Mean values and heat type equations

Grzegorz Łysik

Jan Kochanowski University in Kielce, Poland

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We start the lecture by introducing integral means over spheres and balls and derivation of the Pizzetti formulas for real analytic functions.

Next we give a characterization of real analytic functions in terms of integral means. The characterization justifies introduction of a definition of analytic functions on metric measure spaces.

We also apply the Pizzetti formulas to the study of convergence and Borel summability of formal solutions to the classical heat equation and its some generalizations.

In the second part we introduce integral mean value functions which are averages of integral means over spheres or balls and over their images under the action of a discrete group of complex rotations. In the case of real analytic functions we derive higher order Pizzetti's formulas. As applications we get:

- a maximum principle for polyharmonic functions;
- a characterization of convergent solutions to the initial value problem for higher order heat type equations;
- a Dirichlet type problem for polyharmonic functions.

1. Solid and spherical means value functions

Let Ω be a domain in $\mathbb{R}^n$, $u \in C^0(\Omega)$, $x \in \Omega$, $0 < R < \operatorname{dist}(x, \partial\Omega)$. Define solid and spherical means

$$M(u;\, x, R) = \frac{1}{\sigma(n) R^n} \int_{B(x,R)} u(y)dy, \tag{1a}$$

$$N(u;\, x, R) = \frac{1}{n\sigma(n) R^{n-1}} \int_{S(x,R)} u(y)dS(y), \tag{1b}$$

where $\sigma(n) = |B(0,1)| = \pi^{n/2}/\Gamma(n/2+1)$.

Lemma 1 *Let* $u \in C^0(\Omega)$.
Then for any $x \in \Omega$ *and* $0 < R < \operatorname{dist}(x, \partial\Omega)$

$$\Big(\frac{R}{n}\frac{\partial}{\partial R} + 1\Big)M(u;\, x, R) = N(u;\, x, R). \tag{2a}$$

If $u \in C^2(\Omega)$, *then*

$$\frac{n}{R}\frac{\partial}{\partial R}N(u;\, x, R) = M(\Delta u;\, x, R). \tag{2b}$$

2. Mean-value properties for real-analytic functions

Theorem 1 ([9], Mean-value property). *Let $u \in \mathcal{A}(\Omega)$, $x \in \Omega$. Then $M(u;\, x, R)$ and $N(u;\, x, R)$ are even, analytic functions at the origin and for small $|R|$,*

$$M(u;\, x, R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k \left(\frac{n}{2}+1\right)_k k!} R^{2k}, \tag{3a}$$

$$N(u;\, x, R) = \sum_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k \left(\frac{n}{2}\right)_k k!} R^{2k}. \tag{3b}$$

Here $(a)_k = a(a+1)\cdots(a+k-1)$ is the Pochhammer symbol.

Proof. Assume $x = 0$. Then for $y \in B(0, \rho)$,

$$u(y) = \sum_{\ell \in \mathbb{N}_0^n} \frac{1}{\ell_1! \cdots \ell_n!} \frac{\partial^{|\ell|}}{\partial x^\ell} u(x) y^\ell,$$

Take $R < \rho$. Note that if at least one of the exponents $\ell_1, \ldots, \ell_n$ is odd, then the integral of

$$y^\ell = y_1^{\ell_1} \cdots y_n^{\ell_n}$$

over $B(0, R)$ vanishes.

Next for $\ell = 2\kappa$ we derive

$$\begin{aligned}
&\frac{1}{\sigma(n)R^n}\int_{B(0,R)} y_1^{2\kappa_1}\cdots y_n^{2\kappa_n}\,dy \\
&= \frac{R^{2k}}{\sigma(n)}\int_{B(0,1)} y_1^{2\kappa_1}\cdots y_n^{2\kappa_n}\,dy \\
&= \frac{\left(\frac{1}{2}\right)_{\kappa_1}\cdots\left(\frac{1}{2}\right)_{\kappa_n}}{\left(\frac{n}{2}+1\right)_k}\,R^{2k}.
\end{aligned}$$

$$
\begin{aligned}
M(u;\, x, R) &= \cdots \\
&= \sum_{k=0}^{\infty} \frac{R^{2k}}{4^k \left(\frac{n}{2}+1\right)_k k!} \sum_{\kappa \in \mathbb{N}_0^n, |\kappa|=k} \frac{k!}{\kappa_1! \cdots \kappa_n!} \frac{\partial^{2k} u(x)}{\partial x^{2\kappa}} \\
&= \sum_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k \left(\frac{n}{2}+1\right)_k k!} \, R^{2k}.
\end{aligned}
$$

Clearly, the series converges for $|R|$ small enough. Finally, applying (2a) we get (3b). □

Theorem 2 ([9], Converse to the mean-value property).
Let $u \in C^{\infty}(\Omega)$ *and* $\rho \in C(\Omega, \mathbb{R}_+)$. *If*

$$\widetilde{M}(x,R) = \sum\nolimits_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k \left(\frac{n}{2}+1\right)_k k!} R^{2k}$$

or

$$\widetilde{N}(x,R) = \sum\nolimits_{k=0}^{\infty} \frac{\Delta^k u(x)}{4^k \left(\frac{n}{2}\right)_k k!} R^{2k}$$

is loc. uni. conv. in $\{(x,R) : x \in \Omega, |R| < \rho(x)\}$, *then* $u \in \mathcal{A}(\Omega)$, $M = \widetilde{M}$ *and* $N = \widetilde{N}$.

Proof. Fix a compact set $K \Subset \Omega$ and set $\rho = \operatorname{dist}(K, \partial\Omega)$. Then the assumption implies that

$$\frac{\Delta^k u(x)}{4^k \left(\frac{n}{2}\right)_k k!} R^{2k} \to 0 \quad \text{as} \quad k \to \infty$$

uniformly on $K \times \{|R| \le \rho_1\}$ with any $\rho_1 < \rho$. So for any $\rho_1 < \rho$ there exists a constant $C(\rho_1) < \infty$ such that for $k \in \mathbb{N}_0$

$$\sup_{x \in K} |\Delta^k u(x)| \le C(\rho_1) \cdot 4^k \big(n/2\big)_k k! \, \rho_1^{-2k}.$$

Applying inequalities $(a)_k \leq \big(\max(1, a)\big)^k k!$ for $a > 0$ and $2^k k! k! \leq (2k)!$ we see that for any compact set $K \Subset \Omega$ one can find $C < \infty$ and $L < \infty$ such that for $k \in \mathbb{N}_0$

$$\sup_{x \in K} |\Delta^k u(x)| \leq C(2k)! L^{2k}.$$

But by Komatsu theorem this inequality implies that $u \in \mathcal{A}(\Omega)$. Finally, by Theorem 1 we get $\widetilde{N}(x, R) = N(u; x, R)$ and $\widetilde{M}(x, R) = M(u; x, R)$. $\square$

3. A characterization of real analyticity

Lemma 2 ([11]). *Let* $u \in C^0(\Omega)$. *If there exist functions* $v_{2l} \in C^0(\Omega)$ *for* $l \in \mathbb{N}_0$ *and* $\rho \in C^0(\Omega, \mathbb{R}_+)$ *such that*

$$N(u; x, R) = \sum_{l=0}^{\infty} v_{2l}(x) R^{2l} \tag{4}$$

locally uniformly in $\{(x, R) : x \in \Omega, |R| < \rho(x)\}$, *then* $u \in C^\infty(\Omega)$ *and* $v_{2l} \in C^\infty(\Omega)$ *for* $l \in \mathbb{N}_0$.

Proof. Let $\widetilde{\eta}(r)$ be a smooth function on $[0, \infty)$ supported by $[0, 1]$ with $n\sigma(n) \int_0^1 \widetilde{\eta}(r) r^{n-1}\, dr = 1$. Then

$$\eta^\varepsilon(y) = \frac{1}{\varepsilon^n} \widetilde{\eta}\left(\frac{|y|}{\varepsilon}\right)$$

is a radially symmetric mollifier supported by $\overline{B}(0, \varepsilon)$.

Integrating in the spherical coordinates we get

$$\int_{B(0,\varepsilon)} \eta^{\varepsilon}(y)\, dy = \int_0^1 n\sigma(n)\widetilde{\eta}(r)r^{n-1}\, dr = 1.$$

Since η^{ε} is radially symmetric we have

$$\Delta\eta^{\varepsilon}(y) = \frac{1}{\varepsilon^{n+2}}\widetilde{\eta}''\left(\frac{|y|}{\varepsilon}\right) + \frac{1}{\varepsilon^{n+1}}\frac{n-1}{|y|}\widetilde{\eta}'\left(\frac{|y|}{\varepsilon}\right)$$
$$:= L_{\varepsilon}(\widetilde{\eta})(|y|). \tag{5}$$

For $x \in \Omega$ and $0 < \varepsilon < \rho(x)$ we compute

$$\Delta\big(\eta^\varepsilon * u\big)(x) = \big(\Delta\eta^\varepsilon\big) * u(x) = \int_{B(0,\varepsilon)} \big(\Delta\eta^\varepsilon\big)(y)u(x-y)\,dy$$

$$= \int_0^\varepsilon \bigg(\int_{S(0,1)} \big(\Delta\eta^\varepsilon\big)(rz)u(x-rz)\,dS(z)\bigg) r^{n-1}\,dr$$

$$\overset{(5)}{=} \int_0^\varepsilon \int_{S(0,1)} u(x-rz)\,dS(z) L_\varepsilon(\widetilde{\eta})(r) r^{n-1}\,dr$$

$$= \int_0^\varepsilon n\sigma(n)N(u;\, x, r)L_\varepsilon(\widetilde{\eta})(r)r^{n-1}\,dr$$

$$\overset{(4)}{=} \sum_{l=0}^{\infty} n\sigma(n)v_{2l}(x) \int_0^\varepsilon L_\varepsilon(\widetilde{\eta})(r)r^{2l+n-1}\,dr$$

$$= \sum_{l=0}^{\infty} n\sigma(n) v_{2l}(x)\, \varepsilon^{2l-2} \int_0^1 L_1(\widetilde{\eta})(t) t^{2l+n-1}\, dt$$
$$= \sum_{l=0}^{\infty} v_{2l}(x)\, \varepsilon^{2l-2} \cdot n\sigma(n) \int_{B(0,1)} L_1(\widetilde{\eta})(|y|)|y|^{2l}\, dy$$
$$= \sum_{l=1}^{\infty} v_{2l}(x)\, \varepsilon^{2l-2} \int_{B(0,1)} \Delta\eta^1(y)\, y^{2l}\, dy$$

since $\int_{B(0,1)} \Delta\eta^1(y)\, dy = \int_{S(0,1)} \frac{\partial\eta^1}{\partial n}(y)\, dS(y) = 0.$

So

$$\Delta\big(\eta^{\varepsilon} * u\big)(x) = \sum_{l=0}^{\infty} v_{2l+2}(x) \cdot \varepsilon^{2l} m_{2l+2}(\Delta\eta^1),$$

where $m_{2l}(\eta^1) = \int_{B(0,1)} \eta^1(y) y^{2l}\, dy$ for $l \in \mathbb{N}_0$.
Similarly for $k \in \mathbb{N}_0$ we get

$$\Delta^k\big(\eta^{\varepsilon} * u\big)(x) = \sum_{l=0}^{\infty} v_{2l+2k}(x) \cdot \varepsilon^{2l} m_{2l+2k}(\Delta^k\eta^1).$$

Note that $\Delta^k(\eta^\varepsilon * u)$ is distributionally convergent as $\varepsilon \to 0$. Hence

$$\begin{aligned} m_{2k}(\Delta^k \eta^1) v_{2k} &= \lim_{\varepsilon \to 0} \Delta^k(\eta^\varepsilon * u) \\ &= \Delta^k \big(\lim_{\varepsilon \to 0} \eta^\varepsilon * u \big) = \Delta^k v_0 \in D'(\Omega). \end{aligned}$$

Since $v_{2k} \in C(\Omega)$ applying the Weil lemma we conclude that $u = v_0 \in C^{2k}(\Omega)$.

Next for $0 \leq l \leq k$ we have

$$\Delta^k v_0 = \Delta^{k-l}(\Delta^l v_0) = m_{2l}(\Delta^l \eta^1) \cdot \Delta^{k-l} v_{2l} \in C^0(\Omega).$$

So $v_{2l} \in C^{2k-2l}(\Omega)$.
Since k is arbitrary big we conclude that $v_{2l} \in C^\infty(\Omega)$ for $l \in \mathbb{N}_0$. $\square$

Theorem 3 ([11, Theorem 3]). *Let* $u \in C^0(\Omega)$.
If there exist functions $u_k \in C^0(\Omega)$ *for* $k \in \mathbb{N}_0$ *and* $\rho \in C^0(\Omega, \mathbb{R}_+)$ *such that*

$$M(u;\, x, R) = \sum_{k=0}^{\infty} u_k(x) R^k,$$

locally uniformly in $\{(x, R) : x \in \Omega, |R| < \rho(x)\}$,
then u *is real analytic on* Ω *and for* $l \in \mathbb{N}_0$,

$$u_{2l+1} = 0 \text{ and } u_{2l} = \left(4^l \left(\tfrac{n}{2} + 1\right)_l l!\right)^{-1} \cdot \Delta^l u$$

Proof. Since $M(u;\, x, R)$ is even we get $u_{2l+1} = 0$. Applying the relation between mean value functions M and N, we get for $x \in \Omega$ and $0 < R < \rho(x)$,

$$N(u;\, x, R) = \Big(\frac{R}{n}\frac{\partial}{\partial R} + 1\Big)\Big(\sum\nolimits_{l=0}^{\infty} u_{2l}(x) R^{2l}\Big)$$
$$= \sum\nolimits_{l=0}^{\infty} \Big(\frac{2l}{n} + 1\Big) u_{2l}(x) R^{2l}.$$

Hence the assumptions of Lemma 2 are satisfied with $v_{2l} = \big(\frac{2l}{n} + 1\big) u_{2l}$ and so $u_{2l} \in C^{\infty}(\Omega)$ for $l \in \mathbb{N}_0$. Next we derive that $4^l \big(\frac{n}{2} + 1\big)_l l!\, u_{2l} = \Delta^l u$ and apply Theorem 2. $\square$

4. Analytic functions on metric measure spaces

It is well known that the mean value characterization of harmonic functions can be used to define harmonic functions on metric measure spaces (MMS). Namely, let (X, ρ, μ) be a metric measure space with a metric ρ and a Borel regular measure μ which is positive on open sets and finite on bounded sets.

Then a continuous function $u : \Omega \to \mathbb{R}$ on an open set $\Omega \subset X$ is said to be harmonic on Ω if for every $x \in \Omega$ and any closed ball $B(x, R) \subset \Omega$ it holds

$$u(x) = \frac{1}{\mu(B(x,R))} \int_{B(x,R)} u(y)\, d\mu(y).$$

If the measure is continuous with respect to the metric, then harmonic functions on MMS satisfy maximum principle, the Harnack type inequality and the Weierstrass and Montel convergence theorem, see [5].

Another approach to the theory of harmonic functions on MMS, based on variational methods, was proposed by Shanmugalingam [15].
Recently Alabern, Mateu and Verdera obtained in [2] a characterization of Sobolev spaces on $\mathbb{R}^n$ only in terms of the Euclidean metric and the Lebesgue measure which allowed them to define higher order Sobolev spaces on MMS.

We propose a definition of analytic functions on MMS.

Definition ([11]). Let (X, ρ, μ) be a metric measure space with a metric ρ and a Borel regular measure μ which is positive on open sets and finite on bounded sets. Let Ω be an open subset of X.
For any $x \in \Omega$ and $0 < R < \operatorname{dist}(x, \partial\Omega)$ define a solid mean of a continuous function $u \in C^0(\Omega)$ by

$$M_X(u;\, x, R) = \frac{1}{\mu(B_\rho(x, R))} \int_{B_\rho(x,R)} u(y)\, d\mu(y).$$

Definition ([11]). Let (X, ρ, μ) be a metric measure space and Ω be an open subset of X. Let $u \in C^0(\Omega, \mathbb{C})$. We say that u is (X, ρ, μ)-*analytic on* Ω and write $u \in \mathcal{A}_X(\Omega, \rho, \mu)$ if there exist functions $u_k \in C^0(\Omega)$ for $k \in \mathbb{N}_0$ and $\rho \in C^0(\Omega, \mathbb{R}_+)$ such that

$$M_X(u;\, x, R) = \sum_{k=0}^{\infty} u_k(x) R^k.$$

locally uniformly in $\{(x, R) : x \in \Omega, |R| < \rho(x)\}$.

By the Theorem 3 we get

Corollary 1 *Let* $X = \mathbb{R}^n$ *with the Euclidean metric* ρ *and the Lebesgue measure* λ*. Let* $\Omega \subset X$*.*
Then $u \in C^0(\Omega)$ *is* (X, ρ, λ)*-analytic on* Ω *if and only if it is real analytic on* Ω*.*

Definition ([11]). The metric measure space (X, ρ, μ) is called *analytizable* if for any $x \in X$ there exist open sets $U \subset \mathbb{R}^n$, $\Omega \subset X$ and a homeomorphism $\Phi : U \to^{\text{onto}} \Omega$ such that for $y \in \Omega$ and R small enough

$$\Phi\big(B(\Phi^{-1}(y), R)\big) = B_\rho(y, R)$$

and for Borel sets $A \subset \Omega$

$$\mu(A) = |\Phi^{-1}(A)|.$$

Theorem 4 ([11]). *Under the notations of the last definition let* $u : \Omega \to \mathbb{C}$ *be a continuous function. Then* u *is* (X, ρ, μ)*-analytic on* Ω *if and only if* $u \circ \Phi$ *is real analytic on* $U = \Phi^{-1}(\Omega)$.

Hence if X is locally homeomorphic to $\mathbb{R}^n$, then the metrical properties of X-analytic functions can be derived from the analogous properties of real analytic functions.

5. Functions of Laplacian growth

In order to control the growth of iterated Laplacians of smooth functions Aronszajn et al. [1] introduced the notion of the Laplacian growth.

Definition ([1]). Let $\rho > 0$ and $\tau \geq 0$. A function u smooth on $\Omega \subset \mathbb{R}^n$ is of *Laplacian growth* (ρ, τ) if for every $K \Subset \Omega$ and $\varepsilon > 0$ one can find $C = C(K, \varepsilon) < \infty$ such that for $k \in \mathbb{N}_0$

$$\sup_{x \in K} |\Delta^k u(x)| \leq C (2k)!^{1-1/\rho} (\tau + \varepsilon)^{2k}. \tag{6}$$

Definition ([3]). Let $\rho > 0$ and $\tau \geq 0$. An entire function F is said to be of *exponential growth* (ρ, τ) if for every $\varepsilon > 0$ one can find C_ε such that for any $R < \infty$

$$\sup_{|z| \leq R} |F(z)| \leq C_\varepsilon \exp\{(\tau + \varepsilon) R^\rho\}.$$

The exponential growth of an entire function can be expressed in terms of estimations of its Taylor coefficients.

It appears that a function u of Laplacian growth (ρ, τ) on Ω is in fact real-analytic on Ω (see [1, Theorem 2.2 in Chapter II]). So the spherical and solid means $N(u; x, R)$ and $M(u; x, R)$ are well defined for $x \in \Omega$ and R small enough. However due to estimation (6) both functions $N(u; x, R)$ and $M(u; x, R)$ can be extended to entire functions of exponential growth.

Theorem 5 ([9]). *Let* $\rho > 0$ *and* $\tau \geq 0$.
If u *is of Laplacian growth* (ρ, τ), *then* $N(u; x, R)$ *and* $M(u; x, R)$ *extend holomorphically to entire functions of exponential growth* $(\rho, \tau^\rho/\rho)$ *locally uniformly in* Ω.

Theorem 6 ([9]). *Let* $u \in \mathcal{A}(\Omega)$.
If $M(u; x, R)$ *(resp.* $N(u; x, R)$*) defined for* $x \in \Omega$ *and* $0 \le R < \operatorname{dist}(x, \partial\Omega)$ *extends to an entire function* $\widetilde{M}(u; x, z)$ *(resp.* $\widetilde{N}(u; x, z)$*) of exponential growth* (ρ, τ) *locally uniformly in* Ω,
then u *is of Laplacian growth* $\left(\rho, (\tau\rho)^{1/\rho}\right)$.

6. Convergent solutions of the heat equation

Let us consider the initial value problem for the heat equation

$$\begin{cases} \partial_t u - \Delta_x u &= 0, \\ u\big|_{t=0} &= u_0, \end{cases} \tag{7}$$

where $u_0 \in \mathcal{A}(\Omega)$, $\Omega \subset \mathbb{R}^n$.

Then its formal power series solution is given by

$$\widehat{u}(x,t) = \sum_{k=0}^{\infty} \frac{\Delta^k u_0(x)}{k!} t^k. \tag{8}$$

We ask when the solution u is an analytic function of time variable at $t = 0$. In the dimension $n = 1$ the problem was solved by Kowalevskaya [8].

She proved that the solution u is analytic in time if and only if the initial data u_0 can be analytically extended to an entire function of exponential order 2. In the multidimensional case the solution of the problem was given by Aronszajn at al. [1], but only in terms of the Laplacian growth of the initial data.
Here we give its solution in terms of the mean value functions of the initial data.

Theorem 7 ([9, Theorem 5.1]). *Let* $0 < T \leq \infty$. *If formal power series solution* (8) *of the heat equation* (7) *is convergent for* $|t| < T$ *locally uniformly in* Ω, *then* $M(u_0; x, R)$ *and* $N(u_0; x, R)$ *extend to an entire function of exponential growth* $(2, 1/(4T))$ *locally uniformly in* Ω.

Conversely, if $M(u_0; x, R)$ *or* $N(u_0; x, R)$ *can be extended to an entire function of exponential growth* $(2, 1/(4T))$ *locally uniformly in* Ω*, then the solution* (8) *of* (7) *is convergent for* $|t| < T$ *locally uniformly in* Ω*.*

Proof. Assume that $\widehat{u}(t,x)$ is convergent for $|t| < T$ loc. unif. in Ω. Then $\forall K \Subset \Omega, \varepsilon > 0$ $\exists C$ s.t.

$$\begin{aligned}
\sup_{x\in K} |\Delta^k u_0(x)| &\le C\Big(\frac{1}{T}+\varepsilon\Big)^k \cdot k! \\
&\le C_\varepsilon\Big(\frac{1}{T}+\varepsilon\Big)^k\Big(\frac{1}{2}+\varepsilon\Big)^k \cdot (2k)!^{1/2} \\
&\le C_\varepsilon\Big((2T)^{-1/2}+\varepsilon\Big)^{2k} \cdot (2k)!^{1/2}.
\end{aligned}$$

Hence u_0 is of Laplacian growth $(2, 1/\sqrt{2T})$ and by Theorem 5, $M(u_0; x, R)$ and $N(u_0; x, R)$ extend to entire functions of exponential growth $(2, 1/(4T))$ locally uniformly in Ω.

On the other hand let $M(u_0; x, R)$ or $N(u_0; x, R)$ can be extended to an entire function of exponential growth $(2, 1/(4T))$ loc. unif. in Ω. Then by Theorem 6, u_0 is of Laplacian growth $(2, 1/\sqrt{2T})$ loc. unif. in Ω.

Hence for $|t| < T$ and small $\varepsilon > 0$

$$\sup_{x \in K} \sum_{k=0}^{\infty} \frac{|\Delta^k u_0(x)|}{k!} |t|^k \leq \ldots$$
$$\leq C_\varepsilon \sum_{k=0}^{\infty} \left[\left(\frac{1}{T} + \varepsilon \right) |t| \right]^k < \infty.$$

So $\widehat{u}(t, x)$ is convergent for $|t| < T$ locally uniformly in Ω. $\square$

The problem of Borel summability of formal solutions of the heat equation was solved by Michalik ([12, 13]).

Theorem 8 ([13, Theorem 1]). *Let* $d \in \mathbb{R}$, D^n *a disc in* $\mathbb{C}^n$ *and* $\widehat{u}$ *be the formal power series solution (8) of the heat equation (7) with* $u_0 \in \mathcal{O}(D^n)$. *Then TFCAE*

- $\widehat{u}$ *is* 1*-summable in the direction* d*;*
- $M(u_0; z, R) \in \mathcal{O}\big(D^n; \mathcal{O}^2(\widehat{S}_{d/2} \cup \widehat{S}_{d/2+\pi})\big)$*;*
- $N(u_0; z, R) \in \mathcal{O}\big(D^n; \mathcal{O}^2(\widehat{S}_{d/2} \cup \widehat{S}_{d/2+\pi})\big)$.

Furthermore, the 1*-sum of* $\widehat{u}$ *is given by*

$$u^d(z,t) = \frac{1}{(4\pi t)^{n/2}} \int\limits_{(e^{\imath d/2}\mathbb{R})^n} \exp\left\{\frac{-e^{i\theta}|x|^2}{4t}\right\} u_0(x+z)dx$$

if the integral is well defined.

Here $\widehat{S}_d = D^1 \cup S_d$, S_d is a sector bisected by the direction d and $\mathcal{O}^s(\widehat{S}_d)$ is the space of holomorphic functions F on $\widehat{S}_d$ of exponential order s, i.e. satisfying

$$|F(\zeta)| \leq Ce^{c|\zeta|^s} \quad \text{for} \;\; \zeta \in \widehat{S}_d.$$

7. Second order heat type equations

Let P be a homogeneous, second order PDO with constant coefficients. Then the formal solution of

$$\begin{cases} \partial_t u - P_x u &= 0, \\ u|_{t=0} &= u_0 \in \mathcal{A}(\Omega), \end{cases} \tag{9}$$

is given by

$$\widehat{u}(x,t) = \sum_{k=0}^{\infty} \frac{P^k u_0(x)}{k!} t^k. \tag{10}$$

In [14] S. Michalik has given conditions for convergence and Borel summability of $\widehat{u}(x,t)$ in terms of generalized integral mean value function M_μ with respect to a probability Borel measure μ supported by the ball $B(0,1)$.
For $u \in C^0(\Omega)$, $x \in \Omega$ and $0 \le R < \operatorname{dist}(x, \partial\Omega)$ set

$$M_\mu(u;\, x, R) = \int u(x + Ry)\, d\mu(y).$$

Assuming that M_μ satisfies a Pizzetti type formula with respect to the operator P, i.e. if $u \in \mathcal{A}(\Omega)$, then

$$M_\mu(u;\, x, R) = \sum_{j=0}^{\infty} \frac{P^j u(x)}{m(j)} R^{2j}$$

for some moment function m of order 2, (i.e. $m(j) \approx (2j)!$) he proved the following.

Theorem 9 ([13, Theorem 1]). *Let $d \in \mathbb{R}$ and $\widehat{u}$ be the formal power series solution (10) of the heat type equation (9) with $u_0 \in \mathcal{O}(D^n)$. Then*

- *$\widehat{u}$ is convergent for small t iff* $M_\mu(u_0; z, R) \in \mathcal{O}\big(D^n; \mathcal{O}^2(\mathbb{C})\big)$;
- *$\widehat{u}$ is 1-summable in the direction d iff* $M_\mu(u_0; z, R) \in \mathcal{O}\big(D^n; \mathcal{O}^2(\widehat{S}_{d/2} \cup \widehat{S}_{d/2+\pi})\big)$.

8. Higher order mean value functions

Let $k \in \mathbb{N}$. Denote by $\epsilon = \epsilon_k$ the transformation of $\mathbb{C}^n$ into $\mathbb{C}^n$ given by

$$\epsilon(z_1, \ldots, z_n) = (e^{2\pi\imath/k} z_1, \ldots, e^{2\pi\imath/k} z_n).$$

Let u be a continuous function defined on a complex neighborhood U of an open set $\Omega \subset \mathbb{R}^n$.
For $x \in \Omega$ and $0 < R < \mathrm{dist}(x, \partial U)$ we define

the *spherical* and *solid mean value functions of order* k

$$N_k(u;\, x, R) = \frac{1}{kn\sigma(n)} \sum_{j=0}^{k-1} \int_{S^{n-1}(0,1)} u\big(x + R\epsilon^j(y)\big)\, dS(y),$$

$$M_k(u;\, x, R) = \frac{1}{k\sigma(n)} \sum_{j=0}^{k-1} \int_{B^n(0,1)} u\big(x + R\epsilon^j(y)\big)\, dy,$$

where $\sigma(n) = \pi^{n/2}/\Gamma(n/2+1) = |B(0,1)|$.

Note that if k is odd, then $N_k(u;\, x, R) = N_{2k}(u;\, x, R)$ and $M_k(u;\, x, R) = M_{2k}(u;\, x, R)$. In particular

$$N_1(u; x, R) = N_2(u; x, R) = \frac{1}{n\sigma(n)} \int\limits_{S(0,1)} u(x + Ry)\, dS(y)$$

and

$$M_1(u; x, R) = M_2(u; x, R) = \frac{1}{\sigma(n)} \int\limits_{B(0,1)} u\big(x + Ry\big)\, dy.$$

Assume that $u \in \mathcal{A}(\Omega)$ is a real analytic function on an open set $\Omega \subset \mathbb{R}^n$.

Then u extends to a function $\widetilde{u}$ holomorphic on a complex neighborhood U of Ω and for any $x \in \Omega$ it holds

$$\widetilde{u}(y) = \sum_{\kappa \in \mathbb{N}_0^n} \frac{1}{\kappa!} \frac{\partial^{|\kappa|} u}{\partial x^\kappa}(x)(y-x)^\kappa \quad \text{for } \|y-x\| < \rho(x)$$

with some function $\rho \in C^0(\Omega, \mathbb{R}_+)$.

Theorem 10 ([10, Theorem 1], Higher order Pizzetti's formulas).
Let $k = 2l$ *with* $l \in \mathbb{N}$, $u \in \mathcal{A}(\Omega)$ *and* $x \in \Omega$.
Then the functions

$$R \mapsto N_k(u; x, R) \quad \textit{and} \quad R \mapsto M_k(u; x, R)$$

are real analytic at the origin and for R *small enough it holds*

$$N_k(u;\, x, R) = \sum_{m=0}^{\infty} \frac{\Delta^{lm} u(x)}{4^{lm} \left(\frac{n}{2}\right)_{lm} (lm)!} \, R^{2lm}, \tag{12a}$$

$$M_k(u;\, x, R) = \sum_{m=0}^{\infty} \frac{\Delta^{lm} u(x)}{4^{lm} \left(\frac{n}{2} + 1\right)_{lm} (lm)!} \, R^{2lm}. \tag{12b}$$

The proof is done by expending u into Taylor power series, noting that the integral of y^κ over $B(0,1)$ vanishes if at least one of the coordinates $\kappa_\imath$ of κ is odd, using the following property of the roots of unity

$$\sum_{j=0}^{k-1} e^{2j|\kappa|\pi\imath/k} = \begin{cases} k & \text{if } |\kappa| = km \text{ for some } m \in \mathbb{N}_0, \\ 0 & \text{otherwise,} \end{cases}$$

and the formula [4, formula 676, 11], and finally recognizing in the obtained expression the powers of the laplacian multiplied by numerical factors. □

Real analytic functions can be characterized as those smooth ones for which the higher order Pizzetti's series converge.

Theorem 11 ([10, Theorem 2]). *Let* $l \in \mathbb{N}$, $\rho \in C^0(\Omega; \mathbb{R}_+)$ *and* $u \in C^\infty(\Omega)$. *If the series*

$$\widetilde{N}(x,R) = \sum_{m=0}^{\infty} \frac{\Delta^{lm} u(x)}{4^{lm} \left(\frac{n}{2}\right)_{lm} (lm)!} R^{2lm}$$

is convergent locally uniformly in $\{(x,R) : x \in \Omega, |R| < \rho(x)\}$, *then* $u \in \mathcal{A}(\Omega)$ *and* $N_{2l}(u; x, R) = \widetilde{N}(x,R)$.

Corollary 2 *Under the assumptions of Theorem 11 if the series*

$$\widetilde{M}(x,R)=\sum_{m=0}^{\infty}\frac{\Delta^{lm}u(x)}{4^{lm}\left(\frac{n}{2}+1\right)_{lm}(lm)!}\,R^{2lm}$$

is convergent locally uniformly in $\{(x,R): x\in\Omega, |R|<\rho(x)\}$, *then* $u\in\mathcal{A}(\Omega)$ *and* $M_{2l}(u;x,R)=\widetilde{M}(x,R)$ *for* $x\in\Omega$ *and* $0<R<\min\big(\rho(x),\operatorname{dist}(x,\partial\Omega)\big)$.

9. Maximum principle for polyharmonic functions

It is well known that modulus of a function u harmonic on a connected domain $\Omega \subset \mathbb{R}^n$ cannot attain its maximum at an interior point of Ω unless u is constant. On the other hand this maximum principle does not extends to p-polyharmonic functions, i.e., solutions to $\Delta^p u = 0$ with $p \geq 2$. However due real analyticity of such functions by the formula (12b) we obtain the following maximum principle for polyharmonic functions.

Theorem 12 ([10, Theorem 3]). *Let u be a real valued, p-polyharmonic function on a connected open set $\Omega \subset \mathbb{R}^n$. Denote by $\widetilde{u}$ its holomorphic extension to a connected complex neighborhood U of Ω.*
If for some $x_0 \in \Omega$ and $r_0 > 0$ we have

$$u(x_0) \geq \operatorname{Re} \widetilde{u}(y) \quad \textit{for} \quad y \in x_0 + \sum_{j=0}^{p-1} \epsilon^j \big(B(0, r_0)\big),$$

where $\epsilon(z) = e^{2\pi\imath/(2p)} z$, then u is constant on Ω.

Proof. Since u is p-polyharmonic the series in (12b) terminates at the first term. Hence

$$M_{2p}(u;\, x, R) = \frac{1}{p\sigma(n)} \sum_{j=0}^{p-1} \int_{B^n(0,R)} \widetilde{u}\big(x+\epsilon^j(y)\big)\, dy = u(x).$$

So $M_{2p}(u;\, x_0, R) = u(x_0)$ for $0 < R < \rho(x_0)$ and the assumption implies that $\operatorname{Re} \widetilde{u}(y) = u(x_0)$ for $y \in x_0 + \sum_{j=0}^{p-1} \epsilon^j\big(B(0, r_1)\big)$ with $0 < r_1 < \min(r_0, \rho(x_0))$. It follows that u is constant on Ω. $\square$

10. Convergent solutions of higher order heat equations

For $p \in \mathbb{N}$ let us consider the initial value problem for the p-th order heat type equation

$$\begin{cases} \partial_t u - \Delta_x^p u &= 0, \\ u_{|t=0} &= u_0, \end{cases} \tag{13}$$

where $u_0 \in \mathcal{A}(\Omega)$, $\Omega \subset \mathbb{R}^n$. Clearly, the unique formal power series solution of (13) is given by

$$\widehat{u}(t,x) = \sum_{m=0}^{\infty} \frac{\Delta^{mp} u_0(x)}{m!} t^m. \tag{14}$$

We ask when the solution u is an analytic function of the time variable at $t = 0$.

Theorem 13 ([10, Theorem 6]). *Let* $0 < T \leq \infty$. *The formal power series solution* (14) *of the initial value problem* (13) *is convergent for* $|t| < T$ *locally uniformly in* Ω, *iff* $M_{2p}(u_0; x, R)$ *and/or* $N_{2p}(u_0; x, R)$ *extend holomorphically to entire functions of exponential growth*
$\left(\frac{2p}{2p-1}, \frac{2p-1}{2p}(2pT)^{1-2p}\right)$ *locally uniformly in* Ω.

The problem of summability of formal solutions of the p-th order heat equation was solved by Michalik.

Theorem 14 ([14, Corollary 3]). *Let $\widehat{u}$ be the formal power series solution (14) of the equation (13) with $u_0 \in \mathcal{O}(D)$. Then TFCAE*

- $\widehat{u}$ *is* $\frac{1}{2p-1}$*-summable in a direction* d*;*
- $M_{2p}(u_0; z, R) \in \mathcal{O}\Big(D^n; \mathcal{O}^{\frac{2p}{2p-1}}(\sum_{k=0}^{2p-1} \widehat{S}_{\frac{d+2k\pi}{2p}})\Big)$*;*
- $N_{2p}(u_0; z, R) \in \mathcal{O}\Big(D^n; \mathcal{O}^{\frac{2p}{2p-1}}(\sum_{k=0}^{2p-1} \widehat{S}_{\frac{d+2k\pi}{2p}})\Big)$.

11. A Dirichlet type problem for polyharmonic functions

Let Ω be a domain in $\mathbb{R}^n$ and $p \in \mathbb{N}$. We introduce the following Dirichlet type problem for p-polyharmonic functions. Set $\epsilon = \epsilon_{2p}(z) = e^{2\pi\imath/(2p)}z$.

Given functions f_k on $\epsilon^k\big(\partial\Omega\big)$, $k = 0, \ldots, p-1$, find a function u satisfying

$$\begin{cases} \Delta^p u = 0 & \text{on } \bigcup_{k=0}^{p-1} \epsilon^k\big(\Omega\big), \\ u = f_k & \text{on } \epsilon^k\big(\partial\Omega\big), \quad k = 0, \ldots, p-1. \end{cases}$$

In the case of the unit ball the problem was solved by H. Grzebuła in 2016.

Theorem 15 ([6, Theorem 1]). *The problem*

$$\begin{cases} \Delta^p u = 0 & \text{on} \quad \bigcup_{k=0}^{p-1} \epsilon^k\big(B(0,1)\big), \\ u = f_k & \text{on} \quad \epsilon^k\big(S(0,1)\big), \quad k = 0, \dots, p-1 \end{cases} \tag{15}$$

has a unique solution given by

$$u(x) = \frac{1}{p\sigma(n)} \sum_{k=0}^{p-1} \int_{S(0,1)} \frac{1-|x|^{2p}}{|\epsilon^{-k}x - y|^n} f_k(\epsilon^k y)\, dy.$$

The solution is a holomorphic function on the Lie ball $LB(0,1) = \{z \in \mathbb{C}^n : |z|^2 + \sqrt{|z|^4 - |z^2|^2} < 1\}$.

The proof uses the Almanasi expansion of a polyharmonic function to reduce the problem (15) to the Dirichlet problem for harmonic functions on the unit ball.

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Thank you for your attention!