Tauberian properties for monomial asymptotic expansions

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Asymptotic Analysis and Borel summability in one variable

We have at our disposal a powerful summability theory useful in the study of solutions of analytic differential equations at singular points, solutions of difference equations, conjugacy of diffeomorphisms, singular perturbation problems among others.

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We have at our disposal a powerful summability theory useful in the study of solutions of analytic differential equations at singular points, solutions of difference equations, conjugacy of diffeomorphisms, singular perturbation problems among others.

- Asymptotic expansions, Gevrey asymptotic expansions, $k-$summability.
- Borel and Laplace transformations. Tauberian theorems.
- Ecalle's accelerator operators, Multisummability.

Monomial summability

The concept of *summability w.r.t. a monomial* was introduced and then generalized in the papers:

- Canalis-Durand M.,Mozo-Fernández J., Schäfke R.: *Monomial summability and doubly singular differential equations.* J. Differential Equations, vol. 233, (2007) 485-511.,
- Mozo-Fernádez J., Schäfke R.: *Asymptotic expansions and summability with respect to an analytic germ.* 2017. Available at arxiv.org/pdf/1610.01916v2.pdf

in order to study the formal solutions of the *doubly singular equation*

$$\varepsilon^q x^{p+1}\frac{d\boldsymbol{y}}{dx} = F(x,\varepsilon,\boldsymbol{y}).$$

The method combines the variables x and ε in the new one $t = x^p\varepsilon^q$, corresponding to the source of divergence of the solutions.

Formal setting

Let $\boldsymbol{x} = (x_1, \ldots, x_d)$ be coordinates of $\mathbb{C}^d$. We will work with formal power series in $\mathbb{C}[[\boldsymbol{x}]]$.

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We will restrict our attention to series $\hat{f} = \sum_{n=0}^{\infty} f_{n,j} x_j^n$ such that all $f_{n,j}$ have a common polyradius of convergence and are bounded for all $j = 1, \ldots, d$. Let $\mathcal{C}$ be the space of such series.

Given a monomial $\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_{>0}^d$ and $\hat{f} \in \mathcal{C}$ we can write it uniquely as

$$\hat{f}(\boldsymbol{x}) = \sum_{n=0}^{\infty} f_n(\boldsymbol{x}) \boldsymbol{x}^{n\boldsymbol{\alpha}},$$

where each $f_n \in \mathcal{E}_r^{\boldsymbol{\alpha}}$ (an adequate space of analytic functions, $r > 0$).

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where each $f_n \in \mathcal{E}_r^{\boldsymbol{\alpha}}$ (an adequate space of analytic functions, $r > 0$).

More precisely, $g \in \mathcal{E}_r^{\boldsymbol{\alpha}}$ if it is analytic at the polydisk at the origin with radius r and $\frac{\partial^{|\boldsymbol{\beta}|}}{\partial x^{\boldsymbol{\beta}}}(g)(\mathbf{0}) = 0$ for $\beta_j \geq \alpha_j$, $j = 1, \ldots, d$.

Gevrey series in a monomial

We may consider the linear map

$$\begin{aligned} \hat{T}_{\boldsymbol{\alpha}} : \mathcal{C} &\longrightarrow \left(\bigcup_{r>0} \mathcal{E}_r^{\boldsymbol{\alpha}} \right) [[t]], \\ \hat{f} &\longmapsto \sum_{n=0}^{\infty} f_n t^n. \end{aligned}$$

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Let $\mathbb{C}[[\boldsymbol{x}]]_s^{\boldsymbol{\alpha}}$ be the set of $s-$Gevrey series in the monomial $\boldsymbol{\alpha}$, i.e. series $\hat{f}$ such that for some $r>0$, $\hat{T}_{\boldsymbol{\alpha}}(\hat{f}) \in \mathcal{E}_r^{\boldsymbol{\alpha}}[[t]]$ and it is a $s-$Gevrey series in t.

Lemma

The series $\sum a_{\boldsymbol{\beta}} \boldsymbol{x}^{\boldsymbol{\beta}}$ *is* $s-$*Gevrey in the monomial* $\boldsymbol{x}^{\boldsymbol{\alpha}}$ *if and only if there are constants* $C, A > 0$ *satisfying*

$$|a_{\boldsymbol{\beta}}| \leq C A^{|\boldsymbol{\beta}|} \min\left\{\beta_1!^{s/\alpha_1}, \dots, \beta_d!^{s/\alpha_d}\right\}, \quad \boldsymbol{\beta} = (\beta_1, \dots, \beta_d) \in \mathbb{N}^d.$$

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Corollary

If $\hat{f} \in \mathbb{C}[[\boldsymbol{x}]]_s^{\boldsymbol{\alpha}'}$ *then* $\hat{T}_{\boldsymbol{\alpha}}(\hat{f})$ *is a* $\max_{1\leq j\leq d}\{\alpha_j/\alpha_j'\}s-$*Gevrey series in some* $\mathcal{E}_r^{\boldsymbol{\alpha}}$.

Analytic setting

A *sectors in the monomial* $\boldsymbol{x}^{\boldsymbol{\alpha}}$ is a set defined as

$$\begin{aligned}\Pi_{\boldsymbol{\alpha}}(a,b,r) &= S_{\boldsymbol{\alpha}}(d,b-a,r)\\ &= \left\{\boldsymbol{x}\in\mathbb{C}^d \mid a<\arg(\boldsymbol{x}^{\boldsymbol{\alpha}})<b,\ 0<|x_j|^{\alpha_j}<r, j=1,\dots,d\right\},\end{aligned}$$

where $a,b\in\mathbb{R}$ with $a<b$ and $r>0$. The number r is called the *radius*, $b-a$ the *opening* and $d=(b+a)/2$ the *bisecting direction* of the sector, respectively.

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where $a, b \in \mathbb{R}$ with $a < b$ and $r > 0$. The number r is called the *radius*, $b-a$ the *opening* and $d = (b+a)/2$ the *bisecting direction* of the sector, respectively.

If $\boldsymbol{x} \in \Pi_{\boldsymbol{\alpha}}(a,b,r)$ then
$t = \boldsymbol{x}^{\boldsymbol{\alpha}} \in V(a,b,r^d) := \{z \in \mathbb{C} \mid 0 < |z| < r^d,\ a < \arg(z) < b\}$.

Given a bounded function $f \in \mathcal{O}(\Pi_{\boldsymbol{\alpha}}(a, b, r))$, as in the formal case it is possible to construct an analytic map

$$T_{\boldsymbol{\alpha}}(f)_\rho : V(a, b, \rho^d) \longrightarrow \mathcal{E}_\rho^{\boldsymbol{\alpha}},$$

for all $\rho < r$, satisfying

$$T_{\boldsymbol{\alpha}}(f)_\rho(\boldsymbol{x}^{\boldsymbol{\alpha}})(\boldsymbol{x}) = f(\boldsymbol{x}).$$

In fact, f is completely determined by the map $T_{\boldsymbol{\alpha}}(f)_\rho$.

Asymptotic expansions in a monomial

Definition

Let $f \in \mathcal{O}(\Pi_{\boldsymbol{\alpha}})$, $\Pi_{\boldsymbol{\alpha}} = \Pi_{\boldsymbol{\alpha}}(a, b, r)$ and $\hat{f} \in \mathcal{C}$ with $\hat{T}_{\boldsymbol{\alpha}} \hat{f} = \sum f_n t^n \in \mathcal{E}_{r'}^{\boldsymbol{\alpha}}[[t]]$ for some $0 < r' \leq r$.

We say that f *has* $\hat{f}$ *as asymptotic expansion in* $\boldsymbol{x}^{\boldsymbol{\alpha}}$ *on* $\Pi_{\boldsymbol{\alpha}}$ ($f \sim^{\boldsymbol{\alpha}} \hat{f}$ on $\Pi_{\boldsymbol{\alpha}}$) if for every subsector $\widetilde{\Pi}_{\boldsymbol{\alpha}}$ and $N \in \mathbb{N}$ there is a constant $C_N > 0$ such that:

$$\left| f(\boldsymbol{x}) - \sum_{n=0}^{N-1} f_n(\boldsymbol{x}) \boldsymbol{x}^{n\boldsymbol{\alpha}} \right| \leq C_N |\boldsymbol{x}^{N\boldsymbol{\alpha}}|, \quad \boldsymbol{x} \in \widetilde{\Pi}_{\boldsymbol{\alpha}}. \tag{1}$$

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The asymptotic expansion is said to be of $s-$*Gevrey type* ($f \sim^{\boldsymbol{\alpha}}_s \hat{f}$ on $\Pi_{\boldsymbol{\alpha}}$) if it is possible to choose $C_N = CA^N N!^s$ for some C, A independent of N. In this case $\hat{f} \in \mathbb{C}[[\boldsymbol{x}]]^{\boldsymbol{\alpha}}_s$.

A characterization by passing to one variable

Proposition

Let $f \in \mathcal{O}(\Pi_{\boldsymbol{\alpha}})$ *be an analytic function* ($\Pi_{\boldsymbol{\alpha}} = \Pi_{\boldsymbol{\alpha}}(a, b, r)$), $\hat{f} \in \mathcal{C}$ *and* $0 < r' \leq r$ *such that* $\hat{T}_{\boldsymbol{\alpha}} \hat{f} \in \mathcal{E}^{\boldsymbol{\alpha}}_{r'}[[t]]$. *The following statements are equivalent:*

1. $f \sim^{\boldsymbol{\alpha}} \hat{f}$ *on* $\Pi_{\boldsymbol{\alpha}}$,
2. *For every* $0 < \rho < r'$, $T_{\boldsymbol{\alpha}}(f)_\rho \sim \hat{T}_{\boldsymbol{\alpha}}(\hat{f})$ *on* $V(a, b, \rho^d)$.

The same result is valid for asymptotic expansions of $s-$*Gevrey type.*

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This asymptotics behave well under usual algebraic operations and differentiation. In particular, if $f \sim^{\boldsymbol{\alpha}} \hat{f} = \sum a_{\boldsymbol{\beta}} \boldsymbol{x}^{\boldsymbol{\beta}}$ on $\Pi_{\boldsymbol{\alpha}}$ then

$$a_{\boldsymbol{\beta}} = \lim_{\substack{\boldsymbol{x} \to \boldsymbol{0} \\ \boldsymbol{x} \in \Pi'_{\boldsymbol{\alpha}}}} \frac{1}{\boldsymbol{\beta}!} \frac{\partial^{\boldsymbol{\beta}} f}{\partial \boldsymbol{x}^{\boldsymbol{\beta}}}(\boldsymbol{x}).$$

Monomial summability

Definition

Let $k>0$ and $\hat{f}\in\mathcal{C}$ be given. We say that $\hat{f}$ is $k-$*summable in the monomial* $\boldsymbol{x}^{\boldsymbol{\alpha}}$ *in the direction* d if there is a sector $\Pi_{\boldsymbol{\alpha}}(a,b,r)$ bisected by d with opening $b-a>\pi/k$ and $f\in\mathcal{O}(\Pi_{\boldsymbol{\alpha}}(a,b,r))$ with $f\sim^{\boldsymbol{\alpha}}_{1/k}\hat{f}$ on $\Pi_{\boldsymbol{\alpha}}(a,b,r)$.

We simply say that $\hat{f}$ is $k-$*summable in the monomial* $\boldsymbol{x}^{\boldsymbol{\alpha}}$ if it is $k-$summable in the monomial $\boldsymbol{x}^{\alpha}$ in every direction d, with finitely many exceptions mod. 2π.

- $\mathbb{C}\{\boldsymbol{x}\}^{\boldsymbol{\alpha}}_{1/k,d}$: $k-$summable series in $\boldsymbol{x}^{\boldsymbol{\alpha}}$ in the direction d,
- $\mathbb{C}\{\boldsymbol{x}\}^{\boldsymbol{\alpha}}_{1/k}$: $k-$summable series in $\boldsymbol{x}^{\boldsymbol{\alpha}}$.

Monomial Borel transform with weights

Definition

The $k-$*Borel transform w.r.t.* $\boldsymbol{x}^{\boldsymbol{\alpha}}$ *and a weight* $\boldsymbol{s} \in \sigma_d$, of a map f is defined by the formula

$$\mathcal{B}_{\boldsymbol{\lambda}}(f)(\boldsymbol{\xi}) = \frac{(\boldsymbol{\xi}^{k\boldsymbol{\alpha}})^{-1}}{2\pi i} \int_\gamma f\left(\xi_1 u^{-\frac{s_1}{\alpha_1 k}}, \ldots, \xi_d u^{-\frac{s_d}{\alpha_d k}}\right) e^u du,$$

where $\boldsymbol{\lambda} = \left(\frac{s_1}{\alpha_1 k}, \ldots, \frac{s_d}{\alpha_d k}\right)$ and γ denotes a Hankel's path.

Here and below, $\sigma_d = \{(s_1, \ldots, s_d) \in \mathbb{R}^d_{>0} \mid s_1 + \cdots + s_d = 1\}$.

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- Balser W.: *Summability of power series in several variables, with applications to singular perturbation problems and partial differential equations*. Ann. Fac. Sci. Toulouse Math, vol. XIV, n°4 (2005) 593-608.
- Balser W., Mozo-Fernández J.: *Multisummability of Formal Solutions of Singular Perturbation Problems*. J. Differential Equations, vol. 183, (2002) 526-545.

The *formal* $k-$*Borel transform associated to the monomial* $\boldsymbol{x}^{\boldsymbol{\alpha}}$ *with weight* $\boldsymbol{s}$ is thus defined term-by-term by

$$\begin{aligned}\hat{\mathcal{B}}_{\boldsymbol{\lambda}} : \boldsymbol{x}^{k\boldsymbol{\alpha}}\mathbb{C}[[\boldsymbol{x}]] &\longrightarrow \mathbb{C}[[\boldsymbol{\xi}]]\\ \boldsymbol{x}^{k\boldsymbol{\alpha}+\boldsymbol{\beta}} &\longmapsto \frac{\boldsymbol{\xi}^{\boldsymbol{\beta}}}{\Gamma\left(1+\frac{\beta_1 s_1}{\alpha_1 k}+\cdots+\frac{\beta_d s_d}{\alpha_d k}\right)}.\end{aligned}$$

Monomial Laplace transform with weights

Definition

The $k-$*Laplace transform w.r.t.* $\boldsymbol{x}^{\boldsymbol{\alpha}}$ *with weight* $\boldsymbol{s} \in \sigma_d$ *in direction* θ, $|\theta| < \pi/2$, of a function f is defined by the formula

$$\mathcal{L}_{\boldsymbol{\lambda},\theta}(f)(\boldsymbol{x}) = \boldsymbol{x}^{k\boldsymbol{\alpha}} \int_0^{e^{i\theta}\infty} f\left(x_1 u^{\frac{s_1}{\alpha_1 k}}, \dots, x_d u^{\frac{s_d}{\alpha_d k}}\right) e^{-u} du.$$

We assume that f has an exponential growth of the form

$$|f(\boldsymbol{\xi})| \leq C \exp\left(B \max\{|\xi_1|^{\frac{\alpha_1 k}{s_1}}, \dots, |\xi_d|^{\frac{\alpha_d k}{s_d}}\}\right). \tag{2}$$

Monomial Borel-Laplace summation methods

Definition

Let $\hat{f}$ be a $1/k-$Gevrey series in $\boldsymbol{x}^{\boldsymbol{\alpha}}$, $\boldsymbol{s} \in \sigma_d$ and d a direction. We will say that $\hat{f}$ is $k - \boldsymbol{s}-$*Borel summable in the monomial* $\boldsymbol{x}^{\boldsymbol{\alpha}}$ *in direction* d if:

1. $\hat{\varphi}_{\boldsymbol{s}} = \hat{\mathcal{B}}_{\boldsymbol{\lambda}}(\boldsymbol{x}^{k\boldsymbol{\alpha}}\hat{f})$, $\boldsymbol{\lambda} = \left(\frac{s_1}{\alpha_1 k}, \dots, \frac{s_d}{\alpha_d k}\right)$, can be analytically continued, say as $\varphi_{\boldsymbol{s}}$, to a monomial sector of the form $S_{\boldsymbol{\alpha}}(d, 2\epsilon)$.
2. $\varphi_{\boldsymbol{s}}$ has exponential growth as in (2).

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2. $\varphi_{\boldsymbol{s}}$ has exponential growth as in (2).

In this case the *$k - \boldsymbol{s}-$Borel sum of* $\hat{f}$ in direction d is defined as

$$f(\boldsymbol{x}) = \frac{1}{\boldsymbol{x}^{k\boldsymbol{\alpha}}}\mathcal{L}_{\boldsymbol{\lambda}}(\varphi_{\boldsymbol{s}})(\boldsymbol{x}).$$

Monomial summability and Borel-Laplace method

Theorem

Let $\hat{f}$ be a $1/k-$Gevrey series in the monomial $\boldsymbol{x}^{\boldsymbol{\alpha}}$. Then it is equivalent:

1. $\hat{f} \in \mathbb{C}\{\boldsymbol{x}\}^{\boldsymbol{\alpha}}_{1/k,d}$, *i.e.* $\hat{f}$ *is* $k-$*summable in* $\boldsymbol{x}^{\boldsymbol{\alpha}}$ *in direction* d.
2. *There is* $\boldsymbol{s} \in \sigma_d$ *such that* $\hat{f}$ *is* $k-\boldsymbol{s}-$*Borel summable in the monomial* $\boldsymbol{x}^{\boldsymbol{\alpha}}$ *in direction* d.
3. *For all* $\boldsymbol{s} \in \sigma_d$, $\hat{f}$ *is* $k-\boldsymbol{s}-$*Borel summable in the monomial* $\boldsymbol{x}^{\boldsymbol{\alpha}}$ *in direction* d.

In all cases the corresponding sums coincide.

Monomial summability and blow-ups

Consider the monomial transformations

$$\pi_{ij}(x_1, \ldots, x_d) = (x_1, \ldots, \underbrace{x_i x_j}_{j \text{ position}}, \ldots, x_d), \quad i, j = 1, \ldots, d.$$

Lemma

1. $\hat{f} \in \mathbb{C}\{\boldsymbol{x}\}$ *if and only if* $\hat{f} \circ \pi_{ij} \in \mathbb{C}\{\boldsymbol{x}\}$ *for some* $i, j = 1, \ldots, d$.
2. $\hat{f} \in \mathbb{C}[[\boldsymbol{x}]]_s^{\boldsymbol{\alpha}}$ *if and only if there are* $i, j = 1, ..., d$, $i \neq j$ *such that* $\hat{f} \circ \pi_{ij} \in \mathbb{C}[[\boldsymbol{x}]]_s^{\boldsymbol{\alpha} + \alpha_j \boldsymbol{e}_i}$ *and* $\hat{f} \circ \pi_{ji} \in \mathbb{C}[[\boldsymbol{x}]]_s^{\boldsymbol{\alpha} + \alpha_i \boldsymbol{e}_j}$.

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2. $\hat{f}\in\mathbb{C}[[\boldsymbol{x}]]_s^{\boldsymbol{\alpha}}$ *if and only if there are* $i,j=1,...,d$, $i\neq j$ *such that* $\hat{f}\circ\pi_{ij}\in\mathbb{C}[[\boldsymbol{x}]]_s^{\boldsymbol{\alpha}+\alpha_j\boldsymbol{e}_i}$ *and* $\hat{f}\circ\pi_{ji}\in\mathbb{C}[[\boldsymbol{x}]]_s^{\boldsymbol{\alpha}+\alpha_i\boldsymbol{e}_j}$.

Proposition

If $\hat{f}\in\mathbb{C}\{\boldsymbol{x}\}_{1/k,d}^{\boldsymbol{\alpha}}$ *has* $k-$*sum* f *in direction* d *then* $\hat{f}\circ\pi_{ij}\in\mathbb{C}\{\boldsymbol{x}\}_{1/k,d}^{\boldsymbol{\alpha}+\alpha_j\boldsymbol{e}_i}$ *and have* $k-$*sum* $f\circ\pi_{ij}$ *in direction* d, *for all* $i,j=1,\ldots,d$, $i\neq j$.

Tauberian properties for monomial summability

Proposition

If $\hat{f} \in \mathbb{C}\{\boldsymbol{x}\}_{1/k}^{\boldsymbol{\alpha}}$ *has no singular directions then* $\hat{f}$ *is convergent.*

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Theorem

Let $\boldsymbol{x}^{\boldsymbol{\alpha}}$ *and* $\boldsymbol{x}^{\boldsymbol{\alpha}'}$ *be two monomials and* $k, l > 0$. *The following statements hold:*

1. *If* $\max_{1\leq j\leq d}\{\alpha_j/\alpha_j'\} < 1/k/1/l$ *then* $\mathbb{C}\{\boldsymbol{x}\}^{\boldsymbol{\alpha}}_{1/k} \cap \mathbb{C}[[\boldsymbol{x}]]^{\boldsymbol{\alpha}'}_{1/l} = \mathbb{C}\{\boldsymbol{x}\}$.
2. $\mathbb{C}\{\boldsymbol{x}\}^{\boldsymbol{\alpha}}_{1/k} \cap \mathbb{C}\{\boldsymbol{x}\}^{\boldsymbol{\alpha}'}_{1/l} = \mathbb{C}\{\boldsymbol{x}\}$, *except in the case* $\alpha_j/\alpha_j' = l/k$ *for all* $j = 1, \ldots, d$ *where* $\mathbb{C}\{\boldsymbol{x}\}^{\boldsymbol{\alpha}}_{1/k} = \mathbb{C}\{\boldsymbol{x}\}^{\boldsymbol{\alpha}'}_{1/l}$.

Applications

Singular linear ODEs

Theorem

Consider the singularly perturbed differential equation

$$\boldsymbol{\varepsilon}^{\boldsymbol{\alpha}} x^{p+1} \frac{\partial \boldsymbol{y}}{\partial x} = A(x, \boldsymbol{\varepsilon})\boldsymbol{y} + b(x, \boldsymbol{\varepsilon}),$$

where $\boldsymbol{y} \in \mathbb{C}^l$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_d)$, $\boldsymbol{\alpha} \in \mathbb{N}^d_{>0}$ *and* A *and* b *are analytic in a neighborhood of* $(0, \mathbf{0})$.

If $A(0, \mathbf{0})$ *is invertible then the previous equation has a unique formal solution* $\hat{\boldsymbol{y}}$. *Furthermore it is* $1-$*summable in* $x^p \boldsymbol{\varepsilon}^{\boldsymbol{\alpha}}$.

The induced vector field by the Borel transform

Consider the vector field $X_{\boldsymbol{\lambda}}$ given by

$$X_{\boldsymbol{\lambda}} = \frac{\boldsymbol{x}^{k\boldsymbol{\alpha}}}{k}\left(\frac{s_1}{\alpha_1}x_1\frac{\partial}{\partial x_1} + \cdots + \frac{s_d}{\alpha_d}x_d\frac{\partial}{\partial x_d}\right).$$

If $f \in \mathcal{O}_b(S_{\boldsymbol{\alpha}})$ then

$$\mathcal{B}_{\boldsymbol{\lambda}}(X_{\boldsymbol{\lambda}}(f))(\boldsymbol{\xi}) = \boldsymbol{\xi}^{k\boldsymbol{\alpha}}\mathcal{B}_{\boldsymbol{\lambda}}(f)(\boldsymbol{\xi}).$$

Monomial summability of a family of PDEs

Consider the problem

$$\boldsymbol{x}^{\boldsymbol{\alpha}}\left(\frac{s_1}{\alpha_1}x_1\frac{\partial \boldsymbol{y}}{\partial x_1}+\cdots+\frac{s_d}{\alpha_d}x_d\frac{\partial \boldsymbol{y}}{\partial x_d}\right)=C(\boldsymbol{x})\boldsymbol{y}+b(\boldsymbol{x}),$$

where $\boldsymbol{\alpha}\in\mathbb{N}_{>0}^d$, $(s_1,\ldots,s_d)\in\sigma_d$ and C,b analytic at $\mathbf{0}\in\mathbb{C}^d$.

Theorem

If $C(\mathbf{0})$ is invertible then the previous equation has a unique formal solution $\hat{\boldsymbol{y}}$ and it is $1-$summable in $\boldsymbol{x}^{\boldsymbol{\alpha}}$.

Pfaffian system with normal crossings

Consider the following the system of PDEs:

$$\begin{cases} x_2^q x_1^{p+1} \dfrac{\partial \boldsymbol{y}}{\partial x_1} = f_1(x_1, x_2, \boldsymbol{y}), & \text{(3a)} \\ x_1^{p'} x_2^{q'+1} \dfrac{\partial \boldsymbol{y}}{\partial x_2} = f_2(x_1, x_2, \boldsymbol{y}), & \text{(3b)} \end{cases}$$

where $p, q, p', q' \in \mathbb{N}^*$, $\boldsymbol{y} \in \mathbb{C}^l$, and f_1, f_2 are analytic in a neighborhood of $(0, 0, \mathbf{0})$.

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where $p, q, p', q' \in \mathbb{N}^*$, $\boldsymbol{y} \in \mathbb{C}^l$, and f_1, f_2 are analytic in a neighborhood of $(0, 0, \mathbf{0})$.

It is called *completely integrable* if $f_1(x_1, x_2, \mathbf{0}) = f_2(x_1, x_2, \mathbf{0}) = \mathbf{0}$ and the functions f_1, f_2 satisfy the following identity on their domains of definition:

$$\frac{\partial}{\partial x_2}\left(\frac{1}{x_1^{p+1} x_2^q}\right) f_1 + \frac{1}{x_1^{p+1} x_2^q}\left(\frac{\partial f_1}{\partial x_2} + \frac{\partial f_1}{\partial \boldsymbol{y}} \frac{f_2}{x_1^{p'} x_2^{q'+1}}\right) =$$

$$\frac{\partial}{\partial x_1}\left(\frac{1}{x_1^{p'} x_2^{q'+1}}\right) f_2 + \frac{1}{x_1^{p'} x_2^{q'+1}}\left(\frac{\partial f_2}{\partial x_1} + \frac{\partial f_2}{\partial \boldsymbol{y}} \frac{f_1}{x_1^{p+1} x_2^q}\right).$$

If the system is completely integrable, $f_1 = Ay + h.o.t.$ and $f_2 = By + h.o.t.$ then A and B satisfy

$$x_1^{p'} x_2^{q'} \left(x_2 \frac{\partial A}{\partial x_2} - qA \right) - x_1^p x_2^q \left(x_1 \frac{\partial B}{\partial x_1} - p' B \right) + AB - BA = 0.$$

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From this equation we have deduced that:

1. If $p' < p$ or $q' < q$ then $A(0,0)$ is nilpotent.
2. If $p < p'$ or $q < q'$ then $B(0,0)$ is nilpotent.
3. If $p = p'$ and $q = q'$, for every eigenvalue μ of $B(0,0)$ there is an eigenvalue λ of $A(0,0)$ such that $q\lambda = p\mu$. The number λ is an eigenvalue of $A(0,0)$, when restricted to its invariant subspace $E_\mu = \{v \in \mathbb{C}^n | (B(0,0) - \mu I)^k v = 0$ for some $k \in \mathbb{N}\}$.

Convergence of solutions for different monomials

Theorem (Gérard-Sibuya)

Consider the completely integrable Pffafian system (3a), (3b), with $q = p' = 0$. *If* $\frac{\partial f_1}{\partial \boldsymbol{y}}(0,0,\mathbf{0})$ *and* $\frac{\partial f_2}{\partial \boldsymbol{y}}(0,0,\mathbf{0})$ *are invertible then the Pfaffian system admits a unique analytic solution* y *at the origin such that* $\boldsymbol{y}(0,0) = \mathbf{0}$.

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Theorem

Consider the system (3a), (3b). Suppose the system has a formal solution $\hat{\boldsymbol{y}}$. If $\frac{\partial f_1}{\partial \boldsymbol{y}}(0, 0, \mathbf{0})$ and $\frac{\partial f_2}{\partial \boldsymbol{y}}(0, 0, \mathbf{0})$ are invertible and $x_1^p x_2^q \neq x_1^{p'} x_2^{q'}$ then $\hat{\boldsymbol{y}}$ is convergent.

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Thanks for your attention.