The Quasisolution Method

O Costin, R.D. Costin, R. Donninger, I Glogić, W Schlag, S Tanveer, X. Xia

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Outline

The quasisolution method is designed to address questions of existence of solutions of ODEs, PDEs or difference equations, and in determining their properties, especially global ones, when classical methods are not known to apply. We have used it in a number of previously open questions such as Dubrovin's conjecture for P1, blowup in Wave Maps and Yang-Mills equations. It originated in the study of a spectral problem for NLS (w. Schlag and M. Huang).

By a quasisolution we understand an actual function which satisfies a given equation within "suitable" error bounds. Once the quasisolution is determined, showing the existence of an actual solution follows from standard contractive mapping arguments in adapted Banach spaces; these show the existence of an actual solution, roughly within the same error bounds from the quasisolution, globally, over the region of existence.

Obtaining usable quasisolutions with a low degree of complexity is made possible by resurgent function theory, discovered by Écalle. Resurgence theory *applies to a wide spectrum of problems in analysis*, such as ODEs, PDEs and difference equations. It provides very rich information of functions in nontrivial regions, such as close to essential singularities. With various extensions s.a transasymptotic analysis and KAM analysis it provides the tools for find accurate global representations– quasisolutions.

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Types of representation. Illustration, the Dubrovin conjecture

The Painlevé P1 equation,

$$f'' = 6f^2 + z$$

has a five-fold symmetry (simultaneous rotation by $2\pi/5$ of z and of f by $-\pi/5$ leaves the equation invariant).

There exist 5 special solutions, tritronquées, which are asymptotically free of poles in 4 of the 5 symmetry sectors. The Dubrovin conjecture, important in NLS & Toda lattices stated that the tritronquées are not only asymptotically pole-free but completely pole-free in these sectors, down to $z = 0$. This is a central connection question not solvable by Riemann-Hilbert.

Strategy: Find an approximation of f *globally good* in the four sectors.

Infinity is an irregular singularity, where expansions diverge. Beyond the sector of analyticity, P1, and more generally nonlinear equations develop singularity arrays; these affect the behavior in the analytic sector as well.

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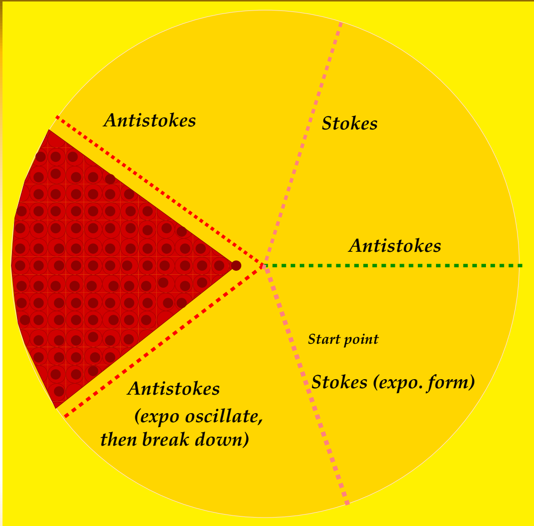

Figure: For the tritronquées there is only one C, and it is zero in 2 out of the five sectors–the sectors opposite to the pole sector. There is a reflection symmetry by the middle antistokes line.

We will also use a standard normalization of P1, similar to the Boutroux form. After the change of variables

$$x = \frac{e^{i\pi/4}}{30}(24z)^{5/4}\; f(z) = i\sqrt{\frac{z}{6}}\left(1 - \frac{4}{25x^2} + y(x)\right) \tag{1}$$

P1 becomes

$$y'' + \frac{1}{x}y' - y = \frac{y^2}{2} + \frac{392}{625x^4}. \tag{2}$$

from which we can derive the asymptotic expansion

$$y(x) = -\frac{392}{625x^4} - \frac{6272}{625x^6} - \frac{141196832}{390625x^8} + O\left(x^{-10}\right)$$

valid for a one parameter family of solutions, the tronquées. To specify the tritronquée we need a constant, the constant beyond all orders.

Behavior at infinity (away from antistokes lines)

Asymptotic expansions and transseries at infinity, in sectors of analyticity. General meromorphic nonlinear ODEs are known to possess **transseries** expansions at infinity (called "multi-instanton expansions" in physics), which for a first order ODE, after normalization, would take the form

$$\sum_{k=0}^{\infty} C^k x^{k\beta} e^{-kx} \tilde{y}_k(x) \quad (*)$$

where C is an arbitrary constant (the constant beyond all orders),

$$\tilde{y}_k(x) = \sum_{j=0}^{\infty} \frac{c_{jk}}{x^k}$$

are *factorially divergent* series. For higher order systems more than one exponential may be present and $C^k x^{k\beta} e^{-kx}$ is replaced by $\prod_{j \le j_0} [C_j x^{\beta_j} e^{-\lambda_j x}]^{k_j}$. Here $x \to \infty$ in such a way that $\mathrm{Re}\,(\lambda_j x) > 0$ (or else we need $C_j = 0$). The lines where $\mathrm{Re}\,(\lambda_j x) = 0$ are called antistokes lines; if $C_j \neq 0$ we call them active.

Each $\tilde{y}_k(x)$ is Écalle-Borel summable, and after Borel summation (*) becomes convergent and represents the general decaying solution of the ODE.

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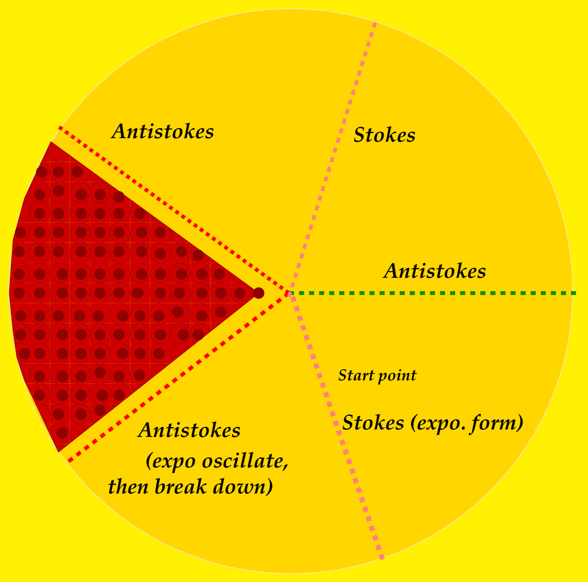

Figure: For the tritronquées there is only one C, and it is zero in 2 out of the five sectors–the sectors opposite to the pole sector.

For the tritronquées there is only one small exponential, and it is absent in the two sectors farthest from the pole sector; $\beta = -1/2$.

Whenever a series is Écalle-Borel summable, it is summable to the least term with exponential accuracy. That means, for $P1$,

$$\sum_{k=A}^{\lfloor x \rfloor} \frac{c_k}{x^k} = o(e^{-|x|})$$

Berry hyperasymptotics, that we improved recently allows for much sharper approximations, not needed here. In fact, the two terms

$$y_{01} :=\sim \sqrt{\frac{z}{6}}\Big(1 - \frac{4}{25x^2}\Big); \quad x = e^{i\pi/4}\frac{(24z)^{5/4}}{30}$$

suffice to obtain relative errors of $\sim 1/200$ for $|z| \geqslant 1.7$ in these two sectors. To show this rigorously, we take $y = y_{01} + \delta$, write the ODE for δ in integral form; the integral equation is contractive for $|z| \geqslant 1.7$ in the two sectors, and the error bound above is the one that follows from this argument (the actual ones are better).

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Whenever a series is Écalle-Borel summable, it is summable to the least term with exponential accuracy. That means, for $P1$,

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Behavior close to active antistokes lines

When a Stokes line (a line where $\mathrm{Im}\,(x\lambda_j) = 0$) is crossed the constant C in

$$\sum_{k=0}^{\infty} C^k x^{k\beta} e^{-kx} \tilde{y}_k(x) \ \ (*)$$

changes from C to $C + S$ where S is the so-called Stokes multiplier. For the tritronquée $C = 0$ before the first Stokes line. Thus its central antistokes line is inactive.

Note that (*) can be thought of as a formal power series in *two variables*, $1/x$ and $\xi = Ce^{-x}x^{\beta}$,

$$\tilde{y} = \tilde{y}(1/x, \xi)$$

Transasymptotic analysis (OC., R.D. Costin, Invent. Math 2001) deals with the behavior of resurgent functions when ξ is not small anymore, or is even large. The natural approach is to combine the terms:

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thought of as a function of two variables, (x^{-1}, ξ), and keep F_k unexpanded. It is shown in the paper above that (**) extends to a valid expansion (divergent in x, convergent in ξ) of the actual solution y in a region containing the first array of singularities, down except for $O(1/x)$ nbds of the actual singularities. These expansions apply in the transseries region as well, and they *extend transseries.*

The singularities of y come in arrays due to the periodicity of e^{-x} in ξ, and are located within $O(1/x)$ of the singular points of $F_0(\xi)$.

Transasymptotic expansions, the resurgent analog of Taylor series, can be matched to the Laurent/Puiseux expansions at singular points and to the Taylor series at zero. For generic second order ODEs, the F_k are explicit and singularities can be calculated in closed form. For $P1$, F_k are some rational functions with $F_0 = \xi/(\xi - 12)^2$.

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The expansion at infinity, truncated for $1/200$ rel. errors when $|z| > 1.7$

Though poles are absent in the four sectors, the behavior of the solution in the bordering sectors is affected by their presence.

We chose the following truncation of f. $f_0 =\sim i\sqrt{z/6}[1 - 4/(25x^2) + y_0]$ with

$$y_0(x) = \left(\xi + \frac{\xi^2}{6} + \frac{\xi^3}{48} + \frac{\xi^4}{432} + \frac{5\xi^5}{20736}\right) - \frac{1}{x}\left(\frac{\xi}{8} + \frac{11}{72}\xi^2 + \frac{43}{1152}\xi^3\right) + \frac{9\xi}{128x^2} \tag{3}$$

Here $\xi = x^{-1/2}e^{-x}$, $\mathrm{Re}\, x > 0$. If x is far from the pole sector (not the case e.g. when $|z| < 3$ or so), the ξ terms can be ignored.

The outer solution is valid within $1/200$ for $|z| \geq 1.7$. At 1.7 it matches a Taylor series about zero with radius of convergence $> 18/10$. For the proof: we write $f = f_0 + \delta$ in integral form and use standard contractive mapping arguments in suitable Banach spaces.

With newer methods A. Adali S. Tanveer computed the solution rigorously within 10^{-5} accuracy, and the **first pole** in the 5th sector within 4.10^{-6}.

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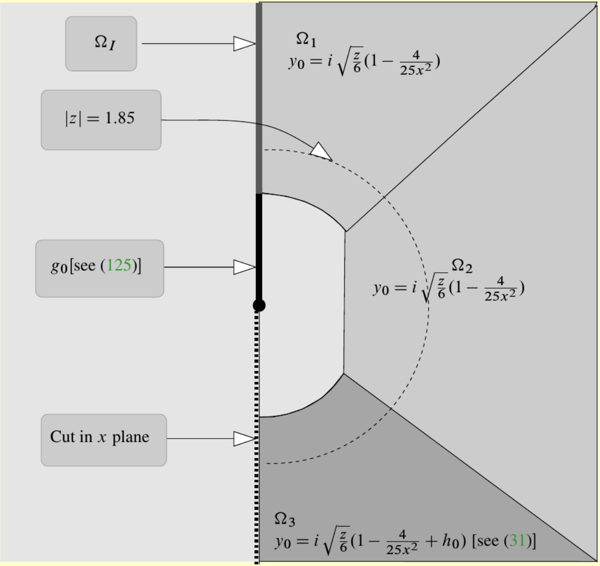
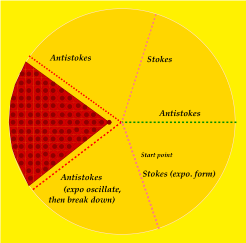


Figure 1. The domains of analysis and the corresponding quasisolutions.

The pole sector and the Stokes constant (O C, R.D. Costin, M. Huang, TAMS 2016)

In the pole sector, we found expansions in asymptotically conserved quantities which we used for determining (in closed form) the Stokes multiplier of P1 without resorting to linearizations (i.e., without isomonodromic deformations, or Riemann-Hilbert).



The idea is that single-valuedness (the “naive” Painlevé property) translates into a consistency condition between the outer transseries + transasymptotic expansions of the tritronquée and the inner KAM one. The consistency equation is an equation for the Stokes multiplier with a unique solution, $S = i\sqrt{\frac{6}{5\pi}}$.

Accurate formula for Blasius' similarity solution

This is an important similarity solution to boundary layer equations past a semi-infinite plate. It satisfies a two point boundary value equation,

$$f'''(x) + f(x)f''(x) = 0 \quad \text{for } x \in (0, \infty) \tag{4}$$

with initial condition at zero and no-slip boundary condition (condition at infinity)

$$f(0) = 0\,, f'(0) = 0\,, \lim_{x\to+\infty} f'(x) = 1 \tag{5}$$

Blasius derived it as an exact solution to Prandtl boundary layer equations. Existence and uniqueness of the solution were first proved by Weyl. In order to optimize various physical quantities, it is important to have accurate formulas over $\mathbb{R}^+$. Again, matching transseries and local expansions at zero

Theorem (OC, S. Tanveer, SIAM 2014)

The solution is within 10^{-5} of the function

$$F_0(x) = \begin{cases} \frac{x^2}{2} + x^4 P(x) \ \text{for } x \in [0, \frac{5}{2}] \\ ax + b + \sqrt{\frac{a}{2t(x)}}\, q_0(t(x)) \ \text{for } x > \frac{5}{2} \end{cases} \tag{6}$$

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where a, b, c are some specific rational numbers, and

$$q_0(t) = 2c\sqrt{t}e^{-t}I_0 + c^2 e^{-2t}\left(2J_0 - I_0 - I_0^2\right), \tag{8}$$

$$t(x) = \frac{a}{2}(x+b/a)^2,\ I_0(t) = 1-\sqrt{\pi t}e^t \mathrm{erfc}(\sqrt{t}),\ J_0(t) = 1-\sqrt{2\pi t}e^{2t}\mathrm{erfc}(\sqrt{2t}), \tag{9}$$

where erfc is the complementary error function and P is a degree 12 polynomial with rational coefficients. Higher accuracy formulas can be similarly obtained.

The same approach was pursued by S. Tanveer, A. Parab, A. Adali, TE Kim and others for a variety of boundary value problems and integro-differential equations in Fluid dynamics.

Spectral problems

An energy-supercritical Yang-Mills model. Let $A_\mu : \mathbb{R}^{1,5} \to \mathfrak{so}(5)$ be five fields on $(1+5)$-d Minkowski space with values in the matrix Lie algebra of $\mathrm{SO}(5)$; for fixed μ and $(t,x) \in \mathbb{R}^{1,5}$, $A_\mu(t,x)$ is real, skew-symmetric. One sets

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

and considers the action functional

$$\int_{\mathbb{R}^{1,5}} \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}). \tag{10}$$

(Y-M can be viewed as a nonlinear generalization of electrodynamics.) The Euler-Lagrange equations associated to the action (10) are

$$\partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0$$

The ansatz

$$A_\mu^{jk}(t,x) = (\delta_\mu^k x^j - \delta_\mu^j x^k)\frac{\psi(t,|x|)}{|x|^2}$$

yields the scalar nonlinear wave equation

$$\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{3\psi(\psi+1)(\psi+2)}{r^2} = 0$$

Does ψ blow up? (Shatah, Schlag, Struwe, Tataru, Donninger)

$$\psi_{tt} - \psi_{rr} - \frac{2}{r}\psi_r + \frac{3\psi(\psi+1)(\psi+2)}{r^2} = 0 \qquad (11)$$

(11) is energy-supercritical and large-data solutions can blow up in finite time ((11) has also been proposed as a model for singularity formation in Einstein's equation). The question of blow up has been open for a decade or so. Bizoń discovered self-similar blowup solutions of the form

$$\psi_0(t,r) = f_0\left(\tfrac{r}{1-t}\right), \qquad f_0(\rho) = -\frac{8\rho^2}{5+3\rho^2}.$$

This is **one** blowup solution, the question is whether solutions do blow up in this way in **some open neighborhood** of ψ_0.

Donninger developed a complete and elegant nonlinear stability theory for this and other types of nonlinear wave equations. However, the theory relies on a spectral condition on a **linear** nonselfadjoint operator described below. In **similarity coordinates** $\tau = -\log(1-t)$, $\rho = \frac{r}{1-t}$, $\varphi(\tau,\rho) = \psi(1-e^{-\tau}, e^{-\tau}\rho)$,

$$\varphi_{\tau\tau} + \varphi_\tau + 2\rho\varphi_{\tau\rho} - (1-\rho^2)(\varphi_{\rho\rho} + \tfrac{2}{\rho}\varphi_\rho) + \frac{3\varphi(\varphi+1)(\varphi+2)}{\rho^2} = 0 \qquad (12)$$

The domain of interest for (12) is the backward lightcone, $\tau \geq 0$, $\rho \in [0,1]$.

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$$\psi_0(t,r) = f_0(\tfrac{r}{1-t}), \qquad f_0(\rho) = -\frac{8\rho^2}{5+3\rho^2}.$$

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The self-similar solution is independent of τ and simply given by $f_0(\rho)$. One takes the **mode ansatz**

$$\varphi(\tau, \rho) = f_0(\rho) + e^{\lambda\tau} u_\lambda(\rho), \qquad \lambda \in \mathbb{C}$$

and linearizes in u_λ.

This yields the ODE spectral problem for u_λ

$$-(1-\rho^2)(u_\lambda'' + \tfrac{2}{\rho} u_\lambda') + 2\lambda\rho u_\lambda' + \lambda(\lambda+1) u_\lambda + \frac{V(\rho)}{\rho^2} u_\lambda = 0 \qquad (13)$$

where the potential V is given by

$$V(\rho) = 6 + 18 f_0(\rho) + 9 f_0(\rho)^2 = 6\frac{25 - 90\rho^2 + 33\rho^4}{(5+3\rho^2)^2}.$$

The singularity at $\rho = 1$ is due to the light cone being a characteristic surface.

Nonlinear stability hinges on the (non)existence of nontrivial *unstable eigenvalues* corresponding to **unstable modes**. These are λs, $\operatorname{Re} \lambda \geq 0$ s.t. there is a $C^\infty[0, 1]$ solution to (13) (which is then real-analytic on $[0, 1]$).

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with $\operatorname{Re}\lambda \geq 0$, where $W(\rho) = 20(15 - 2\rho^2 + 3\rho^4)/(5 + 3\rho^2)^2$.

The self-similar solution ψ_0 is mode stable.

We prove that any C^∞ (implying analytic) solution at zero is singular at one: A power series at zero, $\sum_{n=0}^{\infty} a_n(\lambda)\rho^{2n+3}$, $\quad a_0 \neq 0$ has radius of convergence one, which is shown finding a close approximation to a_n.

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1. The equation is (nontrivially) Heun, and no connection formulas are known. After various substitutions, one is led to a three term recurrence

$$q_2(n)b_{n+2} + q_1(n)b_{n+1} + q_0(n)b_n = 0, \text{ where} \tag{15}$$

$$\begin{aligned} q_2(n) &= -20n^2 - 190n - 390, \\ q_1(n) &= 8n^2 + (20\lambda + 84)n + 5\lambda^2 + 75\lambda + 160, \\ q_0(n) &= 12n^2 + (12\lambda + 42)n + 3\lambda^2 + 21\lambda + 30. \end{aligned}$$

2. Or, with $r_n = b_{n+1}/b_n$ we get the continued fraction

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The quasisolution is obtained analyzing the behavior of the actual solution for large n and small n and interpolating between the two. It is

$$\tilde{r}_n(\lambda) = \frac{\lambda^2}{4n^2 + 31n + 43} + \frac{\lambda}{n+4} + \frac{n+2}{n+4}. \tag{17}$$

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Lemma

r_1 and $(\tilde{r}_n)^{-1}$ for $n \geq 1$, are analytic in the closed RHP.

The denominator of r_1 and the polynomials $\tilde{r}_n(\lambda)$ for $n \geq 1$ have all of their zeros are in the (open) left half-plane; here we use the Routh-Hurwitz criterion.

Now let $\delta_n = \frac{r_n}{\tilde{r}_n} - 1$. Substitution into the recurrence yields

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to which we need to apply fixed point theorems which in turn need estimates.

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Lemma (Contractivity estimates)

The following estimates hold in $\overline{\mathbb{H}}$.

$$|\delta_1| \leq \frac{1}{4}, \;\; |\varepsilon_n| \leq \frac{1}{20}, \; |C_n| \leq \frac{3}{5}, \quad n \geq 1. \tag{20}$$

All three follow similarly. Take C_n: It is analytic in $\mathbb{H}$, and polynomially bounded in $\overline{\mathbb{H}}$. By Phragmén-Lindelöf it suffices to prove the estimate on $i\mathbb{R}$. For t real, we need to show that $|C_{n+1}(it)|^2 = Q_1(n, t^2)/Q_2(n, t^2) \leq 9/25$, or equivalently $9/25 \cdot Q_2 - Q_1 \geq 0$. But $9/25 \cdot Q_2 - Q_1$ is even with positive coefficients. □

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Wave Maps

The method is not limited to this particular equation. In a later paper, we proved blow up of (corotational) wave maps from $\mathbb{R}^{1+d}$ Minkowski spacetime into $\mathbb{S}^d$, the d-dimensional sphere, for any $d \geq 3$ (for $d = 2$ this was known (Struwe).

Let (M, g) be a Lorentzian spacetime and (N, h) a Riemannian manifold. $U : (M, g) \longrightarrow (N, h)$ is called a wave map if it is a critical point of the geometric action functional

$$S_g[U] := \frac{1}{2} \int_M |d_g U|^2 \, d\mu_g.$$

Here,

$$|d_g U(x)|^2 \equiv |d_g U(x)|^2_{T_x^\star M \otimes T_{U(x)} N} := \mathrm{tr}_g \left(U^\star (h) \right)$$

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In local coordinates (x_μ) on (M, g) we have

$$S_g[U] = \int_M g^{\mu\nu} h_{ab}(U)(\partial_\mu U^a)(\partial_\nu U^b)\, d\mu_g \tag{21}$$

where the Einstein summation convention is used. The Euler-Lagrange equations associated to this functional are

$$\Box_g U^a + g^{\mu\nu} \Gamma^a_{bc}(U)(\partial_\mu U^b)(\partial_\nu U^c) = 0 \tag{22}$$

and they constitute a system of semi-linear wave equations. Here, $\Box_g$ is the Laplace-Beltrami operator on (M, g)

$$\Box_g := \frac{1}{|g|}\partial_\mu(g^{\mu\nu}|g|\partial_\nu), \quad |g| := \sqrt{|\det(g_{\mu\nu})|}$$

and Γ^a_{bc} are the Christoffel symbols associated to the metric h on the target manifold. The system (22) is known as the **wave maps equation** (known in the physics literature as non-linear σ-model) and is the analog of harmonic maps between Riemannian manifolds in the case where the domain is a Lorentzian manifold instead.

As in Yang-Mills, from the Euler-Lagrange equations if the target is a hypersphere and there is rotation symmetry, one obtains in similarity variables,

$$\varphi_{\tau\tau} + \varphi_\tau + 2\rho\varphi_{\tau\rho} - (1-\rho^2)\varphi_{\rho\rho} - \left(\frac{d-1}{\rho} - 2\rho\right)\varphi_\rho + \frac{\sin(2\varphi)}{\rho^2} = 0, \quad (23)$$

For all $d \geq 3$, Bizoń-Biernat found self-similar blow-up solutions, now in the form $2\arctan[(d-1)^{-1/2}\rho]$. The supersymmetrically reformulated mode-stability problem is now

$$(1-\rho^2)w_\lambda'' + \left[\frac{d-1}{\rho} - 2(\lambda+1)\rho\right]w_\lambda' - \lambda(\lambda+1)w_\lambda - \frac{2(d-2)}{\rho^2}\frac{\rho^2-d}{\rho^2+d-2}w_\lambda = 0. \quad (24)$$

where now the equation has two parameters.

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As before, we get a three step recurrence, with continued fraction representation $r_{n+1} = -A_n - B_n/r_n$ where now

$$A_n(\lambda, k) = \frac{k\lambda^2 + k(4n+9)\lambda + 4kn^2 + 16nk - 4n^2 + 14k - 16n - 16}{2k(n+2)(2n+k+8)}$$

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$$\tilde{r}_n(\lambda, k) = \frac{1}{2}\frac{\lambda^2}{2n^2 + (k+8)n + k + 5} + \frac{2\lambda}{2n+k+6} + \frac{2n+3}{2n+k+6}, \qquad (25)$$

For any dimension the Bizoń-Biernat solution is mode stable.

The proof is similar to the Y-M one.

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$$\tilde{r}_n(\lambda, k) = \frac{1}{2}\frac{\lambda^2}{2n^2 + (k+8)n + k + 5} + \frac{2\lambda}{2n+k+6} + \frac{2n+3}{2n+k+6}, \qquad (25)$$

For any dimension the Bizoń-Biernat solution is mode stable.

The proof is similar to the Y-M one.

As before, we get a three step recurrence, with continued fraction representation $r_{n+1} = -A_n - B_n/r_n$ where now

$$A_n(\lambda, k) = \frac{k\lambda^2 + k(4n+9)\lambda + 4kn^2 + 16nk - 4n^2 + 14k - 16n - 16}{2k(n+2)(2n+k+8)}$$

and

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Theorem (OC, R. Donninger, I Glogić, CMP, to appear)

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Conclusions

1. The tools of resurgence (transseries, Écalle-Borel summability, Berry hyperasymptotics, transasymptotic matching etc) can now be used to describe in great detail and very good accuracy the behavior at infinity.

2. In many instances when we don't have explicit solutions, we don't need explicit solutions either.

3. and an "accurate enough" rigorous approximation can contain all the needed information.

4. Often there are only two important regions: the region at infinity, where transseries and transasymptotic analysis provide arbitrary accuracy expansions. Their region of validity goes down close to zero or to the first singular point, where they are shown to match the local expansions using standard analysis tools.

5. So far, we found them most useful in global, qualitative or semi-quantitative, analysis of ODEs and difference equations, and we are upgrading them to be used directly in PDEs.

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Thank you