Meromorphic solutions of $P_{4,34}$ and their value distribution

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The basics of Nevanlinna theory: notation, examples and main theorems

$f(z)$: meromorphic function

$T(r,f)$: characteristic function given by

$$T(r,f) = m(r,f) + N(r,f),$$

where

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^i\theta)| d\theta, \ \ \log^+ x = \max(0, \log x), \ x > 0,$$

is a proximity function and

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t} dt + n(0,f)\log r$$

is an integrated counting function. Here $n(r,f)$ is the number of poles of $f(z)$ counting multiplicities in $|z| < r$.

One can further define $m(r,a) = m(r, 1/(f-a)), \ \ N(r,a) = N(r, 1/(f-a))$ for a-points of $f(z)$.

$$\rho(f) := \limsup \frac{\log T(r,f)}{\log r}$$

is the order of growth of $f(z)$.

Examples.

- $f(z) = e^z$ has order 1, $f(z) = e^{e^z}$ has infinite order.
- f is rational iff $T(r,f) = O(\log r)$ as $r \to \infty$.

The first main theorem:

$$T(r,f) = T(r, 1/(f-a)) + O(1).$$

(The characteristic function does not depend on the value a).

Notation: $S(r,f) := o(T(r,f))$

Lemma on the logarithmic derivative:

$$m(r, f'/f) = S(r,f).$$

The defect $\delta(a,f)$ of f at a value $a \in \overline{\mathbb{C}}$ is defined by

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r,a,f)}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r,a,f)}{T(r,f)}.$$

The index of multiplicity $\vartheta(a,f)$ of a value a is defined by

$$\vartheta(a,f) = \liminf_{r\to\infty} \frac{N_1(r,a,f)}{T(r,f)},$$

where $N_1(r,a,f) := N(r,a,f) - \overline{N}(r,a,f)$.

If $\delta(a,f) > 0$, then we say that the value a is defective (in the sense of Nevanlinna), and if $\vartheta(a,f) > 0$ we call a a ramified value of f.

It is known that the set $E_N(f)$ of defective values of a meromorphic function f is at most countable and the following relations are true:

$$0 \le \delta(a,f) + \vartheta(a,f) \le 1,$$

$$\sum_{a\in\overline{\mathbb{C}}} (\delta(a,f) + \vartheta(a,f)) \le 2.$$

Petrenko's theory

In 1969 Petrenko introduced the quantity

$$\beta(a,f) = \liminf_{r\to\infty} \frac{\mathcal{L}(r,a,f)}{T(r,f)}$$

called deviation of a meromorphic function with respect to the value $a \in \overline{\mathbb{C}}$, where

$$\mathcal{L}(r,a,f) := \begin{cases} \max\limits_{|z|=r} \log^+ |f(z)| & \text{for} \quad a = \infty, \\ \max\limits_{|z|=r} \log^+ \left|\frac{1}{f(z)-a}\right| & \text{for} \quad a \neq \infty. \end{cases}$$

For $a \in \overline{\mathbb{C}}$ the inequality

$$\delta(a,f) \leq \beta(a,f)$$

follows easily from the definition of $\beta(a,f)$. Thus we have $E_N(f) \subset E_\Pi(f)$, where $E_\Pi(f) : \{a \in \overline{\mathbb{C}} : \ \beta(a,f) > 0\}$. In general the sets $E_N(f)$ and $E_\Pi(f)$ may differ.

Under certain assumptions on the order of growth of the function f, the set $E_{\Pi}(f)$ of exceptional values in the sense of Petrenko is at most countable and

$$\beta(a,f) \le B(\mu) := \begin{cases} \frac{\pi\mu}{\sin \pi\mu} & \text{if} \quad \mu \le 0.5\,, \\ \pi\mu & \text{if} \quad \mu > 0.5\,. \end{cases}$$

Marchenko and Shcherba proved that

$$\sum_{a\in\overline{\mathbb{C}}} \beta(a,f) \le 2B(\mu).$$

Both estimates are sharp.

Example. Let $f(z) = \exp(z)$. We have $\varrho(f) = 1$, $E_N(f) = E_{\Pi}(f) = \{0, \infty\}$ and for exceptional values:

$$\delta(0,f) = \delta(\infty,f) = 1, \qquad \beta(0,f) = \beta(\infty,f) = \pi,$$

so

$$\sum_{a\in\overline{\mathbb{C}}} \delta(a,f) = 2 \qquad \text{and} \qquad \sum_{a\in\overline{\mathbb{C}}} \beta(a,f) = 2\pi.$$

Clunie's lemma

Let f be a transcendental meromorphic solution of

$$f^n P(z,f) = Q(z,f),$$

where n is a positive integer, $P(z,f), Q(z,f)$ are polynomials in f and its derivatives with meromorphic coefficients $\{a_\lambda : \lambda \in I\}$, such that $m(r,a_\lambda) = S(r,f)$ for all $\lambda \in I$. If the total degree d of $Q(z,f)$ as a polynomial in f and its derivatives is $d \leq n$, then

$$m(r,P(z,f)) = S(r,f).$$

An analogue of Clunie's lemma [GF-Ciechanowicz]

Let f be a transcendental meromorphic solution of

$$f^n P(z,f) = Q(z,f), \tag{1}$$

where n is a positive integer, $P(z,f)$, $Q(z,f)$ are polynomials in f and its derivatives with meromorphic coefficients a_ν, b_ν, respectively, which are small with respect to f in the sense that

$$\mathcal{L}(r,\infty,a_\nu) = S(r,f), \qquad \mathcal{L}(r,\infty,b_\nu) = S(r,f).$$

If the total degree d of $Q(z,f)$ as a polynomial in f and its derivatives is $d \leq n$, then

$$\mathcal{L}(r,\infty,P(z,f)) = S(r,f).$$

Mohon'ko–Mohon'ko's theorem and its analogue

Let

$$P(z, f, f', ..., f^{(n)}) = 0 \tag{2}$$

be an algebraic differential equation ($P(z, u_0, u_1, ..., u_n)$ is a polynomial in all arguments) and let f be its transcendental meromorphic solution. If a constant a does not solve the equation, then $m(r, \frac{1}{f-a}) = S(r, f)$ and $\delta(a, f) = 0$.

[GF-Ciechanowicz] If f is a transcendental meromorphic solution of equation (2) and a constant a does not solve this equation, then $\mathcal{L}(r, a, f) = S(r, f)$ and $\beta(a, f) = 0$.

Painlevé equations and value distribution theory

The six Painlevé equations have many applications in modern mathematics and mathematical physics and a number of remarkable properties.

The second and the fourth Painlevé equations are given by

$$f'' = 2f^3 + zf + \alpha, \qquad (P_2)$$

$$f'' = \frac{f'^2}{2f} + \frac{3f^3}{2} + 4zf^2 + 2(z^2 - \alpha)f + \frac{\beta}{f}, \qquad (P_4)$$

where α, β are arbitrary complex parameters and $f = f(z)$. Their solutions are of finite order.

Known facts about P_2

Transcendental solutions of P_2 fulfill the conditions:

1. $m(r, f) = O(\log r)$ and $\delta(\infty, f) = 0$;

2. if $\alpha \neq 0$, then, for every $a \in \mathbb{C}$, we have $m(r, \frac{1}{f-a}) = O(\log r)$ and $\delta(a, f) = 0$;

3. in the case of $\alpha = 0$ for every $a \in \mathbb{C} \setminus \{0\}$ we have $m(r, \frac{1}{f-a}) = O(\log r)$ and $\delta(a, f) = 0$, and for $a = 0$ we have $m(r, \frac{1}{f}) \leq \frac{1}{2}T(r, f) + O(\log r)$ and $\delta(0, f) \leq \frac{1}{2}$.

4. for every $a \in \mathbb{C} \setminus \{0\}$ we have $N_1(r, \frac{1}{f-a}) \leq \frac{1}{4}T(r, f) + O(\log r)$ and $\vartheta(a, f) \leq \frac{1}{4}$;

5. if $\alpha \neq 0$, then $N_1(r, \frac{1}{f}) \leq \frac{1}{5}T(r, f) + O(\log r)$ and $\vartheta(0, f) \leq \frac{1}{5}$, and if $\alpha = 0$, then $N_1(r, \frac{1}{f}) = 0$ and $\vartheta(0, f) = 0$;

6. $N_1(r, f) = 0$ and $\vartheta(\infty, f) = 0$.

Known facts about P_4

Transcendental solutions of P_4 fulfill the conditions:

1. $m(r, f) = O(\log r)$ and $\delta(\infty, f) = 0$;
2. if $\beta \neq 0$, then for $a \in \mathbb{C}$ we have $m(r, \frac{1}{f-a}) = O(\log r)$ and $\delta(a, f) = 0$;
3. if $\beta = 0$ and $a \neq 0$, then we have $m(r, \frac{1}{f-a}) = O(\log r)$ and $\delta(a, f) = 0$;
4. if $\beta = 0$ and if f does not satisfy the Riccati differential equation $f' = \pm(f^2 + 2zf)$, then $m(r, \frac{1}{f}) \leq \frac{1}{2}T(r, f) + O(\log r)$ and $\delta(0, f) \leq \frac{1}{2}$;
5. for every $a \in \mathbb{C} \setminus \{0\}$, $N_1(r, \frac{1}{f-a}) \leq \frac{1}{4}T(r, f) + O(\log r)$ and $\vartheta(a, f) \leq \frac{1}{4}$;
6. if $\beta \neq 0$, then $N_1(r, \frac{1}{f}) = 0$ and $\vartheta(0, f) = 0$;
7. if $\beta = 0$, then $N_1(r, \frac{1}{f}) = \frac{1}{2}T(r, f) + O(\log r)$ and $\vartheta(0, f) = \frac{1}{2}$;
8. $N_1(r, f) = 0$ and $\vartheta(\infty, f) = 0$.

New facts about P_2 and P_4

Transcendental meromorphic solutions of P_2 and P_4 have the following properties.

1. For solutions of $P_2(\alpha)$ the equalities $\mathcal{L}(r, a, f) = S(r, f)$ and $\beta(a, f) = 0$ hold for all $a \in \overline{\mathbb{C}} \setminus \{0\}$. If $\alpha \neq 0$ we also have $\mathcal{L}(r, 0, f) = S(r, f)$ and $\beta(0, f) = 0$.

2. If f is a solution of $P_4(\alpha, \beta)$, then the equalities $\mathcal{L}(r, a, f) = S(r, f)$ and $\beta(a, f) = 0$ hold for all $a \in \overline{\mathbb{C}} \setminus \{0\}$. If $\beta \neq 0$, then we also have $\mathcal{L}(r, 0, f) = S(r, f)$ and $\beta(0, f) = 0$.

The unified equation of P_4 and P_{34}

Equation P_{34}, also called equation XXXIV, is the second order equation of the form

$$f'' = \frac{(f')^2}{2f} + Bf(2f - z) - \frac{A}{2f}, \tag{3}$$

where A and B are fixed complex parameters.

Y. Ohyama introduced the unified equation

$$f'' = \frac{(f')^2}{2f} - \frac{\alpha}{2f} + \beta f(2f + z) + \gamma f(f + z)(3f + z). \tag{4}$$

If $f(z)$ is a solution of $P_{4,34}(\alpha, \beta, \gamma)$, then $f(cz)/c$ is a solution of $P_{4,34}(\alpha, c^3\beta, c^4\gamma)$. If $\beta = 0$, $\gamma = 0$, then equation (4) can easily be integrated with polynomial solutions

$$f(z) = \frac{(C_1^2 - \alpha)z^2}{4C_2} + C_1 z + C_2.$$

Cases we consider:

$(C1)$ $\gamma = 0$, $\beta \neq 0$;

$(C2)$ $\gamma \neq 0$.

Expansions of solutions around a movable pole z_0

The equation $P_{4,34}$ has the following polar behavior.

1. If $\gamma = 0$, then an arbitrary solution of $P_{4,34}(\alpha, \beta, 0)$ has double poles. Moreover, equation $P_{4,34}(\alpha, \beta, 0)$ can be re-written in the form of a regular system at a pole $z = z_0$ for the variables $u(z)^2 = 1/f(z)$ and $v(z)$ defined by

$$f'(z) = -1 - \frac{\sqrt{2\beta}}{u(z)^3} - \frac{\sqrt{\beta}z}{\sqrt{2}u(z)} - \frac{u(z)(\sqrt{2}\beta z^2 - 120\sqrt{2}v(z))}{24\sqrt{\beta}}$$

such that the functions $u(z)$ and $v(z)$ are analytic in the neighborhood of $z = z_0$ and $u(z_0) = 0$ and $v(z_0) = a_2$, where a_2 is arbitrary.

2. If $\gamma \neq 0$, then an arbitrary solution of $P_{4,34}(\alpha, \beta, \gamma)$ has simple poles. Moreover, equation $P_{4,34}(\alpha, \beta, \gamma)$ can be re-written in the form of a regular system at a pole $z = z_0$ for the variables $u(z) = 1/f(z)$ and $v(z)$ defined by

$$f'(z) = -\frac{\sqrt{2\gamma}}{\varepsilon u(z)^2} - \frac{\beta + 2z\gamma}{\sqrt{2\gamma}\varepsilon u(z)} + \frac{\sqrt{2}\beta^2 - 8\varepsilon\sqrt{\gamma}\gamma}{8\varepsilon\sqrt{\gamma}\gamma} + \frac{(8\sqrt{2\gamma}v(z) - \varepsilon\beta - 2z\varepsilon\gamma)u(z)}{4\varepsilon\gamma}$$

such that the functions $u(z)$ and $v(z)$ are analytic in the neighborhood of $z = z_0$ and $u(z_0) = 0$ and $v(z_0) = a_2$, where a_2 is arbitrary.

Results on the distribution of a-points ($a \in \overline{\mathbb{C}}$) of a transcendental solution of $P_{4,34}$

Transcendental meromorphic solutions of $P_{4,34}(\alpha, \beta, \gamma)$ satisfy the conditions

1. $m(r, f) = S(r, f)$;
2. $m(r, \frac{1}{f-a}) = S(r, f)$ for all $a \in \mathbb{C} \setminus \{0\}$;
3. if $\alpha \neq 0$, then $m(r, \frac{1}{f}) = S(r, f)$;
4. if $\alpha = 0$ and $\gamma \neq 0$, then $m(r, \frac{1}{f}) \leq \frac{1}{2}T(r, f) + S(r, f)$ unless f fulfills the Riccati differential equation
$$f' = \varepsilon\sqrt{2\gamma} f(f + z + \beta/(2\gamma)) \qquad \text{with} \qquad \beta^2 + 4\varepsilon\gamma\sqrt{2\gamma} = 0 \ (\varepsilon^2 = 1), \quad (5)$$
in which case $m(r, \frac{1}{f}) \leq T(r, f) + O(1)$;
5. if $\alpha = 0$ and $\gamma = 0$, then $m(r, \frac{1}{f}) \leq \frac{1}{2}T(r, f) + S(r, f)$.

As a corollary, equation $P_{4,34}$ does not admit transcendental entire solutions.

If f is a transcendental meromorphic solution of $P_{4,34}(\alpha, \beta, \gamma)$ with $\alpha \neq 0$, then both in case $(C1)$ and $(C2)$ for all $a \in \overline{\mathbb{C}}$ we have

$$\delta(a, f) = 0,$$

so the set $E_N(f)$ of Nevanlinna's defective values of f is empty. For $P_{4,34}(0, \beta, \gamma)$, both in case $(C1)$ and $(C2)$, we have $E_N(f) \subseteq \{0\}$. Moreover, $\delta(0, f) \leq 1/2$, unless in case $(C2)$ f fulfills (5) and then $\delta(0, f) = 1$.

For a transcendental meromorphic solution f of $P_{34}(A, B)$, we have $E_N(f) = \emptyset$ if $A \neq 0$ and $E_N(f) \subseteq \{0\}$ with $\delta(0, f) \leq 1/2$ if $A = 0$.

Etimates for deviations of solutions of $P_{4,34}$

Transcendental meromorphic solutions of $P_{4,34}$ satisfy the conditions

1. $\mathcal{L}(r,\infty,f) = S(r,f)$,
2. $\mathcal{L}(r,a,f) = S(r,f)$ for all $a \in \mathbb{C} \setminus \{0\}$.

If $\alpha \neq 0$ we also have $\mathcal{L}(r,0,f) = S(r,f)$.

If f is a transcendental meromorphic solution of $P_{4,34}$, then for all $a \in \overline{\mathbb{C}} \setminus \{0\}$

$$\beta(a,f) = 0.$$

If $\alpha \neq 0$ also $\beta(0,f) = 0$, so in this case the set $E_\Pi(f)$ of Petrenko's exceptional values of f is empty.

A transcendental meromorphic solution f of the equation $P_{34}(A,B)$ does not possess exceptional values in the sense of Petrenko if $A \neq 0$. If $A = 0$ then $E_\Pi(f) \subseteq \{0\}$.

Result on multiplicity of a-points of a solution of $P_{4,34}$

Let f be a transcendental solution of $P_{4,34}$.

1. For $P_{4,34}$ in case $(C1)$, all the poles of f are double and $\vartheta(\infty, f) = 1/2$. For $P_{4,34}(\alpha, \beta, \gamma)$ in case $(C2)$ all the poles of f are simple and $\vartheta(\infty, f) = 0$.

2. For $P_{4,34}(\alpha, \beta, \gamma)$, $(\alpha \neq 0)$ all the zeros of f are simple and $\vartheta(0, f) = 0$. For $P_{4,34}(0, \beta, \gamma)$, the zeros of non-zero solutions are double. Thus we have $\vartheta(0, f) \leq \frac{1}{2}$ in case $(C1)$ and in case $(C2)$ unless f fulfills the equation (5), which then means that $\vartheta(0, f) = 0$.

3. For $a \neq 0$, we have $\vartheta(a, f) \leq \frac{1}{4}$.

Result on multiplicity of a-points of a solution of P_{34}.

A transcendental meromorphic solution f of P_{34} satisfies the conditions:

1. all the poles of f are double and $\vartheta(\infty, f) = 1/2$;

2. for $P_{34}(A, B)$, $(A \neq 0)$ all the zeros of f are simple and $\vartheta(0, f) = 0$, for $P_{34}(0, B)$, the zeros are double and $\vartheta(0, f) \leq \frac{1}{2}$;

3. if $a \in \mathbb{C} \setminus \{0\}$, we have $\vartheta(a, f) \leq \frac{1}{4}$.

Thank you very much for your attention!