Extension to a sector of asymptotic expansions in a direction with strongly regular constraints

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J. Jiménez-Garrido* (joint work with J. Sanz* and G. Schindl†)

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Gevrey asymptotic of order k in a direction

$\mathbb{N}_0 = \{0, 1, 2, ...\}$

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Given $f \in \mathcal{H}(S(d, \gamma, r))$ and $\theta \in \mathbb{R}$ with $|\theta - d| < \pi\gamma/2$, we say that f admits $\hat{f} = \sum_{n=0}^{\infty} a_n z^n$ asymptotic expansion of Gevrey order k and type $R > 0$ in direction θ if for every $\delta > 0$, it exists $C > 0$ such that

$$|f(z) - \sum_{n=0}^{p-1} a_n z^n| \leq C(1/R + \delta)^p (p!)^{1/k} |z|^p, \quad z \in S, \quad \arg(z) = \theta, \quad p \in \mathbb{N}_0,$$

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or equivalently, if for every $\delta > 0$, it exists $K > 0$ such that

$$|f(z) - \sum_{n=0}^{p-1} a_n z^n| \leq K e^{-\left(\frac{R-\delta}{|z|}\right)^k}, \quad z \in S, \quad \arg(z) = \theta, \quad p \in \mathbb{N}_0.$$

Extension theorem with Gevrey constrains of order k

Theorem (A. Fruchard and C. Zhang (1999))

Let f be a function analytic and bounded in a open sector $S=S(d,\gamma,r)$. If f has asymptotic expansion $\hat{f}$ of Gevrey order k and type $R(\theta_0)$ in direction θ_0 of S, then, in every direction θ of S f admits $\hat{f}$ asymptotic expansion of Gevrey order k and type $R(\theta)$,

$$R(\theta)=\begin{cases} R(\theta_0)\left(\frac{\sin(k(\theta-\alpha))}{\sin(k(\alpha'-\alpha))}\right)^{1/k} & \text{if} \quad \theta\in[\alpha,\alpha'] \\ R(\theta_0) & \text{if} \quad \theta\in[\alpha',\beta'] \\ R(\theta_0)\left(\frac{\sin(k(\theta-\beta))}{\sin(k(\beta'-\beta))}\right)^{1/k} & \text{if} \quad \theta\in[\beta',\beta] \end{cases}$$

where $\alpha=d-\frac{\pi\gamma}{2}$, $\alpha'=\min(\theta_0,\alpha+\frac{\pi}{2k})$, $\beta=d+\frac{\pi\gamma}{2}$ and $\beta'=\max(\theta_0,\beta-\frac{\pi}{2k})$.

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Example: Impossible extension to a sector

We consider the function $f(z) = \sin(e^{1/z})e^{-1/z}$. It is easy to check that f is asymptotic to $\hat{0}$ of Gevrey order 1 for $\arg(z) = 0$.

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We see that $\lim_{z>0, z\to 0} f'(z)$ does not exists. Consequently, f can not admit an asymptotic expansion in any sector containing direction 0.

Log-convex sequences

Generalization: we change the powers of the factorials by a suitable, more general sequence $\mathbb{M} = (M_p)_{p\in\mathbb{N}_0}$ of positive real numbers, with $M_0 = 1$.

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Examples:

- $\mathbb{M}_\alpha = (p!^\alpha)_{p\in\mathbb{N}_0}$, Gevrey sequences of order $1/\alpha > 0$.
- $\mathbb{M}_{\alpha,\beta} = \left(p!^\alpha \prod_{m=0}^{p} \log^\beta(e+m)\right)_{p\in\mathbb{N}_0}$, $\alpha > 0$, $\beta \in \mathbb{R}$.
- For $q > 1$, $\mathbb{M} = (q^{p^2})_{p\in\mathbb{N}_0}$.

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We say $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G)$ if f admits the series $\hat{f} = \sum_{p=0}^{\infty} a_p z^p$ as its $\mathbb{M}$-asymptotic expansion at 0, denoted $f \sim_{\mathbb{M}} \hat{f}$: For every bounded proper subsector T of G there exist $C_T, B_T > 0$ such that

$$\Big| f(z) - \sum_{k=0}^{p-1} a_k z^k \Big| \le C_T B_T^p M_p |z|^p, \quad z \in T \quad p \in \mathbb{N}_0. \qquad (*)$$

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We say $f \in \tilde{\mathcal{A}}^u_{\mathbb{M}}(G)$ if f admits the series $\hat{f}$ as its uniform $\mathbb{M}$-asymptotic expansion at 0, denoted $f \sim^u_{\mathbb{M}} \hat{f}$: If there exist $C, B > 0$ such that $(*)$ holds uniformly in G.

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Clearly $\tilde{\mathcal{A}}^u_{\mathbb{M}}(G) \subseteq \tilde{\mathcal{A}}_{\mathbb{M}}(G)$.

The Borel map

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$$\tilde{\mathcal{B}} : \tilde{\mathcal{A}}_{\mathbb{M}}(G) \longrightarrow \mathbb{C}[[z]]$$
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One can show that if $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G)$, then $\tilde{\mathcal{B}}(f) \in \mathbb{C}[[z]]_{\mathbb{M}}$.

$$\mathbb{C}[[z]]_{\mathbb{M}} = \Big\{ \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[[z]] : \exists A > 0 \text{ s.t. } \sup_{p\in\mathbb{N}_0} \frac{|a_p|}{A^p M_p} < \infty \Big\}.$$

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$f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G)$ is said to be $\mathbb{M}$-flat on G if $\tilde{\mathcal{B}}(f)$ is the null formal power series.

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Associated function

Given $\mathbb{M}$ (lc) and $\lim_{p\to\infty} m_p = \infty$, we consider $M : [0,\infty) \to \mathbb{R}$

$$M(t) := \sup_{p\in\mathbb{N}_0} \log\left(\frac{t^p}{M_p}\right) = \begin{cases} p\log t - \log(M_p) & \text{if } t \in [m_{p-1}, m_p),\ p \in \mathbb{N}, \\ 0 & \text{if } t \in [0, m_0). \end{cases}$$

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Theorem (V. Thilliez)

The following are equivalent:

(i) $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(G)$ *and* $\tilde{\mathcal{B}}(f) = \hat{0}$ *(*f *is* $\mathbb{M}$*-flat on* G*).*

(ii) *For every bounded proper subsector* T *of* G *there exist* $c_1, c_2 > 0$ *with*

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Proximate orders

Definition (E. Lindelöf, G. Valiron)

We say $\rho(t) : (a, \infty) \to \mathbb{R}$ is a proximate order if the following hold:

(A) $\rho(t)$ is continous and piecewise continuosly differentiable,

(B) $\rho(t) \geq 0$ for every $r > a > 0$,

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Examples:

- $\rho_{\alpha,\beta}(t) = \dfrac{1}{\alpha} - \dfrac{\beta}{\alpha}\dfrac{\log(\log(t))}{\log(t)}$, $\alpha > 0$, $\beta \in \mathbb{R}$.
- $\rho(t) = \rho + \dfrac{1}{t^\gamma}$ and $\rho(t) = \rho + \dfrac{1}{\log^\gamma(t)}$, $\rho \geq 0$, $\gamma > 0$.
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If $\rho > 0$, we say that $\rho(t)$ is a nonzero proximate order.

Maergoiz classes

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The next result L. S. Maergoiz is key for the construction of holomorphic functions whose growth is controlled by $M(t)$.

Theorem (L.S. Maergoiz (2001))

Let $\rho(t)$ be a nonzero proximate order with index ρ. For every $\gamma > 0$ there exists an analytic function $V(z)$ in S_γ such that:

(I) *For every z in S_γ, $\displaystyle\lim_{t\to\infty} \frac{V(zt)}{V(t)} = z^\rho$, uniformly on compacts.*

(II) $\overline{V(z)} = V(\overline{z})$ *for every $z \in S_\gamma$.*

(III) $V(t)$ *is positive in $(0,\infty)$, strictly increasing and $\lim_{t\to 0} V(t) = 0$.*

(IV) *The function $r \in \mathbb{R} \to V(e^r)$ is strictly convex.*

(V) *The function $\log(V(t))$ is strictly concave in $(0,\infty)$.*

(VI) *The function $\rho_V(t) := \log(V(t))/\log(t)$, $t > 0$, is a proximate order and $\displaystyle\lim_{t\to\infty} V(t)/t^{\rho(t)} = 1$.*

L. S. Maergoiz, Indicator diagram and generalized Borel-Laplace transforms for entire functions of a given proximate order, St. Petersburg Math. J. 12 (2001), 191–232.

Admissibility condition

Given $\gamma > 0$ and $\rho(t)$ a nonzero proximate order, we define the class $\mathfrak{B}(\gamma, \rho(t))$ of the functions $V(z)$ defined in S_γ satisfying the conditions (I)-(VI).

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J. Sanz, Flat functions in Carleman ultraholomorphic classes via proximate orders, J. Math. Anal. Appl. 415 (2014), 623–643.
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Admissibility condition and strong regularty

If $\mathbb{M}$, (lc) with $\lim_{p\to\infty} m_p = \infty$, admits a nonzero proximate order, then $\mathbb{M}$ is strongly regular, i.e., $\mathbb{M}$ is:

- logarithmically convex.
- of moderate growth: There exists a constant $A > 0$ such that

$$M_{l+p} \leq A^{l+p} M_l M_p, \quad l, p \in \mathbb{N}_0.$$

- strongly non-quasianalytic: There exists a constant $B > 0$ such that

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The admissibility of a nonzero proximate order has been characterized by J. J.-G., J. Sanz, G. Schindl. It was also shown that not all the strongly regular sequences admit a nonzero proximate order.

J. J.-G., J. Sanz, G. Schindl, Log-convex sequences and nonzero proximate orders, J. Math. Anal. Appl., 448, (2017), no. 2, 1572–1599.

Growth index $\omega(\mathbb{M})$

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Examples: $\omega\big(\big(p!^\alpha \prod_{m=0}^{p} \log^\beta(e+m)\big)_{p\in\mathbb{N}_0}\big) = \alpha$ and $\omega((q^{p^2})_{p\in\mathbb{N}_0}) = \infty$.

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Proposition (J. J.-G., J. Sanz (2016))

If $\mathbb{M}$ is (lc) with $\lim_{p\to\infty} m_p = \infty$ and admits a nonzero proximate order $\rho(t)$, then for every $\gamma > 0$ and every $V \in \mathfrak{B}(\gamma, \rho(t))$ we have that

$$\lim_{t\to\infty} \frac{\log(V(t))}{\log(t)} = \lim_{t\to\infty} \frac{\log(M(t))}{\log(t)} = \frac{1}{\omega(\mathbb{M})} \in (0,\infty).$$

J. J.-G. J. Sanz, Strongly regular sequences and proximate orders. J. Math. Anal. Appl. 438 (2016), no. 2, 920–945

Main Lemma

Lemma (Extension of $\mathbb{M}$-flatness for small sectors)

Let $\mathbb{M}$ *be a (lc) sequence with* $\lim_{p\to\infty} m_p = \infty$ *admitting a nonzero proximate order. Let* f *be analytic and bounded in* S_γ *and continuous in* $\overline{S_\gamma}\setminus\{0\}$*, with* $\gamma < \omega(\mathbb{M})$*, such that there exist* $c_1, c_2 > 0$ *with*

$$|f(z)| \leq c_1 e^{-M(1/(c_2|z|))}, \qquad \arg(z) = -\pi\gamma/2.$$

Then, for every $0 < \delta < \pi\gamma$*, there exist constants* $k_1(\delta), k_2(\delta) > 0$ *with*

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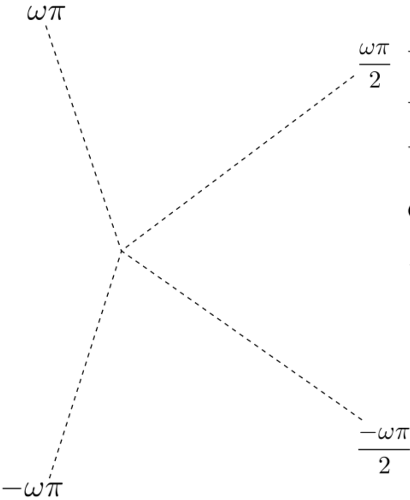
Idea of the proof



We write $\omega = \omega(\mathbb{M})$.

We know that there exist $V \in \mathfrak{B}(2\omega, \rho(r))$ and positive constants A, B such that $AV(t) \leq M(t) \leq BV(t)$.

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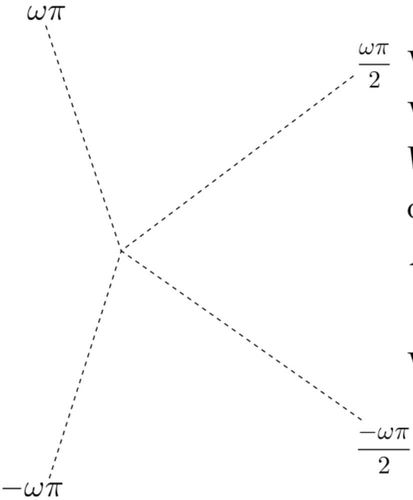

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We consider

$$h(z) = e^{V(1/z)}$$

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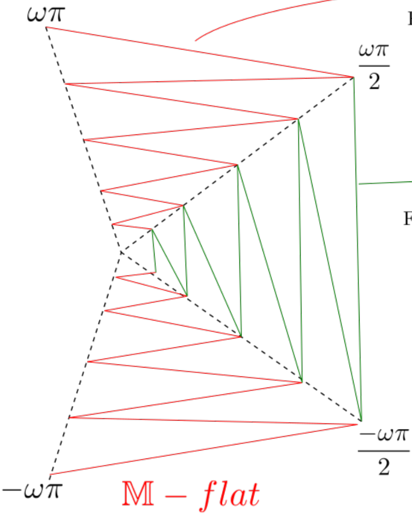

$\mathbb{M}-flat$

For every proper and bounded subsector T of S there exists constants $c_T, a_T > 0$ such that

$$|h(z)| \leq c_T e^{-M(1/a_T|z|)} \qquad z \in T$$

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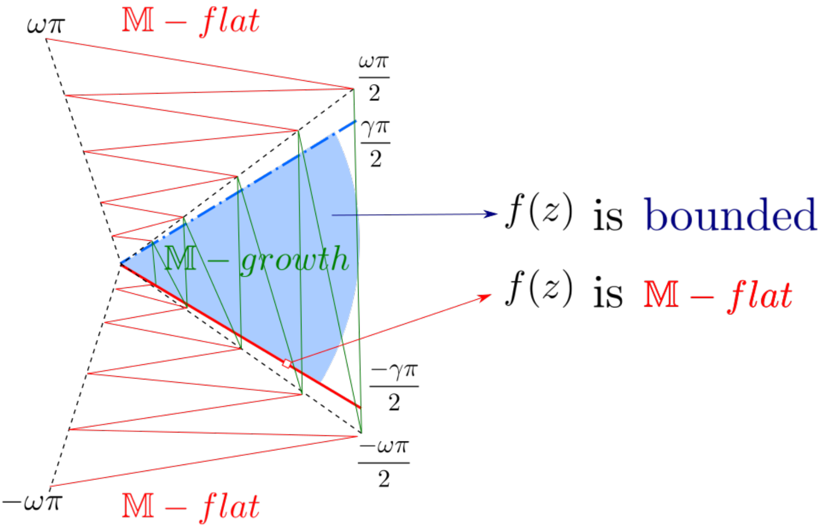

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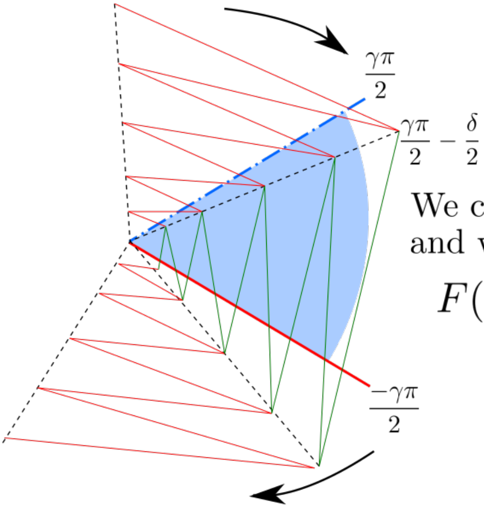
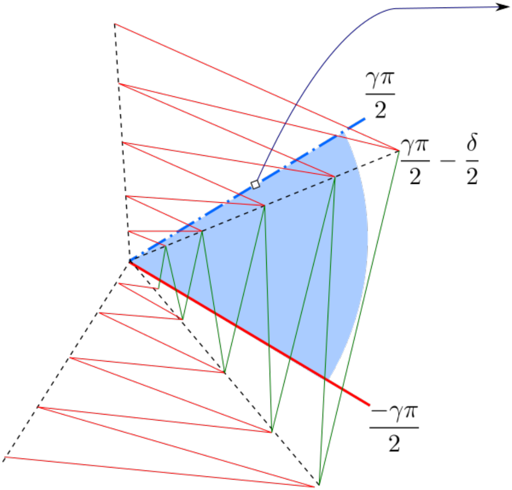


We choose suitably $\arg(a)$ and we define

$$F(z) := f(z)e^{V(a/z)}$$

Idea of the proof



$f(z)$ is bounded

$e^{V(a/z)}$ is $\mathbb{M}-flat$

$\Downarrow$

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Idea of the proof

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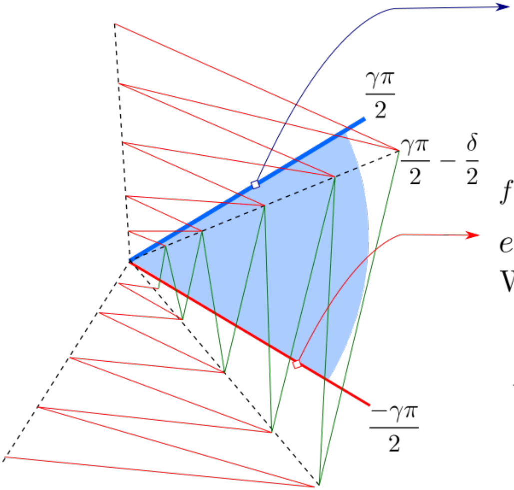

$f(z)$ is $\mathbb{M}-flat$

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We choose suitably $|a|$

$\Downarrow$

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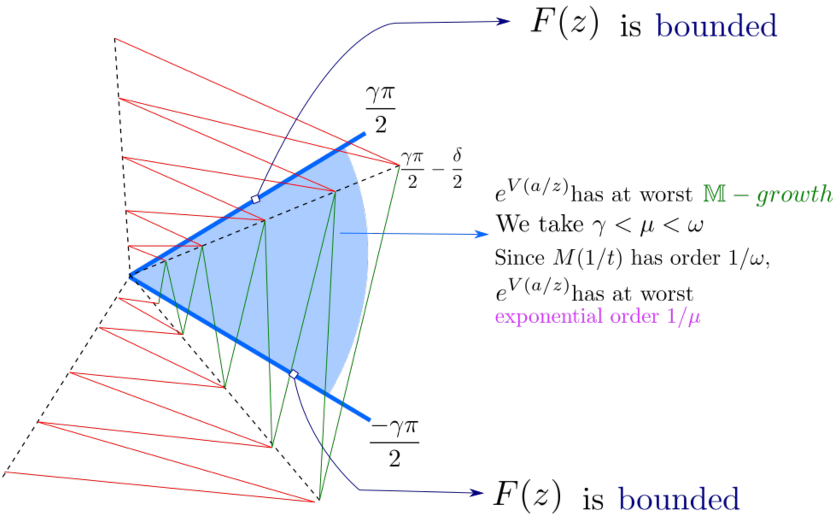
Idea of the proof



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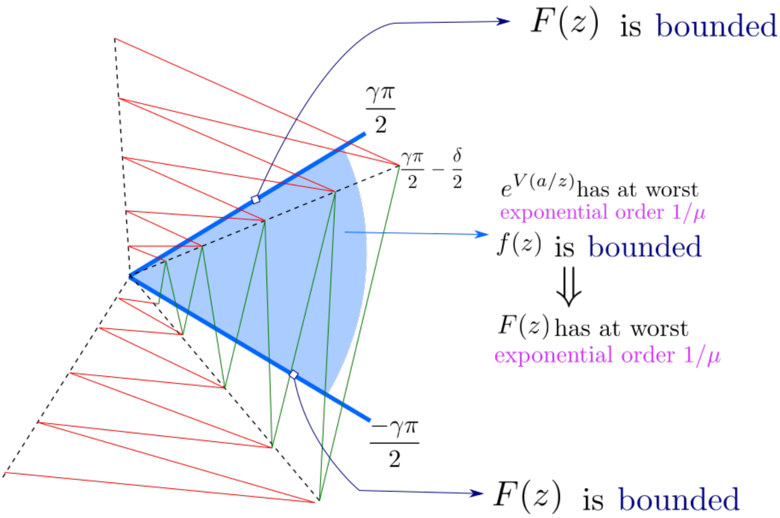

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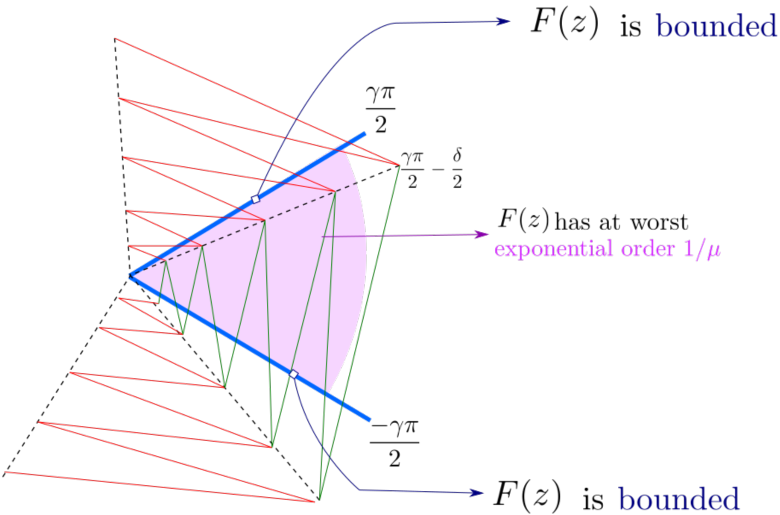

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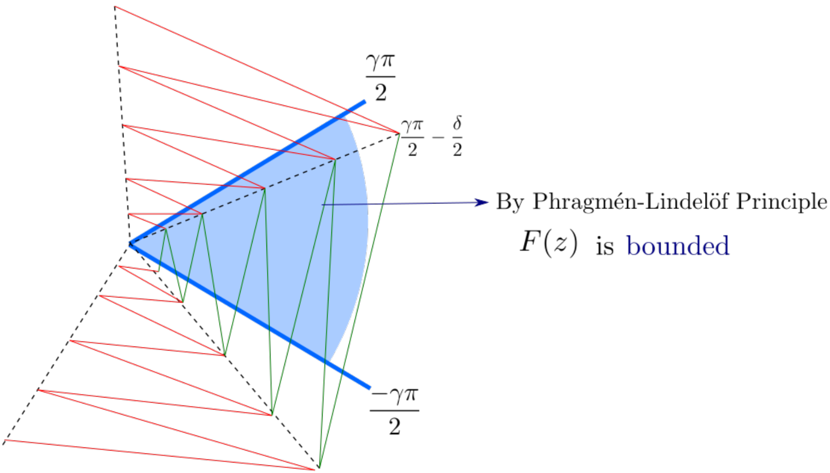

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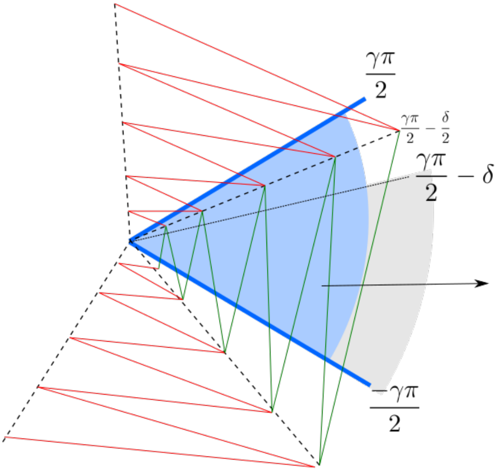


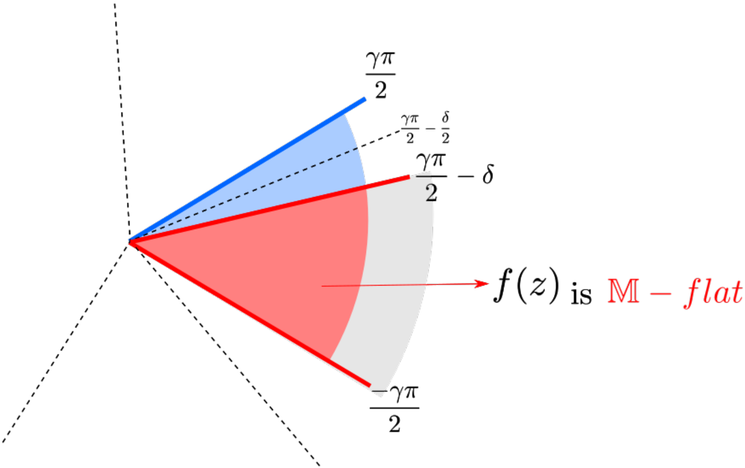
$F(z)$ is bounded

$e^{-V(a/z)}$ is $\mathbb{M}-flat$

$\Downarrow$

$f(z)$ is $\mathbb{M}-flat$

Idea of the proof



Extension of $\mathbb{M}$-flatness for large opening

Lemma (Extension of $\mathbb{M}$-flatness for large opening)

Let $\mathbb{M}$ be a (lc) sequence with $\lim_{p\to\infty} m_p = \infty$ admitting a nonzero proximate order. Let f be analytic and bounded in S_γ and continuous in $\overline{S_\gamma}\setminus\{0\}$, with $\gamma \geq \omega(\mathbb{M})$ such that f is $\mathbb{M}$-flat in direction $d = -\pi\gamma/2$, then for every $0 < \delta < \pi\gamma$, there exist constants $k_1(\delta), k_2(\delta) > 0$ with

$$|f(z)| \leq k_1 e^{-M(1/(k_2|z|))}, \qquad \arg(z) \in [-\pi\gamma/2, \pi\gamma/2 - \delta].$$

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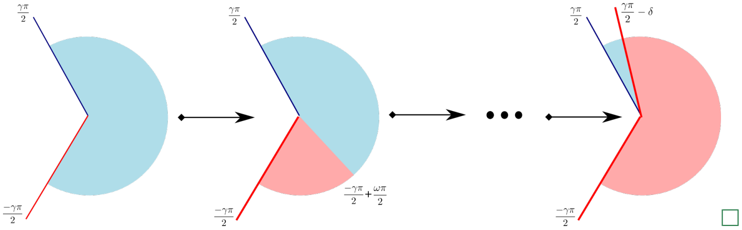
Proof



Extension of $\mathbb{M}$-flatness for sectorial regions

G_γ will denote a sectorial region of opening $\pi\gamma$, bisected by direction 0.

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Proposition (Extension of $\mathbb{M}$-flatness for sectorial regions)

Let $\mathbb{M}$ be a (lc) sequence with $\lim_{p\to\infty} m_p = \infty$ admitting a nonzero proximate order. Let f be analytic and bounded on G_γ, we suppose that there exist $|\theta| < \pi\gamma/2$ and $R > 0$ such that there exist $c_1, c_2 > 0$ with

$$|f(z)| \le c_1 e^{-M(1/(c_2|z|))}, \qquad \arg(z) = \theta, \qquad |z| \le R.$$

Then, for every proper bounded subsector T of G_γ, there exist constants $k_1(T), k_2(T) > 0$ with

$$|f(z)| \le k_1 e^{-M(1/(k_2|z|))}, \qquad z \in T.$$

Watson's Lemma in one direction

Proposition (Partial version of Watson's Lemma in one direction)

Let $\mathbb{M}$ be a (lc) sequence with $\lim_{p\to\infty} m_p = \infty$ admitting a nonzero proximate order. Let f be analytic and bounded in S_γ and continuous in $\overline{S_\gamma}\backslash\{0\}$, with $\gamma > \omega(\mathbb{M})$, or with $\gamma = \omega(\mathbb{M})$ and $\sum_{p=0}^{\infty}(m_p)^{-1/\omega(\mathbb{M})} = \infty$, such that f is $\mathbb{M}$-flat in direction $d = \pi\gamma/2$, then $f \equiv 0$.

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Proof If $\gamma > \omega = \omega(\mathbb{M})$, we take $\omega < \eta < \gamma$. Then f is $\mathbb{M}$-flat in a sector of opening $\pi\eta$.

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Proof If $\gamma > \omega = \omega(\mathbb{M})$, we take $\omega < \eta < \gamma$. Then f is $\mathbb{M}$-flat in a sector of opening $\pi\eta$. We show that $f \equiv 0$ using:

Theorem (S. Mandelbrojt (1952))

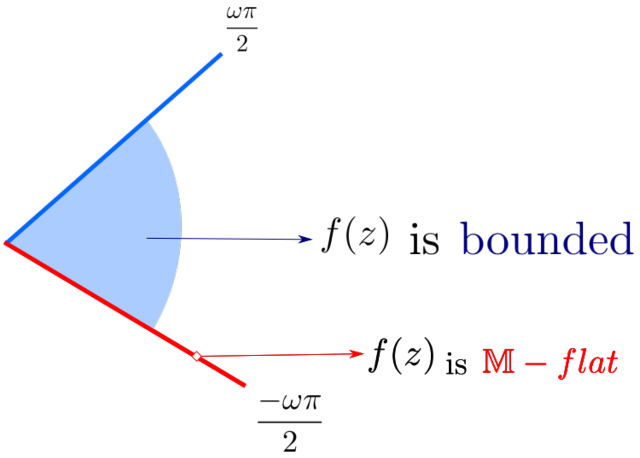
Let $\mathbb{M}$ be a (lc) sequence with $\lim_{p\to\infty} m_p = \infty$ and $\gamma > 0$. The following statements are equivalent:

(i) $\tilde{\mathcal{B}} : \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}}$ *is injective.*

(ii) $\gamma > \omega(\mathbb{M})$ *or* $\gamma = \omega(\mathbb{M})$ *and* $\sum_{p=0}^{\infty}(m_p)^{-1/\omega(\mathbb{M})} = \infty$.

S. Mandelbrojt, *Séries adhérentes, régularisation des suites, applications*, Collection de monographies sur la théorie des fonctions, Gauthier-Villars, Paris, 1952.

Idea of the proof for $\gamma = \omega(\mathbb{M}) = \omega$



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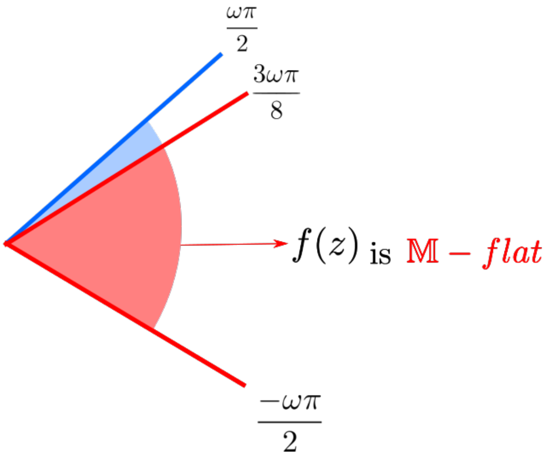

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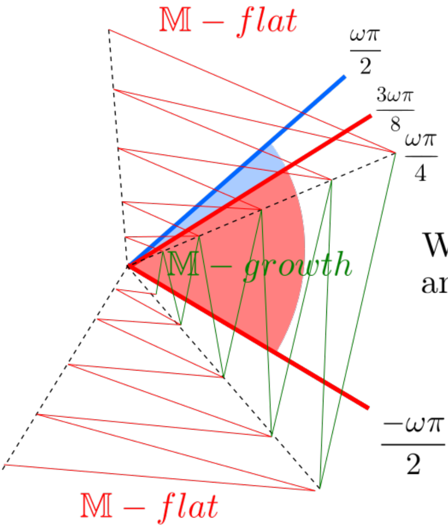

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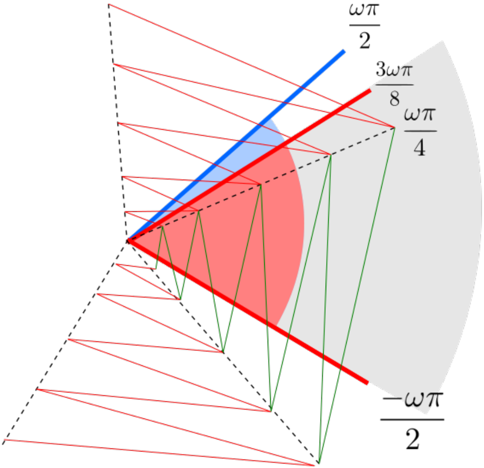

$f(z)$ is $\mathbb{M} - flat$

$e^{V(a/z)}$has at worst $\mathbb{M} - growth$

We choose suitably $|a|$

$$\Downarrow$$

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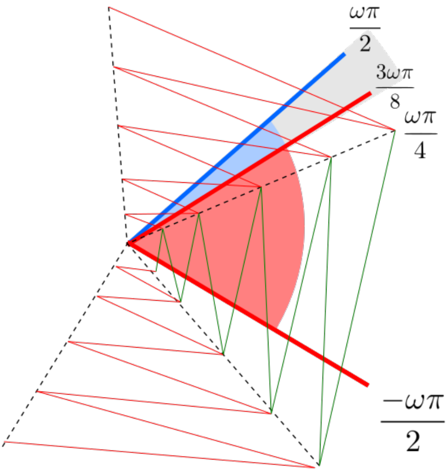
Idea of the proof for $\gamma = \omega(\mathbb{M}) = \omega$



$f(z)$ is bounded

$e^{V(a/z)}$ is $\mathbb{M}-flat$

$$\Downarrow$$

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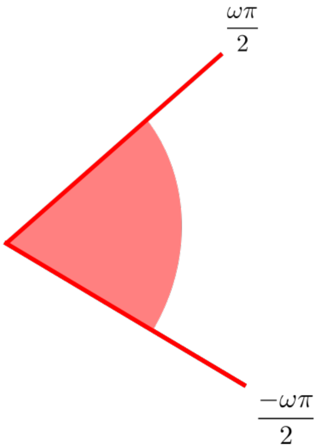
Idea of the proof for $\gamma = \omega(\mathbb{M}) = \omega$



$F(z)$ is $\mathbb{M} - flat$

$$\Big\Downarrow \text{ Mandelbrojt } \sum_{p=0}^{\infty} (m_p)^{-1/\omega} = \infty$$

$$F(z) \equiv 0$$

$$\Downarrow$$

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Partial version

If $\gamma < \omega(\mathbb{M})$, we fix $\gamma < \mu < \omega(\mathbb{M})$. By Mandelbrojt's theorem, there exists a nontrivial $\mathbb{M}$-flat function $f \in \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\mu)$. Then f is analytic and bounded in S_γ and continuous in $\overline{S_\gamma}\backslash\{0\}$ and f is $\mathbb{M}$-flat in direction $-\pi\gamma/2$.

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If $\gamma = \omega(\mathbb{M})$ and $\sum_{p=0}^{\infty}(m_p)^{-1/\omega(\mathbb{M})} < \infty$, we consider

$$\mathcal{A}_{\mathbb{M}}(S_\gamma) = \{f \in \mathcal{H}(S_\gamma); \quad \exists A > 0 \quad s.t. \quad \sup_{p\in\mathbb{N}_0, z\in S_\gamma} \frac{|f^{(p)}(z)|}{A^p p! M_p} < \infty\}$$

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We have that $\mathcal{A}_{\mathbb{M}}(S_\gamma) \subseteq \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma) \subseteq \tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma)$.

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Theorem (B.R. Salinas(1955))

Let $\mathbb{M}$ be a (lc) sequence with $\lim_{p\to\infty} m_p = \infty$ and $\gamma > 0$. The following statements are equivalent:

(i) $\tilde{\mathcal{B}} : \mathcal{A}_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}}$ *is injective.*

(ii) $\gamma > \omega(\mathbb{M})$ *or* $\gamma = \omega(\mathbb{M})$ *and* $\sum_{p=0}^{\infty}((p+1)m_p)^{-1/(\omega(\mathbb{M})+1)} = \infty$.

B. R. Salinas, Funciones con momentos nulos, Rev. Acad. Ci. Madrid 49 (1955), 331–368.

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Watson's Lemma in one direction for sectorial regions

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Thank you for your attention

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