functions on X vanishing at infinity), then $M(\mathsf{A}) \approx C_b(X)$. If $\mathsf{A} \approx K(\mathsf{H})$, the algebra of compact operators on a Hilbert space H, then $M(\mathsf{A}) \approx B(\mathsf{H})$ (the algebra of all bounded operators on H). Apart from the norm topology, $M(\mathsf{A})$ is also equipped with another, so-called *strict* topology: we say that the net $(m_i)_{i\in\mathcal{I}}$ of elements in $M(\mathsf{A})$ tends to $m \in M(\mathsf{A})$ strictly if for each $a \in \mathsf{A}$

$$m_i a \longrightarrow ma, \quad am_i \longrightarrow am \quad \text{ in norm.}$$

Exercise 1.1. Describe the strict topology in $C_b(X)$ viewed as the multiplier algebra of $C_0(X)$ and in $B(\mathsf{H})$ viewed as the multiplier algebra of $K(\mathsf{H})$.

If A, B are C^*-algebras and $T : \mathsf{A} \to M(\mathsf{B})$ is a linear map then it is called *strict* if it is bounded and strictly continuous on bounded subsets. Each such strict map possesses a unique bounded strictly continuous extension to a map $\widetilde{T} : M(\mathsf{A}) \to M(\mathsf{B})$. A *-homomorphism $T : \mathsf{A} \to M(\mathsf{B})$ is said to be *nondegenerate* if the linear span of the elements of the form $T(a)b$ ($a \in \mathsf{A}, b \in \mathsf{B}$) is dense in B. Nondegenerate *-homomorphisms are automatically strict; so are continuous functionals on A.

Continuous maps between locally compact spaces X and Y correspond to nondegenerate morphisms between $C_0(Y)$ and $C_b(X) = M(C_0(X))$.

Exercise 1.2. Find an example showing that if X, Y are locally compact but not compact and $T : X \to Y$ is a continuous map, then the map α_T introduced earlier need not map $C_0(Y)$ into $C_0(X)$.

1.3. **Topological quantum groups and semigroups.** A topological semigroup is a topological space X equipped with a continuous map $\cdot : X \times X \to X$, which is associative. As $C_0(X \times X) \approx C_0(X) \otimes C_0(X)$, the dual point of view suggests the following definition.

Definition 1.6. A compact quantum semigroup is a unital C^*-algebra A equipped with a unital *-homomorphism $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ which is coassociative, that is:

$$(\Delta \otimes \mathrm{id}_{\mathsf{A}})\Delta = (\mathrm{id}_{\mathsf{A}} \otimes \Delta)\Delta.$$

The tensor products above are minimal/spatial tensor products of C^*-algebras.

Definition 1.7. A locally compact quantum semigroup is a C^*-algebra A equipped with a nondegenerate *-homomorphism $\Delta : \mathsf{A} \to M(\mathsf{A} \otimes \mathsf{A})$ which is coassociative, that is:

$$(\Delta \otimes \mathrm{id}_{\mathsf{A}})\Delta = (\mathrm{id}_{\mathsf{A}} \otimes \Delta)\Delta.$$

Note that the equality above in fact uses strict extensions; both sides take values in $M(\mathsf{A} \otimes \mathsf{A} \otimes \mathsf{A})$.

Exercise 1.3. Check that if X is a locally compact space and $T : X \times X \to X$ is continuous, then the dual transformation $\alpha_T : C_0(X) \to C_b(X \times X) \approx M(C_0(X) \otimes C_0(X))$ is coassociative if and only if T is associative.

We will always assume that our (locally) compact quantum semigroups have a *counit*, i.e. a character $\epsilon \in \mathsf{A}^*$ such that

$$(\epsilon \otimes \mathrm{id}_{\mathsf{A}})\Delta = (\mathrm{id}_{\mathsf{A}} \otimes \epsilon)\Delta = \mathrm{id}_{\mathsf{A}}.$$

Exercise 1.4. Show that if G is a (locally) compact semigroup with an identity element e, then the evaluation functional $f \to f(e)$ is the counit of the quantum group $C_0(G)$.

For our purposes it will not be really necessary to work with topological quantum *groups*. In fact finding a good definition for a topological quantum group turned out to be a highly non-trivial task, with satisfactory solutions found in the end for the compact case by Woronowicz in [Wor] and for the general locally compact case by Kustermans and Vaes in [KuV].

2. Lecture 2

2.1. **Convolution semigroups of states.**

We have seen that in the classical theory of Lévy processes an important role was played by the Markov convolution semigroup of the process. Recall that if G is a locally compact space, then the Riesz Theorem implies that bounded regular measures can be identified with functionals on $C_0(G)$. If G is a locally compact semigroup, then the convolution of measures μ and ν is defined via the formula

$$(\mu \star \nu)(f) = \int_G f(st)d\mu(s)d\nu(t), \quad f \in C_0(G).$$

Probability measures correspond to positive functionals on $C_0(G)$ of norm 1.

In fact one can also convolve bounded functionals on a locally compact quantum semigroup A: if $\mu, \nu \in \mathsf{A}^*$ we put:

$$\mu \star \nu = (\mu \otimes \nu)\Delta.$$

We use above the fact that the bounded functional $\mu \otimes \nu \in (\mathsf{A} \otimes \mathsf{A})^*$ is automatically strict. The convolution operation $\star : \mathsf{A}^* \times \mathsf{A}^* \to \mathsf{A}^*$ has the following properties (recall that a *state* on a C^*-algebra is a positive functional of norm 1):

(i) $\|\mu \star \nu\| \leq \|\mu\| \|\nu\|$;
(ii) if μ and ν are states, so is $\mu \star \nu$;
(iii) it is associative: $(\mu \star \nu) \star \omega = \mu \star (\nu \star \omega)$;
(iv) the counit ϵ satisfies $\epsilon \star \mu = \mu \star \epsilon = \mu$.

In particular $(\mathsf{A}^*, \star)$ is a unital Banach algebra.

Definition 2.1. Let A be a locally compact quantum semigroup. A family $\{\lambda_t : t \geq 0\}$ is called a convolution semigroup of states on A if

(i) each λ_t is a state on A;
(ii)
$$\lambda_{s+t} = \lambda_s \star \lambda_t, \quad s, t \geq 0;$$
(iii)
$$\lambda_t(a) \xrightarrow{t \to 0^+} \lambda_0(a) = \epsilon(a), \quad a \in \mathsf{A}.$$

It is said to be norm continuous if $\lambda_t \xrightarrow{t \to 0^+} \epsilon$ in norm.

If $\omega \in \mathsf{A}^*$ we will write $\omega^{\star 0} = \epsilon$, $\omega^{\star 1} = \omega$, $\omega^{\star 2} = \omega \star \omega$ and so on.

Theorem 2.2. *Let $\{\lambda_t : t \geq 0\}$ be a norm continuous convolution semigroup of states on a locally compact quantum semigroup A. Then there exists a unique functional $\gamma \in \mathsf{A}^*$, a so-called generating functional, such that*

$$\lambda_t = \exp_\star(t\gamma) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \gamma^{\star n}, \quad t \geq 0.$$

The functional γ is hermitian ($\gamma(a) = \overline{\gamma(a^*)}, a \in \mathsf{A}$), conditionally positive ($\gamma(a^*a) \geq 0$ if $\epsilon(a) = 0$) and satisfies the condition $\widetilde{\gamma}(1_{M(\mathsf{A})}) = 0$ (note we use the strict extension here).

Proof. We give a proof in the case of unital A. Define for each $t \geq 0$ the map $L_t = (\lambda_t \otimes \mathrm{id}_{\mathsf{A}})\Delta : \mathsf{A} \to \mathsf{A}$. Then $\{L_t : t \geq 0\}$ form a norm continuous semigroup of positive, unital maps on A. The general theory of contractive semigroups implies that there exists a unique bounded map $\Gamma : \mathsf{A} \to \mathsf{A}$ such that

$$L_t = \exp(t\Gamma) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\Gamma^n, \quad t \geq 0.$$

Put $\gamma = \epsilon \circ \Gamma$. Check that γ satisfies all the conditions required in the theorem. □

Exercise 2.1. Show that the generating functional of a norm continuous convolution semigroup of states is hermitian, conditionally positive, and vanishes at the identity.

It is natural to ask if every functional on A which is hermitian, conditionally positive, and vanishes at the identity, is a generating functional of some norm continuous convolution semigroup of states. We will come back to it later.

2.2. **Introduction to quantum probability.**

Definition 2.3. A quantum probability space is a pair (B, ω), where B is a unital C^*-algebra and $\omega : \mathsf{B} \to \mathbb{C}$ is a state on B.

We think of B as the algebra of functions on the 'quantum' probability space and about ω as the integral with respect to the 'quantum' probability measure. We could have also used the definition based on von Neumann algebras or purely algebraic *-algebras.

A *quantum random variable* on a C^*-algebra A over a quantum probability space (B, ω) is a nondegenerate *-homomorphism $j : \mathsf{A} \to \mathsf{B}$; the state $\omega \circ j \in \mathsf{A}^*$ plays the role of the distribution of the quantum random variable j. To speak about quantum Lévy processes we will also need the notion of *independence*. This is in fact rather complicated, as there are several possible quantum notions of independence. For our purposes it will suffice to note that the weakest possible requirement for the independence type condition is the factorization of moments: given 'independent' quantum random variables $j_1, \ldots, j_n : \mathsf{A} \to \mathsf{B}$ we expect to have

$$\omega\left(\prod_{i=1}^{n} j_i(a_i)\right) = \prod_{i=1}^{n} (\omega \circ j_i)(a_i).$$

2.3. **Quantum Lévy processes – definition based on distributions.** The idea of defining quantum Lévy processes dates back to the late 1980s, to the paper [ASW], where they were defined in the purely algebraic context, for so-called *-bialgebras (the detailed treatments of this theory can be found in [Sch] and [Fra]). Analytic difficulties force us in a way to work with the definition focusing on the distributions; thus we will consider *weak* quantum Lévy process.

Definition 2.4. A *(weak) quantum Lévy process* on A over a quantum probability space (B, ω) is a family $\big(j_{s,t} : \mathsf{A} \to \mathsf{B}\big)_{0 \leq s \leq t}$ of nondegenerate *-homomorphisms for which the states $\lambda_{s,t} := \omega \circ j_{s,t}$ satisfy the following conditions, for $0 \leq r \leq s \leq t$:

(i) $\lambda_{r,t} = \lambda_{r,s} \star \lambda_{s,t}$;
(ii) $\lambda_{t,t} = \epsilon$;
(iii) $\lambda_{s,t} = \lambda_{0,t-s}$;
(iv)
$$\omega\left(\prod_{i=1}^{n} j_{s_i,t_i}(a_i)\right) = \prod_{i=1}^{n} \lambda_{s_i,t_i}(a_i)$$
whenever $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathsf{A}$ and the intervals $[s_1, t_1[, \ldots, [s_n, t_n[$ are disjoint;
(v) $\lambda_{0,t} \to \epsilon$ pointwise as $t \to 0^+$.

The family $\{\lambda_{0,t} : t \geq 0\}$ is a convolution semigroup of states on A; we call it a *Markov convolution semigroup* of the process. A quantum Lévy process is called *Markov-regular* if $\lambda_{0,t} \to \epsilon$ in norm, as $t \to 0^+$.

In a sense we do not want to distinguish between the processes which carry the same probabilistic information.

Definition 2.5. Two quantum Lévy processes on A are said to be equivalent if and only if their distribution functionals $\{\lambda_{s,t} : 0 \leq s \leq t\}$ and $\{\lambda'_{s,t} : 0 \leq s \leq t\}$ coincide.

Exercise 2.2. Show that two quantum Lévy processes are equivalent if and only if their Markov convolution semigroups coincide. If the processes in question are Markov regular, they are equivalent if and only if the generating functionals of their Markov convolution semigroups coincide.

Do quantum Lévy processes actually exist? We will construct some examples and provide in fact a characterisation (up to equivalence) of Markov-regular quantum Lévy processes in the next lecture.

3. Lecture 3

3.1. **Quantum stochastics – general notations.** In this lecture we will very quickly introduce the definition of quantum stochastic processes in the sense of Hudson and Parthasarathy and describe a special class of quantum stochastic differential equations.

Given a Hilbert space H, the *symmetric Fock space* over H, denoted by $\mathcal{F}(\mathsf{H})$ is the Hilbert space arising as an infinite Hilbert space sum of Hilbert spaces of symmetric tensors of rank n:

$$\mathcal{F}(\mathsf{H}) = \bigoplus_{n=0}^{\infty} \mathsf{H}^{\otimes_{\mathrm{sym}} n}$$

Here $\mathsf{H}^{\otimes_{\mathrm{sym}} 0}$ is by definition equal to $\mathbb{C}\Omega$, where Ω is a fixed vector of length 1, a so-called *vacuum vector* in $\mathcal{F}(\mathsf{H})$, and for each $n \geq 1$ the space $\mathsf{H}^{\otimes_{\mathrm{sym}} n}$ is the closed subspace of the usual Hilbert space tensor power $\mathsf{H}^{\otimes n}$ spanned by symmetric tensors. Probably the most important property of this construction is a so-called exponential property of the symmetric Fock space:

$$\mathcal{F}(\mathsf{H}_1 \oplus \mathsf{H}_2) \approx \mathcal{F}(\mathsf{H}_1) \otimes \mathcal{F}(\mathsf{H}_2).$$

Given a vector $\xi \in \mathsf{H}$ the associated *exponential vector* in $\mathcal{F}(\mathsf{H})$ is defined as

$$\exp(\xi) := \sum_{n=0}^{\infty} \frac{\xi^{\otimes n}}{\sqrt{n!}},$$

with the convention $\xi^{\otimes 0} = \Omega$. The exponential vectors are linearly independent and total in $\mathcal{F}(\mathsf{H})$; note that $\exp(0) = \Omega$.

Exercise 3.1. Guess how the exponential property of the Fock spaces can be seen on the level of exponential vectors. Describe two possible proofs of the exponential property. What happens to the respective vacuum vectors under the natural isomorphism?

Let k be a fixed Hilbert space (a so-called *noise dimension space*) and let $\mathcal{F}_{\mathsf{k}}$ denote the symmetric Fock space over $L^2(\mathbb{R}_+; \mathsf{k})$. We can also consider the natural subspaces of $\mathcal{F}_{\mathsf{k}}$, such as $\mathcal{F}_{s,t,\mathsf{k}} := \mathcal{F}(L^2([s,t]; \mathsf{k}))$ or $\mathcal{F}_{s,\infty,\mathsf{k}} := \mathcal{F}(L^2([s,\infty); \mathsf{k}))$ (here $0 \leq s < t < \infty$). When the noise dimension space is clear from the context we omit it from the notation, speaking only about $\mathcal{F}$ or $\mathcal{F}_{s,t}$. The exponential property implies that we have

$$\mathcal{F} \approx \mathcal{F}_{0,s} \otimes \mathcal{F}_{s,t} \otimes \mathcal{F}_{t,\infty}.$$

The family of L^2-spaces over the intervals is equipped with the natural 'shift' maps: $s_t : L^2(\mathbb{R}_+; \mathsf{k}) \to L^2([t,\infty); \mathsf{k})$ given by

$$(s_t f)(r) = f(r-t), \quad r \geq t, f \in L^2(\mathbb{R}_+; \mathsf{k}).$$

These naturally lift to bounded maps between respective Fock spaces: for each $f \in L^2(\mathbb{R}_+; \mathsf{k})$ we put

$$S_t(\exp(f)) := \exp(s_t f)$$

and finally induce a semigroup of endomorphisms $\{\sigma_t : t \in \mathbb{R}_+\}$ of $B(\mathcal{F})$ via the formula:

$$\sigma_t(X) = I_{\mathcal{F}_{0,t}} \otimes S_t X S_t^*.$$

Definition 3.1. Let A be a C^*-algebra. By a Fock-space quantum stochastic process on A we understand a family of bounded maps $\{j_t : \mathsf{A} \to B(\mathcal{F}_{0,t}) \otimes I_{\mathcal{F}_{t,\infty}}, t \geq 0\}$.

Note that this definition contains a version of the adaptedness property; one can view j_t as an 'adapted' map taking values in $B(\mathcal{F})$. We need three more notational conventions: if H is a Hilbert space, and $\xi, \eta \in \mathsf{H}$, then $\omega_{\xi,\eta}$ denotes the functional given by the formula

$$\omega_{\xi,\eta}(T) = \langle \xi, T\eta \rangle, \quad T \in B(\mathsf{H}).$$

We will write $\hat{\mathsf{H}} := \mathbb{C} \oplus \mathsf{H}$ and put $\hat{\xi} := \binom{1}{\xi} \in \hat{\mathsf{H}}$. Finally let $S = \operatorname{Lin}\{d1_{s,t} : d \in \mathsf{k}, 0 \leq s < t\} \in L^2(\mathbb{R}_+; \mathsf{k})$.

3.2. **Quantum stochastic differential equations.** Let A be a locally compact quantum semigroup.

Definition 3.2. Let $\varphi : \mathsf{A} \to B(\hat{\mathsf{k}})$ be a bounded linear map. We say that a a Fock-space quantum stochastic process $\{j_t : t \geq 0\}$ satisfies the QSDE (quantum stochastic differential equation)

$$j_0 = \epsilon(\cdot) I_{\mathcal{F}}, \quad dj_t = \varphi \star d\Lambda_t \tag{3.1}$$

if for all functions $f, g \in S$, each $a \in \mathsf{A}$ and $t \geq 0$

$$\langle \exp(f), (j_t(a) - \epsilon(a) I_{\mathcal{F}}) \exp(g) \rangle = \int_0^t ds (\omega_{\exp(f),\exp(g)} \circ j_s) \star (\omega_{\hat{f}(s),\hat{g}(s)} \circ \varphi)(a). \tag{3.2}$$

This definition might seem very strange; for the motivation, explanations of the relation to the usual stochastic integration and many examples of several types of QSDEs in various contexts we refer to the lecture notes [Par] and [Lin]. The QSDE of the form (3.1) is sometimes called *coalgebraic*.

Definition 3.3. We say that $\varphi : \mathsf{A} \to B(\hat{\mathsf{k}})$ is a structure map if for all $a, b \in \mathsf{A}$

$$\varphi(a^*) = \varphi(a)^*,$$
$$\varphi(ab) = \epsilon(a)\varphi(b) + \varphi(a)\epsilon(b) + \varphi(a) \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathsf{k}} \end{bmatrix} \varphi(b),$$
$$\varphi(1_{\mathsf{A}}) = 0$$

(the last condition of course gets modified in the usual way when A is non-unital).

Exercise 3.2. Check what kind of the algebraic relations are satisfied by 'matrix entries' of φ, if we write it as

$$\varphi = \begin{bmatrix} \gamma & \delta \\ \eta & \pi - \epsilon(\cdot) I_{\mathsf{k}} \end{bmatrix}.$$

Theorem 3.4. *Let $\varphi : \mathsf{A} \to B(\hat{\mathsf{k}})$ be a structure map. Then the QSDE* (3.1) *has a unique solution, which we will denote $j^\varphi = \{j_t : \mathsf{A} \to B(\mathcal{F}), t \geq 0\}$. Moreover each j_t is a nondegenerate *-homomorphism.*

The crucial fact for the existence of the solution is that each structure map is automatically bounded; in fact it is *completely bounded.* Usually one considers the quantum stochastic differential equations whose solutions a priori need not be bounded and are only defined on the span of suitable exponential vectors. We refer for a detailed technical treatment to the paper [LS$_2$].

Theorem 3.5. *If $\varphi : \mathsf{A} \to B(\hat{\mathsf{k}})$ is a structure map, then j^φ is a quantum stochastic convolution cocycle, i.e.*

$$j_0(a) = \epsilon(a) I_{\mathcal{F}}, \quad a \in \mathsf{A},$$

$$j_{s+t} = \big(j_s \otimes (\sigma_s \circ j_t)\big)\Delta. \tag{3.3}$$

Note the identifications used in (3.3). The proof of the above result is based on the analysis of so-called associated semigroups of j^φ. We will see an example of them in the next result.

Corollary 3.6. *Suppose that φ is a structure map and j^φ is the solution of the QSDE* (3.1). *Let $\lambda_t(a) = \langle \Omega, j_t(a)\Omega\rangle$, $a \in \mathsf{A}$, $t \geq 0$. Then the family $\{\lambda_t : t \geq 0\}$ is a (norm continuous) convolution semigroup of states, whose generator $\gamma \in \mathsf{A}^*$ is given by the formula $a \to \langle \binom{1}{0}, \varphi(a)\binom{1}{0}\rangle$.*

Proof. Each λ_t is a state as a composition of a state $\omega_{\Omega,\Omega}$ on $B(\mathcal{F})$ (a so-called vacuum state) and a nondegenerate *-homomorphism $j_t : \mathsf{A} \to B(\mathcal{F})$. Consider now the following string of equalities:

$$\begin{aligned}
\lambda_s \star \lambda_t &= (\langle \Omega, j_s(\cdot)\Omega\rangle \otimes \langle \Omega, j_t(\cdot)\Omega\rangle)\Delta \\
&= (\langle \Omega_{0,s}, j_s(\cdot)\Omega_{0,s}\rangle \otimes \langle S_s^*\Omega_{s,t}, j_t(\cdot)S_s^*\Omega_{s,t}\rangle)\Delta \\
&= \langle \Omega_{0,s}, j_s(\cdot)\Omega_{0,s}\rangle \otimes \langle \Omega_{s,t}, \sigma_s \circ j_t(\cdot)\Omega_{s,t}\rangle)\Delta \\
&= \langle \Omega_{0,s} \otimes \Omega_{s,t}, (j_s \otimes \sigma_s \circ j_t)\Delta(\cdot)\Omega_{0,s} \otimes \Omega_{s,t}\rangle \\
&= \langle \Omega_{0,s+t}, j_{s+t}(\cdot)\Omega_{0,s+t}\rangle = \lambda_{s+t}.
\end{aligned}$$

The equation (3.2) for $f = g = 0$ gives

$$\langle \Omega, j_t(a)\Omega - \epsilon(a)\Omega \rangle = \int_0^t ds(\omega_{\Omega,\Omega} \circ j_s) \star (\omega_{\binom{1}{0},\binom{1}{0}} \circ \varphi)(a),$$

so in our notation

$$\lambda_t(a) - \epsilon(a) = \int_0^t ds(\lambda_s \star \gamma)(a).$$

Differentiating of the above yields

$$\gamma = \lim_{t \to 0^+} \frac{\lambda_t - \epsilon}{t}.$$

This ends the proof.

□

4. LECTURE 4

4.1. **Fock space quantum Lévy processes.** As the last result suggests, solutions of quantum stochastic differential equations with the 'stochastic generator' given by a structure map are very closely connected to quantum Lévy processes. This is formalised in the next result. We assume that A is a fixed locally compact quantum semigroup.

Theorem 4.1. *Let $\varphi : \mathsf{A} \to B(\hat{\mathsf{k}})$ be a structure map, and let j^φ be the solution of the QSDE* (3.1)*. For each $0 \leq s \leq t$ put*

$$j_{s,t} = \sigma_s \circ j^\varphi_{t-s} : \mathsf{A} \to B(\mathcal{F}_{s,t}) \subset B(\mathcal{F}).$$

Then the family $\{j_{s,t} : 0 \leq s \leq t\}$ is a Markov-regular quantum Lévy process over the quantum probability space $(B(\mathcal{F}), \omega_\Omega)$. We call it a Fock space quantum Lévy process.

Proof. A direct consequence of Theorems 3.4 and 3.5. □

Are structure maps easy to construct? In fact they all have a remarkably simple form.

Theorem 4.2. *Let $\varphi : \mathsf{A} \to B(\hat{\mathsf{k}})$ be a structure map. Then there exists a nondegenerate representation $\pi : \mathsf{A} \to B(\mathsf{k})$ and a vector $\eta \in \mathsf{k}$ such that:*

$$\varphi(a) \begin{bmatrix} \langle \eta, \pi(a)\eta - \epsilon(a)\eta \rangle & \langle \pi(a^*)\eta - \epsilon(a^*)\eta | \\ |\pi(a)\eta - \epsilon(a)\eta \rangle & \pi(a) - \epsilon(a) I_{\mathsf{k}} \end{bmatrix}.$$

Conversely, each map of the above form is a structure map.

4.2. **Schürmann Reconstruction Theorem.** It turns out that in fact all Markov-regular quantum Lévy processes can be realised on the Fock space. This was first proved by Schürmann for purely algebraic quantum Lévy processes ([Sch], see also [LS$_1$]). The following version comes from the papers [LS$_3$] and [LS$_5$].

Theorem 4.3 (Schürmann Reconstruction Theorem)**.** *Let* $\big(j_{s,t} : \mathsf{A} \to \mathsf{B}\big)_{0\le s\le t}$ *be a Markov regular quantum Lévy process. Then it is equivalent to a Fock space quantum Lévy process.*

Proof. Let $\gamma \in \mathsf{A}^*$ be the generating functional of the process j. Due to properties of γ stated in Theorem 2.2

$$q : (a, b) \mapsto \gamma(a^* b) - \gamma(a)^* \epsilon(b) - \epsilon(a)^* \gamma(b)$$

defines a nonnegative sesquilinear form on A.

Let k and $d : \mathsf{A} \to \mathsf{k}$ be respectively the Hilbert space and induced map obtained by quotienting A by the null space of q and completing, so that

$$\overline{d(\mathsf{A})} = \mathsf{k} \text{ and } \langle d(a), d(b)\rangle = q(a, b), \quad a, b \in \mathsf{A}.$$

Further one can check that there are bounded operators $\pi(a)$ on k satisfying

$$\pi(a)d(b) = d(ab) - \epsilon(b)d(a), \quad a, b \in \mathsf{A}.$$

Some more manipulations, using some automatic boundedness arguments imply that the formula

$$\varphi(a) := \begin{bmatrix} \gamma(a) & \langle d(a^*)| \\ |d(a)\rangle & \pi(a) - \epsilon(a)I_{\mathsf{k}} \end{bmatrix}, \quad a \in \mathsf{A},$$

defines a structure map mapping A into $B(\hat{\mathsf{k}})$. Then the Fock space quantum Lévy process j^φ constructed in Theorem 4.1 is equivalent to the process j (as they have the same generating functionals). $\square$

Corollary 4.4. *Each hermitian, conditionally positive, and vanishing at the identity bounded functional on a locally compact quantum semigroup is a generating functional of a norm continuous convolution semigroup of states.*

4.3. **General convolution semigroups of states – unbounded generators.** The last theorem together with Theorem 4.2 show that we have a good understanding of the quantum Lévy processes which have norm-continuous Markov convolution semigroups. To analyse the general case we need first to understand 'unbounded generating functionals' of convolution semigroups of states. We will just present two theorems from [LS$_4$] in a sense initiating such a study.

Definition 4.5. A locally compact quantum semigroup A is said to satisfy the 'residual vanishing at infinity' property if

$$(\mathsf{A} \otimes 1)\Delta(\mathsf{A}) \subset \mathsf{A} \otimes \mathsf{A} \quad \text{and} \quad (1 \otimes \mathsf{A})\Delta(\mathsf{A}) \subset \mathsf{A} \otimes \mathsf{A}.$$

Note that the above property is trivially satisfied for compact quantum semigroups. Also all locally compact quantum *groups* satisfy it.

Exercise 4.1. Prove that if G is a locally compact group then $C_0(G)$ satisfies the 'residual vanishing at infinity' property.

Theorem 4.6. *Let $\{\mu_t : t \geq 0\}$ be a convolution semigroup of states on a locally compact quantum semigroup A which satisfies the 'residual vanishing at infinity' property. Define $\gamma : Dom\,\gamma \subset \mathsf{A} \to \mathbb{C}$ by*

$$Dom\,\gamma := \Big\{a \in \mathsf{A} : \lim_{t \to 0^+} \frac{\lambda_t(a) - \epsilon(a)}{t} \ \textit{exists}\Big\};$$
$$\gamma(d) := \lim_{t \to 0^+} \frac{\lambda_t(d) - \epsilon(d)}{t}, \quad d \in Dom\,\gamma.$$

Then $Dom\,\gamma$ is dense in A and the (linear) map γ determines $\{\lambda_t : t \geq 0\}$ uniquely.

In some cases the situation becomes much simpler.

Definition 4.7. A locally compact quantum semigroup A is said to be of discrete type if A is isomorphic to a direct (c_0-type) sum of matrix algebras.

Theorem 4.8. *Let $\{\mu_t : t \geq 0\}$ be a convolution semigroup of states on a locally compact quantum semigroup of discrete type. Then it is automatically norm continuous.*

Using some advanced interpretations (see [Fra] or [LS_5]) one may say that the last theorem implies that all quantum Lévy processes on discrete quantum semigroups are of *compound Poisson type.*

4.4. **Further directions of research.** We finish by listing a few further results and directions of recent and current research :

- One can consider a wider class of quantum stochastic convolution cocycles. In fact all sufficiently regular *completely positive and contractive* quantum stochastic convolution cocycles satisfy QSDEs of the type described earlier, with 'coefficients' given by completely bounded maps (but not necessarily having the form of a structure map).
- Each Markov-regular Fock space quantum Lévy process can be approximated in a natural strong sense by 'quantum random walks', which can be thought of as discrete quantum stochastic

evolutions; these can be studied as independent objects, which leads to connections to other areas of 'probability on quantum groups'.

- It is natural to search for quantum versions of Lévy-Khintchine formula. They should describe a general form of a convolution semigroup of states on a given quantum (semi)group and is known for example for $SU_q(2)$ ([ScS]).
- As already mentioned, the most challenging problems are those given by the study of quantum Lévy processes which are not Markov regular. Here we cannot really hope for a general theory; one should rather focus on concrete examples.

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