Modulated elliptic wave and a train of asymptotic solitons in a vicinity of the leading edge for MKdV

A.Minakov
SISSA, Trieste
joint work with V.P.Kotlyaov

Asymptotic and computational aspects of complex differential equations,
Pisa, Italy
February 13 – 17, 2017

Modified Korteweg - de Vries equation

$$q_t(x,t) + 6q^2(x,t)q_x(x,t) + q_{xxx}(x,t) = 0, \quad x \in \mathbb{R}, t \geq 0,$$

Step-like initial data

$$\begin{cases} q(x,0) \to c > 0 & \text{as} \quad x \to -\infty \text{ and} \\ q(x,0) \to 0 & \text{as} \quad x \to +\infty. \end{cases}$$

We are interested in large time asymptotics in the domain

$$4c^2 t - \varepsilon t < x < 4c^2 t - \frac{N+1/2}{c} \log t, \quad \varepsilon > 0.$$

For simplicity, to have analytic continuation of the corresponding reflection coefficient, we take initial data exponentially close to its limits, i.e.

$$\int_{-\infty}^{0} |q_0(x) - c| \mathrm{e}^{2|x|L} \mathrm{d}x + \int_{0}^{+\infty} |q(x,t)| \mathrm{e}^{2xL} \mathrm{d}x < \infty, \qquad \text{for some } L > c.$$

We assume that the solution to the Cauchy problem exists, and satisfies

$$\int_{-\infty}^{0} (1+|x|)|q(x,t) - c| \mathrm{d}x + \int_{0}^{+\infty} (1+|x|)|q(x,t)| \mathrm{d}x < \infty,$$

We assume the absence of usual solitons, i.e. for transmission coefficient $a^{-1}(k)$:

$$a(k) \neq 0 \quad \text{for} \quad \Im k \geq 0.$$

1. A method for the domain

$$4c^2 t - \frac{2N+1}{2c}\log t < x, \quad N \in \mathbb{N}$$

was introduced firstly for KdV $u_t - uu_x + u_{xxx} = 0$ by E. Khruslov 76, and was applied to MKdV $q_t + 6q^2 q_x + q_{xxx} = 0$ by E. Khruslov, V. Kotlyarov 89

"Asymptotic" solitons

$$q(x,t) = q_{as}(x,t) + \mathrm{O}\left(t^{-1/2+\sigma}\right), \quad \sigma > 0,$$

$$q_{as}(x,t) = \sum_{n=1}^{N} \frac{2c}{\cosh\left[2c(x - 4c^2 t) + (2n - 1/2)\log t - \tilde{\alpha}_n\right]},$$

$$\tilde{\alpha}_n = \log\left[\frac{(h^*)^{-2}(2n)!}{2^{12n-7/2} c^{6n-3/2}\sqrt{\pi n}}\right]$$

here constant h^* is determined by expansion of transmission coefficient at the edge of the simple spectrum

$$a(k) = \frac{h^*}{2}\sqrt[4]{\frac{2\mathrm{i}c}{k - \mathrm{i}c}}\left(1 + \mathrm{O}\left(\sqrt{-\mathrm{i}(k - \mathrm{i}c)}\right)\right), \quad k \to \mathrm{i}c.$$

In particular,

$$q(x,t) = \mathrm{O}\left(\mathrm{t}^{-1/2+\sigma}\right) \quad \text{for} \quad x > 4c^2 t.$$

The centers of solitons lie on the lines

$$x = 4c^2 t - \frac{2n - 1/2}{2c} \log t + \frac{\tilde{\alpha}_n}{2c}, \quad n = 1, 2, 3, \ldots.$$

The asymptotics in the domain

$$(-6c^2+\varepsilon)t < x < 4c^2 t - \varepsilon t$$

was studied by V. Kotlyarov, A. M.

$$q(x,t) = q_{el}(x,t) + \mathrm{o}(1), \qquad \xi = \frac{x}{12t},$$

$$q_{el}(x,t) = \sqrt{c^2 - d^2(\xi)}\,\frac{\Theta(\mathrm{i}tB(\xi) + \mathrm{i}\Delta(\xi) + \pi\mathrm{i}\ ,\ \tau(\xi))}{\Theta(\mathrm{i}tB(\xi) + \mathrm{i}\Delta(\xi)\ ,\ \tau(\xi))},$$

Here

$$\begin{cases} \frac{c^2}{2} + \xi = \mu^2(\xi) + \frac{d^2(\xi)}{2}, \\ \int_0^{\mathrm{i}d(\xi)} \frac{(s^2+\mu^2(\xi))\sqrt{s^2+d^2(\xi)}}{\sqrt{s^2+c^2}} = 0. \end{cases}$$

$$B(\xi) = 24 \int\limits_{\mathrm{i}d(\xi)}^{c} \frac{(k^2 + \mu^2(\xi)) \left(\sqrt{k^2 + d^2(\xi)}\right)_+ \mathrm{d}k}{\left(\sqrt{k^2 + d^2(\xi)}\right)_+},$$

$$\Delta(\xi) = \int\limits_{\mathrm{i}d(\xi)}^{\mathrm{i}c} \frac{\log\bigl(a_+(k)a_-(k)\bigr)\mathrm{d}k}{\left(\sqrt{(k^2+c^2)(k^2+d^2(\xi))}\right)_+} \left(\mathrm{i}\int\limits_{0}^{\mathrm{i}d(\xi)} \frac{\mathrm{d}k}{\sqrt{(k^2+c^2)(k^2+d^2(\xi))}}\right)^{-1}.$$

$$\tau(\xi) = -\pi\mathrm{i} \int\limits_{\mathrm{i}d(\xi)}^{\mathrm{i}c} \frac{\mathrm{d}k}{\left(\sqrt{(k^2+c^2)(k^2+d^2(\xi))}\right)_+} \left(\int\limits_{0}^{\mathrm{i}d(\xi)} \frac{\mathrm{d}k}{\sqrt{(k^2+c^2)(k^2+d^2(\xi))}}\right)^{-1},$$

The expression for $q_{el}(x,t)$ makes sense in $-6c^2t < x < 4c^2t$, hence it is natural to ask the question,

whether $q(x,t) - q_{el}(x,t) = \mathrm{o}(1)$ in $4c^2t - \varepsilon t < x < 4c^2t$?

It turned out, that in the domain $4c^2t - \frac{2N+1}{2c}\log t < x < 4c^2t$,

$$q_{el}(x,t) = \sum_{n=1}^{N} \frac{2c}{\cosh\left[2c(x-4c^2t) + (2n-1/2)\log t - \alpha_n(x,t)\right]} + \mathrm{O}\left(t^{-1/2}\right),$$

Here $\alpha_n(x,t)$ is no more a constant, but depends on x, t in the following way:

if x, t lie on a curve $x = 4c^2t - \frac{\gamma}{2c}\log t + \delta$, $\gamma > 0$, then

$$\alpha_n(x,t) = \alpha_n(\gamma) + \mathrm{o}(1),$$

where

$$\alpha_n(\gamma) := \left(2n - \frac{1}{2}\right)\log\frac{\gamma}{8c^3} - \gamma - \left(6n - \frac{7}{2}\right)\log 2 - 2\log|h^*| + \mathrm{O}\left(\frac{\log^2\log t}{\log t}\right).$$

Compare $q(x,t)$ and $q_{el}(x,t)$ in the interval $4c^2t - \frac{N}{c}\log t < x < 4c^2t$.

Question. Can we replace $\alpha_n(x,t)$ by a constant?

Observation. "Elliptic" soliton is supported along the line

$$x = 4c^2t - \frac{2n - 1/2}{2c}\log t + \frac{\alpha_n(x,t)}{2c},$$

hence, probably we can change $\alpha_n(x,t)$ with $\alpha_n(\gamma)|_{\gamma=2n-1/2}$?

However, this is not true.

Consider the curve

$$x = 4c^2 t - \frac{2n - 1/2 - \varepsilon}{2c} \log t + \frac{\delta}{2c}.$$

On it, the "elliptic" soliton with variable phase is

$$\frac{2c}{\cosh\left[2c(x - 4c^2 t) + (2n - 1/2)\log t - \alpha_n(x,t)\right]} =$$

$$= \frac{2c}{\cosh\left[\varepsilon \log t + \delta - \alpha_n(\gamma)|_{\gamma = 2n - 1/2 - \varepsilon}\right]} = \frac{4c(1 + o(1))}{t^{\varepsilon} \exp\left\{\delta - \alpha_n(\gamma)|_{\gamma = 2n - 1/2 - \varepsilon}\right\}},$$

while the "elliptic" soliton with constant phase is

$$\frac{2c}{\cosh\left[2c(x - 4c^2 t) + (2n - 1/2)\log t - \alpha_n(\gamma)|_{\gamma = 2n - 1/2}\right]} =$$

$$= \frac{2c}{\cosh\left[\varepsilon \log t + \delta - \alpha_n(\gamma)|_{\gamma = 2n - 1/2}\right]} = \frac{4c(1 + o(1))}{t^{\varepsilon} \exp\left\{\delta - \alpha_n(\gamma)|_{\gamma = 2n - 1/2}\right\}},$$

The difference is of the order $O(t^{-\varepsilon})$, bigger than admissible error $O(t^{-1/2})$.
Besides,

$$\tilde{\alpha}_n \neq \alpha_n(\gamma)|_{\gamma = 2n - 1/2}.$$

Hence, moreover,

$$q(x,t) \neq q_{el}(x,t) + o(1) \quad \text{in} \quad 4c^2 t - \frac{N}{c}\log t < x < 4c^2 t.$$

Observation. As $n \to \infty$, on the peaks of "elliptic" solitons (denote the peak curve by $x_n(t)$) we have

$$\tilde{\alpha}_n - \alpha_n(x_n(t), t) = \mathrm{O}\left(\frac{1}{n}\right) + \mathrm{O}\left(\frac{n \log\log t}{\log t}\right) + \mathrm{O}\left(\frac{\log^2 \log t}{\log t}\right).$$

This suggests a **hypothesis:**

For $t, N >> 1$ the main term of solution of the Cauchy problem is

$$q(x,t) \sim \begin{cases} q_{el}(x,t), \quad (-6c^2 + \varepsilon)t < x < 4c^2 t - \frac{2N - 3/2}{2c} \log t, \\ q_{as}(x,t), \quad 4c^2 t - \frac{2N + 1/2}{2c} \log t < x < 4c^2 t, \\ 0, x > 4c^2 t. \end{cases}$$

Method for finding asymptotic solitons

▷ For simplicity assume absence of discrete spectrum, i.e. solitons.

$$\begin{cases} K_1(x,z,t) + \int\limits_x^{+\infty} K_2(x,y,t)H(y+z,t)\mathrm{d}y = 0, \\ K_2(x,z,t) + \int\limits_x^{+\infty} K_1(x,y,t)H(y+z,t)\mathrm{d}y = -H(x+z,t), \end{cases}, \quad z > x,$$

where

$$H(x,t) = \frac{1}{2\pi}\int\limits_{\mathbb{R}} r(k)\mathrm{e}^{\mathrm{i}kx+8\mathrm{i}k^3t}\mathrm{d}k + \frac{1}{2\pi}\int\limits_0^{\mathrm{i}c} \frac{\mathrm{i}}{a_-(k)a_+(k)}\mathrm{e}^{\mathrm{i}kx+8\mathrm{i}k^3t}\mathrm{d}k.$$

The solution of MKdV is reconstructed by formulas

$$q(x,t) = 2\mathrm{i}K_2(x,x,t), \qquad \int\limits_x^{+\infty} q^2(y,t)\mathrm{d}y = -2K_1(x,x,t).$$

Make change of variables:

$$x = 4c^2t + \xi, \quad y = 4c^2t + \eta, \quad z = 4c^2t + \zeta,$$

Main input for $H(x+z)$ come from interval $(\mathrm{i}c, \mathrm{i}c - \mathrm{i}\epsilon)$.

$$K_j(x,y,t) = \widehat{K}_j(x - 4c^2t, y - 4c^2t, t), \qquad H_j(y+z,t) = \widehat{H}_j(y+z-8c^2t, t),$$

$$\widehat{H}(\eta+\zeta,t)=\frac{\mathrm{e}^{-c(\eta+\zeta)}}{2\pi}\int\limits_{\mathrm{i}c-\mathrm{i}\epsilon}^{\mathrm{i}c}\frac{\mathrm{i}}{a_-(k)a_+(k)}(k)\mathrm{e}^{\mathrm{i}(k-\mathrm{i}c)(\eta+\zeta)+8\mathrm{i}k(k^2+c^2)t}\mathrm{d}k+\mathrm{O}\left(\frac{1}{x+y}\right).$$

$$H_N(y+z,t)=\mathrm{e}^{-c(\zeta+\eta)}\sum_{n=0}^{N-1}\frac{(\zeta+\eta)^n}{n!}\frac{\omega_k^{(n)}}{t^{n+3/2}}.$$

Approximated system of equations:

$$\begin{cases}\widehat{K}_1(\xi,\zeta,t)+\int\limits_{\xi}^{+\infty}\widehat{K}_2(\xi,\eta,t)\widehat{H}_N(\eta+\zeta,t)\mathrm{d}y=0,\\ \\ \widehat{K}_2(\xi,\zeta,t)+\int\limits_{\xi}^{+\infty}\widehat{K}_1(\xi,\eta,t)\widehat{H}_N(\eta+\zeta,t)\mathrm{d}\eta=-\widehat{H}_N(\xi+\zeta,t),\end{cases}\quad,\quad\zeta>\xi.$$

The solution can be found in the form

$$\begin{cases}\widehat{K}_1(\xi,\eta,t)=\sum\limits_{j=0}^{N-1}X_j(\xi,t)\eta^{j-1}\mathrm{e}^{-c\eta},\\ \widehat{K}_2(\xi,\eta,t)=\sum\limits_{j=0}^{N-1}Y_j(\xi,t)\eta^{j-1}\mathrm{e}^{-c\eta},\end{cases}$$

RH problem for problem for $q_{el}(x,t)$.

1. $M(\xi,t;.)$ is 2x2 matrix-valued function, analytic in $k\in\mathbb{C}\setminus\Sigma$;
2. $M(\xi,t;k)\to I$ as $k\to\infty$; $\quad \xi=\frac{x}{12t}$;
3. $M_-(\xi,t;k)=M_+(\xi,t;k)J(\xi,t;k)$,

$$J=\begin{pmatrix}1 & 0\\ \frac{-r(k)}{F^2(k,\xi)}\mathrm{e}^{2\mathrm{i}tg(k,\xi)} & 1\end{pmatrix},\quad k\in L_1,$$

$$J=\begin{pmatrix}1 & \frac{F^2(k,\xi)}{\hat f(k)}\mathrm{e}^{-2\mathrm{i}tg(k,\xi)}\\ 0 & 1\end{pmatrix},\quad k\in L_7,$$

$$J=\begin{pmatrix}\mathrm{e}^{\mathrm{i}tB(\xi)+\mathrm{i}\Delta(\xi)} & 0\\ 0 & \mathrm{e}^{-\mathrm{i}tB(\xi)-\mathrm{i}\Delta(\xi)}\end{pmatrix},\quad k\in(\mathrm{i}d(\xi),-\mathrm{i}d(\xi)),$$

$$J=\begin{pmatrix}0 & \mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\quad k\in(\mathrm{i}c,\mathrm{i}d(\xi))\cup(-\mathrm{i}d(\xi),-\mathrm{i}c).$$

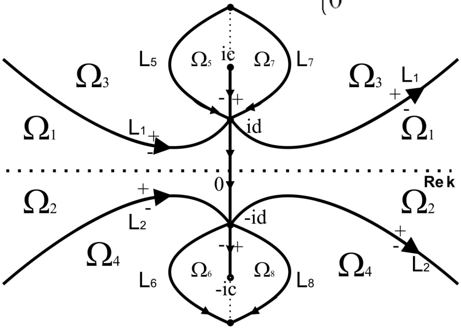
Jumps on other parts of the contour are defined by the symmetry

$$\overline{M(-\overline{k})}=M(k),\qquad \overline{M(\overline{k})}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}M(k)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

Reconstruction of $q(x,t)$ via $M(x,t;k)$

$$q(x,t)=2\mathrm{i}\lim_{k\to\infty}M_{21}(x,t;k).$$

$$J_{L_7} = \begin{pmatrix} 1 & \dfrac{F^2(k,\xi)}{\hat{f}(k)} e^{-2itg(k,\xi)} \\ 0 & 1 \end{pmatrix}$$



$$J_{L_1} = \begin{pmatrix} 1 & 0 \\ \dfrac{-r(k)}{F^2(k,\xi)} e^{2itg(k,\xi)} & 1 \end{pmatrix}$$

Here

$$g(k,\xi) = 12 \int\limits_{\mathrm{i}c}^{k} \frac{(s^2+\mu^2(\xi))\sqrt{s^2+d^2(\xi)}\mathrm{d}s}{\sqrt{s^2+c^2}},$$

$$\begin{cases} F_-(k,\xi)F_+(k,\xi) = (a_+(k)a_-(k))^{-1}, & k \in (\mathrm{i}c, \mathrm{i}d(\xi)), \\ F_-(k,\xi)F_+(k,\xi) = \overline{a_+(\overline{k})a_-(\overline{k})}, & k \in (-\mathrm{i}d(\xi), -\mathrm{i}c), \\ \frac{F_+(k,\xi)}{F_-(k,\xi)} = \mathrm{e}^{\mathrm{i}\Delta(\xi)}, & k \in (\mathrm{i}d(\xi), -\mathrm{i}d(\xi)). \end{cases}$$

and

$$\hat{f}(k) = \frac{1}{a^2(k)r(k)}.$$

Observation:

$$F_+(k,\xi) \sim \frac{2}{h^*}\sqrt[4]{\frac{k-\mathrm{i}c}{2\mathrm{i}c}}, \qquad \Delta(\xi) \to \frac{-\pi}{2},$$

$$\frac{F^2}{\hat{f}} \to 1, \quad \frac{-r(k)}{F^2(k,\xi)} \sim \frac{(h^*)^2}{4}\sqrt{\frac{2\mathrm{i}c}{k-\mathrm{i}c}}.$$

The solution coming from the model problem,

$$q_{el}(x,t) := 2\mathrm{i} \lim_{k\to\infty} (M_{mod})_{21}\,(\xi,t;k), \qquad \xi = \frac{x}{12t},$$

$$q_{el}(x,t) = \sqrt{c^2 - d^2\,(\xi)}\,\frac{\Theta(\mathrm{i}tB(\xi) + \mathrm{i}\Delta(\xi) + \pi\mathrm{i}, \tau(\xi))}{\Theta(\mathrm{i}tB(\xi) + \mathrm{i}\Delta(\xi), \tau(\xi))}, \qquad \xi = \frac{x}{12t}.$$

Asymptotic expansion of $q_{el}(x,t)$ in the domain $4c^2t - \frac{2N+1}{2c}\log t < x < 4c^2t$.

1. Slowly convergent Θ-series.

$$\Theta(z,\tau) = \sum_{m\in\mathbb{Z}} \exp\left\{\frac{\tau}{2}m^2 + zm\right\}$$

As $\xi \to \frac{c^2}{3} - 0$, we have $\tau \to -0$, and series becomes slowly-convergent. But

Poisson summation formula

$$\Theta(z,\tau) = \Theta\left(\frac{2\pi i z}{\tau}, \frac{4\pi^2}{\tau}\right)\sqrt{\frac{2\pi}{-\tau}}\exp\left(\frac{-z^2}{2\tau}\right).$$

Poisson summation formula transforms slowly-covergent series into fast convergent series. Denote

$$\tau^* = \frac{4\pi^2}{\tau}.$$

($b-$period for another choice of a, b-cycles.)

$$q_{el}(x,t) = \sqrt{c^2 - d^2(\xi)}\exp\left\{\frac{-\tau^*}{8} + \frac{\tau^*}{4}(z+1)\right\}\frac{\Theta\left(\frac{\tau^*}{2}(z+1),\tau^*\right)}{\Theta\left(\frac{\tau^*}{2}z,\tau^*\right)}.$$

where

$$z = \frac{1}{\pi}\left(tB(\xi) + \Delta(\xi)\right).$$

2. By direct computation we find that

$$q_{el}(x,t)=\sum_{n=1}^{N}\frac{2c}{\cosh\frac{\tau^*(-1-z+2n)}{4}}+\mathrm{O}\left(\mathrm{e}^{\tau*/4}\right),\qquad 0\le z\le 2N,$$

$$q_{el}(x,t)=\mathrm{O}\left(\mathrm{e}^{\tau*/8}\right),\qquad 0\le z\le 2N.$$

Further, we can find leading terms for the argument of cosh :

$$\frac{\tau^*(-1-z+2n)}{4}=2c(x-4c^2t)+(2n-1/2)\log t-\alpha_n(x,t),$$

The following asymptotics is valid for $\alpha_n(x,t)$:

$$\alpha_n(x,t)=-\left(2n-\frac{1}{2}\right)\log\frac{\log\frac{1}{v}}{vt}-\frac{8c^3tv}{\log\frac{1}{v}}-\left(6n-\frac{7}{2}\right)\log 2-2\log|h^*|+\mathrm{O}\left(\frac{\log^2\log\frac{1}{v}}{\log v}\right),$$

where

$$v=1-\frac{3\xi}{c^2}=1-\frac{x}{4c^2t}.$$

3. Asymptotics of integrals. Asymptotics for integrals $\int\limits_{\mathrm{i}d}^{\mathrm{i}c}$ can be found directly, since $d \to c$. Asymptotics of integrals $\int\limits_{0}^{\mathrm{i}d}$ is more tricky.

$\eta := 1 - \frac{d}{c}$ – introduce small variable η instead of d, $d \to c$.

$$\tau^*(d) = -4\pi \frac{I_1(d)}{I_0(d)}, \quad I_1(d) = \int\limits_0^d \frac{\mathrm{d}y}{\sqrt{(c^2 - y^2)(d^2 - y^2)}}, \quad I_0(d) = \int\limits_d^c \frac{\mathrm{d}y}{\sqrt{(c^2 - y^2)(d^2 - y^2)}}.$$

Expansion of I_0 is straightforward:

$$I_0 = |y = d + (c-d)s| = \int\limits_0^1 \frac{\mathrm{d}s}{\sqrt{s(1-s)}\sqrt{(c + d + (c-d)s)(2d + (c-d)s)}} = \frac{\pi}{2c} + \mathrm{O}(\eta).$$

But asymptotics of $I_1(d)$ is more tricky, and requires identity between Θ - function and the geometry of Riemann surface.

Link between τ^* and d is given by the relation

$$\frac{\Theta(0,\tau)}{\Theta(\pi \mathrm{i},\tau)} = \sqrt{\frac{c+d}{c-d}} \quad \Rightarrow \quad = \frac{\Theta(0,\tau^*)}{\Theta(\frac{\tau^*}{2},\tau^*)\exp\frac{\tau^*}{8}}.$$

Hence,

$$\tau^* = -4\log\frac{8}{\eta} + \mathrm{O}\left(\eta\right).$$

Now, when we know expansion of τ^* in η , we can get asymptotics for $I_1(d)$:

$$I_1(d) = \frac{1}{2c}\log\frac{8}{\eta} + \mathrm{O}\left(\eta\log\eta\right).$$

Evaluation of Δ

$$\Delta(d) = \int_{\mathrm{i}d}^{\mathrm{i}c} \frac{\log\left(a_+(k)a_-(k)\right)\mathrm{d}k}{\left(\sqrt{(k^2+c^2)(k^2+d^2)}\right)_+}\left(-\int_0^d \frac{\mathrm{d}s}{\sqrt{(c^2-s^2)(d^2-s^2)}}\right)^{-1} =: \frac{I_2(d)}{-I_1(d)}.$$

Easy

$$I_2(d) = \int_{\mathrm{i}d}^{\mathrm{i}c} \frac{\log\left(a_+(k)a_-(k)\right)\mathrm{d}k}{\left(\sqrt{(k^2+c^2)(k^2+d^2)}\right)_+} = \frac{\pi}{4c}\log\frac{(h^*)^4}{2\eta} + \mathrm{O}\left(\eta\right).$$

Hence,

$$\frac{1}{\pi}\Delta = \frac{-1}{2}\left(1 - \frac{4\log\frac{2}{|h^*|}}{\log\frac{1}{\eta}} + \mathrm{O}\left(\frac{1}{\log^2\eta}\right)\right).$$

Asymptotics of $\mu(d)$

$$\mu^2(d) = \frac{\int\limits_0^d \frac{y^2\sqrt{d^2-y^2}\,\mathrm{d}y}{\sqrt{c^2-y^2}}}{\int\limits_0^d \frac{\sqrt{d^2-y^2}\,\mathrm{d}y}{\sqrt{c^2-y^2}}} = d^2 - \frac{\int\limits_0^d \frac{(d^2-y^2)^{3/2}\,\mathrm{d}y}{\sqrt{c^2-y^2}}}{\int\limits_0^d \frac{\sqrt{d^2-y^2}\,\mathrm{d}y}{\sqrt{c^2-y^2}}} =: d^2 - \frac{I_4(d)}{I_3(d)},$$

To evaluate

$$I_3(d) = \int\limits_0^d \frac{\sqrt{d^2-y^2}\,\mathrm{d}y}{\sqrt{c^2-y^2}},$$

we notice that

$$I_3'(d) = dI_1(d), \qquad \Rightarrow \qquad I_3(d) = c - \frac{c}{2}\eta\log\frac{8}{\eta} + \mathrm{O}\left(\eta\log\eta\right).$$

Further,

$$I_4'(d) = 3dI_3(d) \qquad \Rightarrow \qquad I_4(d) = \frac{2}{3}c^3 - 3c^3\eta + \mathrm{O}\left(\eta^2\log\eta\right).$$

Hence,

$$\frac{3\mu^2(d)}{c} = 1 - \eta\log\frac{8}{\eta e^2} + \mathrm{O}\left(\eta^2\log^2\eta\right).$$

Once we know asymptotics of μ,asymptotics of B is straightforward:

Easy

$$B(d) = 24 \int_{\mathrm{i}d}^{\mathrm{i}c} \frac{(k^2 + \mu^2(d)) \left(\sqrt{k^2 + d^2}\right)_+ \mathrm{d}k}{\left(\sqrt{k^2 + c^2}\right)} = 8\pi c^3 \eta \left(1 + \mathrm{O}\left(\eta \log \eta\right)\right).$$

Link between $\xi = \frac{x}{12t}$ and d

$$v := 1 - \frac{3\xi}{c^2} = 1 - \frac{x}{4c^2 t}$$ – we use small variable v instead of ξ

$$\xi = \mu^2(d) + \frac{c^2 - d^2}{2} \qquad \Rightarrow \qquad \frac{v}{8e} = \frac{\eta}{8e} \log \frac{8e}{\eta} + \mathrm{O}\left(\eta^2 \log^2 \eta\right).$$

Hence

$$\frac{\eta}{8e} \log \frac{\eta}{8e} = \frac{-v}{8e} + \mathrm{O}\left(v^2\right).$$

We come to Lambert equation

$$we^w = z,\ w < 0, z < 0, \qquad w = \log \frac{\eta}{8e},\ z = \frac{-v}{8e} + \mathrm{O}(v^2).$$

$$w = -L_1 - L_2 - \frac{L_2}{L_1} + \frac{\mathrm{L}_2^2}{\mathrm{L}_1^2},\ z \to -0,$$

$$L_1 = \log \frac{1}{-z}, \quad L_2 = \log\log \frac{-1}{z}.$$

Corless, Gonnet, Hare, Jeffrey and Knuth D 96 On the Lambert W Function.

$$\eta = \frac{v}{\log \frac{1}{v}} \left(1 - \frac{\log 8e + \log\log \frac{1}{v}}{\log 1v} + \mathrm{O}\left(\frac{\log^2 \log \frac{1}{v}}{\log^2 \frac{1}{v}} \right) \right).$$

1. [1] E. Ya. Khruslov: *Asymptotics of the solution of the Cauchy problem for the Korteweg-de Vries equation with initial data of step type*, Matem. Sbornik (New Series) 99(141):2 (1976), 261–281.
2. [2] E. Ya. Khruslov, V. P. Kotlyarov: *Asymptotic solitons of the modified Korteweg-de Vries equation*, Inverse problems 5/6 (1989), 1075-1088.
3. [3] V. Kotlyarov, A. Minakov: *Riemann-Hilbert problem to the modified Korteweg-de Vries equation: long-time dynamics of the steplike initial data*, J. Math. Phys 51, no.9 (2010), 093506, 31pp.
4. [4] V. Kotlyarov, A. Minakov: *Modulated elliptic wave and asymptotic solitons in a shock problem to the modified Korteweg-de Vries equation*, J. Phys. A 13 (2015), 305201, 35pp.