Real solutions of Painlevé VI and special pentagons

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Introduction

Richard Fuchs was a son of Lazarus Fuchs. Father Fuchs is famous for Fuchsian groups, and several (at least three different kinds of) "Fuchs conditions" in the analytic theory of differential equations.

Richard studied in 1905 the following differential equation:

$$w'' - \left(\frac{1}{z-q} + \sum_{j=1}^{3} \frac{\kappa_j - 1}{z - t_j} \right) w' + \left(\frac{p}{z-q} - \sum_{j=1}^{3} \frac{h_j}{z - t_j} \right) w = 0$$

with 5 singularities at $(t_1, t_2, t_3, t_4, q) := (0, 1, x, \infty, q)$.
The singularities at t_j have exponents $\{0, \kappa_j\}$, for $1 \leq j \leq 3$, and the exponents at q are $\{0, 2\}$.
R. Fuchs imposed the following conditions:
a) the singularity at ∞ is regular, and has exponent difference κ_4, and
b) the singularity at q is apparent (has trivial monodromy).
For given $\kappa_j,\ 1 \leq j \leq 4$, and given $p, q, x = t_3$, these conditions determine the rest of the parameters h_j uniquely.

Suppose that all κ_j are fixed, and let us move x continuously. *How should $p(x), q(x)$ change so that the monodromy of this equation remains unchanged?*
Answer: q must satisfy the following non-linear ODE:

$$q_{xx} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-x}\right) q_x^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{q-x}\right) q_x$$
$$+ \frac{q(q-1)(q-x)}{2x^2(x-1)^2} \left\{\kappa_4^2 - \kappa_1^2 \frac{x}{q^2} + \kappa_2^2 \frac{x-1}{(q-1)^2} + (1-\kappa_3^2) \frac{x(x-1)}{(q-x)^2}\right\},$$

which is called Painlevé VI.

This was the first example of "isomonodromic deformation".
All solutions are meromorphic in $\mathbf{C}\setminus\{0,1\}$. The points $x = 0, 1, \infty$ are *fixed singularities* of PVI.
The conditions of Cauchy's theorem are violated at the points where $q(x) \in \{0, 1, x, \infty\}$. These points x are removable singularities or poles. We call them *special points*.
We consider *real* solutions $q(x)$ of PVI with *real* parameters, on an interval of the real line between two adjacent fixed singularities $0, 1, \infty$. WLOG we choose the interval $(1, \infty)$.
We will explain a geometric interpretation of these solutions, and obtain an algorithm which determines the number and mutual position of special points on the interval.
More precisely, the outcome of the algorithm is a sequence of symbols $\{0, 1, x, \infty\}$ which shows the order in which the special points appear on $(1, \infty)$.

How do we select a particular solution $q(x)$? There are the following methods:
a) one can solve the Cauchy problem with some non-special initial conditions $q(x_0) = q_0,\ q'(x_0) = q'_0$.
b) one can specify the second order linear equation with 5 singularities.
c) one can specify the monodromy representation corresponding to the linear equation. At least for generic values of PVI parameters and generic monodromies this specifies the linear equation (that is p_0 and q_0) uniquely.
We will use somewhat different method of assigning the initial conditions, and monodromy representation will be easily computed from our initial conditions. (Initial values of p_0, q_0 are difficult to compute directly from the monodromy).

Linear Fuchsian ODE with real parameters and circular polygons

Suppose that all parameters (singularities, exponents and accessory parameters) in a linear Fuchsian ODE

$$w'' + P(z)w' + Q(z)w = 0$$

are real. *Such equations will be called real.* Consider the ratio $f = w_1/w_2$ of two linearly independent solutions. This function is meromorphic in the upper half-plane H and is locally univalent there. Aat a singular point t we have:

$$f(z) = f(t) + (c + o(1))(z - t)^{\alpha},$$

where α is the absolute value of the exponent difference at t. If $\alpha = 0$ but the singularity at t is not apparent, then

$$f(z) = f(t) + (c + o(1))/\log(z - t).$$

Notice that we measure all angles in half-turns instead of the radians!

Function f is holomorphic in H, locally univalent in $\overline{H}\backslash\{t_j\}$, maps each interval (t_{j-1}, t_j) into some circle C_j, and has conical singularities at t_j.
Such functions are called *developing maps* (of circular polygons).
The formal definition of a circular polygon is

$$Q = (\overline{D}, t_1, \ldots, t_n, f),$$

where $\overline{D}$ is a closed disk, $t_j \in \partial D$ are distinct boundary points, and f is a developing map with conical singularities at t_j. The intervals (t_{j-1}, t_j) are called *sides*, the points t_j *corners* and the α_j are the interior angles at the corners.

Two circular n-gons $Q = (\overline{D}, t_1, \ldots, t_n, f)$ and $Q' = (\overline{D'}, t'_1, \ldots, t'_n, f_1)$ are *equal* if there is a conformal homeomorphism $\phi : \overline{D'} \to \overline{D}$ such that $\phi(t'_j) = t_j$ and

$$f_1 = f \circ \phi. \tag{1}$$

Two circular n-gons are called *equivalent* if instead of (1) we require only $f_1 = L \circ f \circ \phi$, with some linear-fractional transformation L. For polygons which are subsets of the sphere this means that one can be moved onto another by a linear-fractional transformation.

There is a one-to-one correspondence between the equivalence classes of circular n-gons and normalized Fuchsian equations with all parameters real. The developing map defining a polygon is the ratio of two linearly independent solutions.

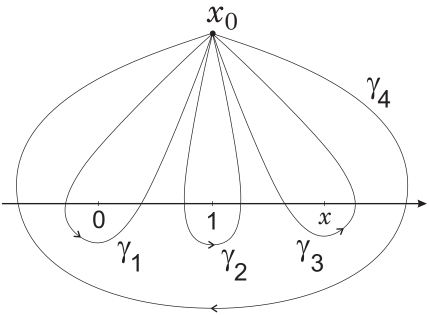
Of course, this fact was well-known to Schwarz and possibly to Riemann.

ÉMILE PICARD

Special pentagons corresponding to the equation of R. Fuchs

This equation with five singularities defines a circular pentagon. But one singularity q is special: it has exponents $0, 2$ and trivial monodromy.
We say in such case that our pentagon Q *has a slit*, and call $f(q)$ the *tip of the slit*. Pentagons of this type (with exactly one slit) will be called *special pentagons*.
There is a one-to-one correspondence between real normalized Fuchsian equations with 5 singularities, one of them apparent with exponent difference 2*, and equivalence classes of special pentagons.*

The sides of a special pentagon are mapped by f into 4 circles. Let C_j be the circle that contains $f([t_{j-1}, t_j])$. If $q \in (t_{k-1}, t_k)$ then two sides (t_{k-1}, q) and (q, t_k) are mapped into the same circle C_k. The intersections $C_j \cap C_{j+1}$ are not empty, they contain $f(t_j)$; and our assumption of non-trivial monodromy implies that $C_j \neq C_{j+1}$ for all $j \in \mathbf{Z}_4$.



$$\gamma_1\gamma_2\gamma_3\gamma_\infty = e,$$

$$f^\gamma = L_\gamma \circ f.$$

$$L_1 \circ L_2 \circ L_3 \circ L_\infty = \mathrm{id}.$$

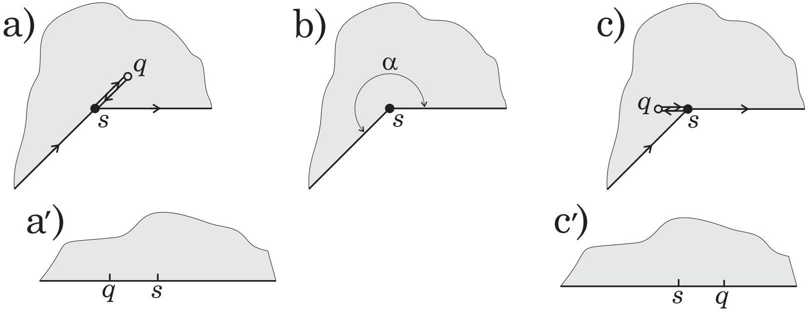
The monodromy transformation L_j at t_j is obtained by the formula

$$L_j = \sigma_j \sigma_{j+1}, \qquad (2)$$

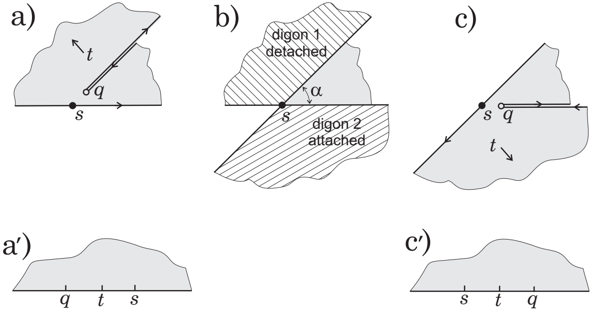
where σ_j is the reflection in the circle C_j. Indeed to perform an analytic continuation along γ_j we first continue f analytically to the lower half-plane by reflection σ_j, and then continue back to H by reflection in the circle $\sigma_j C_{j+1}$. This last reflection is $\sigma_j \sigma_{j+1} \sigma_j$ and applying it after σ_j we obtain

$$\sigma_j \sigma_{j+1} \sigma_j \sigma_j = \sigma_j \sigma_{j+1}.$$

One can show that if a representation (2) of L_j in terms of σ_j exists, then the σ_j are unique, except in the case that all L_j commute. One can also write explicit conditions on the L_j which imply existence of representation (2).

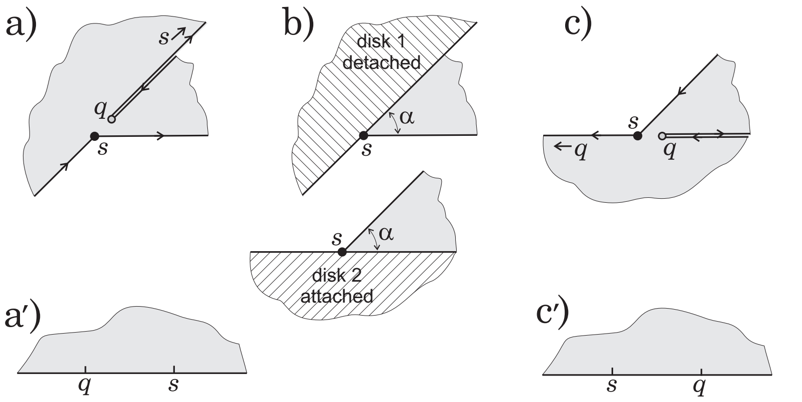


Transformation 1. Let $q \in (t_{k-1}, t_k)$, and as the slit vanishes, q collides with s which is either t_k or t_{k-1}. When $q = s$, we have a quadrilateral without a slit. As x passes s we must have a special pentagon with images of the sides on the same 4 circles, but q and s interchanged their order on ∂H, The slit which was on C_k is now on C_{k+1}, if $s = t_k$, and on C_{k-1} if $s = t_{k-1}$.



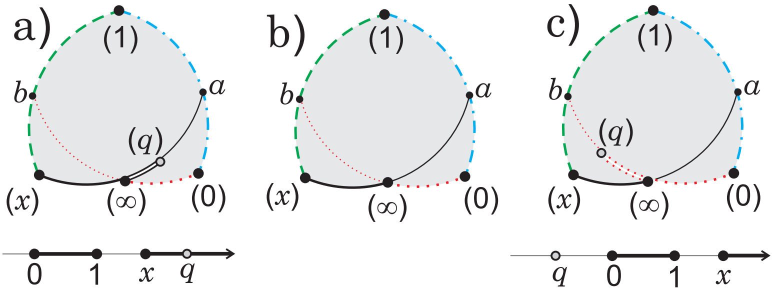
Transformation 2. Suppose that $q \in (t_{k-1}, t_k)$, and the slit lengthens. Then eventually it hits the boundary from inside of Q_x, and becomes a cross-cut. The cross-cut splits the pentagon into two parts. Let $s \in \partial H$ be the point where this collision happens (that is $f(q) \to f(s)$ as the slit lengthens).

The part which splits away is a digon with corners at t and s. This digon is detached in the limit. In the z plane all three points q, t, s collide. Before this collision, it is a small neighborhood of t which is mapped on the would-be digon.
After the collision, we have $s < t < q$, and a new digon is attached. It is easy to see that this new digon has the same angle as the old one, and is bounded by the same two circles. We call it the "vertical digon" to the old one.



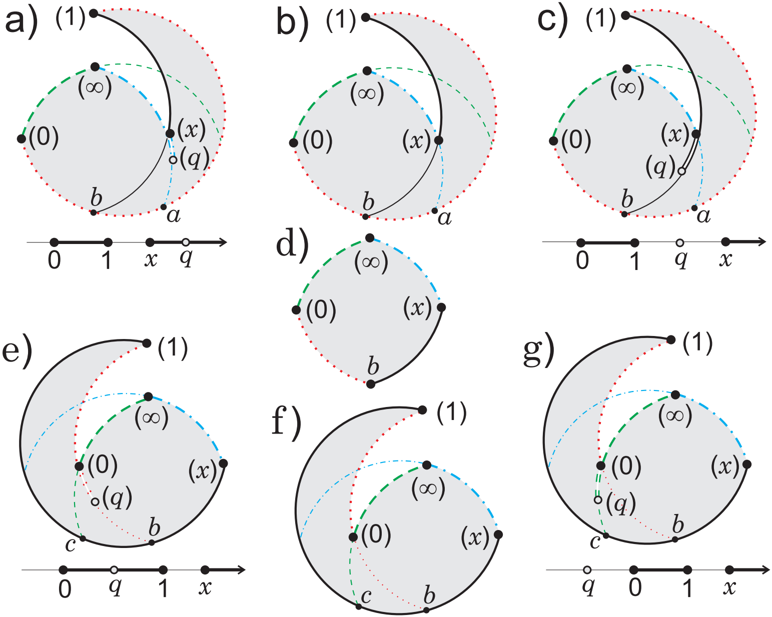
Transformation 3. When the slit lengthens, hits the boundary from inside, and the special pentagon splits, as in Transformation 2, we assume now that the slit hits a corner $s \in \{t_j\}$.

Examples



In Example 1, when the slit in a) lengthens and hits the boundary, we have $x \to 1$. As the slit in a) shortens, x increases. When the slit vanishes we obtain the limit quadrilateral in picture b); at this point $q(x) = \infty$. Then the new slit grows as in picture c) and when it hits the boundary, $x \to +\infty$. Therefore the solution $q(x)$ has only one special point on $(1, +\infty)$, and it is a pole. We had one transformation of type 1 in this example.

In Example 2, when the slit in a) lengthens, x decreases and $x \to 1$ as the slit hits the boundary. When the slit in a) shortens, x increases. Then we have transformation 1 in b), transformation 2 in d) and transformation 1 in f). The solution has 3 special points: $x_0 < x_1 < x_2$ with $x_0 > 1,\ q(x_0) = x_0,\ q(x_1) = 1,\ q(x_2) = 0.$



a)
(1)
(∞)
(0)
(x)
(q)
b
a
0
1
x
q
b)
(1)
(∞)
(0)
(x)
b
a
c)
(1)
(∞)
(0)
(x)
(q)
b
a
0
1
q
x
d)
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c
b
q
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1
x

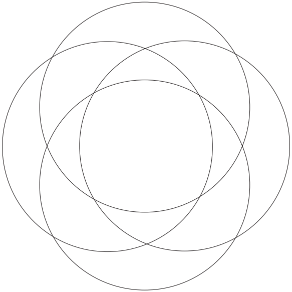
Representation of polygons by nets

Let $Q = (\overline{D}, t_1, \ldots, t_n, f)$ be a circular n-gon. C_j is the circle containing $f((t_{j-1}, t_j))$. Circles C_j define a cell decomposition of the sphere which we call the *lower configuration*. The f-preimage of the lower configuration is a cell decomposition of the closed disk $\overline{D}$ which is called the *net* of our polygon. Vertices of the net at the corners are labeled by t_j.

Two nets are considered the same if there is an orientation-preserving homeomorphism of $\overline{D}$ sending one net to another and labeled vertices to similarly labeled vertices.
Specifying the cells

$$(f(t_1),\ f(e),\ f(T))$$

of the lower configuration will define the polygon uniquely.
So a polygon is completely determined by the lower configuration, the net and the normalization data.
It is difficult to describe intrinsically all possible nets on a given lower configuration. But in the case when $n = 4$ and the lower configuration is homeomorphic to a generic quadruple of great circles, one can give such an intrinsic description.



The corresponding cell decomposition of the sphere has the following property:

a) *any pair of* 2*-cells adjacent along a* 1*-cell consists of a triangle and a quadrilateral.*

This property is inherited by the net. Two additional property of the net are:

b) *every interior vertex has degree* 4, *and every vertex on a side has degree 3*, and
c) *the degrees of the corners (as vertices of the net) are even.*
The last property follows from our assumption that the circles C_j and C_{j+1} are distinct.
One can show that these three properties a), b) and c) characterize the nets over lower configurations homeomorphic to generic configurations of 4 great circles.
This permits to construct many examples of nets, circular quadrilaterals and special circular pentagons.

Transformations 1, 2, 3 above can be explicitly performed on the nets.
Lower configurations of four great circles correspond to $PSU(2)$ monodromy representations.
Properties of special points of real PVI solutions strongly depend on the topological type of the lower configuration. For example, the number of special points can be infinite only if some two circles of the lower configuration are disjoint. We conjecture that this condition is also sufficient for a PVI solution to have infinitely many real special points.