THE STOKES PHENOMENON FOR CERTAIN PARTIAL DIFFERENTIAL EQUATIONS

Bożena Podhajecka (joint work with Sławomir Michalik)

Cardinal Stefan Wyszyński University
Warsaw,Poland

Asymptotic and Computational Aspects of Complex Differential Equations
Piza, Italy 2017

Abstract

We consider the title Stokes phenomenon for the solutions of the 1-dimensional complex heat equation and its generalizations. We focus our attention to find the Stokes lines, the anti-Stokes lines and jumps across these Stokes lines. The important point to note here is that we also show how to describe jumps in terms of hyperfunctions for mentioned equations in cases, where the Cauchy data are holomorphic functions with finitely many singular or branching points.
We emphasize that our fundamental tool which allows us to prove most of the main theorems is the theory of Borel summability.

The Gevrey asymptotics

Definition 1

Let S be a given sector in the complex plane and let $f \in \mathcal{O}(S)$.
A formal power series of Gevrey order s (where $s \in \mathbb{R}$)
$\hat{f}(t) = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{C}[[t]]_s$ is called *Gevrey's asymptotic expansion of order s* of the function f in S if

$$\forall_{S^* \prec S} \; \exists_{A,B<\infty} \; \forall_{N \in \mathbb{N}_0} \; \forall_{t \in S^*} \; |f(t) - \sum_{n=0}^{N} a_n t^n| \leqslant AB^N (N!)^s |t|^{N+1},$$

where S^* is a proper subsector of S.
If this is so, we will use notation $f(t) \sim_s \hat{f}(t)$ in S.

The Gevrey asymptotics

Now we recall two important theorems for Gevrey asymptotics.

The Gevrey asymptotics

Now we recall two important theorems for Gevrey asymptotics.

- Theorem 1. *Ritt's theorem for Gevrey asymptotics*
  Let $\hat{x}(t) \in \mathbb{C}[[t]]_s$, where $s > 0$. Let S be a sector of an opening α, where $0 < \alpha \leqslant s\pi$. Then there exists $x(t) \in \mathcal{O}(S)$ such that

$$x(t) \sim_s \hat{x}(t) \text{ in } S.$$

The Gevrey asymptotics

Now we recall two important theorems for Gevrey asymptotics.

- Theorem 1. *Ritt's theorem for Gevrey asymptotics*
  Let $\hat{x}(t) \in \mathbb{C}[[t]]_s$, where $s > 0$. Let S be a sector of an opening α, where $0 < \alpha \leqslant s\pi$. Then there exists $x(t) \in \mathcal{O}(S)$ such that

$$x(t) \sim_s \hat{x}(t) \text{ in } S.$$

- Theorem 2. *Watson's lemma*
  Let S be a sector of an opening α such that $\alpha > s\pi$ and $s > 0$. Suppose that $x(t) \in \mathcal{O}(S)$ satisfies $x(t) \sim_s 0$ in S. Then

$$x(t) \equiv 0 \ \text{ in } S.$$

Modified k-summability method in a direction d

The title procedure consists in transforming a formal power series $\hat{x}(t) = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{C}[[t]]_{1/k}$ into a holomorphic function. First, we apply *the Borel modified transform of order k* ($k > 0$) defined by

$$(\breve{B}_k \hat{x})(t) := \sum_{n=0}^{\infty} \frac{a_n\, t^n\, n!}{\Gamma(1 + n(1 + \frac{1}{k}))},$$

then we use *the Ecalle's operator of order k in a direction d* i.e.

$$(E_{k,d}\, g)(t) := t^{-k/(1+k)} \int_0^{\infty e^{id}} g(s)\, C_{(k+1)/k}((s/t)^{\frac{k}{1+k}})\, ds^{\frac{k}{1+k}},$$

where $k > 0, d \in \mathbb{R},\ g(s) = (\breve{B}_k \hat{x})(s)$ and for $\alpha > 1$

$$C_\alpha(\tau) := \sum_{n=0}^{\infty} \frac{(-\tau)^n}{n!\,\Gamma(1 - \frac{n+1}{\alpha})}.$$

Modified k-summability method in a direction d

Finally, we obtain so-called k-sum of $\hat{x}(t)$ given by $x(t) := (E_{k,d}g)(t) \in \mathcal{O}(S(d, \epsilon + \pi/k, r))$, where $S(d, \epsilon + \pi/k, r)$ is a bounded sector.

Modified k-summability method in a direction d

Finally, we obtain so-called k-sum of $\hat{x}(t)$ given by $x(t) := (E_{k,d}g)(t) \in \mathcal{O}(S(d, \epsilon + \pi/k, r))$, where $S(d, \epsilon + \pi/k, r)$ is a bounded sector.

- We can use modified k-summability method in the direction d if $E_{k,d}g$ is well defined, i.e. the following conditions are satisfied:
  1. $\hat{x}(t) \in \mathbb{C}[[t]]_{1/k}$,
  2. $(\breve{B}_k \hat{x})(t) \in \mathcal{O}(S_d)$, where $S_d = S(d, \epsilon)$ is an unbounded sector in a direction d,
  3. $|(\breve{B}_k \hat{x})(t)| \leqslant C_1 e^{C_2 |t|^k}$, for some $C_1, C_2 > 0$ as $t \longrightarrow \infty, t \in S_d$.

Modified k-summability method in a direction d

Finally, we obtain so-called k-sum of $\hat{x}(t)$ given by $x(t) := (E_{k,d}g)(t) \in \mathcal{O}(S(d, \epsilon + \pi/k, r))$, where $S(d, \epsilon + \pi/k, r)$ is a bounded sector.

- We can use modified k-summability method in the direction d if $E_{k,d}g$ is well defined, i.e. the following conditions are satisfied:
  1. $\hat{x}(t) \in \mathbb{C}[[t]]_{1/k}$,
  2. $(\breve{B}_k\hat{x})(t) \in \mathcal{O}(S_d)$, where $S_d = S(d, \epsilon)$ is an unbounded sector in a direction d,
  3. $|(\breve{B}_k\hat{x})(t)| \leqslant C_1 e^{C_2|t|^k}$, for some $C_1, C_2 > 0$ as $t \longrightarrow \infty, t \in S_d$.
- In this case we say that $\hat{x}(t)$ is *k-summable in a direction d* $(k > 0, d \in \mathbb{R})$.

Modified k-summability method in a direction d

Observe that by Watson's lemma k-sum $x(t)$ is the unique holomorphic function on $S(d, \epsilon + \pi/k, r)$ satisfying:

$$x(t) \sim_{1/k} \hat{x}(t) \text{ in } S(d, \epsilon + \pi/k, r).$$

The Stokes phenomenon

Definition 2

Assume that $\hat{x}(t) \in \mathbb{C}[[t]]_{1/k}$ is k-summable in all directions $d \in (\phi - \epsilon, \phi + \epsilon)$, but singular direction $d = \phi$ (for some $\epsilon > 0$). Then *the Stokes line* for $\hat{x}$ is a set $\mathcal{L}_\phi = \{t \in \mathbb{C} : \arg t = \phi\}$ and *the anti-Stokes lines* are sets $\mathcal{L}_{\phi \pm \frac{\pi}{2k}} = \{\ t \in \mathbb{C} : \arg t = \phi \pm \frac{\pi}{2k}\}$. Moreover, if $\phi + \epsilon$ and $\phi - \epsilon$ denotes a direction close to ϕ and, respectively, greater or less than ϕ, then the difference $x^{\phi+\epsilon} - x^{\phi-\epsilon}$ is called a *jump across the Stokes line* $\mathcal{L}_\phi$.

Remark

Analogously, we define the Stokes and anti-Stokes lines in case when we replace $\hat{x}(t) \in \mathbb{C}[[t]]_{1/k}$ by $\hat{u}(t,z) \in \mathcal{O}(D)[[t]]_{1/k}$.

Remark

Assume that S is a sector with an opening π/k in a direction ϕ. Let $f(t), g(t) \in \mathcal{O}(S)$ be k-sums of $\hat{x}(t)$ in directions $\phi - \epsilon/2$ and $\phi + \epsilon/2$ respectively. It means that $f(t) \sim_{1/k} \hat{x}(t)$ and $g(t) \sim_{1/k} \hat{x}(t)$ for all $t \in S$. Set $r(t) := |f(t) - g(t)|$ for all $t \in S$. Then $r(t)$ is minimal on the Stokes line $\mathcal{L}_\phi$ for t close to zero and satisfies inequalities $|f(t)| \leqslant r(t)$ or $|g(t)| \leqslant r(t)$ on the anti-Stokes lines $\mathcal{L}_{\phi \pm \frac{\pi}{2k}}$.

The Stokes and anti-Stokes lines for the heat equation

Now we are ready to describe the main part of this presentation.

The Stokes and anti-Stokes lines for the heat equation

Now we are ready to describe the main part of this presentation.

- Let us consider the heat equation $u_t(t,z) = u_{zz}(t,z)$ with initial condition $u(0,z) = \phi(z)$, where $\phi \in \mathcal{O}(D)$. As easily seen, this Cauchy problem has the unique formal solution

$$\hat{u}(t,z) = \sum_{n=0}^{\infty} \frac{\phi^{(2n)}(z)\, t^n}{n!}.$$

The Stokes and anti-Stokes lines for the heat equation

Theorem 3.(*D.A Lutz, M .Miyake and R. Schäfke 1999*)

Suppose that $\hat{u}(t,z)$ is a unique formal solution of the Cauchy problem of heat equation $u_t(t,z) = u_{zz}(t,z)$ with $u(0,z) = \phi(z) \in \mathcal{O}^2\left(D \cup S(\frac{d}{2},\varepsilon) \cup S(\frac{d}{2}+\pi,\varepsilon)\right)$ for some $\varepsilon > 0$.

Then $\hat{u}(t,z)$ is 1-summable in the direction d for every $\theta \in (d-\frac{\varepsilon}{2}, d+\frac{\varepsilon}{2})$ and its 1-sum is represented by:

$$u^\theta(t,z) = E_{1,\theta}\, \breve{B}_1\, \hat{u}(t,z) = \frac{1}{\sqrt{4\pi t}} \int_0^{e^{i\frac{\theta}{2}}\infty} \left(\phi(z+s)+\phi(z-s)\right) e^{\frac{-s^2}{4t}}\, ds,$$

for small t such that $\arg t \in (-\frac{\pi}{2}+\theta; \frac{\pi}{2}+\theta)$ and $z \in D$.

The Stokes and anti-Stokes lines for the heat equation

Theorem 4. (*S.M, B.P 2015*)

Assume that $\phi(z) = \frac{a}{z-z_0} + \tilde{\phi}(z)$ for some $a, z_0 \in \mathbb{C} \setminus \{0\}$ and $\tilde{\phi}(z) \in \mathcal{O}^2(\mathbb{C})$. Let $\delta := 2\arg z_0$. Then Stokes line is the set $\mathcal{L}_\delta$ and anti-Stokes lines are the sets $\mathcal{L}_{-\frac{\pi}{2}+\delta}$ and $\mathcal{L}_{\frac{\pi}{2}+\delta}$.
Moreover, jump across the Stokes line $\mathcal{L}_\delta$ is given by

$$u^{\delta+\varepsilon}(t,z) - u^{\delta-\varepsilon}(t,z) = -i\sqrt{(\pi/t)}ae^{-\frac{(z_0-z)^2}{4t}},$$

for $\arg t \in (-\frac{\pi}{2} + \delta + \varepsilon; \frac{\pi}{2} + \delta - \varepsilon)$.

The Stokes and anti-Stokes lines for the heat equation

The idea of the proof of Theorem 4

According to Theorem 3 we have:

The Stokes and anti-Stokes lines for the heat equation

The idea of the proof of Theorem 4

According to Theorem 3 we have:

- $$u^{\delta+\varepsilon}(t,z)=\frac{1}{\sqrt{4\pi t}}\int_0^{e^{\frac{i(\delta+\varepsilon)}{2}}\infty}\big(\phi(z+\tilde{s})+\phi(z-\tilde{s})\big)e^{-\tilde{s}^2/4t}\,d\tilde{s}$$

  for $\arg t\in(-\frac{\pi}{2}+\delta+\varepsilon;\frac{\pi}{2}+\delta+\varepsilon)$, also:

The Stokes and anti-Stokes lines for the heat equation

The idea of the proof of Theorem 4

According to Theorem 3 we have:

- $$u^{\delta+\varepsilon}(t,z)=\frac{1}{\sqrt{4\pi t}}\int_0^{e^{\frac{i(\delta+\varepsilon)}{2}}\infty}\big(\phi(z+\tilde{s})+\phi(z-\tilde{s})\big)e^{-\tilde{s}^2/4t}\,d\tilde{s}$$

  for $\arg t\in(-\frac{\pi}{2}+\delta+\varepsilon;\frac{\pi}{2}+\delta+\varepsilon)$, also:

- $$u^{\delta-\varepsilon}(t,z)=\frac{1}{\sqrt{4\pi t}}\int_0^{e^{\frac{i(\delta-\varepsilon)}{2}}\infty}\big(\phi(z+\tilde{s})+\phi(z-\tilde{s})\big)e^{-\tilde{s}^2/4t}\,d\tilde{s}$$

  for $\arg t\in(-\frac{\pi}{2}+\delta-\varepsilon;\frac{\pi}{2}+\delta-\varepsilon)$.

The Stokes and anti-Stokes lines for the heat equation

The idea of the proof of Theorem 4

Hence (and based on the residue theorem) we derive:

$$u^{\delta+\varepsilon}(t,z) - u^{\delta-\varepsilon}(t,z) =$$

$$= -2\pi i \operatorname{res}_{\tilde{s}=z_0-z}\left[\frac{1}{\sqrt{4\pi t}}\Big(\phi(z+\tilde{s})+\phi(z-\tilde{s})\Big)e^{-\tilde{s}^2/4t}\right] =$$

$$= -i\sqrt{(\pi/t)}\lim_{\tilde{s}\to z_0-z}(\tilde{s}-(z_0-z))\left[\Big(\phi(z+\tilde{s})+\phi(z-\tilde{s})\Big)e^{-\tilde{s}^2/4t}\right] =$$

The Stokes and anti-Stokes lines for the heat equation

The idea of the proof of Theorem 4

$$= -i\sqrt{(\pi/t)} \lim_{\tilde{s}\to z_0 - z}(\tilde{s}-(z_0-z))\left[\left(\frac{a}{\tilde{s}-(z_0-z)}+\tilde{\phi}(z+\tilde{s})+\right.\right.$$

$$\left.\left.+\frac{a}{z-\tilde{s}-z_0}+\tilde{\phi}(z-\tilde{s})\right)e^{-\frac{\tilde{s}^2}{4t}}\right] = -i\sqrt{(\pi/t)}\, a\, e^{-\frac{(z_0-z)^2}{4t}},$$

for $\arg t \in (-\frac{\pi}{2}+\delta+\varepsilon; \frac{\pi}{2}+\delta-\varepsilon)$.

The idea of the proof of Theorem 4

Notice that for such t that $\arg t \in (-\frac{\pi}{2} + \delta + \varepsilon; \frac{\pi}{2} + \delta - \varepsilon)$ occur $-i\sqrt{(\pi/t)}\, a\, e^{-\frac{(z_0-z)^2}{4t}} \sim_1 0$, $u^{\delta+\varepsilon} \sim_1 \hat{u}$ and $u^{\delta-\varepsilon} \sim_1 \hat{u}$.
Hence the Stokes line is the set $\mathcal{L}_\delta$, the anti-Stokes lines are the sets $\mathcal{L}_{-\frac{\pi}{2}+\delta}$, $\mathcal{L}_{\frac{\pi}{2}+\delta}$.

□

Examples

Consider the heat equation $\partial_t u = \partial_z^2 u,\ u(0,z) = \phi(z)$.

Examples

Consider the heat equation $\partial_t u = \partial_z^2 u,\ u(0,z) = \phi(z)$.

- Suppose that the Cauchy datum of the heat equation is given by $\phi(z) = \operatorname{ctg}(z - z_0) + \tilde{\phi}(z) =$

$$= \frac{1}{z - z_0} + \sum_{n=1}^{\infty} \left(\frac{1}{z - z_0 - n\pi} + \frac{1}{z - z_0 + n\pi} \right) + \tilde{\phi}(z),$$

  for some $z_0 \in \mathbb{R} \setminus \pi\mathbb{Z}$ and $\tilde{\phi}(z) \in \mathcal{O}^2(\mathbb{C})$. Then the Stokes line is $\mathcal{L}_\delta$ and the Anti-Stokes lines are $\mathcal{L}_{\delta \pm \frac{\pi}{2}}$ and the jump is of the form

$$u^{\delta+\varepsilon}(t,z) - u^{\delta-\varepsilon}(t,z) = -i\sqrt{\frac{\pi}{t}} \sum_{n \in \mathbb{Z}} e^{-\frac{(z_0 - z + n\pi)^2}{4t}},$$

  where $\delta = 0$.

Examples

Examples

- Suppose now that the Cauchy datum of the heat equation is given by $\phi(z) = \Gamma(z - z_0) + \tilde{\phi}(z) =$

$$= \lim_{n \longrightarrow \infty} \frac{n! \; n^{(z-z_0)}}{(z-z_0)(z-z_0+1)(z-z_0+2)\cdots(z-z_0+n)},$$

for some $z_0 \in \mathbb{R} \setminus \mathbb{N}_0$ and $\tilde{\phi}(z) \in \mathcal{O}^2(\mathbb{C})$. Then the Stokes line is $\mathcal{L}_\delta$ and the Anti-Stokes lines are $\mathcal{L}_{\delta \pm \frac{\pi}{2}}$ and the jump is of the form

$$u^{\delta+\varepsilon}(t,z) - u^{\delta-\varepsilon}(t,z) = -i\sqrt{\frac{\pi}{t}} \sum_{n \in \mathbb{N}_0} \frac{(-1)^n}{n!} e^{-\frac{(z_0 - z - n)^2}{4t}},$$

where $\delta = 0$.

The Stokes and anti-Stokes lines for generalizations of the heat equation

In this part we will generalize results presented in the previous slides.

The Stokes and anti-Stokes lines for generalizations of the heat equation

In this part we will generalize results presented in the previous slides.

- Let us consider an equation $\partial_t^p u(t,z) = \partial_z^q u(t,z)$, $p,q \in \mathbb{N}$ with the following initial conditions: $u(0,z) = \phi(z) \in \mathcal{O}(D)$ and $\partial_t^j u(0,z) = 0$ for $j = 1, 2, \ldots, p-1$.
  The equation above has a unique formal solution represented by

$$\hat{u}(t,z) = \sum_{n=0}^{\infty} \frac{\phi^{(qn)}(z) t^{pn}}{(pn)!}.$$

The Stokes and anti-Stokes lines for generalizations of the heat equation

Theorem 5.

Suppose that $\hat{u}(t,z)$ is the unique formal solution of the Cauchy problem $\partial_t^p u = \partial_z^q u$ (where $1 \leqslant p < q$) with initial conditions: $u(0,z) = \phi(z) \in \mathcal{O}^{\frac{q}{q-p}}\left(D \cup S(\frac{dp}{q} + \frac{2\pi l}{q}, \varepsilon)\right)$, $l = 0, \dots, q-1$ (for some $\varepsilon > 0$) and $\partial_t^j u(0,z) = 0$, $j = 1, 2, \dots, p-1$.

The Stokes and anti-Stokes lines for generalizations of the heat equation

Theorem 5.

Then $\hat{u}(t,z)$ is $\frac{p}{q-p}$-summable in the direction d and for every $\theta \in (d-\frac{\epsilon}{2}, d+\frac{\epsilon}{2})$ its $\frac{p}{q-p}$-sum is given by

$$u^{\theta}(t,z) = E_{\frac{p}{q-p},\theta}\check{B}_{\frac{p}{q-p}}\hat{u}(t,z) =$$

$$= \frac{1}{q\sqrt[q]{t^p}}\int_0^{e^{\frac{i\theta p}{q}}\infty}\left(\phi(z+\tilde{s})+\dots+\phi(z+e^{\frac{2(q-1)\pi i}{q}}\tilde{s})\right) C_{\frac{q}{p}}(\tilde{s}/\sqrt[q]{t^p})\, d\tilde{s},$$

for small t such that $\arg t \in (-\frac{\pi(q-p)}{2p}+\theta; \frac{\pi(q-p)}{2p}+\theta)$.

The Stokes and anti-Stokes lines for generalizations of the heat equation

Analogously to Theorem 4, we formulate the following results

The Stokes and anti-Stokes lines for generalizations of the heat equation

Analogously to Theorem 4, we formulate the following results

- Theorem 6.(*S.M, B.P 2015*)

  Assume that $\phi = \frac{a}{z-z_0} + \tilde{\phi}(z)$ for some $a, z_0 \in \mathbb{C} \setminus \{0\}$ and $\tilde{\phi}(z) \in \mathcal{O}^{\frac{q}{q-p}}(\mathbb{C})$. Set $\delta := \frac{q}{p}\arg z_0$. Then the Stokes line is the set $\mathcal{L}_\delta$ and the anti-Stokes lines are the sets $\mathcal{L}_{-\frac{\pi(q-p)}{2p}+\delta}$, $\mathcal{L}_{\frac{\pi(q-p)}{2p}+\delta}$.
  Moreover, jump across the Stokes line $\mathcal{L}_\delta$ is given by

$$u^{\delta+\varepsilon}(t,z) - u^{\delta-\varepsilon}(t,z) = \frac{-2\pi i}{q\sqrt[q]{t^p}}\, a\, C_{\frac{q}{p}}\Big(\frac{z_0 - z}{\sqrt[q]{t^p}}\Big),$$

$$\text{for } \arg t \in \Big(-\frac{\pi(q-p)}{2p} + \delta + \varepsilon; \frac{\pi(q-p)}{2p} + \delta - \varepsilon\Big).$$

The Stokes phenomenon for the heat equation via hyperfunctions

Consider again the heat equation $u_t(t,z) = u_{zz}(t,z)$ with $u(0,z) = \phi(z) \in \widetilde{\mathcal{O}^2(\mathbb{C} \setminus \{z_0\})}$. Let $z \in D$, $\theta = \arg z_0$ and $\theta(z) = \arg(z_0 - z)$. Assume that $F_z(s)$ is a hyperfunction such that $F_z(s) \in \mathcal{O}^2(S(\theta,\alpha) \setminus L(\theta(z),r)) / \mathcal{O}^2(S(\theta,\alpha))$ where $L(\theta(z),r) = \{z \in \mathbb{C} : \arg z = \theta(z),\ |z| > r\}$ (for some $\alpha, r > 0$) and $F_z(s) = \Big[\phi(z+s) + \phi(z-s)\Big]_{\theta(z)}$. Then for $\delta = 2\theta$ and sufficiently small z, the jump is given by

$$u^{\delta+\varepsilon}(t,z) - u^{\delta-\varepsilon}(t,z) = \frac{1}{\sqrt{4\pi t}} \int_0^{e^{i\theta(z)}\infty} F_z(s) e^{-\frac{s^2}{4t}} ds.$$

Example 1

Assume (as before) that $\phi(z) = \frac{a}{z-z_0} + \tilde{\phi}(z)$ for some $a, z_0 \in \mathbb{C} \setminus \{0\}$ and $\tilde{\phi}(z) \in \mathcal{O}^2(\mathbb{C})$. Then $F_z(s) = \Big[\phi(z+s) + \phi(z-s)\Big]_{\theta(z)} = \Big[\frac{a}{z+s-z_0}\Big]_{\theta(z)}$ and the jump is of the following form

$$u^{\delta+\varepsilon}(t,z) - u^{\delta-\varepsilon}(t,z) = \frac{1}{\sqrt{4\pi t}} \int_0^{e^{i\theta(z)}\infty} F_z(s) e^{-\frac{s^2}{4t}}\, ds =$$

$$= -i\sqrt{\frac{\pi}{t}}\, a\, \delta(s-(z_0-z)) \Big[e^{-\frac{s^2}{4t}}\Big] = -i\sqrt{\frac{\pi}{t}}\, a\, e^{-\frac{(z_0-z)^2}{4t}}.$$

Example 2

Assume now that $\phi(z) = ln(-z + z_0)$ for some $z_0 \in \mathbb{C} \setminus \{0\}$. Then $F_z(s) = \Big[\phi(z+s) + \phi(z-s)\Big]_{\theta(z)} = \Big[ln(-z - s + z_0)\Big]_{\theta(z)}$ and the jump is given by

$$u^{\delta+\varepsilon}(t,z) - u^{\delta-\varepsilon}(t,z) = \frac{1}{\sqrt{4\pi t}} \int_0^{e^{i\theta(z)}\infty} F_z(s) e^{-\frac{s^2}{4t}} ds =$$

$$= -i\sqrt{\frac{\pi}{t}} H_\theta(s - (z_0 - z)) \Big[e^{-\frac{s^2}{4t}}\Big] =$$

$$= -i\sqrt{\frac{\pi}{t}} \int_{z_0 - z}^{e^{i\theta}\infty} e^{-\frac{s^2}{4t}} ds.$$

Example 3

Assume now that $\phi(z) = (-z + z_0)^\lambda$ for some $z_0 \in \mathbb{C} \setminus \{0\}$ and $\lambda \notin \mathbb{Z}$. Then

$F_z(s) = \Big[\phi(z+s) + \phi(z-s)\Big]_{\theta(z)} = \Big[(-z - s + z_0)^\lambda\Big]_{\theta(z)} =$

$= -2iH_\theta\big(s - (z_0 - z)\big)\big(s - (z_0 - z)\big)^\lambda \sin(\lambda\pi),$

and the jump is given by

$$u^{\delta+\varepsilon}(t,z) - u^{\delta-\varepsilon}(t,z) = \frac{1}{\sqrt{4\pi t}} \int_0^{e^{i\theta(z)}\infty} F_z(s) e^{-\frac{s^2}{4t}}\, ds =$$

$$= -\frac{i}{\sqrt{\pi t}} \int_{z_0 - z}^{e^{i\theta}\infty} e^{-\frac{s^2}{4t}} \big(s - (z_0 - z)\big)^\lambda \sin(\lambda\pi) ds.$$

Remarks

Remarks

- It is worth pointing out that we can consider in Theorems 4 and 6 function $\phi(z)$ with more than one pole and with poles of higher order.

Remarks

- It is worth pointing out that we can consider in Theorems 4 and 6 function $\phi(z)$ with more than one pole and with poles of higher order.
- We also emphasize that presented results can be extend to general linear partial differential equations with constant coefficients and to moment partial differential equations introduced by W.Balser and M.Yoshino [3].

REFERENCES

[1] W. Balser, *Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations*, Springer, New York, 2000.

[2] W. Balser, *From Divergent Power Series to Analytic Functions: Theory and Applications of Multisummable Power Series*, Springer, Verlag, 1994.

[3] W. Balser, M. Yoshino, *Gevrey order of formal series solutions of inhomogeneous partial differential equations with constant coefficients*, Funkcialaj Ekvacioj, 53 (2010), 411–434.

[4] S. Michalik, B. Podhajecka, *The Stokes phenomenon for certain partial differential equations with meromorphic initial data*, Asymptot. Anal. 99 (2016), 163-182

THANK YOU!