Injectivity and surjectivity of the asymptotic Borel map in classes with log-convex constraints

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Motivation

Let $\alpha > 0$ and $\gamma > 0$.

S_γ will denote the unbounded sector of opening $\pi\gamma$ and vertex at 0, bisected by direction 0.

$\tilde{\mathcal{A}}_\alpha(S_\gamma)$ is the space of holomorphic functions in S_γ with (non-uniform) Gevrey asymptotic expansion of order α (so, estimates for the remainders in terms of the sequence $n!^\alpha$) as $z \to 0$ in the sector.

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Theorem (Watson's lemma)

$\mathcal{B} : \tilde{\mathcal{A}}_\alpha(S_\gamma) \longrightarrow \Lambda_{\{\alpha\}}$ *is injective if, and only if,* $\gamma > \alpha$.

Theorem (Borel–Ritt–Gevrey, J. P. Ramis (1978))

$\mathcal{B} : \tilde{\mathcal{A}}_\alpha(S_\gamma) \longrightarrow \Lambda_{\{\alpha\}}$ *is surjective if, and only if,* $\gamma \leq \alpha$.

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Aim: Generalize these results for asymptotic expansions with respect to more general sequences (only in the one variable case).

Logarithmically convex sequences

$\mathbb{N}_0 = \{0, 1, 2, ...\}$

A sequence $\mathbb{M} = (M_n)_{n\in\mathbb{N}_0}$ of positive numbers, with $M_0 = 1$, is said to be logarithmically convex (lc) if $M_n^2 \leq M_{n-1}M_{n+1}, \quad n \geq 1$.
The sequence of quotients of $\mathbb{M}$ is $\boldsymbol{m} = (m_n)_{n\in\mathbb{N}_0}$ given by $m_n := \frac{M_{n+1}}{M_n}$.

Examples:

- $\mathbb{M} = (\prod_{k=0}^{n} \log^{\beta}(e+k))_{n\in\mathbb{N}_0}$, $\beta > 0$, with quotients $m_n = \log^{\beta}(e+n+1)$.
- $\mathbb{M}_\alpha = (n!^{\alpha})_{n\in\mathbb{N}_0}$, Gevrey sequence of order $\alpha > 0$, with quotients $m_n = (n+1)^{\alpha}$.
- $\mathbb{M}_{\alpha,\beta} = \left(n!^{\alpha} \prod_{m=0}^{n} \log^{\beta}(e+m)\right)_{n\in\mathbb{N}_0}$, $\alpha > 0$, $\beta \in \mathbb{R}$, with quotients $m_n = (n+1)^{\alpha} \log^{\beta}(e+n+1)$.
- For $q > 1$, $\mathbb{M} = (q^{n^2})_{n\in\mathbb{N}_0}$, with quotients $m_n = q^{2n+1}$.

We will always assume that $\mathbb{M}$ **is (lc) and $\boldsymbol{m}$ tends to infinity**, and we will associate with it three different spaces of functions admitting an asymptotic expansion in a sector (or sectorial region).

Ultraholomorphic Carleman-Roumieu classes and the Borel map

Given $\mathbb{M}$, $A>0$ and a sector S, we consider

$$\mathcal{A}_{\mathbb{M},A}(S)=\left\{f\in\mathcal{H}(S):\|f\|_{\mathbb{M},A}:=\sup_{z\in S,n\in\mathbb{N}_0}\frac{|f^{(n)}(z)|}{A^n n! M_n}<\infty\right\}.$$

$(\mathcal{A}_{\mathbb{M},A}(S),\|\ \|_{\mathbb{M},A})$ is a Banach space.
$\mathcal{A}_{\{\mathbb{M}\}}(S):=\cup_{A>0}\mathcal{A}_{\mathbb{M},A}(S)$ is an $(LB)-$space.

$$\Lambda_{\mathbb{M},A}=\left\{\boldsymbol{\mu}=(\mu_n)_{n\in\mathbb{N}_0}\in\mathbb{C}^{\mathbb{N}_0}:|\boldsymbol{\mu}|_{\mathbb{M},A}:=\sup_{n\in\mathbb{N}_0}\frac{|\mu_n|}{A^n n! M_n}<\infty\right\}.$$

$(\Lambda_{\mathbb{M},A},|\ |_{\mathbb{M},A})$ is a Banach space.
$\Lambda_{\{\mathbb{M}\}}:=\cup_{A>0}\Lambda_{\mathbb{M},A}$ is an $(LB)-$space.

The Borel map is

$$\mathcal{B}:\mathcal{A}_{\{\mathbb{M}\}}(S)\longrightarrow\Lambda_{\{\mathbb{M}\}}$$
$$f\mapsto(f^{(n)}(0):=\lim_{z\to0,z\in S}f^{(n)}(z))_{n\in\mathbb{N}_0}.$$

Classes of functions with an asymptotic expansion

We say $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(S)$ if $f \in \mathcal{H}(S)$, and f admits the series $\hat{f} = \sum_{n=0}^{\infty} a_n z^n$ as its $\mathbb{M}$-asymptotic expansion at 0, denoted $f \sim_{\mathbb{M}} \hat{f}$: For every bounded proper subsector T of S there exist $C_T, B_T > 0$ such that for every $z \in T$ and every $n \in \mathbb{N}_0$, we have

$$\Big|f(z) - \sum_{k=0}^{n-1} a_k z^k\Big| \leq C_T B_T^n M_n |z|^n.$$

We say $f \in \tilde{\mathcal{A}}^u_{\mathbb{M}}(S)$ if $f \in \mathcal{H}(S)$, and f admits the series $\hat{f} = \sum_{n=0}^{\infty} a_n z^n$ as its $\mathbb{M}$-uniform asymptotic expansion at 0: There exist $C, B > 0$ such that for every $z \in S$ and every $n \in \mathbb{N}_0$, we have

$$\Big|f(z) - \sum_{k=0}^{n-1} a_k z^k\Big| \leq C B^n M_n |z|^n.$$

Remark: The elements of $\tilde{\mathcal{A}}^u_{\mathbb{M}}(S)$ and $\mathcal{A}_{\{\mathbb{M}\}}(S)$ are bounded functions in S.

If we put $f^{(n)}(0) := n! a_n$, $n \in \mathbb{N}_0$, the Borel map makes sense,

$$\mathcal{B} : \tilde{\mathcal{A}}_{\mathbb{M}}(S) \ (\text{or } \tilde{\mathcal{A}}^u_{\mathbb{M}}(S)) \to \Lambda_{\{\mathbb{M}\}}.$$

Questions. Flat functions and quasianalyticity

All the previous spaces may be defined in the same way for sectorial regions.
It is clear that for any sector S,

$$\mathcal{A}_{\{\mathbb{M}\}}(S) \subset \tilde{\mathcal{A}}^{u}_{\mathbb{M}}(S) \subset \tilde{\mathcal{A}}_{\mathbb{M}}(S).$$

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And for what sectors or sectorial regions is the asymptotic Borel map surjective in these classes?

We will concentrate in the classes defined in sectors S_γ.
We first study injectivity, so we look for

$$\tilde{I}_{\mathbb{M}} := \{\gamma > 0 : \tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma) \text{ is quasianalytic}\},$$

and for $\tilde{I}^u_{\mathbb{M}}$ and $I_{\{\mathbb{M}\}}$, defined accordingly.

One clearly has

$$\tilde{I}_{\mathbb{M}} \subset \tilde{I}^u_{\mathbb{M}} \subset I_{\{\mathbb{M}\}}.$$

Interval of quasianalyticity for $\tilde{A}^u_{\mathbb{M}}(S_\gamma)$

S. Mandelbrojt, *Séries adhérentes, régularisation des suites, applications*, Collection de monographies sur la théorie des fonctions, Gauthier-Villars, Paris, 1952.

Theorem

Let $\mathbb{M}$ be (lc) with $\lim_{n\to\infty} m_n = \infty$, $S_1 = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ and $\gamma > 0$. The following statements are equivalent:

(i) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{m_n}\right)^{1/\gamma}$ *diverges,*

(ii) *If $f \in \mathcal{H}(S_1)$ and there exist $A, C > 0$ such that*

$$|f(z)| \le \frac{CA^n M_n}{|z|^{\gamma n}}, \qquad z \in S_1, \quad n \in \mathbb{N}_0,$$

then f identically vanishes.

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then f *identically vanishes.*

Observe that, if we put $g(z) := f(1/z^{1/\gamma})$, then $g \in \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma)$ and g is flat.

So,

$$\tilde{I}^u_{\mathbb{M}} = \{\gamma > 0 : \sum_{n=0}^{\infty} \left(\frac{1}{m_n}\right)^{1/\gamma} \text{ diverges}\}.$$

Interval of quasianalyticity for $\mathcal{A}_{\mathbb{M}}(S_\gamma)$

B. Rodríguez Salinas, Functions with null moments, Rev. Acad. Ciencias, 49 (1955), 331–368.

B. I. Korenbljum, Conditions of nontriviality of certain classes of functions analytic in a sector, and problems of quasianalyticity, Soviet Math. Dokl. 7 (1966), 232–236.

Theorem

Let $\mathbb{M}$ be logarithmically convex and $\gamma > 0$. The following statements are equivalent:

- *The class $\mathcal{A}_{\{\mathbb{M}\}}(S_\gamma)$ is quasianalytic.*
- $\displaystyle\sum_{n=0}^{\infty}\Big(\frac{1}{(n+1)m_n}\Big)^{1/(\gamma+1)} = \infty.$

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$$I_{\{\mathbb{M}\}} = \{\gamma > 0 : \sum_{n=0}^{\infty}\left(\frac{1}{(n+1)m_n}\right)^{1/(\gamma+1)} \text{ diverges}\}.$$

Optimal opening for quasianalyticity, and cases

J. Jiménez-Garrido, J. S., Strongly regular sequences and proximate orders. J. Math. Anal. Appl. 438 (2016), no. 2, 920–945.

Proposition

For $\mathbb{M}$ logarithmically convex with $\lim_{n\to\infty} m_n = \infty$, the value that "puts apart" quasianalyticity from non-quasianalyticity, for both $\mathcal{A}_{\{\mathbb{M}\}}(S_\gamma)$ and $\tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma)$, is the inverse of the exponent of convergence of the sequence $\boldsymbol{m}$, i.e.,

$$\omega(\mathbb{M}) := \liminf_{n\to\infty} \frac{\log(m_n)}{\log(n)}.$$

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$$\omega(\mathbb{M}) := \liminf_{n\to\infty} \frac{\log(m_n)}{\log(n)}.$$

Case A: $\omega(\mathbb{M}) = \infty$. The two previous series never converge. All the intervals of quasianalyticity are empty. In any sector there are nontrivial flat functions of any kind.

Example: For $q > 1$, $\mathbb{M} = (q^{n^2})_{n\in\mathbb{N}_0}$.

Cases (recall $\tilde{I}_{\mathbb{M}} \subset \tilde{I}^u_{\mathbb{M}} \subset I_{\{\mathbb{M}\}}$)

Case B: $\omega(\mathbb{M}) < \infty$. First, observe that $(\omega(\mathbb{M}), \infty) \subset \tilde{I}_{\mathbb{M}}$ by Mandelbrojt's result.

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Case B.1: $\omega(\mathbb{M}) = 0$. Then, $\tilde{I}_{\mathbb{M}} = \tilde{I}^u_{\mathbb{M}} = I_{\{\mathbb{M}\}} = (0, \infty)$. No sector admits nontrivial flat functions of any kind.

Example: $\mathbb{M} = (\prod_{k=0}^{p} \log^{\beta}(e+k))_{p\in\mathbb{N}_0}$.

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Case B.2: $\omega(\mathbb{M}) \in (0, \infty)$, then

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Case B.2.1: $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{(n+1)m_n}\right)^{1/(\omega(\mathbb{M})+1)} < \infty$. Then, all the intervals are $(\omega(\mathbb{M}), \infty)$. There are nontrivial flat functions in $S_{\omega(\mathbb{M})}$ with uniformly bounded derivatives.

Example: Consider $\mathbb{M}_{\alpha,\beta} = \left(n!^{\alpha} \prod_{m=0}^{n} \log^{\beta}(e+m)\right)_{n\in\mathbb{N}_0}$, $\alpha > 0$, $\beta \in \mathbb{R}$. Then $\omega(\mathbb{M}_{\alpha,\beta}) = \alpha$, and B.2.1 holds precisely when $\beta > \alpha + 1$.

Cases

Case B.2.2: $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{(n+1)m_n}\right)^{1/(\omega(\mathbb{M})+1)}=\infty$, but $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{m_n}\right)^{1/\omega(\mathbb{M})}<\infty$.

Then, $\tilde{I}_{\mathbb{M}}=\tilde{I}^u_{\mathbb{M}}=(\omega(\mathbb{M}),\infty)$ and $I_{\{\mathbb{M}\}}=[\omega(\mathbb{M}),\infty)$. There are nontrivial flat functions with uniform $\mathbb{M}-$asymptotic expansion in $S_{\omega(\mathbb{M})}$, but there is no nontrivial flat function with uniformly bounded derivatives.

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In this case, it makes sense to study the quasianalyticity of $\mathcal{A}_{\mathbb{M}}(G)$ for a general sectorial region G of opening $\pi\omega(\mathbb{M})$: results by R. S. Yulmukhametov (1986) for bounded convex domains, and by K. V. Trunov and R. S. Yulmukhametov (2009) for some non-convex bounded domains.

Cases

Case B.2.3: $\sum_{n=0}^{\infty}\left(\frac{1}{m_n}\right)^{1/\omega(\mathbb{M})}=\infty$. Then, $\tilde{I}^u_{\mathbb{M}}=I_{\{\mathbb{M}\}}=[\omega(\mathbb{M}),\infty)$. There is no nontrivial flat function with uniform $\mathbb{M}-$asymptotic expansion in $S_{\omega(\mathbb{M})}$.

Example: $\mathbb{M}_{\alpha,\beta}$ for $\beta\le\alpha$ (in particular for Gevrey sequences and for $\mathbb{M}_{1,-1}$, related to the 1^+ level). This situation is important in applications.

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Little information available: for the Gevrey sequence $\mathbb{M}_\alpha$, $\tilde{I}_{\mathbb{M}_\alpha}=(\alpha,\infty)$ (Watson's Lemma).

Flat functions in $\tilde{\mathcal{A}}_{\mathbb{M}}(S)$

Let $f \in \tilde{\mathcal{A}}_{\mathbb{M}}(S)$ be flat. Then, for every $z \in T$ we have

$$|f(z) - \sum_{k=0}^{n-1} 0 \cdot z^k| \leq C_T B_T^n M_n |z|^n,\ n \in \mathbb{N}_0$$

$$\Leftrightarrow |f(z)| \leq C_T \inf_{n \geq 0} M_n (B_T|z|)^n = C_T h_{\mathbb{M}}(B_T|z|) = C_T e^{-M(1/(B_T|z|))},$$

where $h_{\mathbb{M}}(t) := \inf_{n\geq 0} M_n t^n$, $t > 0$; $M(t) := -\log\big(h_{\mathbb{M}}(1/t)\big)$.

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$$h_{\mathbb{M}}(t) = \begin{cases} M_n t^n & \text{if } t \in \left[\frac{1}{m_n}, \frac{1}{m_{n-1}}\right), \ n = 1, 2, \ldots, \\ 1 & \text{if } t \geq 1/m_0. \end{cases}$$

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Hint: Find a holomorphic function $V(z)$ in the optimal sector $S_{\omega(\mathbb{M})}$ whose growth is suitably governed by $M(t)$, then $\exp(-V(1/z))$ will be flat.

Proximate orders, I

Our solution will come from the theory of growth of holomorphic functions, resting on the concept of proximate order. Results by E. Lindelöf, G. Valiron, M. Cartwright, V. Bernstein, B. Ya. Levin, M. M. Dzhrbashyan, M. A. Evgrafov, A. A. Gol'dberg, I. V. Ostrovskii and, in our regards, mainly L. S. Maergoiz.

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Definition (E. Lindelöf, G. Valiron)

We say $\rho(r) : (0, \infty) \to \mathbb{R}$ is a proximate order if the following hold:

(1) ρ is continuous and piecewise continuously differentiable,

(2) $\rho(r) \geq 0$ for every $r > 0$,

(3) $\lim_{r\to\infty} \rho(r) = \rho < \infty$,

(4) $\lim_{r\to\infty} r\rho'(r)\log(r) = 0$.

In case $\lim_{r\to\infty} \rho(r) \in (0, \infty)$, we say $\rho(r)$ is a nonzero proximate order.

Proximate orders, II

L. S. Maergoiz, Indicator diagram and generalized Borel-Laplace transforms for entire functions of a given proximate order, St. Petersburg Math. J. 12 (2001), no. 2, 191–232.

Theorem

Let $\rho(r)$ be a proximate order with $\rho(r) \to \rho > 0$ as $r \to \infty$. For every $\gamma > 0$ there exists an analytic function $V(z)$ in S_γ such that:

(1) $\displaystyle\lim_{r\to\infty} \frac{V(zr)}{V(r)} = z^\rho$ *uniformly in the compact sets of S_γ (regular variation).*

(2) $\overline{V(z)} = V(\overline{z})$ *for every $z \in S_\gamma$.*

(3) $V(r)$ *is positive in $(0, \infty)$.*

(4) $\displaystyle\lim_{r\to\infty} \frac{V(r)}{r^{\rho(r)}} = 1.$

Proximate orders, II

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Theorem

Let $\rho(r)$ be a proximate order with $\rho(r) \to \rho > 0$ as $r \to \infty$. For every $\gamma > 0$ there exists an analytic function $V(z)$ in S_γ such that:

(1) $\lim_{r\to\infty} \frac{V(zr)}{V(r)} = z^\rho$ *uniformly in the compact sets of S_γ (regular variation).*

(2) $\overline{V(z)} = V(\overline{z})$ *for every $z \in S_\gamma$.*

(3) $V(r)$ *is positive in $(0, \infty)$.*

(4) $\lim_{r\to\infty} \frac{V(r)}{r^{\rho(r)}} = 1.$

We denote by $\mathfrak{B}(\gamma, \rho(r))$ the class of such functions V.

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We denote by $\mathfrak{B}(\gamma,\rho(r))$ the class of such functions V.
If $M(r)$ is comparable to $r^{\rho(r)}$ for some nonzero proximate order $\rho(r)$, we will have constructed flat functions.

General Watson's Lemma

The solution of the problem in full generality comes from the following result in the theory of growth of positive functions (G. Valiron):

If $\omega(\mathbb{M}) \in (0,\infty)$, then $M(r)$ is a function of growth order $1/\omega(\mathbb{M})$, and there exists a nonzero proximate order $\rho(r)$ (with limit $1/\omega(\mathbb{M})$) such that there exist $A>0$ with

$$M(r) \leq Ar^{\rho(r)}, \quad r \text{ large enough.}$$

Theorem (general Watson's lemma)

Suppose $\mathbb{M}$ *is (lc) with* $\lim_{n\to\infty} m_n = \infty$ *and* $\omega(\mathbb{M}) \in (0,\infty)$. *Then, for every* $V \in \mathfrak{B}(2\omega(\mathbb{M}), \rho(r))$ *the function* $e^{-V(1/z)}$ *is flat in* $\tilde{\mathcal{A}}_{\mathbb{M}}(S_{\omega(\mathbb{M})})$, *and so* $\tilde{I}_{\mathbb{M}} = (\omega(\mathbb{M}), \infty)$.

Proof: Use the regular variation of V instead of additional properties of $\mathbb{M}$.

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Proof: Use the regular variation of V instead of additional properties of $\mathbb{M}$.

In the case B.2.3 ($\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{m_n}\right)^{1/\omega(\mathbb{M})} = \infty$), it makes sense to study the quasianalyticity of $\tilde{\mathcal{A}}^u_{\mathbb{M}}(G)$ and $\mathcal{A}_{\mathbb{M}}(G)$ for a general sectorial region G of opening $\pi\omega(\mathbb{M})$: some results by A. D. Sokal, J. Rezende, G. Immink, etc.

Intervals of surjectivity

We now look for

$$\tilde{S}_{\mathbb{M}} := \{\gamma > 0 : \text{the Borel map } \mathcal{B} \text{ in } \tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma) \text{ is surjective}\},$$

and for $\tilde{S}^u_{\mathbb{M}}$ and $S_{\{\mathbb{M}\}}$, defined accordingly.

Since $\mathcal{A}_{\{\mathbb{M}\}}(S_\gamma) \subset \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma) \subset \tilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma)$, one has

$$S_{\{\mathbb{M}\}} \subset \tilde{S}^u_{\mathbb{M}} \subset \tilde{S}_{\mathbb{M}}.$$

Known results for surjectivity, I

A) J. Schmets, M. Valdivia, Extension maps in ultradifferentiable and ultraholomorphic function spaces, Studia Math. 143 (3) (2000), 221–250.

If $\tilde{S}^{u}_{\mathbb{M}} \neq \emptyset$, then $\mathbb{M}$ verifies (snq).
So, in any surjectivity result it is natural to depart from a sequence (lc) and (snq), which together imply that m tends to infinity.

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B) Again by Mandelbrojt's result, if $\gamma > 0$ and $\gamma \in \tilde{S}^u_{\mathbb{M}}$, one always has that $\sum_{n=0}^{\infty} \left(\frac{1}{m_n}\right)^{1/\gamma} < \infty$. So, $\gamma \leq \omega(\mathbb{M})$ and $\tilde{S}^u_{\mathbb{M}} \subset (0, \omega(\mathbb{M})]$.

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C) J. P. Ramis, Dévissage Gevrey, Asterisque 59–60 (1978), 173–204.

Theorem

For the Gevrey sequence of order $\alpha > 0$, $\mathbb{M}_\alpha$, one has $\tilde{S}_{\mathbb{M}_\alpha} = (0, \alpha]$.

Tools: Formal and analytic (truncated) Laplace and Borel transforms with exponential kernels.

Known results for surjectivity, II

D) V. Thilliez, Division by flat ultradifferentiable functions and sectorial extensions, Results Math. 44 (2003), 169–188.

Definition

$\mathbb{M}$ is strongly regular if it is (lc) and:

- of moderate growth (mg): there exists a constant $A > 0$ such that

$$M_{n+p} \leq A^{n+p} M_n M_p, \quad n, p \in \mathbb{N}_0.$$

- strongly non-quasianalytic (snq): there exists $B > 0$ such that

$$\sum_{k \geq n} \frac{M_k}{(k+1)M_{k+1}} \leq B \frac{M_n}{M_{n+1}}, \quad n \in \mathbb{N}_0.$$

.

Known results for surjectivity, III

Definition

Let $\gamma > 0$. We say a strongly regular sequence $\mathbb{M}$ satisfies property (P_γ) if there exist a sequence of real numbers $m' = (m'_p)_{p\in\mathbb{N}_0}$ and a constant $a \geq 1$ such that:

- $a^{-1}m_p \leq m'_p \leq am_p$, $p \in \mathbb{N}$,
- $\left(\dfrac{m'_p}{(p+1)^\gamma}\right)_{p\in\mathbb{N}_0}$ is increasing.

(In other words, the sequence $\left(\dfrac{m_p}{(p+1)^\gamma}\right)_{p\in\mathbb{N}_0}$ is almost increasing.) The *growth index* $\gamma(\mathbb{M})$ is defined by

$$\gamma(\mathbb{M}) := \sup\{\gamma \in \mathbb{R} : (P_\gamma) \text{ is fulfilled}\} \in (0, \infty).$$

Example: $\gamma(\mathbb{M}) = \omega(\mathbb{M})$ for all the examples given at the beginning of the talk.

Known results for surjectivity, IV

Theorem (V. Thilliez (2003))

Let $\mathbb{M}$ *be a strongly regular sequence and* $0<\gamma<\gamma(\mathbb{M})$. *Then:*

- $\mathcal{A}_{\{\mathbb{M}\}}(S_\gamma)$ *is not quasianalytic.*
- $\mathcal{B}:\mathcal{A}_{\{\mathbb{M}\}}(S_\gamma)\longrightarrow\Lambda_{\{\mathbb{M}\}}$ *is surjective.*

Conclusions: For every $\mathbb{M}$ strongly regular one has $\gamma(\mathbb{M})\le\omega(\mathbb{M})$ and $(0,\gamma(\mathbb{M}))\subset S_{\{\mathbb{M}\}}$. No information about the optimality of $\gamma(\mathbb{M})$.

Comment: Although V. Thilliez only works with strongly regular sequences, $\gamma(\mathbb{M})$ makes sense for any (lc) sequence $\mathbb{M}$ with $\boldsymbol{m}\uparrow\infty$, and we still have $\gamma(\mathbb{M})\le\omega(\mathbb{M})$.

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Theorem (V. Thilliez (2003))

Let $\mathbb{M}$ be a strongly regular sequence and $0 < \gamma < \gamma(\mathbb{M})$. Then:

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Comment: Although V. Thilliez only works with strongly regular sequences, $\gamma(\mathbb{M})$ makes sense for any (lc) sequence $\mathbb{M}$ with $\boldsymbol{m} \uparrow \infty$, and we still have $\gamma(\mathbb{M}) \le \omega(\mathbb{M})$.

Tools: (1) Construction of optimal flat functions: Given $0 < \delta < \gamma(\mathbb{M})$, there exists a holomorphic function $G_{\mathbb{M}} : S_\delta \to \mathbb{C}$ such that there exist $k_1, k_2, k_3 > 0$ with

$$k_1 h_{\mathbb{M}}(k_2|z|) \le |G_{\mathbb{M}}(z)| \le h_{\mathbb{M}}(k_3|z|), \qquad z \in S_\delta.$$

(2) Use of Whitney extension results from the ultradifferentiable setting (J. Bruna; H.-J. Petzsche; J. Bonet, R. W. Braun, R. Meise and B. A. Taylor, and J. Chaumat and A.-M. Chollet (1980-1994)).

Admissibility of a proximate order

Definition

We say $\mathbb{M}$ admits a nonzero proximate order if there exists a nonzero proximate order $\rho(r)$ and constants $A, B > 0$ with

$$A \leq \frac{M(r)}{r^{\rho(r)}} \leq B, \quad r \text{ large enough.}$$

In this case, $\omega(\mathbb{M}) \in (0, \infty)$.

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In this case, $\omega(\mathbb{M}) \in (0, \infty)$.

Example: For $\mathbb{M}_{\alpha,\beta}$ one has $M_{\alpha,\beta}(r)$ is comparable to $r^{1/\alpha} \log^{-\beta/\alpha}(r)$, so $\mathbb{M}_{\alpha,\beta}$ admits the proximate order

$$\rho_{\alpha,\beta}(r) = \frac{1}{\alpha} - \frac{\beta}{\alpha} \frac{\log(\log(r))}{\log(r)}.$$

New version of V. Thilliez's result, I

Aim: For sequences admitting a nonzero proximate order, we apply the technique of J. P. Ramis with kernels defined from flat functions in the optimal sector $S_{\omega(\mathbb{M})}$. We will follow the ideas on moment summability, as developed in

W. Balser, *Formal power series and linear systems of meromorphic ordinary differential equations*, Springer, Berlin, 2000.

A. Lastra, S. Malek, J. S., Summability in general Carleman ultraholomorphic classes, J. Math. Anal. Appl. 430 (2015), 1175–1206.

New version of V. Thilliez's result, II

Theorem (Generalized Borel–Ritt–Gevrey theorem, J.S. (2014))

Suppose $\mathbb{M}$ *admits a nonzero proximate order (say* $\rho(r)$*). Then,*
$(0,\omega(\mathbb{M})) \subset S_{\{M\}} \subset \tilde{S}_{\mathbb{M}} = (0,\omega(\mathbb{M})]$.

Proof: Given $V \in \mathfrak{B}(2\omega(\mathbb{M}),\rho(r))$, define the kernel e and its moment function m_e as

$$e(z) := ze^{-V(z)}, \quad z \in S_{\omega(\mathbb{M})}; \quad m_e(u) = \int_0^\infty t^{u-1} e(t)\, dt, \quad \mathrm{Re}(u) \geq 0.$$

$\mathbb{M}$ and $\mathfrak{m} = (m_e(n))_{n\in\mathbb{N}_0}$ are equivalent, i. e. there exist $L, H > 0$ such that

$$L^n M_n \leq m_e(n) \leq H^n M_n, \ n \in \mathbb{N}_0.$$

New version of V. Thilliez's result, III

Given $(f_n)_{n\in\mathbb{N}_0} \in \Lambda_{\mathbb{M},A}$, put $\hat{f} = \sum_{n\geq 0} \frac{f_n}{n!} z^n$, and define its formal Borel transform as

$$\hat{\mathcal{B}}\hat{f} := \sum_{n\geq 0} \frac{f_n}{n! m_e(n)} z^n,$$

which is convergent in a disc $D(0,r)$, with $r > 0$ independent of $\hat{f}$.

Then, choose $0 < R < r$, and define the truncated Laplace transform as

$$f(z) := \int_0^R e(u/z)\hat{\mathcal{B}}\hat{f}(u)\frac{du}{u}, \quad z \in S_{\omega(\mathbb{M})}.$$

Then, for every $\gamma \in (0, \omega(\mathbb{M}))$ there exist $C, c > 0$, independent of $\hat{f}$, such that the restriction of f to S_γ belongs to $\mathcal{A}_{\mathbb{M},cA}(S_\gamma)$, $\mathcal{B}(f) = (f_n)_{n\in\mathbb{N}_0}$, and moreover,

$$\|f\|_{\mathbb{M},cA} \leq C|(f_n)_{n\in\mathbb{N}_0}|_{\mathbb{M},A}. \qquad \square$$

So, we also have linear continuous right inverses for the Borel map between suitable Banach spaces with scaled types.

Remarks on the admissibility condition

J. Jiménez-Garrido, J. S., G. Schindl, Log-convex sequences and nonzero proximate orders, J. Math. Anal. Appl. 448 (2017), no. 2, 1572–1599.

- If $\mathbb{M}$ admits a nonzero proximate order, then it is strongly regular.
- Not every strongly regular sequence admits a nonzero proximate order: again regular variation (for sequences) plays a prominent role.

A natural question is whether the indices $\gamma(\mathbb{M})$ and $\omega(\mathbb{M})$ can be different in this situation.

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A natural question is whether the indices $\gamma(\mathbb{M})$ and $\omega(\mathbb{M})$ can be different in this situation.

Theorem (J. Jiménez-Garrido, J.S.(2015))

Let $\mathbb{M}$ be (lc) with $\boldsymbol{m}$ tending to infinity, and that admits a nonzero proximate order. Then, $\gamma(\mathbb{M})$ and $\omega(\mathbb{M})$ are equal.

So, for sequences admitting a proximate order we have that

$$(0,\gamma(\mathbb{M})) \subset S_{\{\mathbb{M}\}} \subset \tilde{S}^u_{\mathbb{M}} \subset \tilde{S}_{\mathbb{M}} = (0,\omega(\mathbb{M})] = (0,\gamma(\mathbb{M})],$$

improving previous results.

Right inverses

Theorem (Existence of right inverses for the Borel map)

Let $\mathbb{M}$ be a strongly regular sequence, and let $\gamma > 0$ be given. Each of the following assertions implies the next one:

(i) $\gamma < \gamma(\mathbb{M})$.

(ii) *There exists $d \geq 1$ such that for every $A > 0$ there is a right inverse for $\mathcal{B}$,*

$$T_{\mathbb{M},A,\gamma} : \Lambda_{\mathbb{M},A} \to \mathcal{A}_{\mathbb{M},dA}(S_\gamma).$$

(iii) *The Borel map $\mathcal{B} : \mathcal{A}_{\{\mathbb{M}\}}(S_\gamma) \to \Lambda_{\{\mathbb{M}\}}$ is surjective.*

(iv) *The Borel map $\mathcal{B} : \tilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma) \to \Lambda_{\{\mathbb{M}\}}$ is surjective.*

(v) $\sum_{n=0}^\infty \left(\frac{1}{m_n}\right)^{1/\gamma} < \infty$.

(vi) $\gamma \leq \omega(\mathbb{M})$.

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(vi) $\gamma \leq \omega(\mathbb{M})$.

If $\sum_{n=0}^{\infty} \left(\frac{1}{m_n}\right)^{1/\omega(\mathbb{M})} = \infty$ and (v), then $\gamma \neq \omega(\mathbb{M})$. If $\omega(\mathbb{M}) = \gamma(\mathbb{M})$, then all the conditions (i)-(v) are equivalent, and

$$(0, \gamma(\mathbb{M})) = S_{\{\mathbb{M}\}} = \tilde{S}^u_{\mathbb{M}}.$$

Complete result for applications

Theorem

If $\mathbb{M}$ admits a nonzero proximate order and $\sum_{n=0}^{\infty}\left(\frac{1}{m_n}\right)^{1/\omega(\mathbb{M})}=\infty$, we have that $(0,\infty)$ can be written, for the three asymptotics considered, as the disjoint union of the corresponding intervals of surjectivity and injectivity, the first one being open for both $\mathcal{A}_{\{\mathbb{M}\}}(S)$ and $\tilde{\mathcal{A}}^u_{\mathbb{M}}(S)$, and half-closed for $\tilde{\mathcal{A}}_{\mathbb{M}}(S)$.

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Question: Are $\gamma(\mathbb{M})$ and $\omega(\mathbb{M})$ always equal for strongly regular sequences?

NO. We define $\mathbb{M}$ by the sequence of its quotients,

$$m_0 := 1, \qquad m_p := e^{\delta_p/p} m_{p-1} = \exp\Big(\sum_{k=1}^{p} \frac{\delta_k}{k}\Big), \qquad p \in \mathbb{N}.$$

We consider $k_n := 2^{3^n} < q_n := k_n^2 = 2^{3^n 2} < k_{n+1} = 2^{3^{n+1}}$, $n \in \mathbb{N}_0$, and we choose the sequence $(\delta_k)_{k=1}^{\infty}$ as follows: $\delta_1 = \delta_2 = 2$,

$$\delta_k = 3, \quad \text{if} \quad k \in \{k_j+1,\dots,q_j\}, j \in \mathbb{N}_0,$$
$$\delta_k = 2, \quad \text{if} \quad k \in \{q_j+1,\dots,k_{j+1}\}, j \in \mathbb{N}_0.$$

One can prove that $\gamma(\mathbb{M}) = 2$, $\omega(\mathbb{M}) = 5/2$.

Choice between $\gamma(\mathbb{M})$ and $\omega(\mathbb{M})$

It is important to decide which one of the indices, $\gamma(\mathbb{M})$ or $\omega(\mathbb{M})$, plays the main role in the study of surjectivity of the Borel map.

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S. Tikhonov, On generalized Lipschitz classes and Fourier series, J. Analysis Appl. 23 (2004), 745–764.

Important fact: Given a general sequence $\mathbb{M}$ which is (lc) and with $\boldsymbol{m} \uparrow \infty$, we have $\gamma(\mathbb{M}) > 0$ if, and only if, $\mathbb{M}$ is (snq).

Ultraholomorphic Beurling classes and the Borel map

Given $\mathbb{M}$ and a sector S, we consider

$$\mathcal{A}_{(\mathbb{M})}(S) = \left\{ f \in \mathcal{H}(S) : \|f\|_{\mathbb{M},A} < \infty \text{ for every } A > 0 \right\}.$$

$\mathcal{A}_{(\mathbb{M})}(S)$ is a Fréchet space.

$\Lambda_{(\mathbb{M})} = \left\{ \boldsymbol{\mu} = (\mu_n)_{n\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0} : |\boldsymbol{\mu}|_{\mathbb{M},A} < \infty \text{ for every } A > 0 \right\}.$

$\Lambda_{(\mathbb{M})}$ is a Fréchet space.

The Borel map is

$$\mathcal{B} : \mathcal{A}_{(\mathbb{M})}(S) \longrightarrow \Lambda_{(\mathbb{M})}$$
$$f \mapsto (f^{(n)}(0) := \lim_{z\to 0, z\in S} f^{(n)}(z))_{n\in\mathbb{N}_0}.$$

Partial result

Theorem

Let $\mathbb{M}$ be a strongly regular sequence, and let $r \in \mathbb{N}$ be given. The following assertions are equivalent:

(i) $r < \gamma(\mathbb{M})$.

(ii) *The Borel map* $\mathcal{B} : \mathcal{A}_{\{\mathbb{M}\}}(S_r) \to \Lambda_{\{\mathbb{M}\}}$ *is surjective.*

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(ii) $\Longrightarrow$ (iii) V. Thilliez (2003)
(iii) $\Longrightarrow$ (i) J. Schmets and M. Valdivia (2000), S. Tikhonov (2004) and elementary properties for the index $\gamma(\mathbb{M})$.

Example

Let $\mathbb{M}_0$ be the sequence in the aforementioned example, and consider $\mathbb{M} = \mathbb{M}_0^{1/2}$, which is strongly regular and such that $\gamma(\mathbb{M}) = 1 < 5/4 = \omega(\mathbb{M})$.

By the previous results, $S_{\{\mathbb{M}\}} = (0,1)$, but $I_{\{\mathbb{M}\}} \subset [\omega(\mathbb{M}), \infty) = [5/4, \infty)$!!!

In general, if $\gamma(\mathbb{M}) \notin \mathbb{N}$, we deduce that $\lfloor \gamma(\mathbb{M}) \rfloor + 1 \notin S_{\{\mathbb{M}\}}$, and so $S_{\{\mathbb{M}\}} \subset (0, \lfloor \gamma(\mathbb{M}) \rfloor + 1)$, which can be far away from $\omega(\mathbb{M})$.

One can construct examples of (non strongly regular) sequences with $\gamma(\mathbb{M}) = 0$ and $\omega(\mathbb{M}) = \infty$!

So, it seems clear that the **relevant value for surjectivity is** $\gamma(\mathbb{M})$.

Thank you very much for your attention