THE HAAGERUP PROPERTY FOR LOCALLY COMPACT CLASSICAL AND QUANTUM GROUPS

ADAM SKALSKI

Abstract. We will describe various equivalent approaches to the Haagerup property (HAP) for a locally compact group and introduce recent work on analogous property in the framework of locally compact quantum groups.

8th Jikji Workshop
19-23 August 2013, NIMS Daejeon

Plan of the lectures

Lecture 1 **Haagerup property for locally compact groups – basic characterisations**: conditionally negative definite and positive definite functions, mixing representations, isometric affine actions; definition of the Haagerup property (HAP); fundamental examples.

Lecture 2 **Haagerup property for locally compact groups – advanced characterisations**: HAP via 'typical' representations; group von Neumann algebras and Schur multipliers associated to positive definite functions; Choda's characterisation of HAP.

Lecture 3 **Haagerup property for locally compact quantum groups**: extending the study of HAP to the context of locally compact *quantum* groups.

The lectures should be accessible to the audience having a general functional analytic background.

1. Lecture 1

All the information contained in this lecture can be found in 'the little green book' [CCJGV].

The story of the Haagerup property begins with the following fundamental result of [Haa], due to Uffe Haagerup.

Theorem 1.1. *Let $n \in \mathbb{N}$ and let $\mathbb{F}_n$ denote the free group on n generators. The word-length function (with respect to the standard generating set) $l : \mathbb{F}_n \to \mathbb{C}$ is conditionally negative definite.*

Let G be a locally compact group. We will assume it is second countable (so that for example the space $C_0(G)$, of all complex continuous functions on G vanishing at infinity, is separable). Note that if G is compact then $C_0(G) = C(G)$.

Definition 1.2. A function $\psi : G \to \mathbb{C}$ is said to be conditionally negative definite if for any $n \in \mathbb{N}$, $g_1, \ldots, g_n \in \Gamma$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that $\sum_{i=1}^n \lambda_i = 0$ we have

$$\sum_{i,j=1}^{n} \overline{\lambda_i} \psi(g_i^{-1} g_j) \lambda_j \leq 0.$$

In these lectures we will assume that all conditionally negative definite functions are continuous and normalised (i.e. $\psi(e) = 0$).

We say that a conditionally negative definite function is *real* if it is real-valued. In that case it suffices to check the condition above for real scalars $\lambda_1, \ldots, \lambda_n$.

Definition 1.3. A function $\varphi : G \to \mathbb{C}$ is said to be positive definite if for any $n \in \mathbb{N}$, $g_1, \ldots, g_n \in \Gamma$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ we have

$$\sum_{i,j=1}^{n} \overline{\lambda_i} \varphi(g_i^{-1} g_j) \lambda_j \geq 0.$$

In these lectures we will assume that all positive definite functions are continuous and normalised (i.e. $\varphi(e) = 1$).

Note that $\varphi \equiv 1$ is positive definite and so is $\varphi = \delta_e$.

Theorem 1.4 (Schönberg Correspondence)**.** *Let $\psi : G \to \mathbb{C}$. The following conditions are equivalent:*

(i) *ψ is conditionally negative definite;*
(ii) *for each $t \geq 0$ the function $g \mapsto \exp(-t\psi(g))$ is positive definite.*

For the proof we refer to Appendix C of [BHV].

By Schönberg correspondence we can rephrase Haagerup's theorem by saying that on $\mathbb{F}_n$ each function $\gamma \mapsto \exp(-tl(\gamma))$, $t \geq 0$, is positive definite.

Haagerup property via functions on G.

Definition 1.5. A locally compact group has the Haagerup property (HAP) if it admits a sequence of positive definite functions $(\varphi_n)_{n=1}^\infty$ which vanish at infinity and converge to 1 uniformly on compact subsets of G.

Note that $(\exp(-\frac{1}{n}l)_{n=1}^\infty$ is such a sequence on $\mathbb{F}_n$.

Proposition 1.6. *G has HAP if and only if it admits a conditionally negative definite real function ψ which is proper (i.e. pre-images of compact sets with respect to ψ are compact).*

Sketch of the proof. $\Longleftarrow$ Follows from the Schönberg's correspondence.

$\Longrightarrow$ Notice first that one can assume that the functions φ_n given by the fact that G has HAP are real-valued (if necessary replacing them by $|\varphi_n|^2$). Then construct ψ of the form $\psi = \sum_{k=1}^\infty \alpha_k(1 - \varphi_{n_k})$ (the series converges uniformly on compact subsets) for suitably chosen subsequence $(n_k)_{k=1}^\infty$ and coefficients $\alpha_k \geq 0$. $\square$

Haagerup property via representations of G. By a representation of G we will always mean a unitary, strongly continuous representation – in other words a homomorphism π from G to the space $U(\mathsf{H})$ (of all unitaries on some Hilbert space H) such that for each $\xi \in \mathsf{H}$ the map $g \mapsto \pi(g)\xi$ is continuous. We will only consider representations on separable Hilbert spaces.

Theorem 1.7 (GNS construction for positive definite functions)**.** *If π is a representation of G on a Hilbert space H and $\xi \in \mathsf{H}$ is a unit vector, then the function $g \mapsto \langle \xi, \pi(g)\xi \rangle$ is positive definite. Conversely, if $\varphi : G \to \mathbb{C}$ is positive definite, then there exists a (unique up to a unitary equivalence) triple $(\pi_\varphi, \mathsf{H}_\varphi, \xi_\varphi)$ such that H_φ is a Hilbert space, π_φ is a representation of G on H_φ, $\xi_\varphi \in \mathsf{H}_\varphi$ is a unit vector cyclic for π_φ (i.e. $\mathrm{Lin}\{\pi_\varphi(g)\xi_\varphi : g \in G\}$ is dense in H_φ) and*

$$\varphi(g) = \langle \xi_\varphi, \pi_\varphi(g)\xi_\varphi \rangle, \quad g \in G.$$

The representation π_φ is usually called the GNS (Gelfand-Naimark-Segal) representation associated with φ. The construction is of course a special case of the GNS construction for states on C^*-algebras (see for example [Mur]) – positive definite functions can be viewed as states on the full group C^*-algebra $C^*(G)$, the enveloping C^*-algebra of $L^1(G)$.

Definition 1.8. A representation π of G on H is mixing (or a C_0-representation) if for each $\xi, \eta \in G$ the coefficient function $g \mapsto \langle \xi, \pi(g)\eta \rangle$ vanishes at infinity.

The terminology originates in the dynamical systems, see for example [Gla]. It is easy to verify that the left regular representation (of G on $L^2(G)$) is always mixing. Moreover direct sums (even infinite) of mixing representations are mixing and a tensor product of a mixing representation with arbitrary representation is mixing. Finite-dimensional representations of non-compact groups are never mixing.

Remark 1.9. Consider a positive definite function $\varphi : G \to \mathbb{C}$ and the associated representation π_φ. Then φ vanishes at infinity if and only if π_φ is mixing. The backward implication is trivial, the forward one uses the cyclicity of the GNS representation.

Definition 1.10. We say that a representation π of G on H contains almost invariant vectors if there exists a sequence $(\xi_n)_{n=1}^\infty$ of unit vectors in H such that

$$\|\pi(g)\xi_n - \xi_n\| \stackrel{n\to\infty}{\longrightarrow} 0$$

uniformly on compact subsets of G. This is equivalent to the fact that π weakly contains the trivial representation and is written as $1_G \preceq \pi$.

Definition 1.11. We say that a representation π of G on H contains an invariant vector if there exists a unit vector $\xi \in \mathsf{H}$ such that $\pi(g)\xi = \xi$ for all $g \in G$. This is equivalent to the fact that π contains the trivial representation and written as $1_G \leq \pi$.

Theorem 1.12. *G has HAP if and only if it admits a mixing representation containing almost invariant vectors.*

Proof. $\Longleftarrow$ If π is a mixing representation of G with almost invariant vectors $(\xi_n)_{n=1}^\infty$ then $g \mapsto \langle \xi_n, \pi(g)\xi_n \rangle$ yields the required sequence of positive definite functions.

$\Longrightarrow$ Consider the sequence of positive definite functions $(\varphi_n)_{n=1}^\infty$ given by the fact that G has HAP. By Remark 1.9 each π_{φ_n} is mixing; so is thus the direct sum $\pi := \bigoplus_{n\in\mathbb{N}} \pi_{\varphi_n}$. View $\xi_n := \xi_{\varphi_n} \subset \mathsf{H}_{\varphi_n}$ as unit vectors in $\bigoplus_{n\in\mathbb{N}} \mathsf{H}_{\varphi_n}$ and compute:

$$\|\pi(g)\xi_n - \xi_n\|^2 = 2 - \langle \pi(g)\xi_n, \xi_n \rangle - \langle \xi_n, \pi(g)\xi_n \rangle = 2 - \varphi_n(g) - \overline{\varphi_n(g)}$$

and each of the last two factors converges to 1 uniformly on compact subsets of G. Thus $(\xi_n)_{n=1}^\infty$ are almost invariant vectors for π. □

Recall that one of the equivalent characterisations of amenability says that G is amenable if and only if the left-regular representation of G contains almost invariant vectors.

Corollary 1.13. *Amenable (so for example abelian or compact) groups have HAP.*

Definition 1.14. G has the Kazhdan Property 'T' if any representation of G which contains almost invariant vectors must contain an invariant vector.

Corollary 1.15. *G has both HAP and Property 'T' if and only if G is compact.*

Proof. Suppose that π is a mixing representation of G containing almost invariant vectors. If G has Property 'T', then π must contain an invariant vector, say ξ_0. But then the coefficient function corresponding to ξ_0 is a constant function equal to 1, so the fact it belongs to $C_0(G)$ means that G is compact.

On the other hand if G is compact then every representation of G is mixing and contains an invariant vector.

□

HAP via affine isometric actions.

Theorem 1.16. *G has HAP if and only if admits a continuous action α on a real Hilbert space H by affine isometries, which is proper: if $B, C \subset \mathsf{H}$ are bounded then the set $\{g \in G : \alpha(g)(C) \cap B \neq \emptyset\}$ is relatively compact.*

Remark 1.17. Each affine isometric action as above can be written as

$$\alpha(g)\xi = \pi(g)\xi + b(g), \quad \xi \in \mathsf{H}, g \in G,$$

where $\pi : G \to B(\mathsf{H})$ is an orthogonal representation and $b : G \to \mathsf{H}$ is a *cocycle* for π:

$$b(gg') = b(g) + \pi(g)b(g'), \quad g, g' \in G.$$

The idea of the proof of the theorem above is as follows: on one hand, given an affine isometric action one can show directly that the function $g \mapsto \|b(g)\|^2$ is conditionally negative definite, and properness of the action transforms into properness of that function. On the other, given a real conditionally negative definite function a GNS-type construction leads to a real Hilbert space with affine isometric action of G (see [BHV]).

Examples. Free groups and finitely generated Coxeter groups have HAP. So do $SL(2;\mathbb{Z})$, $SO(1,n)$, $SU(1,n)$ (but not $SL(3;\mathbb{Z})$!). For the discussion of the examples we refer to the books [CCJGV] and [BrO].

2. Lecture 2

HAP via 'typical' representations. In the first lecture we showed that G has HAP if and only if it admits a mixing representation with invariant vectors. Now we will consider the case of 'typical' representations, following the classical ideas of Paul Halmos ([Hal]), later developed for example in [BeR], [KeP] or [Kec].

Fix H – an infinite dimensional separable Hilbert space and consider $\mathrm{Rep}_G\mathsf{H}$, the collection of all representations of G on H. The set $\mathrm{Rep}_G\mathsf{H}$ can be topologised via the basis of neighbourhoods ($\pi \in \mathrm{Rep}_G\mathsf{H}$, K – compact subset of G, Ω – finite subset of H, $\epsilon > 0$)

$$V(\pi, K, \Omega, \epsilon) = \{\sigma \in \mathrm{Rep}_G\mathsf{H} : \forall_{g\in K}\forall_{\xi\in\Omega}\ \|\pi(g)\xi - \sigma(g)\xi\| < \epsilon\}.$$

It can be verified that the relevant topology is metrisable (recall that we assumed that G is second-countable and H is separable) and makes $\mathrm{Rep}_G\mathsf{H}$ a Polish space, i.e. a separable complete metric space. It is equipped with certain natural operations: direct sums and tensoring. These are formally defined via fixing some identifications: for example once we fix a unitary $W : \mathsf{H} \otimes \mathsf{H} \to \mathsf{H}$ we can define for $\rho, \pi \in \mathrm{Rep}_G\mathsf{H}$

$$(\rho \boxtimes \pi)(g) = W(\rho(g) \otimes \pi(g))W^*, \quad g \in G.$$

We will soon show that G has HAP if and only if mixing representations are dense in $\mathrm{Rep}_G\mathsf{H}$. Begin by formulating and proving a general lemma.

Lemma 2.1. *Let $\mathcal{R} \subset \mathrm{Rep}_G\mathsf{H}$ satisfy the following conditions:*

(i) *invariance under unitary conjugations: for each unitary $U \in B(\mathsf{H})$ and $\sigma \in \mathcal{R}$ the representation $U\sigma U^*$ belongs to $\mathcal{R}$;*

(ii) *tensor absorption property: for all* $\sigma \in \mathcal{R}$ *and* $\pi \in \mathrm{Rep}_G \mathsf{H}$ *the representation* $\sigma \boxtimes \pi \in \mathcal{R}$*;*
(iii) *there exists* $\sigma_0 \in \mathcal{R}$ *which contains almost invariant vectors.*

Then $\mathcal{R}$ *is dense in* $\mathrm{Rep}_G \mathsf{H}$.

Proof. Let $\pi \in \mathrm{Rep}_G \mathsf{H}$ and fix $\Omega = \{\eta_1, \ldots, \eta_k\} \subset \mathsf{H}$, K – a compact subset of G, $\epsilon > 0$. Assume that $\|\eta_i\| = 1$ and let $(e_n)_{n=1}^{\infty}$ be an orthonormal basis in H. Exploiting compactness of K find $N \in \mathbb{N}$ such that for all $g \in K$, $i = 1, \ldots, k$

$$\|\pi(g)\eta_i - \sum_{n=1}^{N} \langle e_n, \pi(g)\eta_i \rangle e_n\| < \frac{\epsilon}{3}.$$

Let then $\xi \in \mathsf{H}$ be a unit vector such that $\|\sigma_0(g)\xi - \xi\| < \frac{\epsilon}{3}$ for all $g \in K$. There exists a unitary $V : \mathsf{H} \to \mathsf{H} \otimes \mathsf{H}$ such that

$$V(x) = x \otimes \xi, \quad \xi \in \mathrm{Lin}\{\eta_1, \ldots, \eta_k, e_1, \ldots, e_N\}.$$

Then the representation $\rho = V^*(\pi \otimes \rho_0)V$ belongs to R (as it is unitarily equivalent to $\rho_0 \boxtimes \pi$) and for $g \in K$, $i = 1, \ldots, k$

$$\begin{aligned}
\|V^*(\pi(g) \otimes \rho_0(g))V\eta_i - \pi(g)\eta_i\| &= \|V^*(\pi(g)\eta_i \otimes \rho_0(g)\xi) - \pi(g)\eta_i\| \\
&< \|V^*(\pi(g)\eta_i \otimes \xi) - \pi(g)\eta_i\| + \frac{\epsilon}{3} < \|V^*\left(\sum_{n=1}^{N} \langle e_n, \pi(g)\eta_i \rangle e_n\right) \otimes \xi - \pi(g)\eta_i\| + \frac{2\epsilon}{3} \\
&= \|\sum_{n=1}^{N} \langle e_n, \pi(g)\eta_i \rangle e_n - \pi(g)\eta_i\| + \frac{2\epsilon}{3} < \epsilon.
\end{aligned}$$

This ends the proof. □

Theorem 2.2 ([DFSW])**.** *G has HAP if and only if mixing representations are dense in* $\mathrm{Rep}_G \mathsf{H}$.

Proof. $\Longleftarrow$ Choose $\pi \in \mathrm{Rep}_G \mathsf{H}$ which contains an invariant vector, say ξ_0 (it suffices for example to take the trivial representation on H). Let $(\rho_n)_{n=1}^{\infty}$ be a sequence of mixing representations in $\mathrm{Rep}_G \mathsf{H}$ converging to π. Put $\rho = \bigoplus_{n \in \mathbb{N}} \rho_n \in \mathrm{Rep}_G \mathsf{H}$ (identifying H with $\mathsf{H}^{\oplus\infty}$). The representation ρ is mixing. For each $n \in \mathbb{N}$ put $\xi_n = \xi_0 \otimes \delta_n \in \mathsf{H}^{\oplus\infty}$. Further for a compact set $K \subset G$ and $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$ and $g \in K$ we have $\|\pi(g)\xi_0 - \rho_n(g)\xi_0\| < \epsilon$. Thus for $g \in K$ and $n \geq N_0$

$$\|\rho(g)\xi_n - \xi_n\| = \|\rho_n(g)\xi_0 - \xi_0\| = \|\rho_n(g)\xi_0 - \pi(g)\xi_0\| < \epsilon,$$

and ρ is a mixing representation containing almost invariant vectors. Hence G has HAP.

$\Longrightarrow$ It suffices to observe that if $\mathcal{R}$ denotes the set of mixing representations of G on H, then the first two conditions of Lemma 2.1 are automatically satisfied; if G has HAP then the third condition also holds. Thus Lemma 2.1 ends the proof. □

Definition 2.3. A representation of G is said to be weakly mixing if it contains no non-trivial finite dimensional subrepresentations.

Weak mixing can be viewed as a strong version of *ergodicity* – a representation is said to be ergodic if it contains no invariant vectors. As mixing representations of non-compact groups cannot be finite dimensional, and mixing property passes to subrepresentations, mixing representations of non-compact groups are always weakly mixing.

Theorem 2.4 ([KeP])**.** *The following conditions are equivalent:*

(i) *G does not have Property 'T';*
(ii) *weakly mixing representations are dense in* $\mathrm{Rep}_G \mathsf{H}$.

There are other characterisations of Property 'T' and of HAP via 'typical' actions of G on $[0, 1]$ or even on the hyperfinite II_1 factor ([BeR], [Hjo], [KeP]).

The group von Neumann algebra and Schur multipliers. Let Γ be a discrete group. Then Γ acts on $\ell^2(\Gamma)$ via the left regular representation:

$$\lambda_\gamma \delta_{\gamma'} = \delta_{\gamma\gamma'}, \quad \gamma, \gamma' \in \Gamma.$$

The *group von Neumann algebra of* Γ is the von Neumann algebra generated in $B(\ell^2(\Gamma))$ via this action:

$$\mathrm{VN}(\Gamma) = \{\lambda_\gamma : \gamma \in \Gamma\}'' \subset B(\ell^2(\Gamma)).$$

It is equipped with a faithful normal trace τ given by the formula

$$\tau(x) = \langle \delta_e, x\delta_e \rangle, \ \ x \in \mathrm{VN}(\Gamma);$$

the space $L^2(\mathrm{VN}(\Gamma), \tau)$ is defined as the completion of $\mathrm{VN}(\Gamma)$ with respect to the L^2-norm:

$$\|x\|_2 := \tau(x^*x)^{\frac{1}{2}}, \ \ x \in \mathrm{VN}(\Gamma).$$

The definition of $L^2(M, \tau)$ for arbitrary von Neumann algebra equipped with a faithful normal trace τ is analogous. It is easy to see that $\mathbb{C}[\Gamma] \subset \mathrm{VN}(\Gamma) \subset L^2(\mathrm{VN}(\Gamma), \tau)$ and that for a finite linear combination $x = \sum_\gamma a_\gamma \lambda\gamma$ there is $\|x\|_2^2 = \sum_\gamma |a_\gamma|^2$.

Lemma 2.5. *Let $\varphi : \Gamma \to \mathbb{C}$ be positive definite. Then there exists a unique unital completely positive normal, trace preserving map $M_\varphi : \mathrm{VN}(\Gamma) \to \mathrm{VN}(\Gamma)$,* the Schur multiplier associated to *φ, such that*

$$M_\varphi \lambda_\gamma = \varphi(\gamma)\lambda_\gamma, \quad \gamma \in \Gamma.$$

Proof. Let $(\pi_\varphi, \mathsf{H}_\varphi, \xi_\varphi)$ be the GNS triple associated to φ and choose an orthonormal basis $(e_i)_{i=1}^\infty$ in H_φ. Define then for each $i \in \mathbb{N}$ the function $a_i \in \ell^\infty(\Gamma)$ via

$$a_i(\gamma) = \langle \xi_\varphi, \pi_\varphi(\gamma)e_i \rangle, \quad \gamma \in \Gamma.$$

Then for each $\gamma \in \Gamma$ there is

$$\sum_{i\in\mathbb{N}} |a_i(\gamma)|^2 = \sum_{i\in\mathbb{N}} \langle \pi_\varphi(\gamma^{-1})\xi_\varphi, e_i \rangle \langle e_i, \pi_\varphi(\gamma^{-1})\xi_\varphi \rangle = \|\pi_\varphi(\gamma^{-1})\xi_\varphi\|^2 = 1.$$

The last fact means that if we view a_i as elements of $B(\ell^2(\Gamma))$, acting by multiplication, then $\sum_{i\in\mathbb{N}} a_i^* a_i = I_{B(\ell^2(\Gamma))}$ (strong operator topology convergence) and the map T attaching to each $x \in B(\ell^2(\Gamma))$ the strongly convergent sum $\sum_{i\in\mathbb{N}} a_i x a_i^*$ is a normal unital completely positive map on $B(\ell^2(\Gamma))$. It remains to check that for each $\gamma, h \in \Gamma$ we have

$$\begin{aligned}
(\sum_{i\in\mathbb{N}} a_i \lambda_\gamma a_i^*)\delta_h &= \sum_{i\in\mathbb{N}} a_i(\gamma h)\overline{a_i(h)}\delta_{\gamma h} = \sum_{i\in\mathbb{N}} \langle \xi_\varphi, \pi_\varphi(\gamma h)e_i \rangle \langle e_i, \pi_\varphi(h^{-1})\xi_\varphi \rangle \delta_{\gamma h} \\
&= \sum_{i\in\mathbb{N}} \langle \pi_\varphi(h^{-1}\gamma^{-1})\xi_\varphi, e_i \rangle \langle e_i, \pi_\varphi(h^{-1})\xi_\varphi \rangle \delta_{\gamma h} = \langle \pi_\varphi(h^{-1}\gamma^{-1})\xi_\varphi, \pi_\varphi(h^{-1})\xi_\varphi \rangle \delta_{\gamma h} \\
&= \langle \xi_\varphi, \pi_\varphi(\gamma)\xi_\varphi \rangle \delta_{\gamma h} = \varphi(\gamma)\delta_{\gamma h} = \varphi(\gamma)\lambda_\gamma \delta_h.
\end{aligned}$$

This proves the existence of unital completely positive map M_φ on $\mathrm{VN}(\Gamma)$ as required in the lemma. The facts that M_φ is trace preserving and unique are easy to check. □

Schur multipliers can be characterised abstractly as certain maps on $\mathrm{VN}(\Gamma)$ 'commuting' with the relevant coproduct. Moreover if $\Phi : \mathrm{VN}(\Gamma) \to \mathrm{VN}(\Gamma)$ is given by the formula $\Phi(\lambda_\gamma) = \varphi(\gamma)\lambda_\gamma$, $\gamma \in \Gamma$, for a certain function $\varphi : \Gamma \to \mathbb{C}$, then Φ is completely positive and unital if and only if φ is positive definite. Thus completely positive Schur multipliers can be viewed as abstract incarnations of positive definite functions. Each M_φ as above induces a contraction T_φ on $L^2(\mathrm{VN}(\Gamma), \tau)$ (as can be seen directly or via the Kadison-Schwarz inequality).

We will need another lemma related to Schur multipliers.

Lemma 2.6. *Let $\varphi : \Gamma \to \mathbb{C}$ be positive definite. Then the following conditions are equivalent:*

(i) *φ vanishes at infinity;*
(ii) *T_φ is compact.*

Proof. $\Longleftarrow$ Suppose that φ does not vanish at infinity. This means that there exists $\epsilon > 0$ and a sequence $\{\gamma_n : n \in \mathbb{N}\}$ of elements of Γ such that for all $n \in \mathbb{N}$ there is $|\varphi(\gamma_n)| > \epsilon$. This means however that the image of the closed ball of radius ϵ^{-1} in $L^2(\mathrm{VN}(\Gamma), \tau)$ via T_φ contains an infinite orthonormal set given by $\{\lambda_{\gamma_n} : n \in \mathbb{N}\}$, so cannot be compact.

$\Longrightarrow$ Assume that φ vanishes at infinity, choose $\epsilon > 0$ and a finite set $F \subset \Gamma$ such that $|\varphi(\gamma)| < \epsilon$ for $\gamma \in \Gamma \backslash F$. One can verify that the map T_F defined on $\mathbb{C}[\Gamma]$ inside $L^2(\mathrm{VN}(\Gamma), \tau)$ via the formula:

$$T_F(\lambda_\gamma) = \begin{cases} \varphi(\gamma)\lambda_\gamma & \gamma \in F \\ 0 & \gamma \notin F \end{cases}$$

extends to a bounded finite rank map on $L^2(\mathrm{VN}(\Gamma), \tau)$. Moreover for arbitrary $x \in \mathbb{C}[\Gamma]$, $x = \sum_{\gamma \in \Gamma} a_\gamma \lambda_\gamma$ there is

$$\|T_F x - T_\varphi x\|_2^2 = \| \sum_{\gamma \notin F} \varphi(\gamma)\alpha_\gamma \lambda_\gamma\|_2^2 = \sum_{\gamma \notin F} |\varphi(\gamma)\alpha_\gamma|^2 \leq \epsilon^2 \sum_{\gamma \notin F} |\alpha_\gamma|^2 \leq \epsilon^2 \sum_{\gamma} |\alpha_\gamma|^2 = \|x\|_2^2.$$

Hence $\|M_\varphi - M_F\| \leq \epsilon$ and M_φ is compact.

□

HAP via the group von Neumann algebra.

Definition 2.7 ([Cho], [Jol])**.** Let M be a von Neumann algebra equipped with a faithful normal trace τ. Then M has the von Neumann algebraic Haagerup property (the vNa HAP) if there exists a sequence $(\Phi_n)_{n=1}^\infty$ of completely positive, unital, normal maps on M such that

(i) for all $n \in \mathbb{N}$ we have $\tau \circ \Phi_n = \tau$, so that Φ_n (again due to the Kadison-Schwarz inequality) induces a contractive map T_n on $L^2(M, \tau)$;
(ii) for all $n \in \mathbb{N}$ the map T_n is compact;
(iii) for all $z \in L^2(M, \tau)$ we have $\lim_{n\to\infty} \|T_n(z) - z\|_2 = 0$.

In [Jol] Paul Jolissaint showed that in fact the above property does not depend on the choice of a faithful normal trace on M. Using the fact that each T_n is a contraction it suffices to check the last condition on a linearly dense subset of $L^2(M, \tau)$. The interest in the vNa HAP originates in some work of Alain Connes, but for us the key point is the following characterisation of HAP for discrete groups due to Marie Choda.

Theorem 2.8 ([Cho])**.** *Let Γ be a discrete group. The following conditions are equivalent:*

(i) Γ *has HAP;*
(ii) $\mathrm{VN}(\Gamma)$ *has the vNa HAP.*

Proof. $\Longleftarrow$ Let $M : \mathrm{VN}(\Gamma) \to \mathrm{VN}(\Gamma)$ be unital and completely positive. Define

$$\varphi(\gamma) = \tau(M(\lambda_\gamma)\lambda_{\gamma^{-1}}), \quad \gamma \in \Gamma.$$

Then the function φ is positive definite. Indeed, $M(1) = 1$ implies that $\varphi(e) = 1$ and if $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{C}$ then

$$\begin{aligned}\sum_{i,j=1}^n \overline{c_i} c_j \varphi(\gamma_i^{-1}\gamma_j) &= \sum_{i,j=1}^n \tau(\overline{c_i} c_j T(\lambda_{\gamma_i^{-1}\gamma_j})\lambda_{\gamma_j^{-1}\gamma_i}) = \sum_{i,j=1}^n \tau(\overline{c_i} c_j \lambda_{\gamma_i} M(\lambda_{\gamma_i^{-1}\gamma_j})\lambda_{\gamma_j^{-1}}) \\ &= \sum_{i,j=1}^n \left\langle \delta_e, \overline{c_i} c_j \lambda_{\gamma_i} M(\lambda_{\gamma_i^{-1}\gamma_j})\lambda_{\gamma_j^{-1}}\delta_e \right\rangle = \sum_{i,j=1}^n \left\langle \xi_i, M(\lambda_{\gamma_i^{-1}\gamma_j})\xi_j \right\rangle \\ &= \sum_{i,j=1}^n \left\langle \xi_i, M(\lambda_{\gamma_i}{}^* \lambda_{\gamma_j})\xi_j \right\rangle \geq 0,\end{aligned}$$

where $\xi_i = c_i \delta_{\gamma_i^{-1}}$, $i = 1, \ldots, n$, and we first used the tracial property of τ, then the definition of τ and finally the assumption that M is completely positive.

Consider then a sequence $(\Phi_n)_{n=1}^\infty$ of the approximating maps featuring in the definition of the vNa HAP and let $(\varphi_n)_{n=1}^\infty$ be corresponding positive definite functions. Then for each $\gamma \in \Gamma$

$$\varphi_n(\gamma) = \tau(\Phi_n(\lambda_\gamma)\lambda_{\gamma^{-1}}) = \langle \lambda_\gamma, T_n(\lambda_\gamma)\rangle_2, \overset{n\to\infty}{\longrightarrow} \langle \lambda_\gamma, \lambda_\gamma \rangle_2 = 1.$$

Moreover for each $n \in \mathbb{N}$

$$\limsup_{\gamma\to\infty} |\varphi_n(\gamma)| \leq \limsup_{\gamma\to\infty} \|T_n(\lambda_\gamma)\|_2 = 0.$$

The last equality follows from compactness of T_n (see a similar argument in the proof of Lemma 2.6).

$\Longrightarrow$ Choose a sequence of positive definite functions $(\varphi_n)_{n\in\mathbb{N}}$ in $c_0(\Gamma)$ such that $\varphi_n(\gamma) \stackrel{n\to\infty}{\longrightarrow} 1$ for each $\gamma \in \Gamma$. By Lemmas 2.5 and 2.6 the associated Schur multipliers M_{φ_n} are unital, completely positive and normal maps on $\mathrm{VN}(\Gamma)$ and induce compact contractions T_{φ_n} on the L^2-space. Moreover for each $\gamma \in \Gamma$ we have

$$\|T_{\varphi_n}(\lambda_\gamma) - \lambda_\gamma\|_2^2 = \|(\varphi_n(\gamma) - 1)\lambda_\gamma\|_2^2 = |\varphi_n(\gamma) - 1|^2 \stackrel{n\to\infty}{\longrightarrow} 0.$$

□

The idea of the proof of the first implication above can be viewed as 'averaging' the approximating maps into Schur multipliers. Recently Zhe Dong introduced in [Don] an analogue of the vNa HAP for a C^*-algebra equipped with a faithful trace. It is easy to deduce from the above proof that a discrete group Γ has HAP if and only if the reduced group C^*-algebra $C_r^*(\Gamma)$ (i.e. the norm closure of the algebra $\mathbb{C}[\Gamma]$ inside $B(\ell^2(\Gamma))$) has the Haagerup property in the sense of Dong – the key property lies in the fact that Schur multipliers leave $C_r^*(\Gamma)$ invariant.

Corollary 2.9. *If Γ_1, Γ_2 are discrete groups which have HAP, the free product $\Gamma_1 \star \Gamma_2$ has HAP.*

Proof. This follows from the last theorem and the result of Florin Boca ([Boc]), who showed that the free product of von Neumann algebras which have the vNa HAP has the vNa HAP. □

3. Lecture 3

This section will be presented as slides: it is based mainly on a recent preprint [DFSW].

References

[BHV] B. Bekka, P. de la Harpe and A. Valette, "Kazhdan's property (T)" New Mathematical Monographs, 11. (Cambridge University Press, Cambridge, 2008).

[BeR] V. Bergelson and J. Rosenblatt, Mixing actions of groups, *Illinois J. Math.* **32** (1988), no. 1, 65-80.

[Boc] F. Boca, On the method of constructing irreducible finite index sub factors of Popa, *Pacific J. Math.* **161** (1993), no. 2, 201–231.

[BrO] N. Brown and N. Ozawa, "C^*-Algebras and finite dimensional approximations", Graduate Studies in Mathematics, 88. American Mathematical Society, Providence, RI, 2008.

[CCJGV] P.A. Cherix, M. Cowling, P. Jolissaint, P. Julg and A. Valette, "Groups with the Haagerup property. Gromovs a-T-menability", Progress in Mathematics, 197, Basel, 2001.

[Cho] M. Choda, Group factors of the Haagerup type, *Proc. Japan Acad. Ser. A Math. Sci.* **59** (1983) 174–177.

[DFSW] M. Daws, P. Fima, A. Skalski and S. White, Haagerup property for locally compact quantum groups, *preprint* available at http://arxiv.org/abs/1303.3261.

[Don] Z. Dong, Haagerup property for C^*-algebras, *J. Math. Anal. Appl.* **377** (2011), no. 2, 631-644.

[Gla] E. Glasner, " Ergodic theory via joinings," Mathematical Surveys and Monographs, 101. American Mathematical Society, Providence, RI, 2003.

[Haa] U. Haagerup, An example of a nonnuclear C^*-algebra, which has the metric approximation property, *Invent. Math.* **50** (1978/79), no. 3, 279-293.

[Hal] P.R. Halmos, "Lectures on ergodic theory," Chelsea Publishing Co., New York 1960.

[Hjo] G. Hjorth, Mixing actions of groups with the Haagerup approximation property, *Fund. Math.* **203** (2009), no. 1, 47-56.

[Jol] P. Jolissaint, Haagerup approximation property for finite von Neumann algebras, *J. Operator Theory* **48** (2002), no. 3, 549–571.

[Kec] A. Kechris, " Global aspects of ergodic group actions," Mathematical Surveys and Monographs, 160. American Mathematical Society, Providence, RI, 2010.

[KeP] D. Kerr and M. Pichot, Asymptotic Abelianness, weak mixing, and property T, *J. Reine Angew. Math.* **623** (2008), 213–235.

[Mur] G.J. Murphy, "C^*-algebras and operator theory," Academic Press, Inc., Boston, 1990.