Stokes' phenomenon arising from colliding poles in the confluence from P_{VI} to P_V

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Asymptotic and computational aspects of complex differential equations

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Aims of my PhD research project

To understand how Stokes' phenomenon arises when confluencing the linear systems associated to the Painlevé equations.

$$\begin{tikzcd}
 & & P_{III}^{D_6} \arrow[r] \arrow[dr] & P_{III}^{D_7} \arrow[r] \arrow[dr] & P_{III}^{D_8} \\
P_{VI} \arrow[r] & P_V \arrow[ur] \arrow[r] \arrow[dr] & P_V^{\mathrm{deg}} \arrow[ur] \arrow[dr] & P_{II}^{JM} \arrow[r] & P_I \\
 & & P_{IV} \arrow[ur] \arrow[r] & P_{II}^{FN} \arrow[ur] &
\end{tikzcd}$$

JM: Jimbo-Miwa, FN: Flaschka-Newell.

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Existing literature

Kitaev proposed a procedure to deal with the confluence $P_{VI} \to P_V$ using a surviving Fuchsian singularity, this approach does not apply to $P_V \to P_{III}^{D_6}$ nor $P_{IV} \to P_{II}^{JM}$.

Glutsyuk proved an existence theorem about the limits of monodromy data under a generic confluence of simple poles.

Glutsyuk, A, J. Dynam. Control Systems **5** (1999), no. 1, 101-135.
Kitaev, A., J. Phys. A **39** (2006) 12033-12072.

Plan for today's talk

Review the standard (2×2) auxiliary linear systems for P_{VI} and P_V and their monodromy data.

Show how the P_V linear system arises from confluencing 1 and ∞ in the P_{VI} one.

Calculate explicitly the Stokes' matrices of the P_V linear system as limits of the connection matrix of the P_{VI} linear system between 1 and ∞.

Auxiliary linear systems for the Painlevé equations

Theorem (R. Fuchs, Schlesinger, Flaschka-Newell, Jimbo-Miwa)

For each Painlevé equation, there exist $A(w, w_t; t, \lambda), B(w, w_t; t, \lambda) \in \mathfrak{gl}_2(\mathbb{C})$ *rational in* λ *such that,*

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial \lambda} = [B, A] \Rightarrow w(t) \text{ satisfies that Painlevé equation.}$$

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For $A, B \in \mathfrak{gl}_2(\mathbb{C})$, we consider linear systems of the form

$$\frac{\partial Y}{\partial \lambda} = AY, \quad \frac{\partial Y}{\partial t} = BY$$

where the latter equation implies that the monodromy data, including Stokes' data, of the solution of the first equation does not depend on t.

The natures of fundamental solutions of linear ODEs

Fuchsian	Irregular
Analytic in neighbourhoods	Analytic in sectors
Expressible as convergent series	Asymptotic to divergent series
Power-like behavior	Exponential behavior

Auxiliary linear systems for P_{VI} and P_V

Painlevé VI

$$\frac{\partial Y_6}{\partial \lambda_6} = \left(\frac{A_{06}}{\lambda_6} + \frac{A_{t6}}{\lambda_6 - t_6} + \frac{A_{16}}{\lambda_6 - 1} \right) Y_6, \qquad \frac{\partial Y_6}{\partial t_6} = \frac{-A_{t6}}{\lambda_6 - t_6} Y_6$$

We assume:

- $A_{06} + A_{t6} + A_{16} = -\frac{\Theta_{\infty 6}}{2}\sigma_3$
- $\exists R_{k6} : R_{k6}^{-1} A_{k6} R_{k6} = \frac{\Theta_{k6}}{2}\sigma_3, \;\; k = 0, t, 1$
- $\Theta_{k6} \in \mathbb{C}\backslash\mathbb{Z}, \;\; k = 0, t, 1, \infty$

Compared with the Jimbo-Miwa linear system for P_{VI}, ours differs by
$A_{k6} = A_{k6}^{JM} - \frac{\Theta_{k6}}{2}\sigma_3$

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Compared with the Jimbo-Miwa linear system for P_{VI}, ours differs by $A_{k6} = A_{k6}^{JM} - \frac{\Theta_{k6}}{2}\sigma_3$

Painlevé V

$$\frac{\partial Y_5}{\partial \lambda_5} = \left(\frac{\sigma_3}{2} + \frac{A_{05}}{\lambda_5} + \frac{A_{t5}}{\lambda_5 - t_5} \right) Y_5, \qquad \frac{\partial Y_5}{\partial t_5} = \frac{-A_{t5}}{\lambda_5 - t_5} Y_5$$

We assume:

- $\mathrm{diag}\,(A_{05} + A_{t5}) = -\frac{\Theta_{\infty 5}}{2}\sigma_3$
- $\exists R_{k5} : R_{k5}^{-1} A_{k5} R_{k5} = \frac{\Theta_{k5}}{2}\sigma_3, \;\; k = 0, t$
- $\Theta_{k5} \in \mathbb{C}\backslash\mathbb{Z}, \;\; k = 0, t$

Compared with Jimbo-Miwa, our linear system differs by $Y_5 = Y_5^{JM} \lambda_5^{-\frac{\Theta_{05}}{2}} (\lambda_5 - t_5)^{-\frac{\Theta_{t5}}{2}} t_5^{-\frac{\Theta_{\infty 5}}{2}}$ and $\lambda_5 = \lambda_5^{JM} t_5$.

Auxiliary linear systems for P_{VI} and P_V

Painlevé VI

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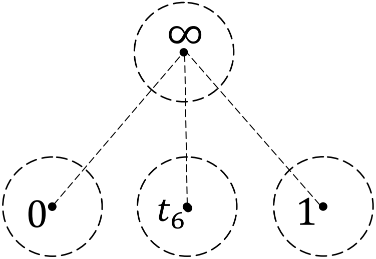

Local solutions

$Y_6^{(\infty)}(\lambda_6) = \left(\mathbb{I} + \mathcal{O}\left(\lambda_6^{-1}\right)\right) \lambda_6^{-\frac{\Theta_{\infty 6}}{2}\sigma_3}$

$Y_6^{(k)}(\lambda_6) = \left(R_{k6} + \mathcal{O}\left(\lambda_6 - a_k\right)\right) \left(\lambda_6 - a_k\right)^{\frac{\Theta_{k6}}{2}\sigma_3}$

$a_k \equiv k, k = 0, t, 1.$

Auxiliary linear systems for P_{VI} and P_V

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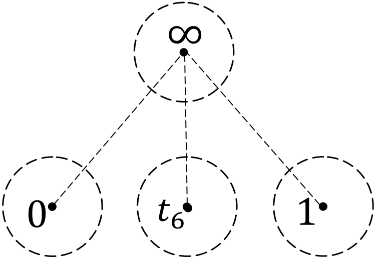


Painlevé V

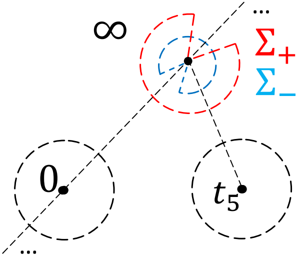
$$\frac{\partial Y_5}{\partial \lambda_5} = \left(\frac{\sigma_3}{2} + \frac{A_{05}}{\lambda_5} + \frac{A_{t5}}{\lambda_5 - t_5} \right) Y_5$$

Local solutions $Y_{5,\pm}^{(\infty)}(\lambda_5)$ on $\Sigma_\pm$ satisfying

$Y_{5,\pm}^{(\infty)}(\lambda_5) \sim \left(\mathbb{I} + \mathcal{O}\left(\lambda_5^{-1}\right)\right) \lambda_5^{-\frac{\Theta_{\infty 5}}{2}\sigma_3} e^{\frac{\lambda_5}{2}\sigma_3}$

as $\lambda_5 \to \infty$ on $\lambda_5 \in \Sigma_+$ or Σ_- respectively

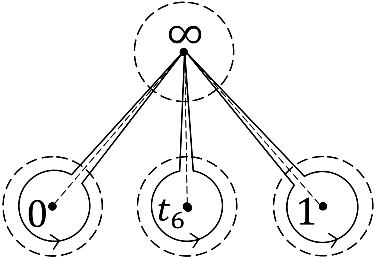
$Y_5^{(k)}(\lambda_5) = \left(R_{k5} + \mathcal{O}\left(\lambda_5 - a_k\right)\right)\left(\lambda_5 - a_k\right)^{\frac{\Theta_{k5}}{2}\sigma_3}$



Monodromy data of P_{VI} and P_V auxiliary linear systems

Painlevé VI

$$\frac{\partial Y_6}{\partial \lambda_6} = \left(\frac{A_{06}}{\lambda_6} + \frac{A_{t6}}{\lambda_6 - t_6} + \frac{A_{16}}{\lambda_6 - 1} \right) Y_6$$



$\mathcal{M}_6 := \{M_{06}, M_{t6}, M_{16}, M_{\infty 6} | \prod M_{k6} = \mathbb{I}\}$
$\simeq \{C_{06}, C_{t6}, C_{16}, \Theta_{06}, \Theta_{t6}, \Theta_{16}, \Theta_{\infty 6} | \prod M_{k6} = \mathbb{I}\}$
where $M_{\infty 6} = e^{i\pi\Theta_{\infty 6}\sigma_3}$ and
$M_{k6} = C_{k6}^{-1} e^{i\pi\Theta_{k6}\sigma_3} C_{k6}, \; k = 0, t, 1 \qquad (\Theta_{k6} \notin \mathbb{Z} \; \forall k)$.

Monodromy data of P_{VI} and P_V auxiliary linear systems

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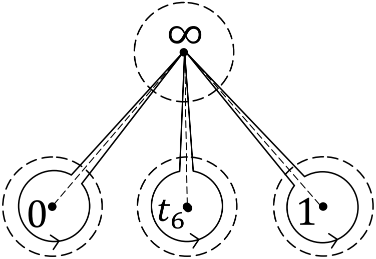

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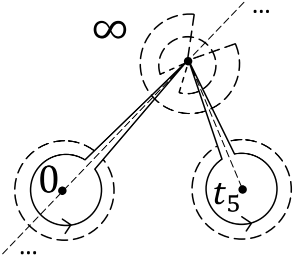

$$\mathcal{M}_5 := \{ M_{05}, M_{t5}, M_{\infty 5} | \prod M_{k5} = \mathbb{I} \}$$
$$\simeq \{ C_{05}, C_{t5}, \Theta_{05}, \Theta_{t5}, \Theta_{\infty 5}, S_1, S_2 | \prod M_{k5} = \mathbb{I} \}$$

where $M_{\infty 5} = e^{i\pi \Theta_{\infty 5} \sigma_3} S_2 S_1$ and

$M_{k5} = C_{k5}^{-1} e^{i\pi \Theta_{k5} \sigma_3} C_{k5}, \ k = 0, t \qquad (\Theta_{k5} \notin \mathbb{Z} \ k = 0, t).$

Formal confluence procecure from P_{VI} to P_V

$$\frac{\partial Y_6}{\partial \lambda_6} = \left(\frac{A_{06}}{\lambda_6} + \frac{A_{t6}}{\lambda_6 - t_6} + \frac{A_{16}}{\lambda_6 - 1} \right) Y_6 \tag{1}$$

$$\frac{\partial Y_5}{\partial \lambda_5} = \left(\frac{\sigma_3}{2} + \frac{A_{05}}{\lambda_5} + \frac{A_{t5}}{\lambda_5 - t_5} \right) Y_5 \tag{2}$$

The following substitutions provide a limit passage from (1) to (2) as $\varepsilon \to 0$:

$$\lambda_6 = \varepsilon\lambda_5,\ t_6 = \varepsilon t_5,\ A_{06} = A_{05} + \mathcal{O}(\varepsilon),\ A_{t6} = A_{t5} + \mathcal{O}(\varepsilon),$$
$$\Theta_{06} = \Theta_{05},\ \Theta_{t6} = \Theta_{t5},\ \Theta_{16} = -\varepsilon^{-1},\ \Theta_{\infty 6} = \varepsilon^{-1} + \Theta_{\infty 5}.$$

The Fuchsian singularities $\lambda_6 = 0, t_6, 1, \infty$ of (1) are mapped to $\lambda_5 = 0, t_5, \varepsilon^{-1}, \infty$ respectively, as $\varepsilon \to 0$ two poles collide.

Formal confluence procecure from P_{VI} to P_V

$$\frac{\partial Y_6}{\partial t_6} = -\frac{A_{t6}}{\lambda_6 - t_6} Y_6 \tag{3}$$

$$\frac{\partial Y_5}{\partial t_5} = -\frac{A_{t5}}{\lambda_5 - t_5} Y_5 \tag{4}$$

The following substitutions provide a limit passage from (3) to (4) as $\varepsilon \to 0$:

$$\lambda_6 = \varepsilon\lambda_5,\ t_6 = \varepsilon t_5,\ A_{06} = A_{05} + \mathcal{O}(\varepsilon),\ A_{t6} = A_{t5} + \mathcal{O}(\varepsilon),$$
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Confluence on fundamental solutions

$$Y_6^{(\infty)}(\lambda_6) = \left(\mathbb{I} + \sum_{n=1}^{\infty} G_n \lambda_6^{-n}\right) \lambda_6^{-\frac{\Theta_{\infty 6}}{2}\sigma_3}$$

$$Y_{5,\pm}^{(\infty)}(\lambda_5) \sim \left(\mathbb{I} + \sum_{n=1}^{\infty} H_n \lambda_5^{-n}\right) \lambda_5^{-\frac{\Theta_{\infty 5}}{2}\sigma_3} e^{\frac{\lambda_5}{2}\sigma_3} \text{ as } \lambda_5 \to \infty, \lambda_5 \in \Sigma_\pm$$

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Note the term $(1-\lambda_6)^{\frac{\Theta_{16}}{2}\sigma_3} = (1-\varepsilon\lambda_5)^{-\frac{\sigma_3}{2\varepsilon}} \to e^{\frac{\lambda_5}{2}\sigma_3}$ as $\varepsilon \to 0$.

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Lemma (C.H. & M.M.)

Under certain generic conditions, $\lim_{\varepsilon\to 0} \widehat{G}_n \varepsilon^{-n} = H_n$

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Under certain generic conditions, $\lim_{\varepsilon\to 0} \widehat{G}_n \varepsilon^{-n} = H_n$

$$Y_6^{(1)}(\lambda_6) = \left(R_{16} + \sum_{n=1}^{\infty} g_n (\lambda_6 - 1)^n\right) (1-\lambda_6)^{\frac{\Theta_{16}}{2}\sigma_3}$$

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Under certain generic conditions, $\lim_{\varepsilon\to 0} \widehat{G}_n \varepsilon^{-n} = H_n = \lim_{\varepsilon\to 0} \widehat{g}_n \varepsilon^{-n}$

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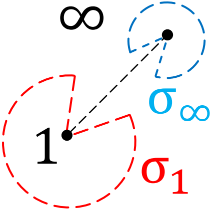
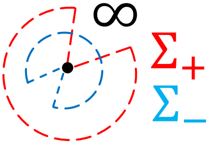
Existence theorem of Glutsyuk

Let $Y^{(\infty)}_{5,\pm}(\lambda_5)$ be the true solutions to the P_V linear system defined in $\Sigma_\pm$ with the prescribed asymptotic behavior.



Define sectors σ_∞ and σ_1 on the λ_6-Riemann sphere as shown, which satisfy $\sigma_\infty \to \Sigma_-$ and $\sigma_1 \to \Sigma_+$ as $\varepsilon \to 0$ along a ray $\varepsilon \in \mathcal{R}$.



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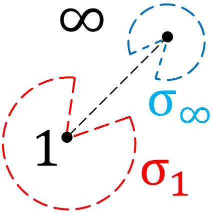

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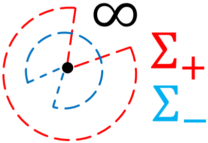

Theorem (Glutsyuk)

There exist diagonal $K_\infty, K_1 \in SL_2(\mathbb{C})$ *such that*

$$\lim_{\varepsilon\to 0} Y^{(\infty)}_6(\lambda_6)\Big|_{\sigma_\infty} K_\infty = Y^{(\infty)}_{5,-}(\lambda_5)$$

$$\lim_{\varepsilon\to 0} Y^{(1)}_6(\lambda_6)\Big|_{\sigma_1} K_1 = Y^{(\infty)}_{5,+}(\lambda_5) \qquad (\varepsilon \in \mathcal{R}).$$

Hence, $\lim_{\varepsilon\to 0} K_1^{-1} C_1 K_\infty = S_1$ *where* C_1 *is the connection matrix.*

Glutsyuk, A, J. Dynam. Control Systems **5** (1999), no. 1, 101-135.

Finding explicit formulae

Theorem (C.H. & M.M.)

The matrices of Glutsyuk's existence theorem are:

$$K_\infty = \begin{pmatrix} \varepsilon^{\frac{\Theta_{\infty 6}+\Theta_{16}}{2}} & 0 \\ 0 & \varepsilon^{-\frac{\Theta_{\infty 6}+\Theta_{16}}{2}} \end{pmatrix} = K_1$$

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We thus have an explicit computation of the P_V Stokes' matrices as a limit of the P_{VI} monodromy data,

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon\in\mathcal{R}}} \varepsilon^{-\frac{\Theta_{\infty 6}+\Theta_{16}}{2}\sigma_3} C_1 \varepsilon^{\frac{\Theta_{\infty 6}+\Theta_{16}}{2}\sigma_3} = S_1.$$

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To obtain the other Stokes' matrix S_2 we take the confluence parameter ε along a certain different ray $\widetilde{\mathcal{R}}$,

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon\in\widetilde{\mathcal{R}}}} \varepsilon^{-\frac{\Theta_{\infty 6}+\Theta_{16}}{2}\sigma_3} C_1 \varepsilon^{\frac{\Theta_{\infty 6}+\Theta_{16}}{2}\sigma_3} = S_2.$$

Outlook

To understand how Stokes' phenomenon arises in the confluence from P_V to $P_{III}^{D_6}$.

An open problem is to understand the confluence in the "ramified" cases (i.e. where leading matrix at the irregular singularity is not diagonalisable).

To generalise these results to the higher order analogues of the Painlevé equations arising from the Schlesinger equations and their confluences.

Extra

$$Y_6^{(\infty)}(\lambda_6) = \left(\mathbb{I} + \sum_{n=1}^{\infty} G_n \lambda_6^{-n}\right) \lambda_6^{-\frac{\Theta_{\infty 6}}{2}\sigma_3}$$
$$\equiv \left(\mathbb{I} + \sum_{n=1}^{\infty} \widehat{G}_n \lambda_6^{-n}\right) \lambda_6^{-\frac{\Theta_{16}+\Theta_{\infty 6}}{2}\sigma_3} (1-\lambda_6)^{\frac{\Theta_{16}}{2}\sigma_3}$$

$$Y_{5,\pm}^{(\infty)}(\lambda_5) \sim \left(\mathbb{I} + \sum_{n=1}^{\infty} H_n \lambda_5^{-n}\right) \lambda_5^{-\frac{\Theta_{\infty 5}}{2}\sigma_3} e^{\frac{\lambda_5}{2}\sigma_3} \text{ as } \lambda_5 \to \infty, \lambda_5 \in \Sigma_\pm$$

Setting $G_0 := I =: \widehat{G}_0 =: H_0$, we have the following recursive formulae for $n \geq 1$:

$$-nG_n + \left[G_n, -\frac{\Theta_{\infty 6}}{2}\sigma_3\right] = \sum_{l=0}^{n-1} \left(A_{16} + t_6^{n-l} A_{t6}\right) G_l$$

$$-n\widehat{G}_n + \left[\widehat{G}_n, -\frac{\Theta_{\infty 6}}{2}\sigma_3\right] = \sum_{l=0}^{n-1} \left(A_{16} + t_6^{n-l} A_{t6}\right) \widehat{G}_l - \sum_{l=0}^{n-1} \widehat{G}_l \frac{\Theta_{16}}{2}\sigma_3$$

$$\left[H_n, \frac{\sigma_3}{2}\right] = (n-1)H_{n-1} + H_{n-1}\frac{\Theta_{\infty 5}}{2}\sigma_3 + (A_{05} + A_{t5})H_{n-1} + \sum_{l=0}^{n-2} t_5^{n-1-l} A_{t5} H_l.$$

Extra

$$w_6(t_6) = \varepsilon t_5 \frac{\Theta_{\infty 5} - \Theta_{05} - \Theta_{t5} - 2(t_5 - 2\Theta_{05} + \Theta_{\infty 5}) w_5(t_5) + (\Theta_{\infty 5} - 3\Theta_{05} + \Theta_{t5}) w_5(t_5)^2 + 2 t_5 \frac{dw_5}{dt_5}}{(w_5(t_5) - 1)\left(\Theta_{05} - \Theta_{\infty 5} + \Theta_{t5} + 2(t_5 - \Theta_{05} - \Theta_{t5}) w_5(t_5) + (\Theta_{05} + \Theta_{\infty 5} + \Theta_{t5}) w_5(t_5)^2 - 2 t_5 \frac{dw_5}{dt_5}\right)} + \mathcal{O}\left(\varepsilon^2\right)$$