Rational solution of Painleve IV

Davide Masoero, FCT Investigator

Grupo de Fisica Matematica, Universidade de Lisboa

13-17/1/2017

Asymptotic and Computational Aspects of Complex ODEs
CRM PISA

⟨φ|μ⟩ *Grupo de Física Matemática da Universidade de Lisboa*

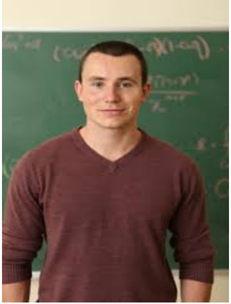
FCT Fundação para a Ciência e a Tecnologia
MINISTÉRIO DA CIÊNCIA, TECNOLOGIA E ENSINO SUPERIOR

Talk based on:

- A. Eremenko, D. Masoero *Poles of rational solutions of Painleve II* (unpublished 2013)
- D. Masoero, P. Roffelsen : *Poles and Zeroes of rational solutions of Painleve IV* (work in progress)

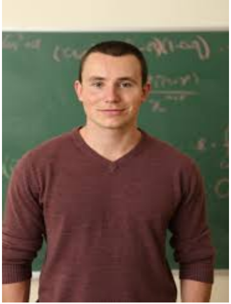
Talk based on:

- A. Eremenko, D. Masoero *Poles of rational solutions of Painleve II* (unpublished 2013)
- D. Masoero, P. Roffelsen : *Poles and Zeroes of rational solutions of Painleve IV* (work in progress)



Talk based on:

- A. Eremenko, D. Masoero *Poles of rational solutions of Painleve II* (unpublished 2013)
- D. Masoero, P. Roffelsen : *Poles and Zeroes of rational solutions of Painleve IV* (work in progress)



Some background

- Painleve equations $P_I, \dots, P_{VI}$: $y''(z) = R_K(y, y', z; \theta)$ such that all movable singularities are poles

Some background

- Painleve equations $P_I, \dots, P_{VI}$: $y''(z) = R_K(y, y', z; \theta)$ such that all movable singularities are poles
- Painleve equations can be realised as isomonodromic deformation of a linear system $\Psi' = A(y(z), y'(z), z; \theta)\Psi$

$$\mathcal{M} : \{ \text{ solutions of } P_K \} \to \{ \text{ space of } \textit{monodromy} \text{ data of } A \}$$

Some background

- Painleve equations $P_I, \dots, P_{VI}$: $y''(z) = R_K(y, y', z; \theta)$ such that all movable singularities are poles
- Painleve equations can be realised as isomonodromic deformation of a linear system $\Psi' = A(y(z), y'(z), z; \theta)\Psi$

$$\mathcal{M} : \{ \text{ solutions of } P_K \} \to \{ \text{ space of } \textit{monodromy} \text{ data of } A \}$$

- Standard inversion: Fix a set of monodromy data $\mathcal{M}(y)$ and a point $z \in \mathbb{C}$ and compute $y(z)$. (ex: PI,PII , the problem has at most one solution and it has no solution iff z is a pole).

Some background

- Painleve equations $P_I, \ldots, P_{VI}$: $y''(z) = R_K(y, y', z; \theta)$ such that all movable singularities are poles
- Painleve equations can be realised as isomonodromic deformation of a linear system $\Psi' = A(y(z), y'(z), z; \theta)\Psi$

$$\mathcal{M} : \{ \text{ solutions of } P_K \} \to \{ \text{ space of } \textit{monodromy} \text{ data of } A \}$$

- Standard inversion: Fix a set of monodromy data $\mathcal{M}(y)$ and a point $z \in \mathbb{C}$ and compute $y(z)$. (ex: PI,PII , the problem has at most one solution and it has no solution iff z is a pole).
- Alternative: Fix a set of monodromy data $\mathcal{M}(y)$ and a point in $w \in \bar{\mathbb{C}}$ and compute the set $y^{-1}(w)$. Generically, the set is infinite (more infos in G. Filipuk's talk).

Some background

- Painleve equations $P_I, \ldots, P_{VI}$: $y''(z) = R_K(y, y', z; \theta)$ such that all movable singularities are poles
- Painleve equations can be realised as isomonodromic deformation of a linear system $\Psi' = A(y(z), y'(z), z; \theta)\Psi$

$$\mathcal{M} : \{ \text{ solutions of } P_K \} \to \{ \text{ space of } \textit{monodromy} \text{ data of } A \}$$

- Standard inversion: Fix a set of monodromy data $\mathcal{M}(y)$ and a point $z \in \mathbb{C}$ and compute $y(z)$. (ex: PI,PII , the problem has at most one solution and it has no solution iff z is a pole).
- Alternative: Fix a set of monodromy data $\mathcal{M}(y)$ and a point in $w \in \bar{\mathbb{C}}$ and compute the set $y^{-1}(w)$. Generically, the set is infinite (more infos in G. Filipuk's talk).

Some background. Classification

An observation (based on PI,PII,PIV): If all fuchsian singularities of the linear system are *apparent*, the set $y^{-1}(w)$ is classified by topologically inequivalent ramified coverings of the Riemann sphere, with the branch points determined by $\mathcal{M}(y)$ (i.e. the Stokes multipliers).

Some background. Classification

An observation (based on PI,PII,PIV): If all fuchsian singularities of the linear system are *apparent*, the set $y^{-1}(w)$ is classified by topologically inequivalent ramified coverings of the Riemann sphere, with the branch points determined by $\mathcal{M}(y)$ (i.e. the Stokes multipliers).

- Write the 2×2 linear equation in scalar form: $\psi''(\lambda) = Q(\lambda)\psi(\lambda)$, where Q is the sum of a polynomial and a finite collection of apparent fuchsian singularities.

Some background. Classification

An observation (based on PI,PII,PIV): If all fuchsian singularities of the linear system are *apparent*, the set $y^{-1}(w)$ is classified by topologically inequivalent ramified coverings of the Riemann sphere, with the branch points determined by $\mathcal{M}(y)$ (i.e. the Stokes multipliers).

- Write the 2×2 linear equation in scalar form: $\psi''(\lambda) = Q(\lambda)\psi(\lambda)$, where Q is the sum of a polynomial and a finite collection of apparent fuchsian singularities.
- For any pair of solutions $\psi_{1,2}$, $f = \frac{\psi_1}{\psi_2}$ is a meromorphic function with a finite number of singularities:

Some background. Classification

An observation (based on PI,PII,PIV): If all fuchsian singularities of the linear system are *apparent*, the set $y^{-1}(w)$ is classified by topologically inequivalent ramified coverings of the Riemann sphere, with the branch points determined by $\mathcal{M}(y)$ (i.e. the Stokes multipliers).

- Write the 2×2 linear equation in scalar form: $\psi''(\lambda) = Q(\lambda)\psi(\lambda)$, where Q is the sum of a polynomial and a finite collection of apparent fuchsian singularities.
- For any pair of solutions $\psi_{1,2}$, $f = \frac{\psi_1}{\psi_2}$ is a meromorphic function with a finite number of singularities: critical points are the fuchsian singularities,

Some background. Classification

An observation (based on PI,PII,PIV): If all fuchsian singularities of the linear system are *apparent*, the set $y^{-1}(w)$ is classified by topologically inequivalent ramified coverings of the Riemann sphere, with the branch points determined by $\mathcal{M}(y)$ (i.e. the Stokes multipliers).

- Write the 2×2 linear equation in scalar form: $\psi''(\lambda) = Q(\lambda)\psi(\lambda)$, where Q is the sum of a polynomial and a finite collection of apparent fuchsian singularities.
- For any pair of solutions $\psi_{1,2}$, $f = \frac{\psi_1}{\psi_2}$ is a meromorphic function with a finite number of singularities: critical points are the fuchsian singularities, the transcendental singularities 'are the Stokes sectors' [Nevanlinna,Elfving, Bergweiler-Eremenko].

Some background. Classification

An observation (based on PI,PII,PIV): If all fuchsian singularities of the linear system are *apparent*, the set $y^{-1}(w)$ is classified by topologically inequivalent ramified coverings of the Riemann sphere, with the branch points determined by $\mathcal{M}(y)$ (i.e. the Stokes multipliers).

- Write the 2×2 linear equation in scalar form: $\psi''(\lambda) = Q(\lambda)\psi(\lambda)$, where Q is the sum of a polynomial and a finite collection of apparent fuchsian singularities.
- For any pair of solutions $\psi_{1,2}$, $f = \frac{\psi_1}{\psi_2}$ is a meromorphic function with a finite number of singularities: critical points are the fuchsian singularities, the transcendental singularities 'are the Stokes sectors' [Nevanlinna,Elfving, Bergweiler-Eremenko].

Some background. Classification

An observation (based on PI,PII,PIV): If all fuchsian singularities of the linear system are *apparent*, the set $y^{-1}(w)$ is classified by topologically inequivalent ramified coverings of the Riemann sphere, with the branch points determined by $\mathcal{M}(y)$ (i.e. the Stokes multipliers).

- Write the 2×2 linear equation in scalar form: $\psi''(\lambda) = Q(\lambda)\psi(\lambda)$, where Q is the sum of a polynomial and a finite collection of apparent fuchsian singularities.
- For any pair of solutions $\psi_{1,2}$, $f = \frac{\psi_1}{\psi_2}$ is a meromorphic function with a finite number of singularities: critical points are the fuchsian singularities, the transcendental singularities 'are the Stokes sectors' [Nevanlinna,Elfving, Bergweiler-Eremenko].
- Conversely, if f is any meromorphic function with a a finite number of singularities then $f = \frac{\psi_1}{\psi_2}$ where the $\psi_{1,2}$ solve a linear ODE $\psi''_{1,2} = P\psi_{1,2}$ for some P as above [Nevanlinna, Elfving]

Some background. Classification

An observation (based on PI,PII,PIV): If all fuchsian singularities of the linear system are *apparent*, the set $y^{-1}(w)$ is classified by topologically inequivalent ramified coverings of the Riemann sphere, with the branch points determined by $\mathcal{M}(y)$ (i.e. the Stokes multipliers).

- Write the 2×2 linear equation in scalar form: $\psi''(\lambda) = Q(\lambda)\psi(\lambda)$, where Q is the sum of a polynomial and a finite collection of apparent fuchsian singularities.
- For any pair of solutions $\psi_{1,2}$, $f = \frac{\psi_1}{\psi_2}$ is a meromorphic function with a finite number of singularities: critical points are the fuchsian singularities, the transcendental singularities 'are the Stokes sectors' [Nevanlinna,Elfving, Bergweiler-Eremenko].
- Conversely, if f is any meromorphic function with a a finite number of singularities then $f = \frac{\psi_1}{\psi_2}$ where the $\psi_{1,2}$ solve a linear ODE $\psi''_{1,2} = P\psi_{1,2}$ for some P as above [Nevanlinna, Elfving]
- Topological classification: two meromorphic functions are equivalent iff they are topologically equivalent as covering maps of the sphere.

Some background. Classification II

- The 'values' of the transcendental singularities of f can be read off the Stokes multipliers [M.]

Some background. Classification II

- The 'values' of the transcendental singularities of f can be read off the Stokes multipliers [M.]
- The location of the (movable) critical points of f are related to the values of y.

Some background. Classification II

- The 'values' of the transcendental singularities of f can be read off the Stokes multipliers [M.]
- The location of the (movable) critical points of f are related to the values of y.
- Poles of y actually simplifies the monodromy problem!

Some background. Classification II

- The 'values' of the transcendental singularities of f can be read off the Stokes multipliers [M.]
- The location of the (movable) critical points of f are related to the values of y.
- Poles of y actually simplifies the monodromy problem!
- Consequences: A-priori characterization of the set $y^{-1}(w)$ by combinatorics (topology) $\rightarrow$ simple proof of surjectivity of the monodromy map (ex: PI).

Some background. PIV

- Main example for this talk: Painleve IV

$$y_{zz} = \frac{1}{2y} y_z^2 + \frac{3}{2} y^3 + 4zy^2 + 2(z^2 + 1 - 2\theta_\infty) y - \frac{8\theta^2}{y}, \quad \theta, \theta_\infty \in \mathbb{C}.$$

Some background. PIV

- Main example for this talk: Painleve IV

$$y_{zz} = \frac{1}{2y} y_z^2 + \frac{3}{2} y^3 + 4zy^2 + 2(z^2 + 1 - 2\theta_\infty) y - \frac{8\theta^2}{y}, \quad \theta, \theta_\infty \in \mathbb{C}.$$

- There are at least three different isomonodromic representations for PIV. I use here the Garnier-Jimbo-Miwa: one fuchsian singularity at the origin and one irregular singularity at ∞.
- The monodromy map is rather involved. The resonant case is considered in [Kapaev, unpublished].

Rational solutions of PIV

P_{IV} has a rational solution iff either

Hermite $$\theta = \frac{m}{2} + n, \qquad \theta_\infty = \frac{m}{2},$$

or

Okamoto $$\theta = \frac{m}{2} + n \pm \frac{2}{3}, \qquad \theta_\infty = \frac{m}{2}$$

for some $m, n \in \mathbb{Z}$, furthermore for any such parameter values the associated rational solution is unique.

Hermite solutions

Three types of Hermite solutions

$$\omega_{m,n}^{(I)} = \frac{H'_{m+1,n}}{H_{m+1,n}} - \frac{H'_{m,n}}{H_{m,n}},$$
$$\omega_{m,n}^{(II)} = \frac{H'_{m,n}}{H_{m,n}} - \frac{H'_{m,n+1}}{H_{m,n+1}},$$
$$\omega_{m,n}^{(III)} = -2z + \frac{H'_{m,n+1}}{H_{m,n+1}} - \frac{H'_{m+1,n}}{H_{m+1,n}},$$

$H_{m,n}(z)$ is the generalized Hermite polynomial.

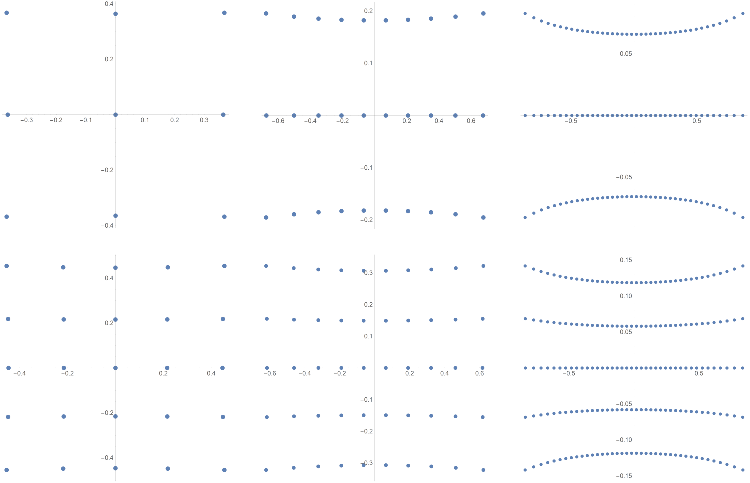
Hermite solutions

Three types of Hermite solutions

$$\omega_{m,n}^{(\mathrm{I})} = \frac{H'_{m+1,n}}{H_{m+1,n}} - \frac{H'_{m,n}}{H_{m,n}},$$
$$\omega_{m,n}^{(\mathrm{II})} = \frac{H'_{m,n}}{H_{m,n}} - \frac{H'_{m,n+1}}{H_{m,n+1}},$$
$$\omega_{m,n}^{(\mathrm{III})} = -2z + \frac{H'_{m,n+1}}{H_{m,n+1}} - \frac{H'_{m+1,n}}{H_{m+1,n}},$$

$H_{m,n}(z)$ is the generalized Hermite polynomial.
Note: poles (residue ± 1) and zeroes coincide with zeroes of Hermite polynomials.

Zeroes of generalized Hermite polynomials $H_{(m,n)}$



Problem [P. Clarkson]: explain the pictures!

Okamoto solutions

For $m, n \in \mathbb{Z}$,

$$\widetilde{\omega}^{(\mathrm{I})}_{m,n} = -\frac{2}{3}z + \frac{Q'_{m+1,n}}{Q_{m+1,n}} - \frac{Q'_{m,n}}{Q_{m,n}}, \tag{1a}$$

$$\widetilde{\omega}^{(\mathrm{II})}_{m,n} = -\frac{2}{3}z + \frac{Q'_{m,n}}{Q_{m,n}} - \frac{Q'_{m,n+1}}{Q_{m,n+1}}, \tag{1b}$$

$$\widetilde{\omega}^{(\mathrm{III})}_{m,n} = -\frac{2}{3}z + \frac{Q'_{m,n+1}}{Q_{m,n+1}} - \frac{Q'_{m+1,n}}{Q_{m+1,n}}, \tag{1c}$$

where $Q_{m,n}$ are the generalized Okamoto polynomials [Okamoto, Noumi and Yamada].

Okamoto solutions

For $m, n \in \mathbb{Z}$,

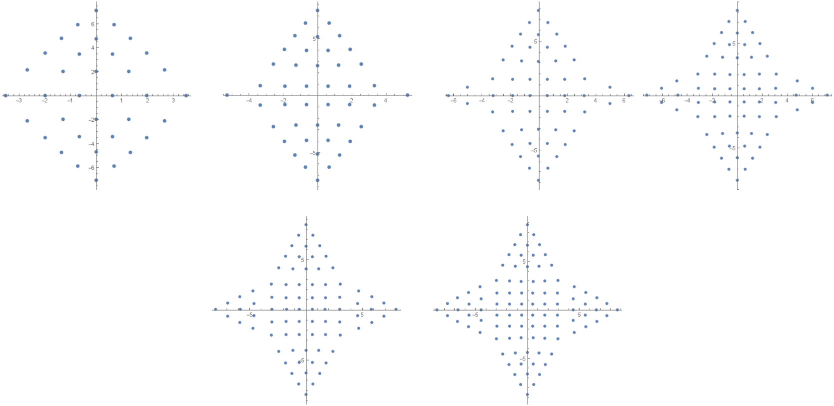
$$\widetilde{\omega}^{(\mathrm{I})}_{m,n} = -\frac{2}{3}z + \frac{Q'_{m+1,n}}{Q_{m+1,n}} - \frac{Q'_{m,n}}{Q_{m,n}}, \tag{1a}$$

$$\widetilde{\omega}^{(\mathrm{II})}_{m,n} = -\frac{2}{3}z + \frac{Q'_{m,n}}{Q_{m,n}} - \frac{Q'_{m,n+1}}{Q_{m,n+1}}, \tag{1b}$$

$$\widetilde{\omega}^{(\mathrm{III})}_{m,n} = -\frac{2}{3}z + \frac{Q'_{m,n+1}}{Q_{m,n+1}} - \frac{Q'_{m+1,n}}{Q_{m+1,n}}, \tag{1c}$$

where $Q_{m,n}$ are the generalized Okamoto polynomials [Okamoto, Noumi and Yamada].
Note: poles have residue ± 1 and coincide with zeroes of Okamoto polynomials.

Zeroes of generalized Okamoto polynomials $Q_{(m,n)}$



Problem: explain the pictures!

Our strategy to explain the pictures

Two aspects to take care of:

Our strategy to explain the pictures

Two aspects to take care of:

- Theory: (Topological) Characterization of the monodromy problems for the poles of rational solutions of Hermite type

Our strategy to explain the pictures

Two aspects to take care of:

- Theory: (Topological) Characterization of the monodromy problems for the poles of rational solutions of Hermite type
- Practice: Asymptotic analysis of these problems

Our strategy to explain the pictures

Two aspects to take care of:

- Theory: (Topological) Characterization of the monodromy problems for the poles of rational solutions of Hermite type
- Practice: Asymptotic analysis of these problems

Remark: 3 families of Hermite solutions $\rightarrow$ 3 'inequivalent' monodromy problems for the same zeroes of the Hermite polynomials.

Isomonodromic equation at a pole. Hermite-I case.

Suppose $a \in \mathbb{C}$ is a pole of residue -1 of $\omega^{(I)}_{m,n}$ and

$$\omega^{(I)}_{m,n}(z) = \frac{-1}{z-a} + \cdots + b(z-a)^2 + \ldots$$

Isomonodromic equation at a pole. Hermite-I case.

Suppose $a \in \mathbb{C}$ is a pole of residue -1 of $\omega^{(I)}_{m,n}$ and

$$\omega^{(I)}_{m,n}(z) = \frac{-1}{z-a} + \cdots + b(z-a)^2 + \ldots$$

The isomonodromic equation evaluated at $z = a$ is

$$\psi''(\lambda) = \left((\lambda + a)^2 - (2m+n) - \frac{\kappa(a,b)}{\lambda} + \frac{n^2-1}{\lambda^2}\right)\psi(\lambda)$$

- The resonant singularity at $\lambda = 0$ is apparent: no logarithms.
- The Stokes multiplier σ_1 vanishes: $\psi_0 \sim \psi_2$.
- The subdominant solution ψ_0 vanishes at zero and has $m-1$ further simple zeroes.

Characterization. 'IF' part

If a is a pole of $H_{m,n}$, the ratio $f = \frac{\psi}{\tilde{\psi}}$ of any two solutions of the linear problem belongs to a finite family of meromorphic functions:

Characterization. 'IF' part

If a is a pole of $H_{m,n}$, the ratio $f = \frac{\psi}{\tilde{\psi}}$ of any two solutions of the linear problem belongs to a finite family of meromorphic functions:

1. f is a meromorphic function with only one critical point, of order $n-1$, at $\lambda = 0$.

Characterization. 'IF' part

If a is a pole of $H_{m,n}$, the ratio $f = \frac{\psi}{\tilde{\psi}}$ of any two solutions of the linear problem belongs to a finite family of meromorphic functions:

1. f is a meromorphic function with only one critical point, of order $n-1$, at $\lambda = 0$.
2. f has 4 transcendental singularities but 2 of the 4 asymptotic values coincide: w_0, w_1, w_{-1}, w_2 with $w_2 = w_0$ ($\leftrightarrow \sigma_1 = 0$).

Characterization. 'IF' part

If a is a pole of $H_{m,n}$, the ratio $f = \frac{\psi}{\tilde{\psi}}$ of any two solutions of the linear problem belongs to a finite family of meromorphic functions:

1. f is a meromorphic function with only one critical point, of order $n-1$, at $\lambda = 0$.
2. f has 4 transcendental singularities but 2 of the 4 asymptotic values coincide: w_0, w_1, w_{-1}, w_2 with $w_2 = w_0$ ($\leftrightarrow \sigma_1 = 0$).
3. the critical value coincides with the double asymptotic value: $f(0) = w_0$ (3 distinct singular values $\leftrightarrow$ 0 dimensional mouduli).

Characterization. 'IF' part

If a is a pole of $H_{m,n}$, the ratio $f = \frac{\psi}{\tilde{\psi}}$ of any two solutions of the linear problem belongs to a finite family of meromorphic functions:

1. f is a meromorphic function with only one critical point, of order $n-1$, at $\lambda = 0$.
2. f has 4 transcendental singularities but 2 of the 4 asymptotic values coincide: w_0, w_1, w_{-1}, w_2 with $w_2 = w_0$ ($\leftrightarrow \sigma_1 = 0$).
3. the critical value coincides with the double asymptotic value: $f(0) = w_0$ (3 distinct singular values $\leftrightarrow$ 0 dimensional mouduli).
4. The equation $f(z) = w_0$ has $m + n - 1$ solutions counting multiplicities!

Characterization. 'IF' part

If a is a pole of $H_{m,n}$, the ratio $f = \frac{\psi}{\tilde{\psi}}$ of any two solutions of the linear problem belongs to a finite family of meromorphic functions:

1. f is a meromorphic function with only one critical point, of order $n-1$, at $\lambda = 0$.
2. f has 4 transcendental singularities but 2 of the 4 asymptotic values coincide: w_0, w_1, w_{-1}, w_2 with $w_2 = w_0$ ($\leftrightarrow \sigma_1 = 0$).
3. the critical value coincides with the double asymptotic value: $f(0) = w_0$ (3 distinct singular values $\leftrightarrow$ 0 dimensional mouduli).
4. The equation $f(z) = w_0$ has $m + n - 1$ solutions counting multiplicities!

There are exactly $m \times n$ inequivalent functions f satisfying these properties.

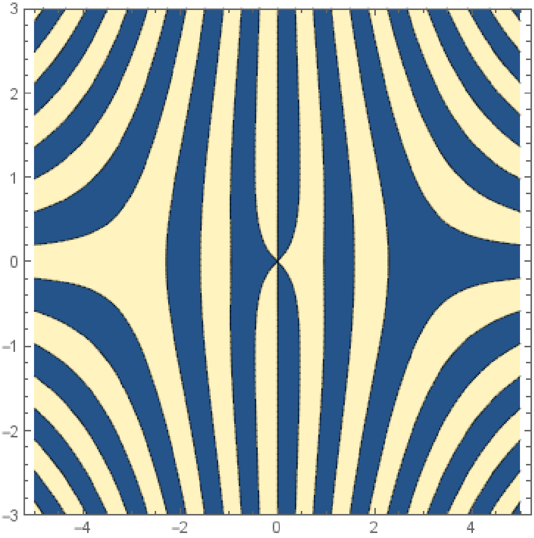
Characterization. 'IF' part

If a is a pole of $H_{m,n}$, the ratio $f = \frac{\psi}{\tilde{\psi}}$ of any two solutions of the linear problem belongs to a finite family of meromorphic functions:

1. f is a meromorphic function with only one critical point, of order $n-1$, at $\lambda = 0$.
2. f has 4 transcendental singularities but 2 of the 4 asymptotic values coincide: w_0, w_1, w_{-1}, w_2 with $w_2 = w_0$ ($\leftrightarrow \sigma_1 = 0$).
3. the critical value coincides with the double asymptotic value: $f(0) = w_0$ (3 distinct singular values $\leftrightarrow$ 0 dimensional mouduli).
4. The equation $f(z) = w_0$ has $m + n - 1$ solutions counting multiplicities!

There are exactly $m \times n$ inequivalent functions f satisfying these properties. Remark: $m \times n =$ order of $H_{m,n}$.

A pictorial example. f for $m = n = 3$



Characterization ✓ [Roffelsen,M.]

The point $a \in \mathbb{C}$ is a pole of residue -1 of $\omega^{(I)}_{m,n}$

$$\omega^{(I)}_{m,n}(z) = \frac{-1}{z-a} + \cdots + b(z-a)^2 + \ldots$$

Characterization ✓ [Roffelsen,M.]

The point $a \in \mathbb{C}$ is a pole of residue -1 of $\omega^{(I)}_{m,n}$

$$\omega^{(I)}_{m,n}(z) = \frac{-1}{z-a} + \cdots + b(z-a)^2 + \ldots$$

if and only if (Obrigado Nevanlinna!) the equation

$$\psi''(\lambda) = \left((\lambda+a)^2 - (2m+n) - \frac{\kappa(a,b)}{\lambda} + \frac{n^2-1}{\lambda^2}\right)\psi(\lambda)$$

has the following properties

Characterization ✓ [Roffelsen,M.]

The point $a \in \mathbb{C}$ is a pole of residue -1 of $\omega^{(I)}_{m,n}$

$$\omega^{(I)}_{m,n}(z) = \frac{-1}{z-a} + \dots + b(z-a)^2 + \dots$$

if and only if (Obrigado Nevanlinna!) the equation

$$\psi''(\lambda) = \left((\lambda + a)^2 - (2m+n) - \frac{\kappa(a,b)}{\lambda} + \frac{n^2-1}{\lambda^2}\right)\psi(\lambda)$$

has the following properties

1. The resonant singularity at $\lambda = 0$ is apparent: no logarithms.
2. The solution ψ_0 vanishes at $\lambda = 0$.

Characterization ✓ [Roffelsen,M.]

The point $a \in \mathbb{C}$ is a pole of residue -1 of $\omega^{(I)}_{m,n}$

$$\omega^{(I)}_{m,n}(z) = \frac{-1}{z-a} + \cdots + b(z-a)^2 + \ldots$$

if and only if (Obrigado Nevanlinna!) the equation

$$\psi''(\lambda) = \left((\lambda + a)^2 - (2m+n) - \frac{\kappa(a,b)}{\lambda} + \frac{n^2-1}{\lambda^2}\right)\psi(\lambda)$$

has the following properties

1. The resonant singularity at $\lambda = 0$ is apparent: no logarithms.
2. The solution ψ_0 vanishes at $\lambda = 0$.

Indeed the further conditions

- $\psi(0)$ has $m-1$ simple zeroes.
- The Stokes multiplier σ_1 vanishes: $\psi_0 \sim \psi_2$

are satisfied because of the parametrization of the potential by m, n.

Asymptotic analysis

On the equation

$$\psi''(\lambda) = \left((\lambda + a)^2 - (2m + n) - \frac{\kappa(a, b)}{\lambda} + \frac{n^2 - 1}{\lambda^2}\right)\psi(\lambda)$$

we must impose two conditions in order to find the poles

1. The resonant singularity at $\lambda = 0$ is apparent: no logarithms.
2. The solution ψ_0 vanishes at 0.

Asymptotic analysis. Apparent singularity

We compute the location of zeroes of Hermite polynomials in the limit $E = 2m + n \to \infty$ with n bounded.

Asymptotic analysis. Apparent singularity

We compute the location of zeroes of Hermite polynomials in the limit $E = 2m + n \to \infty$ with n bounded.

We define $\alpha = E^{-\frac{1}{2}} a$ and restrict to $|1 - \alpha^2| > c > 0$.

Asymptotic analysis. Apparent singularity

We compute the location of zeroes of Hermite polynomials in the limit $E = 2m + n \to \infty$ with n bounded.

We define $\alpha = E^{-\frac{1}{2}} a$ and restrict to $|1 - \alpha^2| > c > 0$.

First step: For any fixed α, we look for κ such that the fuchsian singularity of

$$\psi''(\lambda) = \left(\lambda^2 + 2\alpha E^{\frac{1}{2}} - (1 - \alpha^2)E - \frac{\kappa}{\lambda} + \frac{n^2 - 1}{\lambda^2}\right)\psi(\lambda)$$

is apparent. A complicated algebraic equation of order n for κ.

Asymptotic analysis. Apparent singularity

We compute the location of zeroes of Hermite polynomials in the limit $E = 2m + n \to \infty$ with n bounded.

We define $\alpha = E^{-\frac{1}{2}} a$ and restrict to $|1 - \alpha^2| > c > 0$.

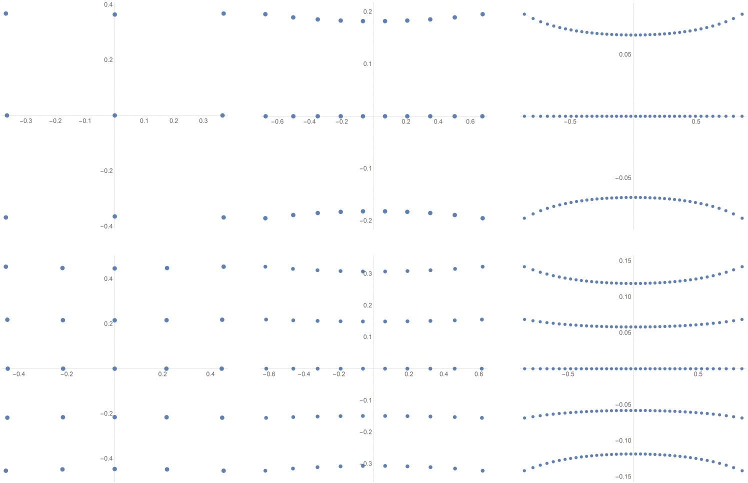
First step: For any fixed α, we look for κ such that the fuchsian singularity of

$$\psi''(\lambda) = \big(\lambda^2 + 2\alpha E^{\frac{1}{2}} - (1-\alpha^2)E - \frac{\kappa}{\lambda} + \frac{n^2-1}{\lambda^2}\big)\psi(\lambda)$$

is apparent. A complicated algebraic equation of order n for κ.
For $E \to +\infty$, we find:

$$\kappa = \sqrt{-1} j (1-\alpha^2)^{\frac{1}{2}} E^{\frac{1}{2}} + O(E^{-\frac{1}{2}}) \text{ with } |j| \leq \frac{n-1}{2} \text{ integer }.$$

Zeroes of generalized Hermite polynomials $H_{(m,n)}$



In the pictures $n = 3, 5$. The integer j parametrizes the lines of poles!

WKB expansion of ψ_0

We got rid of κ:

$$\psi''(\lambda) = \big(\lambda^2 + 2\alpha E^{\frac{1}{2}} - (1-\alpha^2)E - \sqrt{-1}\frac{j(1-\alpha^2)^{\frac{1}{2}}E^{\frac{1}{2}}}{\lambda} + \frac{n^2-1}{\lambda^2}\big)\psi(\lambda)$$

WKB expansion of ψ_0

We got rid of κ:

$$\psi''(\lambda) = \left(\lambda^2 + 2\alpha E^{\frac{1}{2}} - (1-\alpha^2)E - \sqrt{-1}\frac{j(1-\alpha^2)^{\frac{1}{2}}E^{\frac{1}{2}}}{\lambda} + \frac{n^2-1}{\lambda^2}\right)\psi(\lambda)$$

We have to find α such that the solution ψ_0 vanish at 0

WKB expansion of ψ_0

We got rid of κ:

$$\psi''(\lambda) = \big(\lambda^2 + 2\alpha E^{\frac{1}{2}} - (1-\alpha^2)E - \sqrt{-1}\frac{j(1-\alpha^2)^{\frac{1}{2}}E^{\frac{1}{2}}}{\lambda} + \frac{n^2-1}{\lambda^2}\big)\psi(\lambda)$$

We have to find α such that the solution ψ_0 vanish at 0
For $O(E^{-\frac{1}{2}+\varepsilon}) \lesssim \lambda \lesssim (1-\alpha^2)E^{\frac{1}{2}}$, the WKB asymptotic holds

$$\psi_0 \sim \sin\{\frac{\pi}{4} - (1-\alpha^2)^{\frac{1}{2}}E^{\frac{1}{2}}\lambda - \frac{\sqrt{-1}j}{2}\log\lambda +$$
$$+ \frac{E}{2}\big(-\alpha\sqrt{1-\alpha^2} + \arccos\alpha\big) + \sqrt{-1}\frac{j}{2}\log 2(1-\alpha^2)E^{\frac{1}{2}})\}$$

Asymptotic analysis of ψ_+

For $\lambda \to 0$ one can transform the original equation into a Whittaker equation

Asymptotic analysis of ψ_+

For $\lambda \to 0$ one can transform the original equation into a Whittaker equation
The subdominant solution ψ_+ vanishing at 0 has the asymptotic

$$\psi_+(\lambda) \sim M\Big[\frac{j}{2}, \frac{n}{2}, 2\sqrt{-1}(1-\alpha^2)^{\frac{1}{2}} E^{\frac{1}{2}}\lambda\Big], \ \lambda = O(E^{-\gamma}), \gamma > 0$$

where $M\big[\frac{j}{2}, \frac{n}{2}, \mu\big]$ is the Whittaker function.

Asymptotic analysis. Matchings

We need to solve $Wr[\psi_0, \psi_+] = 0$.

Asymptotic analysis. Matchings

We need to solve $Wr[\psi_0, \psi_+] = 0$.
Using the known asymptotics of the Whittaker function, we find

Asymptotic analysis. Matchings

We need to solve $Wr[\psi_0, \psi_+] = 0$.
Using the known asymptotics of the Whittaker function, we find

$$Wr[\psi_0, \psi_+] = \sin\left\{ \frac{E}{2}\left(-\alpha\sqrt{1-\alpha^2} + \arccos\alpha\right) + \sqrt{-1}j \log\left[2(1-\alpha^2)E^{\frac{1}{2}}\right] + \frac{\sqrt{-1}}{2}F_{n,j} \right\} + O(E^{-\frac{1}{3}}), \; F_{n,j} = \log\frac{\Gamma[(1+n-j)]}{\Gamma[(1+n+j)]}$$

Asymptotic analysis. Matchings

We need to solve $Wr[\psi_0, \psi_+] = 0$.
Using the known asymptotics of the Whittaker function, we find

$$Wr[\psi_0, \psi_+] = \sin\left\{ \frac{E}{2}\left(-\alpha\sqrt{1-\alpha^2} + \arccos\alpha\right) + \sqrt{-1} j \log\left[2(1-\alpha^2)E^{\frac{1}{2}}\right] + \frac{\sqrt{-1}}{2} F_{n,j} \right\} + O(E^{-\frac{1}{3}}), \; F_{n,j} = \log \frac{\Gamma[(1+n-j)]}{\Gamma[(1+n+j)]}$$

Solving the approximate equation, we find the α up to an error $O(E^{-\frac{4}{3}})$.
Since the spacing among them is $O(E^{-1})$, all roots such that $|1-\alpha^2| > c > 0$ are resolved.

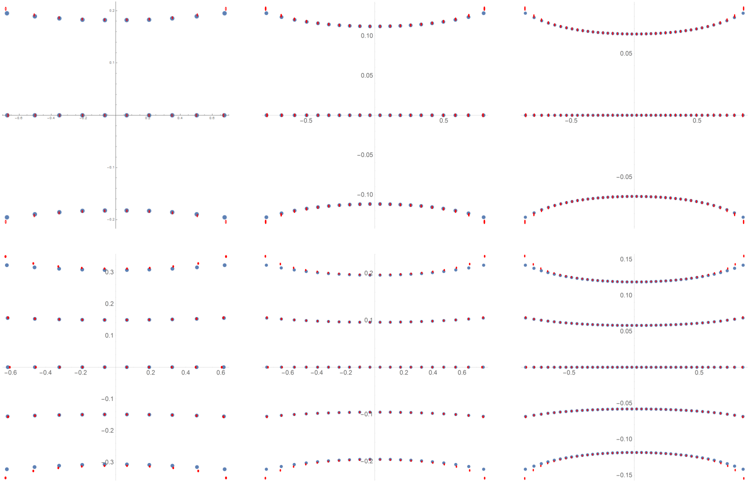
Asymptotic analysis. Matchings

We need to solve $Wr[\psi_0, \psi_+] = 0$.
Using the known asymptotics of the Whittaker function, we find

$$Wr[\psi_0, \psi_+] = \sin\left\{ \frac{E}{2}\left(-\alpha\sqrt{1-\alpha^2} + \arccos\alpha\right) + \sqrt{-1} j \log\left[2(1-\alpha^2)E^{\frac{1}{2}}\right] + \frac{\sqrt{-1}}{2} F_{n,j} \right\} + O(E^{-\frac{1}{3}}), \; F_{n,j} = \log\frac{\Gamma[(1+n-j)]}{\Gamma[(1+n+j)]}$$

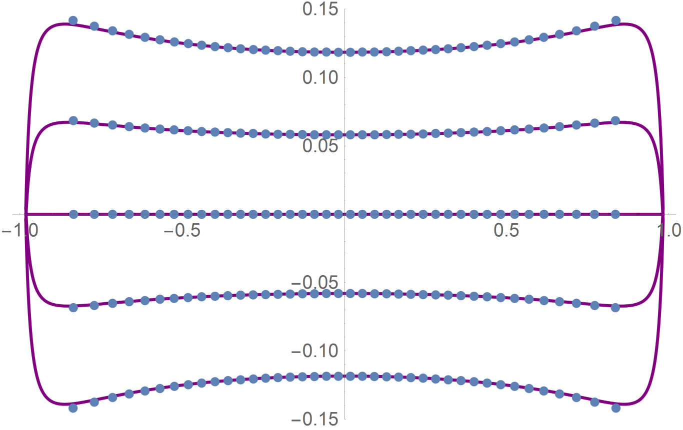
Solving the approximate equation, we find the α up to an error $O(E^{-\frac{4}{3}})$.
Since the spacing among them is $O(E^{-1})$, all roots such that $|1-\alpha^2| > c > 0$ are resolved.
We also computed the behavior for $\alpha \to \pm 1$ but there is no simple formula.

Asymptotics Vs. Numerics.



Layers of zero

In the large m limit, zeroes condensate on the curves $\Im\alpha = f_{j,n,m}(\Re\alpha)$



Conclusions and Outlook

- We charatcterised poles and zero of Hermite solutions and computed its asymptotic distribution for $m \to \infty$ and n bounded (in case $|1-\alpha^2| > 0$ and $1-\alpha^2 \sim 0$).
- Computations for $m \to \infty$ and $m/n \to c > 0$ are underway

Conclusions and Outlook

- We charatcterised poles and zero of Hermite solutions and computed its asymptotic distribution for $m \to \infty$ and n bounded (in case $|1-\alpha^2| > 0$ and $1-\alpha^2 \sim 0$).
- Computations for $m \to \infty$ and $m/n \to c > 0$ are underway
- Preliminary results for rational solutions of Okamoto type: related to finite families of Nevanlinna functions with 12 asymptotic values (anharmonic oscillators of order 10).

Conclusions and Outlook

- We charatcterised poles and zero of Hermite solutions and computed its asymptotic distribution for $m \to \infty$ and n bounded (in case $|1-\alpha^2| > 0$ and $1-\alpha^2 \sim 0$).
- Computations for $m \to \infty$ and $m/n \to c > 0$ are underway
- Preliminary results for rational solutions of Okamoto type: related to finite families of Nevanlinna functions with 12 asymptotic values (anharmonic oscillators of order 10).

Conclusions and Outlook

- We charatcterised poles and zero of Hermite solutions and computed its asymptotic distribution for $m \to \infty$ and n bounded (in case $|1-\alpha^2| > 0$ and $1-\alpha^2 \sim 0$).
- Computations for $m \to \infty$ and $m/n \to c > 0$ are underway
- Preliminary results for rational solutions of Okamoto type: related to finite families of Nevanlinna functions with 12 asymptotic values (anharmonic oscillators of order 10).
- Rational solutions of PII are related to finite families of functions with 6 asymptotic values (oscillators of order 4) [Eremenko, M].

Conclusions and Outlook

- We charatcterised poles and zero of Hermite solutions and computed its asymptotic distribution for $m \to \infty$ and n bounded (in case $|1-\alpha^2| > 0$ and $1-\alpha^2 \sim 0$).
- Computations for $m \to \infty$ and $m/n \to c > 0$ are underway
- Preliminary results for rational solutions of Okamoto type: related to finite families of Nevanlinna functions with 12 asymptotic values (anharmonic oscillators of order 10).
- Rational solutions of PII are related to finite families of functions with 6 asymptotic values (oscillators of order 4) [Eremenko, M].
- Rational solutions and finite families of meromorphic functions: Is there any reason for this phenomenon?

The end

MANY THANKS FOR THE ATTENTION!