# Orthogonal polynomials for the quartic potential: a case study for phases and transitions.

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**Abstract** In this talk I will review a detailed analysis of the orthogonal polynomials and corresponding recurrence coefficients for the varying weight $\exp\left[-N\left(\frac{1}{2}z^2 + \frac{1}{4}tz^4\right)\right]$ with integration supported on several arcs in the complex plane.

While some general results (Rakhmanov, Bertola, Kuijlaars-de Silva) are known for the equilibrium measure in the case of polynomial potentials of more general type, the detailed study of the "phase regions" in the complex parameter $t$ can only be carried out on a case by case basis, and numerical computations are also (almost) necessary.

I will discuss these results (in collaboration with Alexander Tovbis, UCF, Florida) as well as some detailed information on the behaviour of the recurrence coefficients near the critical transition points, of which there are three of substantially different nature (two of them linked

## Based on


- *On Asymptotic Regimes of Orthogonal Polynomials with Complex Varying Quartic Exponential Weight*, Marco Bertola and Alexander Tovbis SIGMA **12** (2016), 118, 50 pages.
- *Asymptotics of orthogonal polynomials with complex varying quartic weight: global structure, critical point behaviour and the first Painlevé equation*, M. Bertola and A.Tovbis, Constr. Approx. (2015) **41**, 3 , 529–587
- *Asymptotics of complex orthogonal polynomials on the cross with varying quartic weight: critical point behaviour and the second Painlevé transcendents*, M. Bertola, A. Tovbis, to appear (2017).

In the 90's Random matrix models were proposed as a discretization of 2D Quantum Gravity by [Gross-Migdal '90]

$$Z_N(\beta) = \int \mathrm{d}\Phi \exp\left[-N\mathrm{Tr}\,V(\Phi)\right] \quad \Phi = \Phi^\dagger \in Mat(N \times N). \tag{1}$$

The partition function $Z_N$ can be computed with the aid of *orthogonal polynomials*

$$\int \exp(-NV(\phi))p_n(\phi)p_m(\phi)d\phi = \mathbf{h}_n\delta_{nm} \ , \qquad h_n := \frac{Z_{n+1}}{Z_n} \tag{2}$$

Soon after, [Fokas-Its-Kitaev '92] showed how to modify the contours of integration and obtain the Painlevé I equation for the *double scaling limit* of $\alpha_n = \frac{\mathbf{h}_{n+1}}{\mathbf{h}_n}$.

We pursue the analiysis of the model in the case $V(\phi) = \frac{t}{4}\phi^4 + \frac{g}{2}\phi^2$, with (nonperturbative) analysis in the neighbourhood of the **poles** of the relevant Painlevé solution, as well as the global behaviour in the phase space of the complex parameter $t$.

## Definitions and motivations

The goal is the study of the **asymptotic** of following (nonHermitean) orthogonal polynomials $p_n(z) = z^n + \ldots$ for the pairing

$$\langle p, q\rangle_{\vec{\varrho}} = \sum_{j=1}^{4} \varrho_j \int_{\Omega_j} p(z)q(z)e^{-NV(z,t)}dz, \; , \qquad V(z) := \frac{t}{4}z^4 + \frac{g}{2}z^2 \tag{3}$$

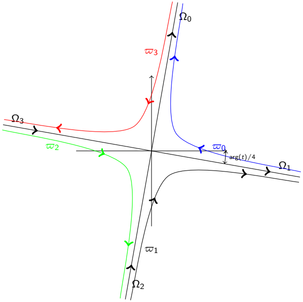
$$\varrho_1 + \varrho_0 = \varrho_3 + \varrho_2, \tag{4}$$

$$n \to \infty, N \to \infty \; , \quad \frac{N}{n} = x \in \mathbb{R}_+ \; , \quad \langle p_n, p_m\rangle_{\vec{\rho}} = \mathbf{h}_n\delta_{nm} \tag{5}$$

Up to rescaling $z \mapsto \sqrt{g}z$ we assume $g = 1$.

for **all** $t \in \mathbb{C}$ and all possible (fixed) values of the parameters $\vec{\rho} \in \mathbb{C}^4$. The main focus si on the asymptotics of the **recurrence coefficients** (which encodes the combinatorics)

$$zp_n = p_{n+1}(z) + \beta_n(t)p_n(z) + \alpha_n(t)p_{n-1}(z) \tag{6}$$



They satisfy general Freud system:

$$
\begin{aligned}
0 &= \beta_n + t\left[(2\beta_n + \beta_{n+1})\alpha_{n+1} + (\beta_n^2 + 2\alpha_n(1-\delta_{n0}))\beta_n + \alpha_n\beta_{n-1}(1-\delta_{n0})\right], \quad (7)\\
\frac{n}{N} &= \alpha_n + t\left[\alpha_n\alpha_{n-1} + \alpha_n^2(1-\delta_{n0}) + \alpha_{n+1}\alpha_n + \beta_n^2\alpha_n + \alpha_n\beta_{n-1}(\beta_n + \beta_{n-1})\right]. (8)
\end{aligned}
$$

Assuming that $\beta_n \to \beta$ and $\alpha_n \to \alpha$ as $n \to \infty$, we obtain two leading order algebraic equations

$$
\beta(1 + 6t\alpha + t\beta^2) = 0\ , \qquad \alpha\left(1 + 3t\alpha + 3t\beta^2\right) = 1, \tag{9}
$$

which have two solutions

$$
\beta = 0, \qquad \alpha = \frac{\sqrt{1+12t} - 1}{6t}, \tag{10}
$$

and

$$
\beta^2 = -6\alpha - \frac{1}{t}\ , \qquad \alpha = \frac{\sqrt{1-15t} - 1}{15t}. \tag{11}
$$

For $t > 0, g > 0$ and integration on the real axis $(\rho_1 = \rho_2 = -1,\ \rho_3 = \rho_4 = 0)$, these are ordinary OPs and the asymptotic is determined first and foremost by the equilibrium measure i.e. the minimizer of the functional

$$\mathcal{F}[\rho] = \int_{\mathbb{R}} V(z)\rho(z)\,\mathrm{d}z + \frac{1}{2}\int_{\mathbb{R}^2} \ln\frac{1}{|z-w|}\rho(z)\rho(w)\,\mathrm{d}z\,\mathrm{d}w \tag{12}$$

The standard potential theory shows that there exists a unique measure $\rho_{eq}(z)\,\mathrm{d}z$ supported on a finite interval like so:

$$\rho_{eq} = \mathrm{argmin}\ \mathcal{F}[\rho] \tag{13}$$

$$\rho_{eq}(z) = \frac{1}{\pi}\left(tz^2 + \frac{g\sqrt{1+12\frac{t}{g^2}}}{3} + \frac{2g}{3}\right)\sqrt{b^2 - z^2}\ , \tag{14}$$

$$b^2 = \frac{2g}{3t}\left(\sqrt{1+12\frac{t}{g^2}} - 1\right) \tag{15}$$

**Remark**

Clearly only important ratio $\frac{t}{g^2}$. From here on: $g = 1$. The case $g < 0$ is equivalent to rotating $z \mapsto iz$.

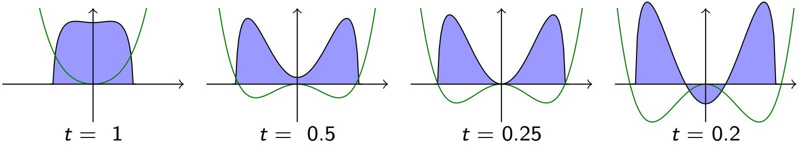



Figure: The density of equilibrium $d\mu_0$ and the potential $\frac{tz^4}{4} - \frac{1}{2}z^2$. For $t > \frac{1}{4}$ the density becomes negative near the origin, which signals that the Ansatz is incorrect and we must look for a two-cut solution.

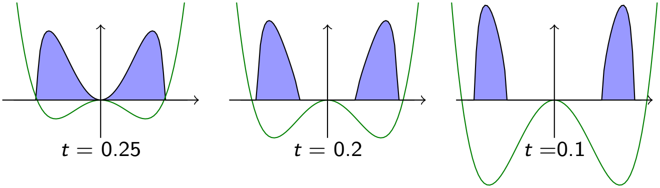



Figure: The density of equilibrium $d\mu_0$ and the potential $\frac{tz^4}{4} - \frac{1}{2}z^2$ in the two-cut assumption, plotted for various values of $\mu$.

# Known transitions

For $g = -1$ and (integration on Real axis), [Bleher-Its '03] it is known that the recurrence coefficients $\beta_n = 0$ and

$$R_n(t) \simeq \frac{1 + (-1)^n \sqrt{1 - 4t}}{2t} , \quad t < \frac{1}{4} \tag{16}$$

$$R_n(t) \simeq \frac{1 + \sqrt{1 + 12t}}{6t}, \quad t > \frac{1}{4} \tag{17}$$

(note that they coincide for $t = \frac{1}{4}$).

## Double scaling asymptotics

Setting $t = \frac{1}{4} - \frac{v}{4n^{\frac{2}{3}}}$

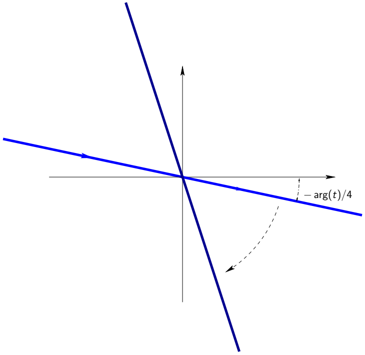
$$R_n = 2 + c_1 \frac{(-1)^{n+1}}{n^{\frac{1}{3}}} y(v) + c_2 \frac{v + y^2(v)}{2n^{\frac{2}{3}}} \tag{18}$$

(for some appropriate constants $c_1, c_2$) where $y(v)$ is the Hastings–McLeod solution of Painlevé II.

$$y''(v) = vy(v) + 2y(v)^3 , \qquad y(v) \simeq \mathrm{Ai}(v), \ v \to +\infty \tag{19}$$

# Analytic continuation in parameter space

To analytically continue for $t \in \mathbb{C}$ we need to rotate the contours of integration by $\vartheta = -\frac{\arg(t)}{4}$, so that –in general– as a result of a loop $t \mapsto t e^{2i\pi}$ the real axis would become the imaginary axis.




$$V(z) := \frac{t}{4} z^4 + \frac{g}{2} z^2 \tag{20}$$

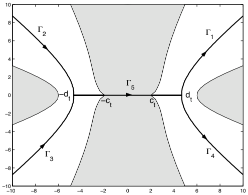
It is known that for **generic values** of $\vec{\rho} = (\varrho_0, \ldots, \varrho_3)$ there is a continuation of the above equilibrium measure around $t = 0$ (either ways) and to the segment

$$-\frac{1}{12} < t < 0$$

given by the obvious continuation of the formulæ above [Duits-Kuijlaars, 2006]

$$\rho_{eq}(z) = \frac{1}{\pi}\left(tz^2 + \frac{\sqrt{1+12t}}{3} + \frac{2}{3}\right)\sqrt{b^2 - z^2}\ , \tag{21}$$

$$b^2 = \frac{2}{3t}\left(\sqrt{1+12t} - 1\right) \tag{22}$$




The support is the horizontal segment: this is also where the roots of the OPs asymptotically accumulate. The contours of integration are asymptotic to the rays $\arg(z) = \frac{\pi}{4} + k\frac{\pi}{2}$ and are deformed as shown. (Picture from [Duits-Kuijlaars '06]). The technique of the nonlinear steepest descent of Deift-Zhou implies that the contours cannot intersect the dark regions

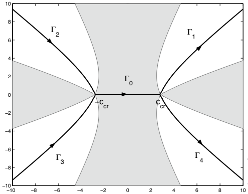
$$\varphi(z) = \Re V(z) + \int \ln \frac{1}{|z-w|}\rho_{eq}(w)\,\mathrm{d}w + \ell \leq 0 \tag{23}$$

Asymptotic behavior of the recurrence coefficients $\alpha_n(t,N)$ $(n=N)$:

$$\alpha_n(t) = \frac{\sqrt{1+12t}-1}{6t} + O(N^{-1})\ , \quad \beta_n(T) = \mathcal{O}(N^{-\infty}) \qquad (24)$$

Near $t_0 = -\frac{1}{12}$ something is gotta happen!
In the double scaling limit $v = N^{\frac{4}{5}} 2^{\frac{9}{5}} 3^{\frac{6}{5}} (t - t_0)$
[Duits-Kuijlaars '06, Fokas-Its-Kitaev '92]

$$\alpha_N(t) = 2 - \frac{2^{\frac{3}{5}} 3^{\frac{2}{5}}}{N^{\frac{2}{5}}} (y_{\varkappa_1}(v) + y_{\varkappa_2}(v)) + O(N^{-\frac{3}{5}}). \qquad (25)$$




Here $y_{\varkappa_1}(v)$, $y_{\varkappa_2}(v)$ are two **tronquée** solutions of the Painlevé I equation

$$y''(v) = y^2(v) + v \qquad (26)$$

The asymptotics is **uniform** as $v$ varies over a compact set $V$ that does not include any of the poles of the two functions (of which there are $\infty$-ly many).

# The global picture in the $t$–plane; $g$–function

- For $t \in \mathbb{C}$ there is no (simple) variational principle; one needs to use the notion of "Boutroux" (algebraic) curve a.k.a. $S$–curve (Rakhmanov).
- Equivalently (in the *nonlinear steepest descent Deift–Zhou* parlance) we need to find the $g$–function $\equiv$ logarithmic potential of the optimal measure.

**The problem**

Describe it in detail in different regions; not known a priori the number of connected components of the accumulation set of zeroes of the OPs.

1. $g(z;t)$ is analytic in $\mathbb{C} \setminus \Sigma$, $\Sigma = (\mathfrak{M} \cup \mathfrak{C})$, and

$$g(z;t) = \ln z + \mathcal{O}(z^{-1}) \ , \quad z \to \infty; \tag{27}$$

$$g_+(z) - g_-(z) = 2i\pi \ , \quad z \in \gamma_0 \subset \mathfrak{C} \tag{28}$$

2. on *bounded* complementary arc $\gamma_{c,j}$, $j = 1, 2, \cdots, L$, the function $g(z)$ has a constant jump

$$g_+(z) - g_-(z) = 2\pi i \eta_j \ , \quad \eta_j \in \mathbb{R} \ , \quad z \in \gamma_{c,j} \subset \mathfrak{C} \tag{29}$$

3. across each main arc $\gamma_{m,j}$, $j = 0, 1, 2, \cdots, L$, the function $g(z)$ has a jump

$$g_+(z) + g_-(z) = \frac{tz^4}{4} + \frac{z^2}{2} + \ell + 2\pi i \varpi_j, \quad \varpi_j \in \mathbb{R} \ , \quad z \in \gamma_{m,j} \subset \mathfrak{M}, \tag{30}$$

# $g$–function, cont'd

In the next few slides I will show these **phase portraits** which show where the zeroes of the OPs accumulate: the nonlinear steepest descent method allows to recover the asymptotics in each of these regions of the $t$–plane.

When the support of zeroes consists of **one arc** we are in the **genus zero** case: then the asymptotic is given by algebraic expression (not reported here). If there are more arcs then we have genus 1, 2 etc. and the asymptotic requires (hyper)elliptic theta functions.

**Dictionary**

**"genus zero"**: the recurrence coefficients have a limit;
**"higher genus"**: the recurrence coefficients are asymptotically (quasi) periodic functions of $n$.

# Global phase transitions (breaking curves)

There are six different situations [Explain on board....]

1. "Generic" case: $\varrho_j \neq 0$ and $\varrho_0 \neq -\varrho_1$, $\varrho_0 \neq \varrho_3$;
2. "Real axis": $\varrho_1 = \varrho_3$ and $\rho_0 = 0 = \varrho_2$ (or viceversa) ;
3. "Single Wedge": $\varrho_1 = 0 = \varrho_2$ and $\varrho_0 = -\varrho_3$ (or viceversa or cyclic);
4. "Consecutive Wedges": $\varrho_0 = 0$, $\varrho_{1,2,3} \neq 0$ (or cyclic);
5. "Opposite Wedges, generic": $\varrho_0 + \varrho_1 = 0 = \varrho_2 + \varrho_3$;
6. "Opposite Wedges, symmetric": $\varrho_0 = -\varrho_1 = \varrho_2 = -\varrho_3$.

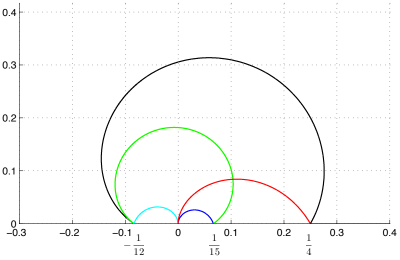



Figure: All breaking curves, summarized: they are symmetric about the real $t$–axis. Recall that in general these phase portraits span several (up to four) sheets of the $t$ plane branched around $t = 0$.

The steepest descent analysis revolves around the "$g$" function (or the effective electrostatic potential); in short a holomorphic function on $\mathbb{C}$ minus certain cuts with behavior

$$h(z) = \frac{t}{4}z^4 + \frac{1}{2}z^2 - 2g(z) = \frac{t}{4}z^4 + \frac{1}{2}z^2 - 2\ln z + \mathcal{O}(z^{-1}) \tag{31}$$

and **constant imaginary jumps** along the cuts. In particular it is **essential** that

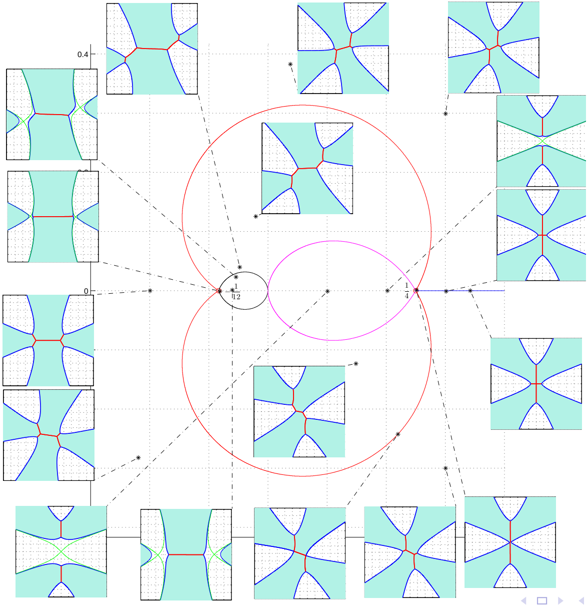
$$\Re h(z) \geq 0 \ , \quad z \in \gamma \quad \Re h < 0 \Leftrightarrow \text{ the } \textbf{sea} \tag{32}$$

over the contours of integration, where the real part is **zero** along the cuts. As $t$ changes a **saddle point** of $\Re h$ may sink/emerge and pinch a contour: a **genus (phase) transition occurs**. The curves in the previous slide are thus obtained by solving the implicit system (but the resulting formulas are rather explicit)

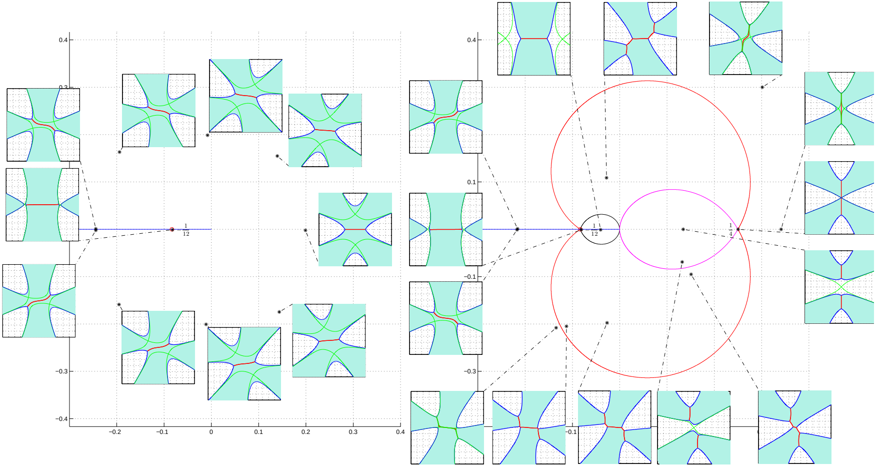
$$\begin{cases} h'(z) = 0 & \text{(saddle point of the (sub)harmonic function } \Re h) \\ \Re h(z) = 0 & \text{(transition at sea-level)} \end{cases} \tag{33}$$

The next slides (and animations if time permits) show the process.

# Generic; only one sheet

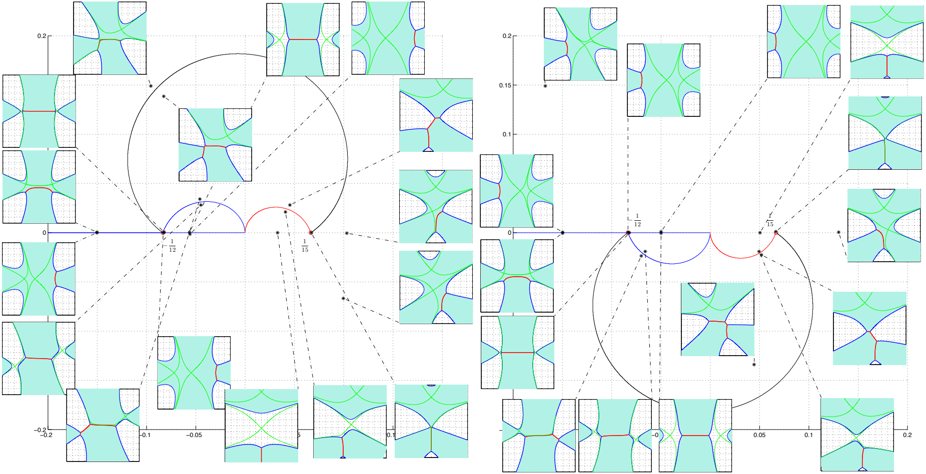


# Real Axis: two sheets glued along $\mathbb{R}_-$



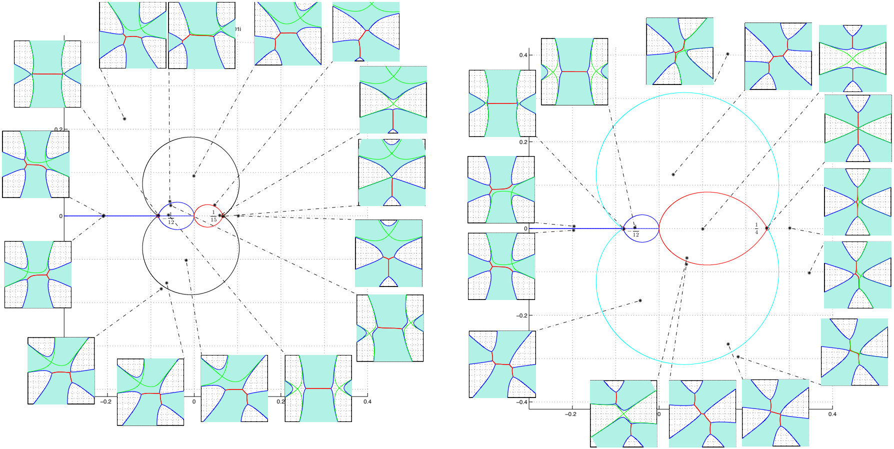
# Single Wedge: four sheets glued along $\mathbb{R}_-$

Only sheets 1 and 2 shown: the others are copies where $z \mapsto -z$. Note that there at the critical point $t = \frac{1}{15}$ on all four sheets we have a transition of type Painlevé I (and also at $t = -\frac{1}{12}$).



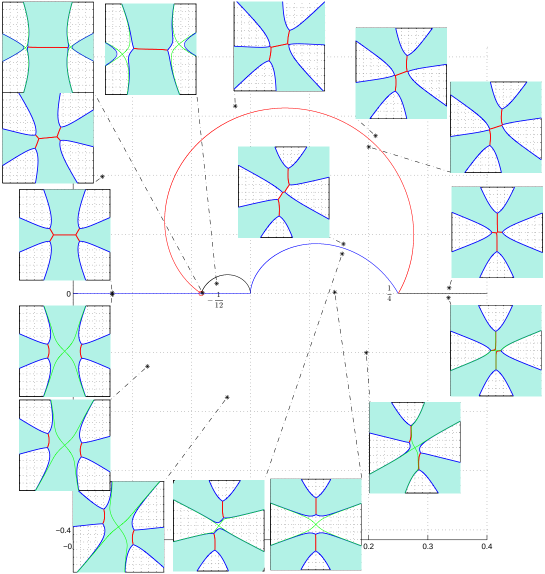
# Consecutive Wedges: four sheets glued along $\mathbb{R}_-$

Only sheet 1 and 2 shown, because the others are copies where $z \mapsto -z$. Note the Painlevé I transition at both $t = -\frac{1}{12}$ and $t = \frac{1}{15}$.

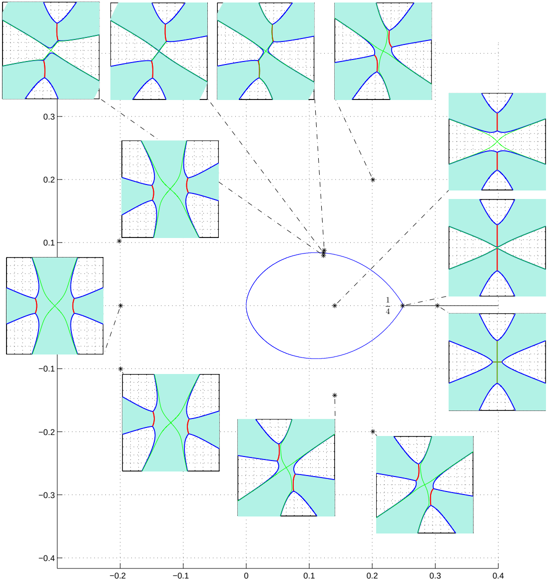


# Opposite Wedges, generic: two sheets glued along $\mathbb{R}_-$

Only sheet 1 shown, because sheet 2 is a copy where $z \mapsto \bar{z}$. Note the Painlevé I transition at $t = -\frac{1}{12}$ and Painlevé II transition at $t = \frac{1}{4}$.



# Opposite Wedges, Symmetric: only one sheet

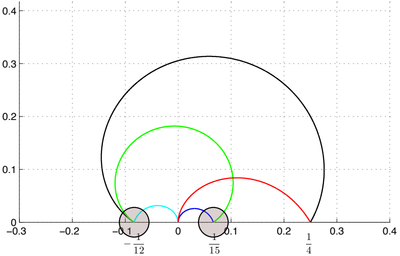



Note the Painlevé II transition at $t = \frac{1}{4}$. The spectral curve is always of genus 1 except at the point $t = \frac{1}{4}$.

[B.-Tovbis '16] The global study in parameter space for matrix models (OPs) is essentially a case study: general results are available in [ B.-Mo 2010, B. 2011, Kuijlaars-Silva '13] but each situation has its own beauty and peculiarity.
Two problems arise:

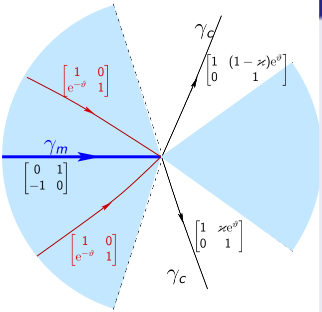
1. **Global analysis**: best performed by computer assistance;
2. **Local analysis** (i.e. Universality); near the transition regions the subleading (and sometimes even the dominant) behavior is expressed in terms of universal objects typically related to Painlevé transcendents;
3. **Ultralocal analysis**: near poles of these transcendents (i.e. when the OPs are badly ill-conditioned) even simpler universality emerges, without any special function:

We now address the second and third point above.


0.4
0.3
0.2
0.1
0
−0.3
−0.2
−0.1
0
0.1
0.2
0.3
0.4
$-\frac{1}{12}$
$\frac{1}{15}$
$\frac{1}{4}$

## What you need to know about Painlevé I

The **tronquée** family of solutions is associated to the Riemann Hilbert problem




$$\vartheta(z) := \frac{4}{5}\xi^{\frac{5}{2}} - v\sqrt{\xi}$$

**Problem (Tronquée family Painlevé 1 RHP [Kapaev])**

*The matrix* $\mathbf{P}(\xi;\varkappa)$ *is locally bounded, admits boundary values on the rays shown and satisfies*

$$\mathbf{P}_+ = \mathbf{P}_- M,$$

$$\mathbf{P}(\xi;\varkappa) = \frac{\xi^{\sigma_3/4}}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \cdot \left(\mathbf{1} - \frac{H_I\sigma_3}{\sqrt{\xi}} + \frac{H_I^2\mathbf{1} + y_\varkappa\sigma_2}{2\xi} + \dots\right)$$

$$H_I := \frac{1}{2}(y'_\varkappa)^2 + y_\varkappa v - 2y_\varkappa^3,$$

*where the jump matrices* $M = M(\xi; v, \vec{\omega})$ *are the matrices indicated on the corresponding ray in the figure.*

It is known since [Boutroux 1913] that any solution of P1

$$y'' = y^2 + v \tag{34}$$

has infinitely many poles. The tronquée family has no poles in a distal sector of amplitude $\frac{2\pi}{5}$; the exceptional case of the **tritronquée** ($\varkappa = 0, 1$) has a larger sector asymptotically free of poles ($\frac{4\pi}{5}$). However, poles are *inevitable*.

The poles of the function $y(v)$ correspond to the values of $v = v_p$ for which the solution of the RHP above **does not exist** (as stated).

**Question**

*What happens near those poles?*

# Beware of poles!


Double Poles!

# Beware of poles!

What happens if we decide to explore the neighborhood of those poles? Let's focus on the **symmetric case** (i.e. the moment functional is symmetric): then $y_{\nu_1} = y_{\nu_2}$ and if $v_p$ is one of those poles then:

**Theorem (Symmetric case: P1 transition only at $t = -\frac{1}{12}$)**

*Let $v_p$ be a pole of $y(v) := y^{(1)}(v) = y^{(0)}(v)$ and let $t$ vary so that*

$$t + \frac{1}{12} = -\frac{v_p}{2^{\frac{9}{5}} 3^{\frac{6}{5}} N^{\frac{4}{5}}} + \frac{s}{2^3 3N}, \tag{35}$$

*where $s = \mathcal{O}(N^{-\rho})$ with an arbitrary $\rho \in [0, \frac{1}{5})$. Then the following holds:*

$$\alpha_n = 2\frac{9 - s^2 + \mathcal{O}(N^{-\frac{1}{5}})}{1 - s^2 + \mathcal{O}(N^{-\frac{1}{5}})}, \qquad \beta_n = 0, \tag{36}$$

$$\mathbf{h}_n = \pi\sqrt{8}\, 2^N \exp\left[-\frac{3N}{2} + \frac{N^{\frac{1}{5}} v_p}{3^{\frac{1}{5}} 2^{\frac{4}{5}}} - \frac{s}{4}\right] \left(\frac{3-s}{1+s} + \mathcal{O}(N^{-\frac{1}{5}}(s^2-1)^{-1})\right). \tag{37}$$

*The variable $s$ may approach the points $s = \pm 1$ at some rate (a* quadruple scaling*) as long as the corresponding error indicated in the formulæ above terms are infinitesimal.*

Note that exactly at the pole there is no divergence, $\alpha_n(\widetilde{t}) = 9\alpha_n(t_0)$, but the asymptotic diverges in two places near the pole; in fact one can use a **quadruple scaling argument** (i.e. let $s$ depend on $N$) to see that those divergencies are genuine.

There is a similar Painlevé I transition at $t = \frac{1}{15}$ (for different configuration of contours)

**Theorem (Nonsymmetric case)**

*Let $t$ approach $t_0 = -\frac{1}{12}$ or $t_1 = \frac{1}{15}$ in such a way that it satisfies respectively*

$$t + \frac{1}{12} = -\frac{v_p}{N^{\frac{4}{5}} 3^{\frac{6}{5}} 2^{\frac{9}{5}}} - \frac{s}{3\sqrt{2}N} \qquad \text{or} \qquad t - \frac{1}{15} = -\frac{v_p \mathrm{e}^{-\frac{3i\pi}{5}}}{3^{\frac{6}{5}} 2^{\frac{1}{4}} 5 N^{\frac{4}{5}}} - i\frac{s}{2N}, \tag{38}$$

$$\alpha_n(t) = \frac{b_0^2}{4} - \frac{1}{4s^2} + \mathcal{O}\left(N^{-\frac{1}{5}} s^{-1}\right), \qquad \beta_n(t) = a_0 + \frac{1}{2s(1 - b_0 s) + \mathcal{O}(N^{-\frac{1}{5}})}, \tag{39}$$

*where $a_0 = 0$, $b_0 = \sqrt{8}$ and $a_0 = -3i$ , $b_0 = 2i$ in the cases $t \sim t_0$ and $t \sim t_1$ respectively. The numbers $\mathbf{h}_n$ satisfy:*

$$\mathbf{h}_n = \pi 2^N \exp\left[-\frac{3N}{2} + \frac{N^{\frac{1}{5}} v_p}{3^{\frac{1}{5}} 2^{\frac{4}{5}}} + \sqrt{2}\, s\right] \left(\sqrt{8} - \frac{1}{s} + \mathcal{O}(N^{-\frac{1}{5}} s^{-1})\right), \tag{40}$$

$$\mathbf{h}_n = \pi(-1)^N \exp\left[\frac{9N}{4} - \frac{13}{4}\frac{N^{\frac{1}{5}} v_p \mathrm{e}^{-\frac{3i\pi}{5}}}{3^{\frac{1}{5}} 2^{\frac{1}{5}}} + \frac{13}{2} is\right] \left(2i - \frac{1}{s} + \mathcal{O}(N^{-\frac{1}{5}} s^{-1})\right). \tag{41}$$

*These formulæ hold uniformly for bounded values of $s$ as long as the indicated error terms remain infinitesimal.*

The results are based on the Deift–Zhou steepest descent method, refined to handle the non-existence of the local parametrix (i.e. the RHP for Painlevé I). The technical tool is the following

**Theorem (B-Tovbis)**

*When $v$ is near a pole of $y = \frac{1}{(v-v_p)^2} + \mathcal{O}(1)$ there is a (unique) solution of the same jump conditions, but with asymptotic behavior ($p_2, \xi_0$ some constants)*

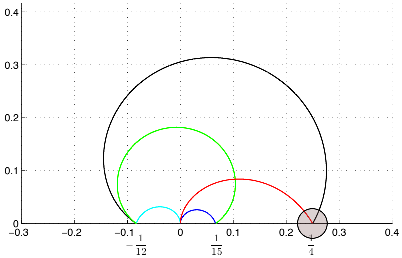
$$\hat{\mathbf{P}}(\xi, v) = \xi^{-\frac{3}{4}\sigma_3} \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} 1 & i \\ 1 & -i \end{array} \right] \left( \mathbf{1} + \mathcal{O}\left( \xi^{-\frac{1}{2}}, y^{-4}, \mathrm{e}^{-p_2 \frac{|y|^{5/2}}{|\xi_0|^{5/2}}} \right) \right) \left( \frac{\sqrt{\xi} + \sqrt{y}}{\sqrt{\xi - y}} \right)^{\sigma_3} \quad (42)$$

The result above holds **uniformly** in a neighborhood of $v = v_p$ (i.e. $y = \infty$); to be compared with the usual asymptotic behavior

$$\mathbf{P}(\xi; \varkappa) = \xi^{\frac{1}{4}\sigma_3} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \left( \mathbf{1} + \mathcal{O}(\xi^{-\frac{1}{2}}) \right) \quad (43)$$

where the $\mathcal{O}(1/\sqrt(\xi))$ term is not uniform for large $y$.

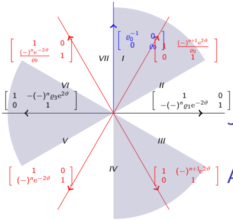
# The Painlevé II transition: a new phenomenon




While for the critical transitions at the critical points $t = -\frac{1}{12}, \frac{1}{15}$ the Painlevé transcendents appear at the **subleading** order, at $t = \frac{1}{4}$ (for special choices of traffics) we get a **leading order phenomenon**.

We consider the "generic" phase diagram, and without loss of generality $\varrho_2 = 1$ by overall scaling of the bilinear pairing (does not affect orthogonality relations)

To explain I need to introduce the general PII equation




## Problem

*Let* $\mathbf{P}(\xi) : \mathbb{C} \to Mat(2 \times 2; \mathbb{C})$ *be a locally bounded, piecewise analytic function on the complement of the oriented rays indicated in the figure and such that*

**Jump** $\mathbf{P}_+(\xi) = \mathbf{P}_-(\xi)M$ *where* $\xi$ *belongs to one of the rays and* $M$ *as in figure.*

**Asym** $\mathbf{P}(\xi)$ *in each connected sector as* $\xi \to \infty$

$$\mathbf{P}(\xi) = \left(\mathbf{1} + \frac{1}{\xi}\begin{bmatrix} -H_{II}(v) & -\frac{u(v)}{2} \\ -\frac{z(v)}{u(v)} & H_{II}(v) \end{bmatrix} + \mathcal{O}(\xi^{-2})\right)\xi^{-\theta\sigma_3}$$

$$\vartheta(\xi; v) := \frac{\xi^3}{3} + \frac{v}{2}\xi$$

$$H_{II} := \frac{1}{2}z^2 + \left(y^2 + \frac{v}{2}\right)z + \theta y = \frac{(y')^2}{2} - \frac{1}{2}\left(y^2 + \frac{v}{2}\right)^2 + \theta y, \quad (44)$$

Consequence of the isomonodromic method is that

$$z = y' - y^2 - v/2, \qquad \frac{d}{dv}\ln u = -y\ ; \qquad y'' = 2y^3 + vy + \frac{1}{2} - \theta. \tag{45}$$

$$H_{II} := \frac{1}{2}z^2 + \left(y^2 + \frac{v}{2}\right) z + \theta y = \frac{(y')^2}{2} - \frac{1}{2}\left(y^2 + \frac{v}{2}\right)^2 + \theta y, \tag{46}$$

In our case we need the special family of solutions of the RHP. 5, and therefore of (45) with

$$\theta = -\frac{\ln \varrho_0}{2i\pi} \qquad \Re\theta \in (-\frac{1}{2}, \frac{1}{2}] \tag{47}$$

The set of poles of $u(v)$ is, in general, infinite (except when $u$ is a rational function) and very complicated (and little studied)

$$\mathcal{P}(\vec{\varrho}) := \{\text{ poles of } \ u(v) = u(v;\vec{\varrho})\} \tag{48}$$

**Theorem**

*Let*

$$t = \frac{1}{4} - \frac{v}{4n^{\frac{2}{3}}}, \qquad \text{where} \qquad v \in \mathbb{C}. \tag{49}$$

*Let* $\delta = \operatorname{dist}(v, \boldsymbol{\mathcal{P}}(\vec{\rho}))$ *and assume that* $\delta = \mathcal{O}(n^{-\epsilon})$ *with* $0 \leq \epsilon < \frac{1}{3}$*. Then the recurrence coefficients and the norms* $\boldsymbol{h}_n$ *have the following asymptotic behavior:*

$$\begin{aligned} \alpha_n &= -2\left(\frac{4n^{\frac{1}{3}} \pm c_{\pm}(v)\mathrm{e}^{\pm 2\mathfrak{a}}}{4n^{\frac{1}{3}} \mp c_{\pm}(v)\mathrm{e}^{\pm 2\mathfrak{a}}} + \mathcal{O}_1\right)^2; \\ \beta_n &= \frac{-16\sqrt{2}c_{\pm}(v)\mathrm{e}^{\pm 2\mathfrak{a}}n^{\frac{1}{3}}}{\left(4n^{\frac{1}{3}} \mp c_{\pm}(v)\mathrm{e}^{\pm 2\mathfrak{a}}\right)\left(4n^{\frac{1}{3}} \pm c_{\pm}(v)\mathrm{e}^{\pm 2\mathfrak{a}} + \mathcal{O}_0\right)} + \mathcal{O}_1; \\ \boldsymbol{h}_n &= i\pi\sqrt{8\varrho_0}(-2)^n\,\mathrm{e}^{\frac{n}{2}+\frac{3}{2}n^{\frac{1}{3}}v}\left(\frac{4n^{\frac{1}{3}} \pm c_{\pm}(v)\mathrm{e}^{\pm 2\mathfrak{a}}}{4n^{\frac{1}{3}} \mp c_{\pm}(v)\mathrm{e}^{\pm 2\mathfrak{a}}} + \mathcal{O}_1\right). \end{aligned} \tag{50}$$

*Here*

$$\mathrm{e}^{2\mathfrak{a}} = (\mathrm{e}^{i\pi}64n^{\frac{2}{3}})^{\theta}(-1)^n, \tag{51}$$

*($\mathcal{O}$'s are suitable error terms) the sign* $\pm$ *is to be chosen according to the sign of* $\Re\theta$ *and*

$$c_+(v) = \frac{u(v)}{2}, \qquad c_- = \frac{z(v)}{u(v)} = \frac{y'(v) - y^2(v) - v/2}{u(v)} \tag{52}$$

# Comments

**Example ($\Re\theta = 0$ or $\varrho_0 = 1$: )**

$$\alpha_n = -2+\mathcal{O}(n^{-\frac{1}{3}+\epsilon}), \quad \beta_n = \mathcal{O}(n^{-\frac{1}{3}+\epsilon}), \quad \boldsymbol{h}_n = i\pi\sqrt{8\varrho_0}(-2)^n \mathrm{e}^{\frac{n}{2}+\frac{3}{2}n^{\frac{1}{3}}v}\left(1+\mathcal{O}(n^{-\frac{1}{3}+\epsilon})\right). \tag{53}$$

Subleading terms contain Painlevé transcendents (as in Bleher-Its'03);

**Example ($\Re\theta = \frac{1}{2}$ or $\varrho_0 = -1$)**

Painlevé transcendents affect the *leading order*.

$$\begin{aligned} \alpha_n &= -2\left(\frac{1+(-1)^n iu(v)}{1-(-1)^n iu(v)} + \mathcal{O}_1\right)^2, \quad \beta_n = \frac{(-1)^{n+1}4\sqrt{2}iu(v)}{(1-(-1)^n iu(v))(1+(-1)^n iu(v)+\mathcal{O}_0)} + \mathcal{O}_1, \\ \boldsymbol{h}_n &= \pi\sqrt{8}(-2)^n \mathrm{e}^{\frac{n}{2}+\frac{3}{2}n^{\frac{1}{3}}v}\left(\frac{1+iu(v)(-1)^n}{1-iu(v)(-1)^n} + \mathcal{O}_1\right). \end{aligned} \tag{54}$$

- **Recurrence coefficients acquire a dependence on $u(v)$ at leading order**
- singularities at some points of the double-scaling variable $v$ that have nothing to do with the poles of $u(v)$. No singularity at the poles; this analysis is the content of our second main theorem.

## Theorem (Triple scaling near poles)

*Let $v_p \in \boldsymbol{\mathcal{P}}(\vec{\varrho})$ so that*

$$u(v) = -\frac{2}{\varkappa(v - v_p)} + \mathcal{O}(v - v_p), \quad \varkappa \in \mathbb{C}^{\times}. \tag{55}$$

$$t = \frac{1}{4} - \frac{v_p}{4n^{\frac{2}{3}}} - \frac{s}{2\sqrt{2}n} + \mathcal{O}(n^{-\frac{4}{3}}), \qquad \Leftrightarrow \qquad v = v_p + \frac{\sqrt{2}}{\sqrt[3]{n}} s \tag{56}$$

*Then*

$$\alpha_n = -2\left[\frac{\cosh(2\widetilde{\mathfrak{a}}) - \sqrt{8}s - 2}{\cosh(2\widetilde{\mathfrak{a}}) + \sqrt{8}s} + \mathcal{O}_2\right]\left[\frac{\cosh(2\widetilde{\mathfrak{a}}) - \sqrt{8}\,s + 2}{\cosh(2\widetilde{\mathfrak{a}}) + \sqrt{8}\,s} + \mathcal{O}_2\right];$$

$$\beta_n = \frac{-\sqrt{8}\mathrm{e}^{-2\widetilde{\mathfrak{a}}}\left(1 - \mathrm{e}^{4\widetilde{\mathfrak{a}}}\right)(1 - \sqrt{8}\,s)}{\left(\cosh(2\widetilde{\mathfrak{a}}) + \sqrt{8}\,s\right)\left(\cosh(2\widetilde{\mathfrak{a}}) - \sqrt{8}\,s + 2 + \mathcal{O}_0\right)} + \mathcal{O}_2;$$

$$\boldsymbol{h}_n = \pi i\sqrt{8\varrho_0}(-2)^n \exp\left[\frac{n}{2} + \frac{3v_p n^{\frac{1}{3}}}{2} + \frac{3s}{\sqrt{2}}\right]\left(1 - \frac{4\cosh^2(\widetilde{\mathfrak{a}})}{\cosh(2\widetilde{\mathfrak{a}}) + \sqrt{8}\,s} + \mathcal{O}_2\right).$$

*($\mathcal{O}$'s are suitable error terms) Here*

$$\mathrm{e}^{2\widetilde{\mathfrak{a}}} = \frac{(\mathrm{e}^{i\pi}64n^{\frac{2}{3}})^{\theta}(-1)^n}{\varkappa} \tag{57}$$

# Comments

The introduction of the triple scaling parameter $s$ allows us to investigate a neighborhood of size $n^{-\frac{1}{3}}$ of the $v$–plane around a pole $v_p$; consider again

**Example ($\Re\theta = 0$ or $\varrho_0 = 1$: )**

We have a *bounded* (in $n$) expression $\cosh(2\widetilde{\mathfrak{a}}) = \frac{(-1)^n}{2}\left(\varkappa + \frac{1}{\varkappa}\right)$; The singularity in the coefficients is not at the pole $v_p$ ($s = 0$) but at some (oscillating with $n$) points in a $\mathcal{O}(n^{-\frac{1}{3}})$–neighborhood of $v_p$.

**Example ($\Re\theta = \frac{1}{2}$ or $\varrho_0 = -1$)**

We have an *unbounded* (in $n$) expression $\cosh(2\widetilde{\mathfrak{a}}) = (-1)^n \frac{4i}{\varkappa} n^{\frac{1}{3}} + \mathcal{O}(n^{-\frac{1}{3}})$. The singularity of the recurrence coeffs "escapes" to $s = \infty$ faster than the maximum allowed rate of $|s| = n^{\frac{1}{12}}$; in other words *the recurrence coefficients do not have any singularity at the poles of $u(v)$, but at some other points where $u(v)$ is otherwise regular*.

Further analysis (in the paper!) shows that the singularities "migrate" from a $\mathcal{O}(n^{-\frac{1}{3}})$ nghbd of the poles of $u(v)$ away to region at distance of order $\mathcal{O}(1)$, as $\Re\theta$ moves from 0 to $\frac{1}{2}$ (i.e. the traffic turns "head on").

The focusing Nonlinear Schrödinger (NLS) equation [see Stephanos Venakides' talk of yesterday]

$$i\varepsilon\partial_t q = -\varepsilon^2\partial_x^2 q - 2|q|^2 q \tag{58}$$

$$q(x,0,\varepsilon) = A(x)e^{i\Phi(x)/\varepsilon} \tag{59}$$

models self-focusing and self-modulation (*optical fibers*). It is **integrable** by inverse scattering methods (Zakharov–Shabat). We study $\varepsilon \to 0$; in different regions of spacetime, there are different asymptotic behaviors (*phases*) separated by **breaking curves** (or **nonlinear caustics**).

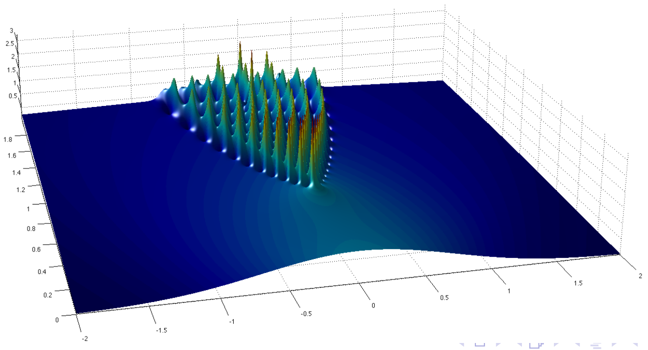



Figure: The case $A(x) = e^{-x^2}$, $\Phi'(x) = \tanh x$ and $\varepsilon = 0.03$

The tip-point of the braking curves is called a point of **gradient catastrophe**, or **elliptic umbilical singularity** [Dubrovin-Grava-Klein].

## Main goal

Leading order asymptotic $q(x, t, \varepsilon)$ on and around the gradient catastrophe point $(x_0, t_0)$.

The behavior in the bulk is described in terms of slow modulation of exact quasi-periodic solutions (**genus 2**), while outside by slow modulation equations for the amplitude. There are (generically) two types of **transitional regions**

- A strip region of scale $\mathcal{O}(\varepsilon \ln \varepsilon)$ around the *breaking curves* (nonlinear caustics);
- a circular region of scale $\mathcal{O}(\varepsilon^{\frac{4}{5}})$ around the gradient catastrophe point.

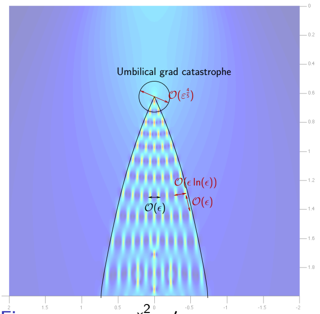



Figure: $A(x) = e^{-x^2}$, $\Phi'(x) = \tanh x$ and $\varepsilon = 0.03$

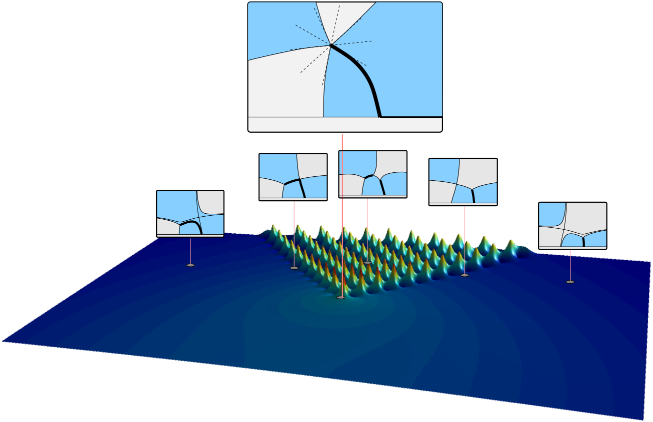


Figure: The physical $(x, t)$ plane and depiction of the fibration in spectral planes. In each spectral plane the negative sign regions of $\Im(h)$ are in blue. At the gradient catastrophe the topology is different.

The physical plane here plays the rôle of the $t$-plane in the previous case.

# The gradient catastrophe point

Separating amplitude and phase

$$q(x,t) = b(x,t)e^{\frac{i}{\varepsilon}\Phi(x,t)} \ , \quad U := |q|^2, V = \Phi_x \tag{60}$$

the equation is recast

$$U_t + (UV)_x = 0 \ , \qquad V_t + VV_x - U_x + \frac{\varepsilon^2}{2}\left(\frac{1}{2}\frac{U_x^2}{U^2} - \frac{U_{xx}}{U}\right)_x = 0 \tag{61}$$

Neglecting the green term yields an elliptic system, with a finite lifespan; they develop singularities in the derivatives at $(x_0, t_0)$.

What is the behavior in the vicinity of $(x_0, t_0)$?

# The gradient catastrophe point

Separating amplitude and phase

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Neglecting the green term yields an elliptic system, with a finite lifespan; they develop singularities in the derivatives at $(x_0, t_0)$.

## Conjecture (Dubrovin-Grava-Klein (2007))

*Let* $x = x_0 + \varepsilon^{\frac{4}{5}}X \ , \quad t = t_0 + \varepsilon^{\frac{4}{5}}T$*; then* $(b^2 = U, a = -2V, \alpha := a + ib)$

$$U + ib_0 V = b_0^2 + ib_0 a_0 + \varepsilon^{\frac{2}{5}}\frac{4ib_0}{C}y(v) + \mathcal{O}(\varepsilon^{\frac{3}{5}}) \tag{62}$$

*where*

$$v = -i\sqrt{\frac{2ib_0}{C}}\,(X + 2(\alpha_0 + a_0)T)\,(1 + \mathcal{O}(\varepsilon^{\frac{1}{5}})) \tag{63}$$

*and* $y(v)$ *is the* **tritronquée solution** *of the Painlevé I equation*

$$y'' = 6y^2 - v \tag{64}$$

# Zooming in on a peak (a.k.a. triple-scaling limit)

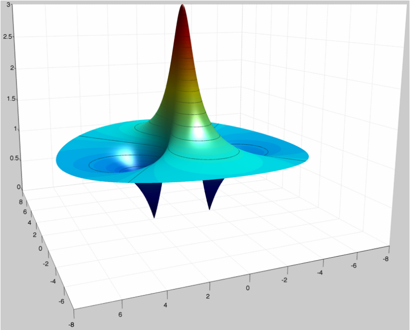
If we scale by $\varepsilon$ around each peak we find the **Peregrine breather**

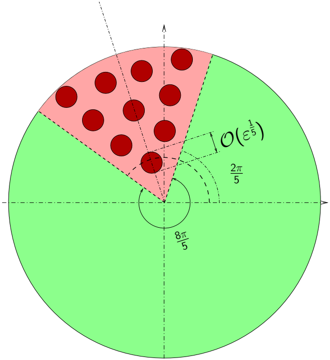
$$q(x,t,\mathrm{e}) = \mathrm{e}^{\frac{i}{\varepsilon}\Phi(x_p,t_p)} Q_{br}\left(\frac{x-x_p}{\varepsilon}, \frac{t-t_p}{\varepsilon}\right)(1+\mathcal{O}(\varepsilon^{\frac{1}{5}})), \tag{65}$$

where the rational breather

$$Q_{br}(\xi,\eta) = \mathrm{e}^{-2i\left(a\xi+(2a^2-b^2)\eta\right)} b\left(1 - 4\frac{1+4ib^2\eta}{1+4b^2(\xi+4a\eta)^2+16b^4\eta^2}\right) \tag{66}$$

$$i\partial_\eta Q_{br} + \partial_\xi^2 Q_{br} + 2|Q_{br}|^2 Q_{br} = 0 \tag{67}$$



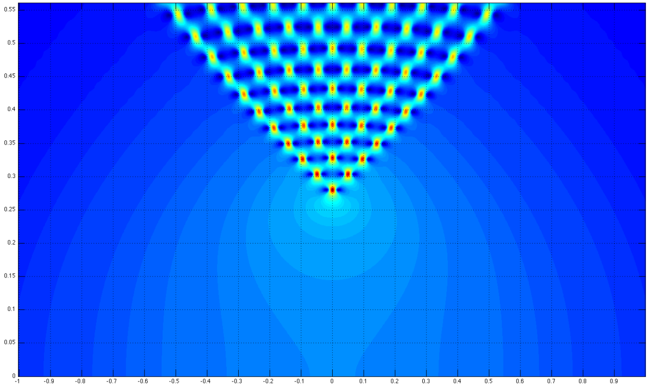


In our case it is obtained from the "stationary" breather

$$Q^0_{br}(\xi,\eta) = \mathrm{e}^{2i\eta}\left(1 - 4\frac{1+4i\eta}{1+4\xi^2+16\eta^2}\right) \tag{68}$$

by applying the transformations (mapping solutions into solutions)

$$\widetilde{Q}(\xi,\eta) = \lambda Q(\lambda\xi, \lambda^2\eta), \qquad \widehat{Q}(\xi,\eta) = \mathrm{e}^{i(kx-k^2\eta)}Q(\xi - 2k\eta, \eta). \tag{69}$$



Ideally these peaks will get very " sparse" near the gradient catastrophe:

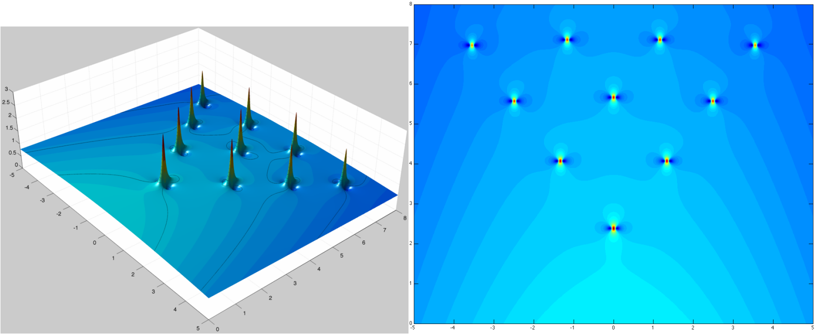


Figure: A mock-up of what would happen for very small $\varepsilon$ (location of peaks modeled after numerics for the poles of the tritronquée)

# Summarizing: B.-Tovbis (2010)

1. **poles of tritronquée** $\Leftrightarrow$ spikes of amplitude of $q$; can be used to find location in spacetime of the peaks after the grad. cat.;

$$v(x,t,\mathrm{e}) = \frac{e^{-i\pi/4}}{\varepsilon^{\frac{4}{5}}}\sqrt{\frac{2b}{C}}\,[\delta x + 2(2a+ib)\delta t]\left(1+\mathcal{O}(\varepsilon^{\frac{2}{5}})\right) \qquad (70)$$

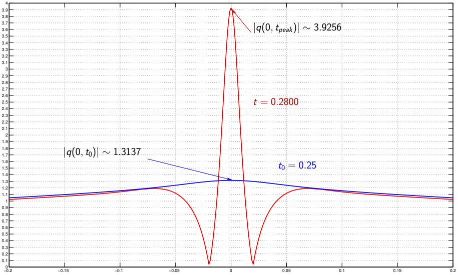



Figure: $q(x,0) = \frac{1}{\cosh(x)}$ and $\varepsilon = \frac{1}{33}$; note that $3|q_0| = 3.9411$. In this case $\mu = 0$ and $t_0 = \frac{1}{4}$. The time of the first peak (numerically 0.2800) matches the prediction from the Tritronquée (0.2791260482)

# Summarizing: B.-Tovbis (2010)

2. Height of each spike $= 3|q_0(x_0, t_0)| + \mathcal{O}(\varepsilon^{1/5})$;

# Summarizing: B.-Tovbis (2010)

3. **Universal shape**

$$q(x,t,\mathrm{e}) = \mathrm{e}^{\frac{i}{\varepsilon}\Phi(x_p,t_p)} Q_{br}\left(\frac{x-x_{p,j}}{\varepsilon}, \frac{t-t_{p,j}}{\varepsilon}\right)(1+\mathcal{O}(\varepsilon^{\frac{1}{5}})), \qquad (71)$$

The two "roots" and the maximum are synchronous. **This is a nonperturbative result, in the sense that it is beyond perturbation theory.**

4. Away from the spikes

$$q(x,t,\mathrm{e}) = \left(b - 2\varepsilon^{\frac{2}{5}}\Im\left(\frac{y(v)}{C}\right) + \mathcal{O}(\varepsilon^{\frac{3}{5}})\right)\times$$
$$\exp\frac{2i}{\varepsilon}\left[\frac{1}{2}\Phi(x_0,t_0) - \left(a\,\delta x - (2a^2-b^2)\delta t\right) + \varepsilon^{\frac{6}{5}}\Re\left(\sqrt{\frac{2i}{Cb}}H_I(v)\right)\right] \qquad (72)$$

$H_I = \frac{1}{2}(y'(v))^2 + vy(v) - 2y^3(v)$. Equation (72) is consistent with the conjecture.

# Final remarks

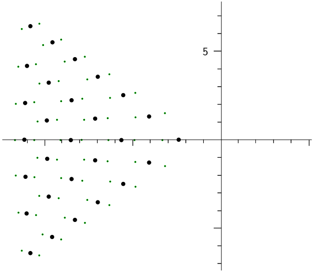



Figure: Numerics of poles (and zeroes in green) of the tritronquée based on Padé approximations following [Novokshenov]

- The main technical tool is the same, namely, the behavior of the Painlevé I parametrix near each of its poles: in this case only the tritronquée solution is relevant and since the spikes are in 1-1 correspondence with the poles, it is important (but a separate issue) to know whether there are any poles outside of the sector. It is a **conjecture** of Dubrovin et al. that there are none (well supported numerically).
- There should be a **quantization** of the possible gain factors of the peaks; 3 for Painlevé I, $5, 7, 9, \ldots$ for the higher degenerate cases (hierarchy of Painlevé I).

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