Connection problem for Painlevé I τ-function

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- Fist rigorous solution of a constant problem for Painlevé τ-function: special case of Painlevé III [Tracy '91].
- We consider here the general case of Painlevé I.

Outline

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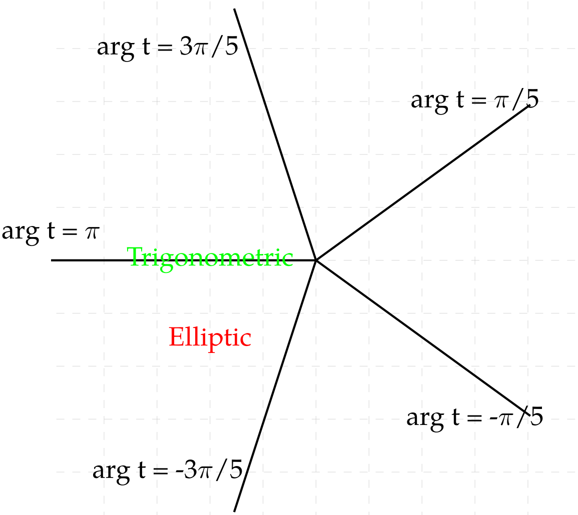
- Painlevé equations = 2nd order ODE with the property that their solutions have no movable branch points.
- Can be written as non-autonomous hamiltonian system

$$q_t = H_p, \qquad p_t = -H_q,$$

$$H = \frac{p^2}{2} - 2q^3 - tq,$$

$$\partial_t \log \tau = H$$

Asymptotics of q and τ for $t \to \infty$

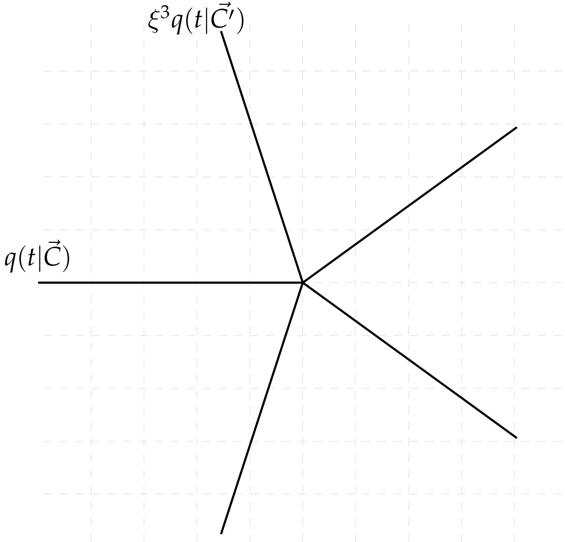


$\mathbb{Z}_5$ symmetry

- Denote $\vec{C}$ as the set of integration constants associated to a solution $q(t|\vec{C})$.
- Painlevé I has no free parameter, but a finite $\mathbb{Z}_5$ symmetry. If $q(t|\vec{C'})$ is a solution, then $\xi^2 q(\xi t|\vec{C})$

$$q(\xi t|\vec{C}) = \xi^3 q(t|\vec{C'})$$

$$\xi = e^{-\frac{2i\pi}{5}}$$


$\xi^3 q(t|\vec{C}')$
$q(t|\vec{C})$

$\mathbb{Z}_5$ symmetry for the τ-function

What is the equivalent for the τ-function?
Because τ is defined as a logarithmic derivative,

$$\tau(\xi t|\vec{C}) = \Upsilon(\vec{C})\tau(t|\vec{C'})$$

$\Rightarrow$ Connection problem for τ: compute $\Upsilon(\vec{C})$, and $\vec{C'} = f(\vec{C})$

Summary of questions

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2. $\Upsilon(\vec{C})$=? Main challenge: describe the monodromy dependence of τ

Outline

1. Introduction of the problem
2. Isomonodromic deformations
3. Monodromy dependence of τ
4. Connection constant

$$\begin{cases} \partial_z \Phi = A(z,t)\Phi \\ \partial_t \Phi = B(z,t)\Phi, \end{cases}$$

with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z^2 + \begin{pmatrix} 0 & q \\ 4 & 0 \end{pmatrix} z + \begin{pmatrix} -p & q^2 + t/2 \\ -4q & p \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} 0 & q \\ 2 & 0 \end{pmatrix}.$$

The non-linear equation

$$\partial_t A = \partial_z B + [B, A]$$

is equivalent to Painlevé I equation

$$\frac{d^2 q(t)}{dt} = 6q^2(t) + t.$$

The highest-polar part is non-diagonalizable. We would prefer a diagonalizable one.

$$\Psi(\xi,t)=K(\xi)^{-1}\Phi(\xi^2,t), \qquad K(\xi)=\begin{pmatrix}\sqrt{\xi}/2 & \sqrt{\xi}/2\\ 1/\sqrt{\xi} & -1/\sqrt{\xi}\end{pmatrix}$$

The new fundamental solution $\Psi(\xi,t)$ satisfies

$$\begin{cases}\partial_\xi\Psi=\tilde{A}(\xi,t)\Psi\\ \partial_t\Psi=\tilde{B}(\xi,t)\Psi,\end{cases}$$

where

$$\tilde{A}(\xi,t)=(4\xi^4+2q^2+t)\sigma_z-(2p\xi+\frac{1}{\xi})\sigma_x-(4q\xi2+2q^2+t)i\sigma_y,$$

$$\tilde{B}(\xi,t)=(\xi+\frac{q}{\xi})\sigma_z-\frac{iq}{\xi}\sigma_y$$

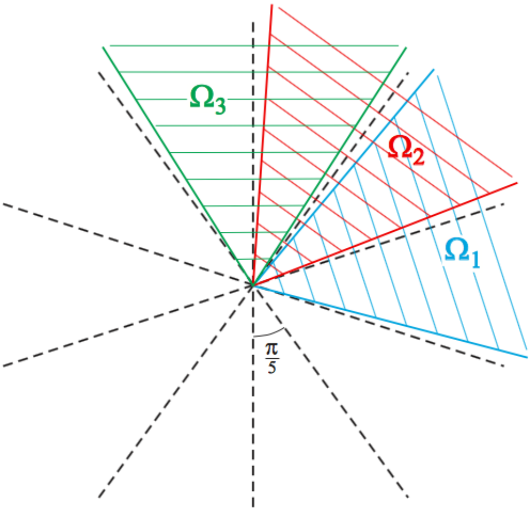
Stokes data

In the neiborhood of ∞, the equation $\partial_\xi \Psi = \tilde{A}(\xi, t)\Psi$ has a unique formal solution of the form

$$\Psi_{\text{form}}(\xi) = G(\xi) e^{(\frac{4}{5}\xi^5 + t\xi)\sigma_z}, \qquad G(\xi) = I_2 + \sum_{k=1}^{\infty} g_k \xi^{-k}$$

- Asymptotics of Ψ exhibits Stokes phenomenon
- Specified by 10 sectors Ω_k around ∞:

Stokes sectors around $z \to \infty$



Stokes data

- Canonical solutions are related by Stokes matrices

$$S_k := \Psi_k(\xi)^{-1}\Psi_{k+1}(\xi).$$

$$S_{2k-1} = \begin{pmatrix} 1 & s_{2k-1} \\ 0 & 1 \end{pmatrix}, \qquad S_{2k} = \begin{pmatrix} 1 & 0 \\ s_{2k} & 1 \end{pmatrix}, \qquad k = 1, \ldots, 5.$$

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- Only 2 independent Stokes parameters:

$$s_{k+3} = i(1 + s_k s_{k+1}), \qquad s_{k+5} = s_k$$

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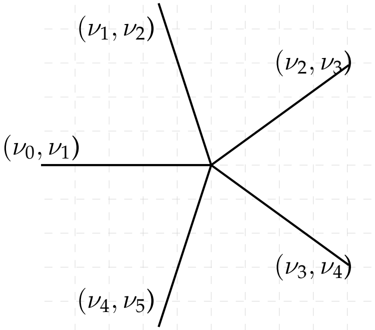
- Remark: Define $v_k = -is_{2k}$. Then,

$$v_{k-1}v_{k+1} = 1 - v_k.$$

v_k satistfy recurrence relations in cluster algebra of type A_2.

Stokes data as the set of integration constants

- Theorem [Kapaev, '88]: Stokes parameters uniquely define Painlevé function $q(t) \equiv q(t|\vec{v})$.
- Define $v_k = e^{2i\pi\nu_k}$. It will turn out that the most convenient set of local coordinates on $\mathcal{S}$ associated to the ray k is provided by $(\nu_k, \nu_{k+1}.)$.



- Main challenge for the connection problem of τ: describe the monodromy dependence of $\tau(t)$ on a ray k around ∞ in terms of canonical pairs of coordinates (ν_k, ν_{k+1})

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- It will turn out that $\Upsilon_{k,k'}$ is the generating function of the canonical transformation between (ν_k, ν_{k+1}) and $(\nu_{k'}, \nu_{k'+1})$.

Outline

Extended Painlevé I τ-function

- We know the t-dependence of τ:

$$\partial_t \log \tau = H.$$

What about

$$\partial_{\nu_k} \log \tau = ?$$

- Problem: find a closed 1-form $\hat{\omega}$ on $\tau \times \mathcal{S}$

$$\hat{\omega} = H dt + X_a dm_a + Y_b dm_b \qquad \text{such that}$$

$$d\hat{\omega} = 0, \qquad d = \frac{d}{dt} + \frac{d}{dm_a} + \frac{d}{dm_b}$$

$$\to \tau(t|\nu_k) \equiv e^{\int \hat{\omega}}$$

Extended Painlevé I τ-function

Strategy [Bertola, '09], [Its, Prokhorov, '15], [Its, Lisovyi, Prokhorov, '16]:

1. Construct a one-form ω such that $\Omega = d\omega$ be a closed 2-form on $\mathcal{S}$, independent of t. It turns out that ω can be written in terms of q and its derivatives.

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3. Construct $\omega_{0,k}$ such that $d\omega_{0,k} = \Omega$
4. $\hat{\omega}_k = \omega - \omega_{0,k}$ is a closed 1-form on $\tau \times \mathcal{S}$

$$\Rightarrow \hat{\omega}_k = d \log \tau_k$$

$$\omega = 2(Hdt + Q_a dm_a + Q_b dm_b),$$

$$Q_k = \frac{1}{5}(4tH_{m_k} + 3q_t q_{m_k} - 2qq_{tm_k}), \qquad k = a, b.$$

$$\to \Omega = 2(q_{tm_a} q_{m_b} - q_{tm_b} q_{m_a}) dm_a \wedge dm_b$$

- Using $q_{tt} = 6q^2 + t$, easy to show that $\partial_t \Omega = 0$.
- We want the explicit monodromy dependence of $\Omega \to$. It can be obtained from the leading order asymptotics of q.

Leading order asymptotics of q

- For $|\mathrm{Re}\ \nu_k| < \frac{1}{6}$, $\quad t \to \infty$, $\quad \arg t = \frac{2k\pi}{5} - i\pi$

$$q(t|\vec{\nu}) = e^{\frac{4i\pi k}{5}} \sqrt{e^{\frac{2i\pi k}{5} - i\pi} t}\Big[-\frac{1}{\sqrt{6}}$$

$$+ \sum_{\epsilon=\pm} \alpha_\epsilon(\nu_k) x^{-\frac{1+2\epsilon\nu_k}{2}} e^{\frac{4i\epsilon x}{5} + 2i\pi\epsilon\nu_{k+1}} + o(x^{-1+2|\mathrm{Re}\ \nu_k|})\Big],$$

where $x = 24^{1/4}\left(e^{\frac{2i\pi k}{5} - i\pi}\right)^{\frac{5}{4}}$ and

$$\alpha_+(\nu) = \frac{48^{-\nu} e^{-\frac{i\pi(1+2\nu)}{4}} \Gamma(1+\nu)}{2\sqrt{\pi}}, \qquad \alpha_-(\nu) = \frac{48^{\nu} e^{-\frac{i\pi(1-2\nu)}{4}} \sqrt{\pi}}{\Gamma(\nu)}.$$

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- Plugging this into $\Omega = 2(q_{tm_a} q_{m_b} - q_{tm_b} q_{m_a}) dm_a \wedge dm_b$, we find

$$\Omega = 4i\pi d\nu_k \wedge d\nu_{k+1}, \qquad k \in \mathbb{Z}/5\mathbb{Z}$$

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Now, we want $\omega_{0,k}$ such that $d\omega_{0,k} = \Omega$.

- Ω defines $\omega_{0,k}$ up to a closed 1-form on the space of Stokes data: this is the choice of normalization for τ.
- Let's define $\omega_{0,k} := 4i\pi\nu_k d\nu_{k+1}$. The extended τ-function $\tau_k(t|\vec{\nu})$ is defined by

$$d \log \tau_k(t|\vec{\nu}) = \frac{\omega - \omega_{0,k}}{2}.$$

Outline

Connection constant

- Once the normalization $\omega_{0,k}$ is fixed, the connection constant becomes a well-defined function of the monodromy:

$$\Upsilon_{kk'}(\vec{\nu}) = \frac{\tau_{k'}(t|\vec{\nu})}{\tau_k(t|\vec{\nu})}.$$

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- $\Rightarrow \Upsilon_{kk'}$ is the generating function of the canonical transformation between the pairs (ν_k, ν_{k+1}) and $(\nu_{k'}, \nu_{k'+1})$.

Theorem

The connection constant $\Upsilon_{k-1,k}(\vec{\nu})$ is expressed in terms of Stokes parameters ν_{k-1}, ν_k as

$$\Upsilon_{k-1,k}(\vec{\nu}) = e^{\frac{i\pi}{30}}(2\pi)^{-\nu_k} e^{2i\pi\nu_{k-1}\nu_k - \frac{i\pi\nu_k^2}{2}} \hat{G}(\nu_k),$$

where

$$\hat{G}(z) = \frac{G(1+z)}{G(1-z)}, \quad k \in \mathbb{Z}/5\mathbb{Z},$$

$$G(1+z) = \Gamma(z)G(z).$$

Properties of the connection constant

1. Quasiperiodicity relations

$$\Upsilon_{k-1,k}(\nu_{k-1}+1,\nu_k) = e^{2i\pi\nu_k}\Upsilon_{k-1,k}(\nu_{k-1},\nu_k)$$

$$\Upsilon_{k-1,k}(\nu_{k-1},\nu_k+1) = e^{-2i\pi\nu_{k+1}}\Upsilon_{k-1,k}(\nu_{k-1},\nu_k)$$

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2.

$$\tau(t|\vec{\nu}) \simeq \mathcal{C}_k(\nu)x^{-\frac{1}{60}}e^{\frac{x^2}{45}}\sum_{n\in\mathbb{Z}} e^{2i\pi n\nu_{k+1}}\mathcal{B}(\nu_k+n,x),$$

$$\mathcal{B}(\nu,x) \simeq C(\nu)x^{-\frac{\nu^2}{2}}e^{\frac{4}{5}i\nu x}\left[1+\sum_{k=1}^{\infty}\frac{B_k(\nu)}{x^k}\right],$$

$$C(\nu) = 48^{-\frac{\nu^2}{2}}(2\pi)^{-\frac{\nu}{2}}e^{-\frac{i\pi\nu^2}{4}}G(1+\nu).$$

1. $\hat{G}(\nu_k)$ satisfies an Abel type 5-terms relation

$$\Pi_{k=1}^{5}\hat{G}(\nu_k) = \sum_{k=1}^{5} L(\nu_k) = \frac{\pi^2}{2},$$

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2. $\Rightarrow$ Pentagonal relation

$$\Pi_{k=1}^{5}\Upsilon_{k-1,k}(\vec{\nu}) = 1$$

Conclusion and perspectives

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- We can in principle apply this method to other Painlevé equations
- Can we describe the Fourier structure of τ_I in terms of "irregular" conformal blocks?

Thank you!