Painlevé functions, Fredholm determinants and combinatorics

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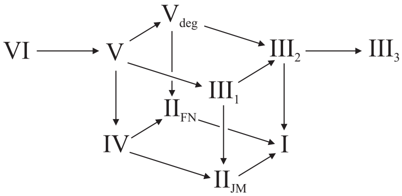
Gavrylenko, OL, 1608.00958

Painlevé VI:

$$\left(t(t-1)\zeta''\right)^2 = -2\det\begin{pmatrix} 2\theta_0^2 & t\zeta'-\zeta & \zeta'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 \\ t\zeta'-\zeta & 2\theta_t^2 & (t-1)\zeta'-\zeta \\ \zeta'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 & (t-1)\zeta'-\zeta & 2\theta_1^2 \end{pmatrix}$$

- $\zeta(t) = t(t-1)\frac{d}{dt}\ln\tau$, where $\tau(t)$ is Painlevé VI tau function

Coalescence diagram:



	PVI	PV	PIII_1	PIII_2	PIII_3	PIV	PII	PI
♯(parameters)	4	3	2	1	0	2	1	0

(Special) solutions of Painlevé VI:

1. Hypergeometric Riccati family

$$\tau_{\text{Riccati}}(t) = (1-t)^{-\frac{N\left(N+\nu+\nu'\right)}{2}} \det\left[A_{j-k}(t)\right]_{j,k=0}^{N-1},$$

$$A_m(t) = \frac{\Gamma(1+\nu')\, t^{\frac{\eta-m}{2}} (1-t)^{\nu}}{\Gamma(1+\eta-m)\,\Gamma(1-\eta+m+\nu')}\, {}_2F_1\left[\begin{array}{c} -\nu, 1+\nu' \\ 1+\eta-m \end{array}\middle| \frac{t}{t-1}\right] + \frac{\xi\, \Gamma(1+\nu)\, t^{\frac{m-\eta}{2}} (1-t)^{\nu'}}{\Gamma(1-\eta+m)\,\Gamma(1+\eta-m+\nu)}\, {}_2F_1\left[\begin{array}{c} 1+\nu, -\nu' \\ 1-\eta+m \end{array}\middle| \frac{t}{t-1}\right]$$

- PVI parameters $(\theta_0, \theta_t, \theta_1, \theta_\infty) = \frac{1}{2}(\eta, N, -N-\nu-\nu', \nu-\nu'+\eta)$ depend on $\nu, \nu', \eta \in \mathbb{C}$ and $N \in \mathbb{Z}_{\geq 0}$
- 1-parameter family of initial conditions depending on $\xi \in \mathbb{C}$
- [Forrester, Witte, '02]

2. Elliptic Picard family

$$\tau_{\mathrm{Picard}}(t) = \frac{e^{i\pi\sigma^2\bar{\tau}}}{t^{\frac{1}{8}}(1-t)^{\frac{1}{8}}} \frac{\vartheta_3(\sigma\pi\bar{\tau}+\sigma'\pi|\bar{\tau})}{\vartheta_3(0|\bar{\tau})}, \qquad \bar{\tau} = \frac{iK'(t)}{K(t)}$$

- PVI parameters $(\theta_0, \theta_t, \theta_1, \theta_\infty) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$
- 2-parameter family of initial conditions depending on $\sigma, \sigma' \in \mathbb{C}$
- [Kitaev, Korotkin, '98]

3. Algebraic solutions

$$\tau_{H_3'}(t) = \frac{(1-s)^{\frac{1}{20}}\, s^{\frac{1}{20}}\, (1+3s)^{\frac{1}{12}}}{(1+s)^{\frac{3}{20}}\, (1-3s)^{\frac{11}{300}}\, (1+4s-s^2)^{\frac{1}{25}}},$$

$$t = \frac{(s-1)^5(3s+1)^3(s^2+4s-1)}{(s+1)^5(3s-1)^3(s^2-4s-1)}.$$

- $(\theta_0, \theta_t, \theta_1, \theta_\infty) = \left(0, 0, 0, -\frac{1}{5}\right)$, 10 branches
- no parameters in the initial conditions
- great icosahedron solution from [Dubrovin, Mazzocco, '98]

4. Fredholm determinant solutions

$$\tau_{\mathrm{BD}}(t) = \det\left(\mathbf{1} - \lambda K\big|_{(0,t)}\right),$$

where continuous ${}_2F_1$ kernel $K(x,y) = \dfrac{\psi(x)\varphi(y) - \varphi(x)\psi(y)}{x-y}$ is defined by

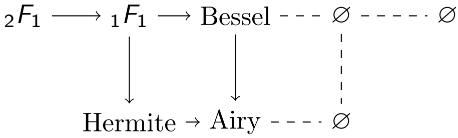
$$\varphi(x) = \quad x^{\theta_0}(1-x)^{\theta_1}\,{}_2F_1\left[\begin{array}{c}\theta_0+\theta_1+\theta_\infty, \theta_0+\theta_1-\theta_\infty \\ 2\theta_0\end{array}; x\right],$$

$$\psi(x) = x^{1+\theta_0}(1-x)^{\theta_1}\,{}_2F_1\left[\begin{array}{c}1+\theta_0+\theta_1+\theta_\infty, 1+\theta_0+\theta_1-\theta_\infty \\ 2+2\theta_0\end{array}; x\right].$$

- PVI parameters $(\theta_0, \theta_t = 0, \theta_1, \theta_\infty)$
- 1-parameter family of initial conditions depending on $\lambda \in \mathbb{C}$
- [Borodin, Deift, '01]

Solutions:

- Riccati: classical special functions



- elliptic (PVI)
- algebraic
- transcendental (almost all solutions!)

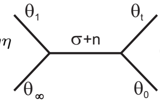
Question 1:

Can the general solution of Painlevé VI be expressed in terms of a Fredholm determinant?

General solution of PVI:

[Gamayun, Iorgov, OL, 1207.0787]

PVI tau function is a Fourier transform of $c = 1$ Virasoro conformal block:

$$\tau(t) = \sum_{n\in\mathbb{Z}} e^{in\eta}\,\mathcal{B}(\vec{\theta}, \sigma + n, t) = \sum_{n\in\mathbb{Z}} e^{in\eta}$$  (t)

- $\mathcal{B}(\vec{\theta}, \sigma, t) = t^{\alpha} \sum_{k=0}^{\infty} B_k(\vec{\theta}, \sigma)\, t^k$, with B_k rational in $\vec{\theta}, \sigma$ and determined by commutation relations of Vir
- as $c \to \infty$ (Vir $\to \mathfrak{sl}_2$), conformal block $\mathcal{B}(t) \sim {}_2F_1(t)$
- all 4 parameters $(\theta_0, \theta_t, \theta_1, \theta_\infty) \Longleftrightarrow$ external momenta
- 2 integration constants $(\sigma, \eta) \Longleftrightarrow$ internal momentum + Fourier conjugate variable

CFT derivations:

[Iorgov, OL, Teschner, 1401.6104]

- understood in the framework of Liouville CFT and generalized to an arbitrary number of punctures (Garnier system)
- uses quantum monodromy of conformal blocks with additional level 2 degenerate insertions

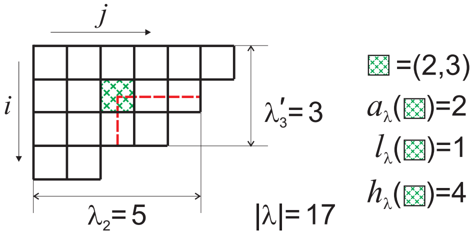
[Bershtein, Shchechkin, 1406.3008]

- bilinear differential-difference equations for conformal blocks coming from an embedding $\mathrm{Vir} \oplus \mathrm{Vir} \subset \mathrm{NSR} \oplus \mathrm{F}$
- extends to arbitrary values of central charge c

AGT correspondence [Alday, Gaiotto, Tachikawa, '09]

$$\mathcal{B}(t) = \mathcal{Z}_{\text{inst}}(t) = \begin{array}{c}\text{combinatorial sum}\\ \text{over tuples of partitions}\end{array} \quad \text{[Nekrasov, '04]}$$

- coefficients of $\mathcal{B}(t)$ are explicit rational functions of parameters determined by geometry of appropriate Young diagram
- proved in [Alba, Fateev, Litvinov, Tarnopolsky, '10]
- provides series representation for general Painlevé VI function!



Young diagram associated to partition
$\lambda = \{6, 5, 4, 2\}$.

Conjecture [Gamayun, Iorgov, OL, 1207.0787]

Complete expansion of Painlevé VI tau function at $t = 0$ is given by

$$\tau(t) = \sum_{n\in\mathbb{Z}} e^{in\eta}\mathcal{B}(\vec{\theta}, \sigma + n; t),$$

where the function $\mathcal{B}(\vec{\theta}, \sigma; t)$ is explicitly given by

$$\mathcal{B}\left(\vec{\theta}, \sigma; t\right) = N^{\theta_1}_{\theta_\infty,\sigma} N^{\theta_t}_{\sigma,\theta_0} t^{\sigma^2-\theta_0^2-\theta_t^2}(1-t)^{2\theta_t\theta_1} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}\left(\vec{\theta}, \sigma\right) t^{|\lambda|+|\mu|},$$

$$\mathcal{B}_{\lambda,\mu}\left(\boldsymbol{\theta}, \sigma\right) = \prod_{(i,j)\in\lambda} \frac{\left((\theta_t+\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1+\sigma+i-j)^2-\theta_\infty^2\right)}{h^2_\lambda(i,j)\left(\lambda'_j-i+\mu_i-j+1+2\sigma\right)^2} \times$$

$$\times \prod_{(i,j)\in\mu} \frac{\left((\theta_t-\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1-\sigma+i-j)^2-\theta_\infty^2\right)}{h^2_\mu(i,j)\left(\mu'_j-i+\lambda_i-j+1-2\sigma\right)^2},$$

$$N^{\theta_2}_{\theta_3,\theta_1} = \frac{\prod_{\epsilon=\pm} G\left(1+\theta_3+\epsilon(\theta_1+\theta_2)\right) G\left(1-\theta_3+\epsilon(\theta_1-\theta_2)\right)}{G(1-2\theta_1)G(1-2\theta_2)G(1+2\theta_3)}.$$

Question 2:

How to understand this combinatorial structure without reference to CFT/gauge theory ?

$$\tau(t) \sim \sum_{n\in\mathbb{Z}} e^{in\eta} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}(\sigma+n)\, t^{(\sigma+n)^2+|\lambda|+|\mu|}$$

Question 2:

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Remark: Similar Fourier transform structure also appears for

- regular type expansions for PV, $\mathrm{PIII}_{1,2,3}$ at $t=0$ (irregular CBs)
- irregular expansions for PI–PV on Stokes rays at $t=\infty$

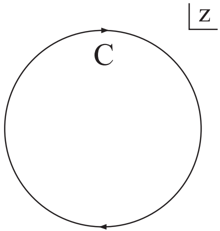
A digression on (block) Toeplitz determinants

- symbol $J(z) \in \mathrm{Hom}(\mathcal{C}, \mathrm{GL}_N(\mathbb{C}))$
- continues into an annulus $\mathcal{A} \supset \mathcal{C}$

$$J(z) = \sum_{k \in \mathbb{Z}'} J_k z^{k-\frac{1}{2}},$$

  with $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$

- $\det J(z)$ has no winding



(Block) Toeplitz and Hankel matrices associated to J :

$$T_K[J] = (J_{k-k'}), \quad H_K[J] = (J_{k+k'+1}), \quad \bar{H}_K[J] = (J_{-k-k'-1}),$$

with $k, k' = 0, \ldots, K-1$.

Theorem [Widom, '76] The limit $W[J] = \lim_{K \to \infty} \det T_K[J]$ exists and is equal to

$$W[J] = \det\left(\mathbf{1} - H_\infty[J]\bar{H}_\infty[J^{-1}]\right) = \det\left(\mathbf{1} + \tilde{U}\right)$$

with $\tilde{U} = \begin{pmatrix} 0 & \tilde{a} \\ \tilde{d} & 0 \end{pmatrix} \in \mathrm{End}\,\mathcal{V}$ and $\tilde{a} : \mathcal{V}_- \to \mathcal{V}_+$, $\tilde{d} : \mathcal{V}_+ \to \mathcal{V}_-$ given by

$$\tilde{a} = \Pi_+ J^{-1} \Pi_-, \qquad \tilde{d} = \Pi_- J \Pi_+.$$

- $N = 1 \implies$ strong Szegö limit theorem

von Koch formula

Write $\tilde{U}$ in the Fourier basis and expand Fredholm determinant using

$$\det\left(\mathbf{1}+\tilde{U}\right)=\sum_{\mathfrak{Y}\in 2^{\mathfrak{X}}}\det\tilde{U}_{\mathfrak{Y}}, \qquad \tilde{U}\in\mathbb{C}^{\mathfrak{X}\times\mathfrak{X}}$$

- multi-indices of principal minors

$$\det\tilde{U}_{\mathfrak{Y}}=\det\begin{pmatrix} 0 & \tilde{\mathsf{a}}^I_J \\ \tilde{\mathsf{d}}^J_I & 0\end{pmatrix}$$

  incorporate color indices $\alpha=1,\dots N$ and (half-)integer Fourier indices

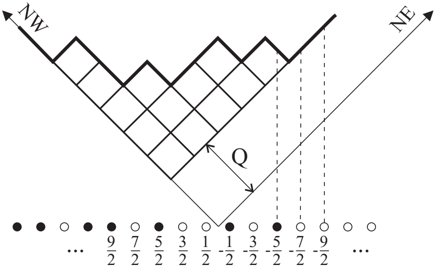
- combinatorial expansion

$$\det\left(\mathbf{1}+\tilde{U}\right)=\sum_{(I,J)\in\mathsf{Conf}_+}\det\tilde{\mathsf{a}}^I_J\det\tilde{\mathsf{d}}^J_I,$$

  with balance condition $|I|=|J|$

- Fourier indices in I and J are resp. positive and negative

- A Maya diagram is a map $\mathrm{m} : \mathbb{Z}' \to \{-1, 1\}$ subject to the condition $\mathrm{m}\,(p) = \pm 1$ for all but finitely many $p \in \mathbb{Z}'_{\pm}$ (positions of particles and holes)
- $\mathrm{charge(m)} = \sharp(\mathrm{particles}) - \sharp(\mathrm{holes})$



- here the charge $Q(\mathrm{m}) = 2$ and the positions of particles and holes are given by $\mathrm{p}(\mathrm{m}) = \left(\frac{13}{2}, \frac{7}{2}, \frac{3}{2}, \frac{1}{2}\right)$ and $\mathrm{h}(\mathrm{m}) = \left(-\frac{5}{2}, -\frac{1}{2}\right)$
- elements of Conf_+ are in bijection with N-tuples of Young diagrams of zero total charge
- $\mathbb{M}_0^N \cong \mathbb{Y}^N \times \mathfrak{Q}_N$, where $\mathfrak{Q}_N$ denotes the A_{N-1} root lattice:

$$\mathfrak{Q}_N := \left\{ \vec{Q} \in \mathbb{Z}^N \,\middle|\, \sum_{\alpha=1}^{N} Q^{(\alpha)} = 0 \right\}.$$

- in the case $N = 2$

$$\det\left(\mathbf{1} + \tilde{U}\right) = \sum_{(I,J) \in \mathbb{Y}^2 \times \mathbb{Z}} \det \tilde{\mathrm{a}}_J^I \det \tilde{\mathrm{d}}_I^J$$

Assume that J admits factorizations

$$J(z) = \Psi^{int}(z)^{-1}\Psi^{ext}(z) = \tilde{\Psi}_+(z)\,\tilde{\Psi}_-(z)^{-1}$$

where $\Psi^{int}(z)$, $\tilde{\Psi}_-(z)$ and $\Psi^{ext}(z)$, $\tilde{\Psi}_+(z)$ are analytic, respectively, outside and inside $\mathcal{C}$.

Corollary. For symbols admitting 1st factorization, the Widom's constant $W[J]$ may be rewritten as

$$W[J] = \det(\mathbf{1} + U), \qquad U = \begin{pmatrix} 0 & \mathsf{a} \\ \mathsf{d} & 0 \end{pmatrix} \in \mathrm{End}(\mathcal{V}_+ \oplus \mathcal{V}_-),$$

where the operators $\mathsf{a} : \mathcal{V}_- \to \mathcal{V}_+$, $\mathsf{d} : \mathcal{V}_+ \to \mathcal{V}_-$ are defined by

$$\mathsf{a} = \Psi^{ext}\Pi_+\Psi^{ext\,-1}\Big|_{\mathcal{V}_-}, \qquad d = \Psi^{int}\Pi_-\Psi^{int\,-1}\Big|_{\mathcal{V}_+}.$$

They thus have integrable kernels

$$(\mathsf{a}g)(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \mathsf{a}(z,z')\,g(z')\,dz', \quad \mathsf{a}(z,z') = \frac{\Psi^{ext}(z)\,\Psi^{ext}(z')^{-1} - \mathbf{1}}{z - z'},$$

$$(\mathsf{d}g)(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \mathsf{d}(z,z')\,g(z')\,dz', \quad \mathsf{d}(z,z') = \frac{\mathbf{1} - \Psi^{int}(z)\,\Psi^{int}(z')^{-1}}{z - z'}.$$

2nd Widom's theorem ['74]

Suppose that $J(z)$ smoothly depends on an additional parameter t.

For symbols admitting left and right factorizations, the log-derivatives of the Widom's constant wrt parameters are given by

$$\partial_t \ln W[J] = \frac{1}{2\pi i} \oint_C \mathrm{Tr} \left(J^{-1} \partial_t J \left[\partial_z \left(\tilde{\Psi}_- \right) \tilde{\Psi}_-^{-1} + \Psi^{ext\,-1} \partial_z \left(\Psi^{ext} \right) \right] \right) dz.$$

- Widom's constant $=$ tau function

Monodromy preserving deformation

Consider rank N Fuchsian system on $\mathbb{P}^1$:

$$\partial_z \Phi = \Phi A(z),$$
$$A(z) = \sum_{\nu=1}^{n-1} \frac{A_\nu}{z - a_\nu}, \qquad A_\nu \in \mathfrak{sl}_N$$

- n regular singular points $a_1, \ldots, a_{n-1}, \infty$

Monodromy representation:

$$\rho : \pi_1(\mathbb{P}^1 \backslash \{a\}) \to SL_N(\mathbb{C})$$

- different choices of the basis of solutions $\Longrightarrow$ equivalent representations

Riemann-Hilbert correspondence:

$$\mathcal{RH}: \begin{array}{c}\text{parameter set } \mathcal{P} \\ \text{of the linear system}\end{array} \longrightarrow \begin{array}{c}\text{space } \mathcal{M} \\ \text{of monodromy data}\end{array}$$

Schlesinger equations:

$$\partial_{a_\mu} A_\nu = \frac{[A_\mu, A_\nu]}{a_\nu - a_\mu}, \qquad \mu \neq \nu$$

- non-autonomous hamiltonian system
- a_ν's play the role of times
- tau function generates hamiltonians of isomonodromic flows:

$$H_\mu := \partial_{a_\nu} \ln \tau(a) = \frac{1}{2} \operatorname{res}_{z=a_\nu} \operatorname{tr} A^2(z)$$

- $\operatorname{sp}(A_\nu)$ conserved due to Lax form

Consider moduli space of representations with fixed local monodromies.

$$\mathcal{M}_{\boldsymbol{\theta}} := \mathrm{Hom}(\pi_1(\mathbb{P}^1 \backslash \{a\}, SL(N, \mathbb{C}))/\sim$$

Example: $N = 2$

- Schlesinger $\Longrightarrow$ Garnier system $\mathcal{G}_{n-3}$
- $\dim \mathcal{M}_{\boldsymbol{\theta}} = 3(n-1) - 3 - n = 2(n-3)$
  (complete set of conserved quantities for $\mathcal{G}_{n-3}$!)
- $n = 4 \Longrightarrow$ Painlevé VI; $a = \{0, t, 1, \infty\}$

Monodromy provides a convenient labeling of Painlevé functions.

solution of Painlevé equations	=	construction of inverse map $\mathcal{RH}^{-1}$

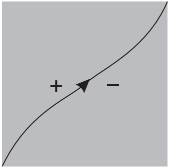
Scheme of the proof

Step 1: Represent the tau function of the Schlesinger system in the form of Fredholm determinant

- arbitrary rank N, arbitrary number n of regular singularities

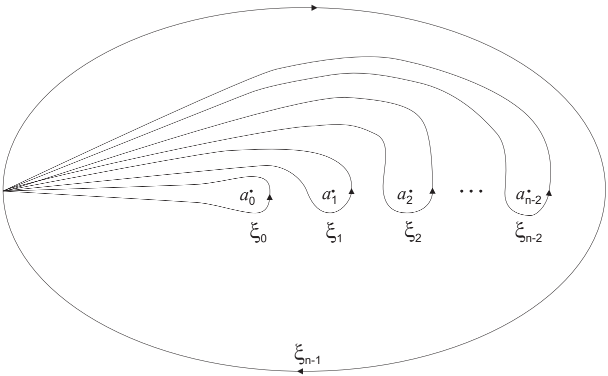
Riemann-Hilbert setup

- contour Γ on a Riemann surface Σ
- jump matrix $J : \Gamma \to \mathrm{GL}\,(N, \mathbb{C})$



RHP defined by (Γ, J) is to find analytic invertible matrix function $\Psi : \Sigma \backslash \Gamma \to \mathrm{GL}\,(N, \mathbb{C})$ whose boundary values satisfy

$$\Psi_+ = J \Psi_-$$

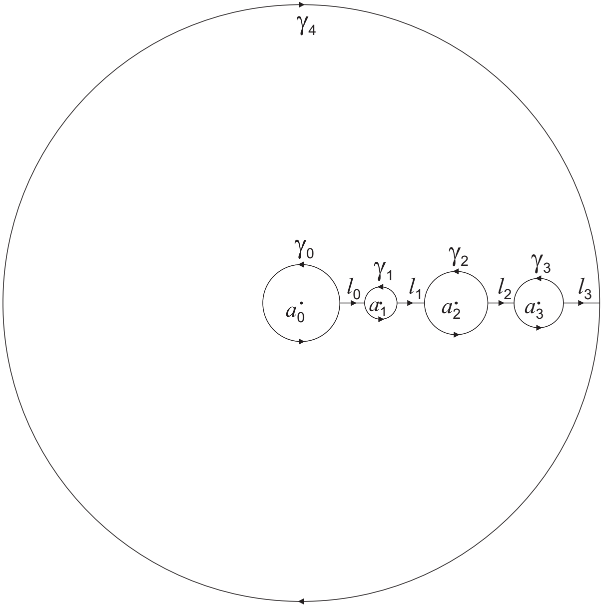


Monodromy representation $\rho : \pi_1\left(\mathbb{P}^1 \backslash a\right) \to \mathrm{GL}\left(N, \mathbb{C}\right)$ generated by

$$M_k = \rho\left(\xi_k\right) = {M_{1\to k-1}}^{-1} M_{1\to k}$$

Assume that all $M_{1\to k} = M_1 \dots M_k$ are diagonalizable,

$$M_{1\to k} = S_k e^{2\pi i \mathfrak{S}_k} S_k^{-1}, \qquad \mathfrak{S}_k = \mathrm{diag}\left\{\sigma_{k,1}, \dots, \sigma_{k,N}\right\}.$$



Contour Γ for $n = 5$

Fundamental matrix solution

$$\Phi(z) = \begin{cases} \Psi(z), & z \text{ outside } \gamma_{1\ldots n}, \\ C_k (a_k - z)^{\Theta_k} \Psi(z), & z \text{ inside } \gamma_k, \quad k = 1, \ldots, n-1, \\ C_n (-z)^{-\Theta_n} \Psi(z), & z \text{ inside } \gamma_n. \end{cases}$$

- only piecewise constant jumps on $\mathbb{R}_{>0}$
- matrix $\Phi^{-1}\partial_z\Phi$ meromorphic on $\mathbb{P}^1$ with poles only possible at $a_1, \ldots, a_n$
- local analysis shows that

$$\partial_z\Phi = \Phi A(z), \qquad A(z) = \sum_{k=1}^{n} \frac{A_k}{z - a_k}$$

with $A_k = \Psi(a_k)^{-1} \Theta_k \Psi(a_k)$

Jump data

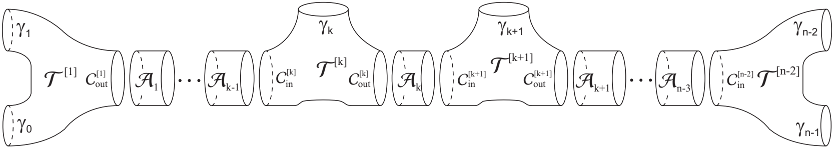
- local exponents: n diagonal non-resonant $N \times N$ matrices $\Theta_k = \mathrm{diag}\{\theta_{k,1}, \dots, \theta_{k,N}\}$ $(k = 1, \dots, n)$ satisfying a consistency relation $\sum_{k=1}^{n} \mathrm{Tr}\, \Theta_k = 0$
- $2n$ connection matrices $C_{k,\pm} \in \mathrm{GL}(N, \mathbb{C})$ satisfying the constraints

$$\begin{aligned} M_{1\to k} &:= C_{k,-} e^{2\pi i \Theta_k} C_{k,+}^{-1} = C_{k+1,-} C_{k+1,+}^{-1}, \qquad k = 1, \dots n-2, \\ M_{1\to n-1} &:= C_{n-1,-} e^{2\pi i \Theta_{n-1}} C_{n-1,+}^{-1} = C_{n,-} e^{-2\pi i \Theta_n} C_{n,+}^{-1}, \\ M_{1\to n} &:= \mathbf{1} = C_{n,-} C_{n,+}^{-1} = C_{1,-} C_{1,+}^{-1}, \end{aligned}$$

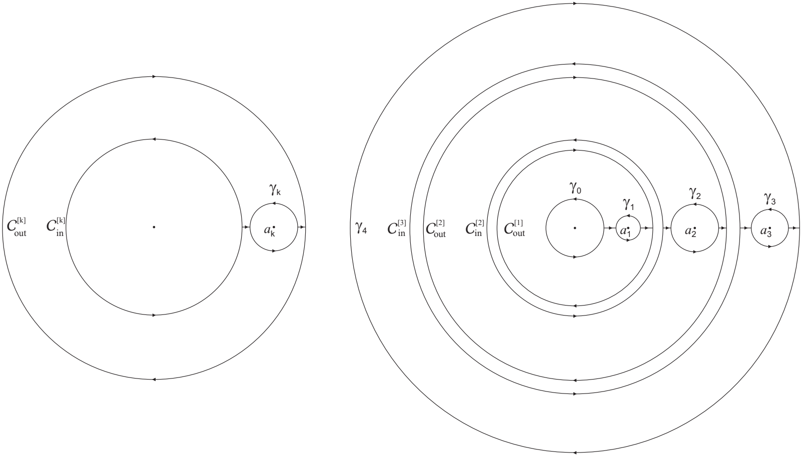
Jump matrix J

$$\begin{aligned} J(z)\Big|_{\ell_k} &= M_{1\to k}^{-1}, && k = 1, \dots, n-1, \\ J(z)\Big|_{\gamma_k} &= (a_k - z)^{-\Theta_k} C_{k,\pm}^{-1}, && \Im z \gtrless 0, \quad k = 1, \dots, n-1, \\ J(z)\Big|_{\gamma_n} &= (-z)^{\Theta_n} C_{n,\pm}^{-1}, && \Im z \gtrless 0. \end{aligned}$$

Auxiliary 3-point RHPs



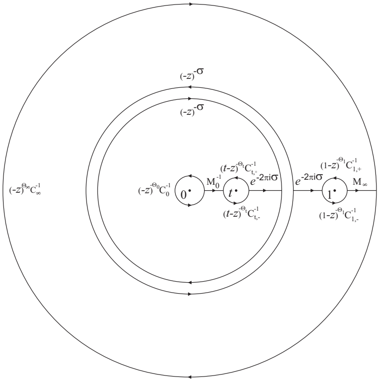
- we are going to associate to the n-point RHP $n-2$ 3-point RHPs assigned to different trinions



Contour $\Gamma^{[k]}$ (left) and $\hat{\Gamma}$ for $n = 5$ (right)

- $\hat{\Psi}(z) = \begin{cases} \Psi(z) & \text{outside the annuli,} \\ (-z)^{-\mathfrak{S}_k} S_k^{-1} \Psi(z) & \text{inside.} \end{cases}$
- jumps on the boundary circles $\mathcal{C}_{k-1}^{\text{out}}$, $\mathcal{C}_k^{\text{in}}$ mimic regular singularities characterized by counterclockwise monodromies $M_{1 \to k}$

Example ($n = 4$)



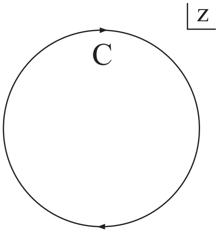
- For a circle $\mathcal{C} \subset \mathcal{A}$ define

$$\tilde{\Psi}(z) = \begin{cases} \Psi^{ext}(z)^{-1} \hat{\Psi}(z), & \text{outside } \mathcal{C}, \\ \Psi^{int}(z)^{-1} \hat{\Psi}(z), & \text{inside } \mathcal{C}. \end{cases}$$

- contour $\tilde{\Gamma} = \mathcal{C}$ (single circle !!!), jump $J : \mathcal{C} \to \mathrm{GL}(N, \mathbb{C})$ is

$$J(z) = \Psi^{int}(z)^{-1} \Psi^{ext}(z) = \tilde{\Psi}_+(z) \tilde{\Psi}_-(z)^{-1}$$

Example ($n = 4$)



2nd Widom's theorem then implies that

$$\tau(t) = t^{\frac{1}{2}\,\mathrm{Tr}\left(\mathfrak{S}^2 - \Theta_0^2 - \Theta_t^2\right)} \det(\mathbf{1} + K),$$

with

$$K = \begin{pmatrix} 0 & \mathsf{a} \\ \mathsf{d} & 0 \end{pmatrix} \in \mathrm{End}(\mathcal{V})$$

where the operators $\mathsf{a} : \mathcal{V}_- \to \mathcal{V}_+$ and $\mathsf{d} : \mathcal{V}_+ \to \mathcal{V}_-$ are

$$(\mathsf{a}g)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \mathsf{a}(z, z')\, g(z')\, dz', \quad \mathsf{a}(z, z') = \frac{\Psi^{ext}(z)\,\Psi^{ext}(z')^{-1} - \mathbf{1}}{z - z'},$$

$$(\mathsf{d}g)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \mathsf{d}(z, z')\, g(z')\, dz', \quad \mathsf{d}(z, z') = \frac{\mathbf{1} - \Psi^{int}(z)\,\Psi^{int}(z')^{-1}}{z - z'}.$$

For $N = 2$:

$$a(z,z') = \frac{(1-z')^{2\theta_1}\begin{pmatrix} K_{++}(z) & K_{+-}(z) \\ K_{-+}(z) & K_{--}(z) \end{pmatrix}\begin{pmatrix} K_{--}(z') & -K_{+-}(z') \\ -K_{-+}(z') & K_{++}(z') \end{pmatrix} - \mathbf{1}}{z-z'},$$

$$d(z,z') = \frac{\mathbf{1} - \left(1-\frac{t}{z'}\right)^{2\theta_t}\begin{pmatrix} \bar{K}_{++}(z) & \bar{K}_{+-}(z) \\ \bar{K}_{-+}(z) & \bar{K}_{--}(z) \end{pmatrix}\begin{pmatrix} \bar{K}_{--}(z') & -\bar{K}_{+-}(z') \\ -\bar{K}_{-+}(z') & \bar{K}_{++}(z') \end{pmatrix}}{z-z'},$$

with

$$K_{\pm\pm}(z) = {}_2F_1\left[\begin{matrix} \theta_1+\theta_\infty\pm\sigma, \theta_1-\theta_\infty\pm\sigma \\ \pm 2\sigma \end{matrix} ; z\right],$$

$$K_{\pm\mp}(z) = \pm\frac{\theta_\infty^2-(\theta_1\pm\sigma)^2}{2\sigma(1\pm 2\sigma)} z\, {}_2F_1\left[\begin{matrix} 1+\theta_1+\theta_\infty\pm\sigma, 1+\theta_1-\theta_\infty\pm\sigma \\ 2\pm 2\sigma \end{matrix} ; z\right],$$

$$\bar{K}_{\pm\pm}(z) = {}_2F_1\left[\begin{matrix} \theta_t+\theta_0\mp\sigma, \theta_t-\theta_0\mp\sigma \\ \mp 2\sigma \end{matrix} ; \frac{t}{z}\right],$$

$$\bar{K}_{\pm\mp}(z) = \mp t^{\mp 2\sigma} e^{\mp i\eta}\frac{\theta_0^2-(\theta_t\mp\sigma)^2}{2\sigma(1\mp 2\sigma)} \frac{t}{z}\, {}_2F_1\left[\begin{matrix} 1+\theta_t+\theta_0\mp\sigma, 1+\theta_t-\theta_0\mp\sigma \\ 2\mp 2\sigma \end{matrix} ; \frac{t}{z}\right].$$

Step 2: Write K in the Fourier basis and expand Fredholm determinant using von Koch formula:

$$\det(\mathbf{1}+K) = \sum_{\mathfrak{Y}\in 2^{\mathfrak{X}}} \det K_{\mathfrak{Y}}, \qquad K \in \mathbb{C}^{\mathfrak{X}\times\mathfrak{X}}$$

- multi-indices of principal minors

$$\det K_{\mathfrak{Y}} = \det \begin{pmatrix} 0 & \mathsf{a}^I_J \\ \mathsf{d}^J_I & 0 \end{pmatrix}$$

  incorporate color indices $\alpha = 1,\ldots N$ and (half-)integer Fourier indices

- combinatorial expansion

$$\det(\mathbf{1}+K) = \sum_{(I,J)\in\mathrm{Conf}_+} \det \mathsf{a}^I_J \det \mathsf{d}^J_I,$$

  with balance condition $|I| = |J|$

- elements of Conf_+ are in bijection with N-tuples of Young diagrams of zero total charge
- in the case $N = 2$

$$\det(\mathbf{1}+K) = \sum_{(I,J)\in\mathbb{Y}^2\times\mathbb{Z}} \det \mathsf{a}^I_J \det \mathsf{d}^J_I$$

Step 3: Explicit computation of elementary determinants $\det \mathrm{a}^{I}_{J}$, $\det \mathrm{d}^{J}_{I}$ of Plemelj operators

- in the case $N = 2 \Longrightarrow$ Cauchy determinants $\det \dfrac{1}{x_i - y_j}$
- rewrite resulting factorized expressions using lengths of rows/columns instead of positions of particles/holes of different colors

Cauchy-Plemelj operators

- associate to every trinion $\mathcal{T}_k$ with $k = 2, \ldots, n-3$ the spaces of vector-valued functions

$$\mathcal{H}^{[k]} = \bigoplus_{\epsilon=\mathrm{in},\mathrm{out}} \left(\mathcal{H}^{[k]}_{\epsilon,+} \oplus \mathcal{H}^{[k]}_{\epsilon,-} \right), \qquad \mathcal{H}^{[k]}_{\epsilon,\pm} = \mathbb{C}^N \otimes \mathcal{V}_{\pm}\left(\mathcal{C}^{\epsilon}_k\right).$$

- elements $f^{[k]} \in \mathcal{H}^{[k]}$ will be written as

$$f^{[k]} = \begin{pmatrix} f^{[k]}_{\mathrm{in},-} \\ f^{[k]}_{\mathrm{out},+} \end{pmatrix} \oplus \begin{pmatrix} f^{[k]}_{\mathrm{in},+} \\ f^{[k]}_{\mathrm{out},-} \end{pmatrix}.$$

- define an operator $\mathcal{P}^{[k]} : \mathcal{H}^{[k]} \to \mathcal{H}^{[k]}$ by

$$\mathcal{P}^{[k]} f^{[k]}(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{in}}_k \cup \mathcal{C}^{\mathrm{out}}_k} \frac{\Psi^{[k]}_{+}(z)\, \Psi^{[k]}_{+}(z')^{-1} f^{[k]}(z')\, dz'}{z - z'}$$

Lemma. We have $\left(\mathcal{P}^{[k]}\right)^2 = \mathcal{P}^{[k]}$ and $\ker \mathcal{P}^{[k]} = \mathcal{H}^{[k]}_{\mathrm{in},+} \oplus \mathcal{H}^{[k]}_{\mathrm{out},-}$. Moreover, $\mathcal{P}^{[k]}$ can be explicitly written as

$$\mathcal{P}^{[k]} : \begin{pmatrix} f^{[k]}_{\mathrm{in},-} \\ f^{[k]}_{\mathrm{out},+} \end{pmatrix} \oplus \begin{pmatrix} f^{[k]}_{\mathrm{in},+} \\ f^{[k]}_{\mathrm{out},-} \end{pmatrix} \mapsto \begin{pmatrix} f^{[k]}_{\mathrm{in},-} \\ f^{[k]}_{\mathrm{out},+} \end{pmatrix} \oplus \begin{pmatrix} \mathsf{a}^{[k]} & \mathsf{b}^{[k]} \\ \mathsf{c}^{[k]} & \mathsf{d}^{[k]} \end{pmatrix} \begin{pmatrix} f^{[k]}_{\mathrm{in},-} \\ f^{[k]}_{\mathrm{out},+} \end{pmatrix},$$

where the operators $\mathsf{a}^{[k]}$, $\mathsf{b}^{[k]}$, $\mathsf{c}^{[k]}$, $\mathsf{d}^{[k]}$ are defined by

$$\left(\mathsf{a}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{in}}_k} \left[\Psi^{[k]}_+(z)\, \Psi^{[k]}_+(z')^{-1} - \mathbf{1}\right] \frac{g(z')\, dz'}{z - z'}, \qquad z \in \mathcal{C}^{\mathrm{in}}_k,$$

$$\left(\mathsf{b}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{out}}_k} \Psi^{[k]}_+(z)\, \Psi^{[k]}_+(z')^{-1} \frac{g(z')\, dz'}{z - z'}, \qquad z \in \mathcal{C}^{\mathrm{in}}_k,$$

$$\left(\mathsf{c}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{in}}_k} \Psi^{[k]}_+(z)\, \Psi^{[k]}_+(z')^{-1} \frac{g(z')\, dz'}{z - z'}, \qquad z \in \mathcal{C}^{\mathrm{out}}_k,$$

$$\left(\mathsf{d}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{out}}_k} \left[\Psi^{[k]}_+(z)\, \Psi^{[k]}_+(z')^{-1} - \mathbf{1}\right] \frac{g(z')\, dz'}{z - z'}, \qquad z \in \mathcal{C}^{\mathrm{out}}_k.$$

- introduce the total space

$$\mathcal{H} := \bigoplus_{k=1}^{n-2} \mathcal{H}^{[k]}.$$

- there is a splitting

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-,$$

$$\mathcal{H}_\pm := \mathcal{H}^{[1]}_{\mathrm{out},\pm} \oplus \left(\mathcal{H}^{[2]}_{\mathrm{in},\mp} \oplus \mathcal{H}^{[2]}_{\mathrm{out},\pm}\right) \oplus \ldots \oplus \left(\mathcal{H}^{[n-3]}_{\mathrm{in},\mp} \oplus \mathcal{H}^{[n-3]}_{\mathrm{out},\pm}\right) \oplus \mathcal{H}^{[n-2]}_{\mathrm{in},\mp}.$$

- combine the 3-point projections $\mathcal{P}^{[k]}$ into an operator $\mathcal{P}_\oplus : \mathcal{H} \to \mathcal{H}$ given by the direct sum

$$\mathcal{P}_\oplus = \mathcal{P}^{[1]} \oplus \ldots \oplus \mathcal{P}^{[n-2]}.$$

- similarly, define another projection $\mathcal{P}_\Sigma : \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{P}_\Sigma f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}_\Sigma} \frac{\hat{\Psi}_+(z)\, \hat{\Psi}_+(z')^{-1} f(z')\, dz'}{z - z'}, \quad \mathcal{C}_\Sigma := \bigcup_{k=1}^{n-3} \mathcal{C}_k^{\mathrm{out}} \cup \mathcal{C}_{k+1}^{\mathrm{in}}.$$

- it is easy to show that $\mathcal{P}_\Sigma \mathcal{P}_\oplus = \mathcal{P}_\oplus$ and $\mathcal{P}_\oplus \mathcal{P}_\Sigma = \mathcal{P}_\Sigma$
- the space

$$\mathcal{H}_\mathcal{T} := \operatorname{im} \mathcal{P}_\oplus = \operatorname{im} \mathcal{P}_\Sigma .$$

  can be thought of as the subspace of functions on the union of boundary circles $\mathcal{C}_k^{\mathrm{in}}$, $\mathcal{C}_k^{\mathrm{out}}$ that can be continued inside $\bigcup_{k=1}^{n-2} \mathcal{T}_k$ with monodromy and singular behavior of the n-point fundamental matrix solution $\Phi(z)$
- varying the positions of singular points, one obtains a trajectory of $\mathcal{H}_\mathcal{T}$ in the infinite-dimensional Grassmannian $\mathrm{Gr}(\mathcal{H})$ defined with respect to the splitting $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$
- each of the subspaces $\mathcal{H}_\pm$ may be identified with $N(n-3)$ copies of the space $L^2(S^1)$ of functions on a circle; the factor $n-3$ corresponds to the number of annuli and N is the rank of the appropriate RHP

- introduce operators $\mathcal{P}_{\oplus,+} : \mathcal{H}_+ \to \mathcal{H}_\mathcal{T}$ and $\mathcal{P}_{\Sigma,+} : \mathcal{H}_+ \to \mathcal{H}_\mathcal{T}$ given by restrictions of $\mathcal{P}_\oplus$ and $\mathcal{P}_\Sigma$ to $\mathcal{H}_+$
- define $L \in \mathrm{End}\,(\mathcal{H}_+)$ defined by

$$L := {\mathcal{P}_{\oplus,+}}^{-1}\mathcal{P}_{\Sigma,+}$$

- there exists a basis in which $L^{-1} = \mathbf{1} - K$, with

$$K = \begin{pmatrix} U_1 & V_1 & 0 & . & 0 \\ W_1 & U_2 & V_2 & . & 0 \\ 0 & W_2 & U_3 & . & . \\ . & . & . & . & V_{n-4} \\ 0 & 0 & . & W_{n-4} & U_{n-3} \end{pmatrix}, \ \vec{g} = \begin{pmatrix} \tilde{g}_1 \\ \tilde{g}_2 \\ \vdots \\ \tilde{g}_{n-3} \end{pmatrix}, \ \tilde{g}_k = \begin{pmatrix} g^{[k]}_{\mathrm{out},+} \\ g^{[k+1]}_{\mathrm{in},-} \end{pmatrix},$$

$$U_k = \begin{pmatrix} 0 & \mathrm{a}^{[k+1]} \\ \mathrm{d}^{[k]} & 0 \end{pmatrix}, \ V_k = \begin{pmatrix} \mathrm{b}^{[k+1]} & 0 \\ 0 & 0 \end{pmatrix}, \ W_k = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{c}^{[k+1]} \end{pmatrix}$$

Definition

The tau function associated to the Riemann-Hilbert problem for Ψ is defined as

$$\tau(a) := \det\left(L^{-1}\right)$$

Theorem

We have

$$\tau(a) = \Upsilon(a)^{-1}\tau_{\mathrm{JMU}}(a),$$

where $\tau_{\mathrm{JMU}}(a)$ is defined up to a prefactor independent of a by

$$d_a \ln \tau_{\mathrm{JMU}} = \sum_{1\leq k<l\leq n-1} \operatorname{Tr} A_k A_l \, d\ln(a_k - a_l),$$

and $\Upsilon(a) = \prod_{k=2}^{n-2} a_k^{\bar{\Delta}_k - \bar{\Delta}_{k-1} - \Delta_k}$, with $\Delta_k = \frac{1}{2}\operatorname{Tr}\Theta_k^2$, $\bar{\Delta}_k = \frac{1}{2}\operatorname{Tr}\mathfrak{S}_k^2$

Fourier basis

Let us represent the elements of $\mathcal{H}_{\mathcal{C}}$ by their Laurent series inside $\mathcal{A}$,

$$f(z) = \sum_{p \in \mathbb{Z}'} f^p z^{-\frac{1}{2}+p}, \qquad f^p \in \mathbb{C}^N,$$

and write integral kernels of 3-point projection operators $\mathrm{a}^{[k]}$, $\mathrm{b}^{[k]}$, $\mathrm{c}^{[k]}$, $\mathrm{d}^{[k]}$ as

$$\mathrm{a}^{[k]}(z,z') := \frac{\Psi_+^{[k]}(z)\,\Psi_+^{[k]}(z')^{-1} - \mathbf{1}}{z-z'} = \sum_{p,q\in\mathbb{Z}'_+} \mathrm{a}^{[k]}{}_{-q}^{\;p}\, z^{-\frac{1}{2}+p} z'^{-\frac{1}{2}+q}, \qquad z,z' \in \mathcal{C}_k^{\mathrm{in}},$$

$$\mathrm{b}^{[k]}(z,z') := -\frac{\Psi_+^{[k]}(z)\,\Psi_+^{[k]}(z')^{-1}}{z-z'} = \sum_{p,q\in\mathbb{Z}'_+} \mathrm{b}^{[k]}{}_{q}^{p}\, z^{-\frac{1}{2}+p} z'^{-\frac{1}{2}-q}, \quad z \in \mathcal{C}_k^{\mathrm{in}}, z' \in \mathcal{C}_k^{\mathrm{out}}$$

$$\mathrm{c}^{[k]}(z,z') := \frac{\Psi_+^{[k]}(z)\,\Psi_+^{[k]}(z')^{-1}}{z-z'} = \sum_{p,q\in\mathbb{Z}'_+} \mathrm{c}^{[k]}{}_{-q}^{-p}\, z^{-\frac{1}{2}-p} z'^{-\frac{1}{2}+q}, \; z \in \mathcal{C}_k^{\mathrm{out}}, z' \in \mathcal{C}_k^{\mathrm{in}}$$

$$\mathrm{d}^{[k]}(z,z') := \frac{\mathbf{1} - \Psi_+^{[k]}(z)\,\Psi_+^{[k]}(z')^{-1}}{z-z'} = \sum_{p,q\in\mathbb{Z}'_+} \mathrm{d}^{[k]}{}^{-p}_{\;\;q} z^{-\frac{1}{2}-p} z'^{-\frac{1}{2}-q}, \quad z,z' \in \mathcal{C}_k^{\mathrm{out}}.$$

Von Koch's formula

Let $A \in \mathbb{C}^{\mathfrak{X}\times\mathfrak{X}}$ be a matrix indexed by a discrete and possibly infinite set $\mathfrak{X}$. The basic tool for expanding $\tau(a)$ is the formula

$$\det(\mathbf{1}+A) = \sum_{\mathfrak{Y}\in 2^{\mathfrak{X}}} \det A_{\mathfrak{Y}},$$

where $\det A_{\mathfrak{Y}}$ denotes the $|\mathfrak{Y}|\times|\mathfrak{Y}|$ principal minor obtained by restriction of A to a subset $\mathfrak{Y}\subseteq\mathfrak{X}$.

In our case : A is K in the Fourier basis. Elements of $\mathfrak{X}$ are multi-indices which encode the following data:

- positions of the blocks $\mathrm{a}^{[k]}$, $\mathrm{b}^{[k]}$, $\mathrm{c}^{[k]}$, $\mathrm{d}^{[k]}$ in K
- a half-integer Fourier index of the appropriate block;
- a color index in $\{1,\dots,N\}$.

Combine Fourier and color indices into one multi-index

$$\imath = (p,\alpha) \in \mathfrak{N} := \mathbb{Z}' \times \{1,\dots,N\}$$

Unordered sets $\{\imath_1,\dots,\imath_m\}\in 2^{\mathfrak{N}}$ of such multi-indices are denoted by I or J. Given $M\in\mathbb{C}^{\mathfrak{N}\times\mathfrak{N}}$, we denote by M_I^J its restriction to rows I and columns J.

Principal minor

$$
\begin{pmatrix}
0 & \left(a^{[2]}\right)^{I_1}_{J_1} & \left(b^{[2]}\right)^{I_1}_{I_2} & 0 & 0 & \cdot & \cdot & 0 & 0 \\
\left(d^{[1]}\right)^{J_1}_{I_1} & 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\
0 & 0 & 0 & \left(a^{[3]}\right)^{I_2}_{J_2} & \left(b^{[3]}\right)^{I_2}_{I_3} & \cdot & \cdot & 0 & 0 \\
0 & \left(c^{[2]}\right)^{J_2}_{J_1} & \left(d^{[2]}\right)^{J_2}_{I_2} & 0 & 0 & \cdot & \cdot & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \left(c^{[3]}\right)^{J_3}_{J_2} & \left(d^{[3]}\right)^{J_3}_{I_3} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \left(b^{[n-3]}\right)^{I_{n-2}}_{I_{n-3}} & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot & 0 & 0 & \\
0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 & \left(a^{[n-2]}\right)^{I_{n-3}}_{J_{n-3}} \\
0 & 0 & 0 & 0 & \cdot & \cdot & \left(c^{[n-3]}\right)^{J_{n-3}}_{J_{n-4}} & \left(d^{[n-3]}\right)^{J_{n-3}}_{I_{n-3}} & 0
\end{pmatrix}
$$

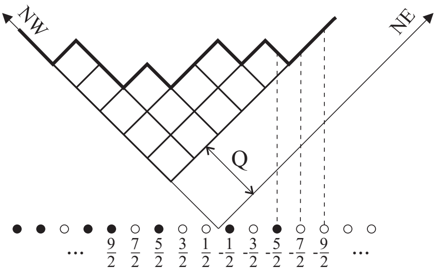
- vanishes unless balance condition $|I_k| = |J_k|$ is satisfied
- factorization into a product of elementary determinants

$$
Z^{I_{k-1},J_{k-1}}_{I_k,J_k}\left(\mathcal{T}^{[k]}\right) := (-1)^{|I_k|} \det \begin{pmatrix} \left(a^{[k]}\right)^{I_{k-1}}_{J_{k-1}} & \left(b^{[k]}\right)^{I_{k-1}}_{I_k} \\ \left(c^{[k]}\right)^{J_k}_{J_{k-1}} & \left(d^{[k]}\right)^{J_k}_{I_k} \end{pmatrix}
$$

Corollary: Fredholm determinant $\tau(a)$ is given by

$$\tau(a) = \sum_{(\vec{I},\vec{J})\in\mathsf{Conf}_+} \prod_{k=1}^{n-2} Z_{I_k, J_k}^{I_{k-1}, J_{k-1}} \left(\mathcal{T}^{[k]}\right)$$

- The set Conf_+ of proper balanced configurations $\left(\vec{I},\vec{J}\right)$ may be described in terms of Maya diagrams and charged partitions
- A Maya diagram is a map $\mathrm{m} : \mathbb{Z}' \to \{-1, 1\}$ subject to the condition $\mathrm{m}(p) = \pm 1$ for all but finitely many $p \in \mathbb{Z}'_{\pm}$ (positions of particles and holes)
- $\mathrm{charge}(\mathrm{m}) = \sharp(\mathrm{particles}) - \sharp(\mathrm{holes})$
- balanced configurations (I_k, J_k) are in one-to-one correspondence with N-tuples of Maya diagrams of zero total charge



- here the charge $Q(\mathrm{m}) = 2$ and the positions of particles and holes are given by $\mathrm{p}(\mathrm{m}) = \left(\frac{13}{2}, \frac{7}{2}, \frac{3}{2}, \frac{1}{2}\right)$ and $\mathrm{h}(\mathrm{m}) = \left(-\frac{5}{2}, -\frac{1}{2}\right)$
- $\mathbb{M}_0^N \cong \mathbb{Y}^N \times \mathfrak{Q}_N$, where $\mathfrak{Q}_N$ denotes the A_{N-1} root lattice:

$$\mathfrak{Q}_N := \left\{ \vec{Q} \in \mathbb{Z}^N \,\middle|\, \sum_{\alpha=1}^{N} Q^{(\alpha)} = 0 \right\}.$$

Alternative combinatorial notation :

$$Z_{\vec{Y}_k, \vec{Q}_k}^{\vec{Y}_{k-1}, \vec{Q}_{k-1}}\left(\mathcal{T}^{[k]}\right) := Z_{I_k, J_k}^{I_{k-1}, J_{k-1}}\left(\mathcal{T}^{[k]}\right),$$

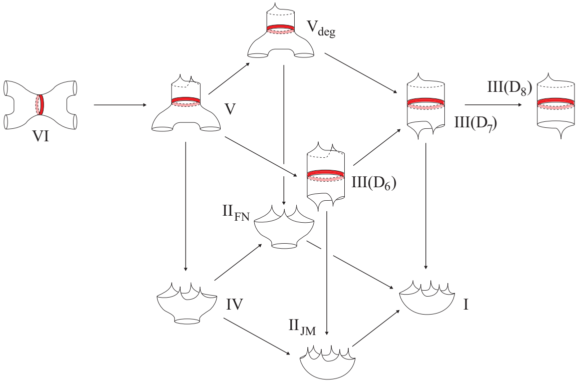
Theorem

Fredholm determinant $\tau(a)$ can be written as a combinatorial series

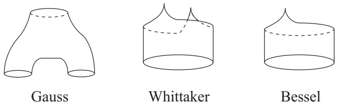
$$\tau(a) = \sum_{\vec{Q}_1,\ldots\vec{Q}_{n-3}\in\mathfrak{Q}_N} \sum_{\vec{Y}_1,\ldots\vec{Y}_{n-3}\in\mathbb{Y}^N} \prod_{k=1}^{n-2} Z^{\vec{Y}_{k-1},\vec{Q}_{k-1}}_{\vec{Y}_k,\vec{Q}_k}\left(\mathcal{T}^{[k]}\right)$$

- elementary determinants $Z^{\vec{Y}_{k-1},\vec{Q}_{k-1}}_{\vec{Y}_k,\vec{Q}_k}$ are constructed from matrix elements of 3-point Plemelj operators in Fourier basis
- in rank $N = 2$, they are given by Cauchy matrices conjugated by diagonal factors $\Rightarrow$ explicitly computable !!!
- the result coincides with dual Nekrasov partition function for $U(2)$ linear quiver gauge theory with $\epsilon_1 + \epsilon_2 = 0$
- series representation for general solution of PVI/Garnier system
- rank $N \Rightarrow$ a sum of $N-1$ Cauchy matrices (unless additional spectral conditions are imposed)

Other Painlevé equations



Chekhov-Mazzocco-Rubtsov confluence diagram



Some solvable RHPs in rank $N = 2$

Conclusions

1. Isomonodromic tau functions of Fuchsian systems can be written as block Fredholm determinants whose kernels are built of fundamental solutions of 3-point Fuchsian systems
2. Expanding these determinants in Fourier basis leads to combinatorial series over tuples of partitions
3. The coefficients of the series can be computed explicitly when 3-point solutions have hypergeometric representations (in particular for $N = 2$)