Stokes geometry of higher order ODEs and middle convolution

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“Asymptotic and computational aspects of
complex differential equations”
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Overview

- **Joint work with T. Moteki (to appear in *Adv. Math.*).**
- **Purpose of the talk :**

  To discuss the relationship between the exact WKB analysis for higher order ODEs & middle convolution.

- **Keywords :**

  Stokes geometry of higher order ODEs

  middle convolution

  exact steepest descent method

Plan of the talk

§**1. Stokes geometry of higher order ODEs**

§**2. Middle convolution**

§**3. Problem and main result**

§**4. Outline of the proof**

§**5. Examples**

1 Stokes geometry of higher order ODEs

Exact WKB analysis for higher order ODEs

$$P\psi = \left[\left(\eta^{-1}\frac{d}{dx}\right)^m + a_1(x)\left(\eta^{-1}\frac{d}{dx}\right)^{m-1} + \cdots + a_m(x)\right]\psi = 0 \qquad (*)$$

$$\psi = \exp\left(\eta\int^x \zeta\, dx\right)\sum_{n=0}^{\infty}\eta^{-(n+1/2)}\psi_n(x)$$: **WKB solution**

where $\eta > 0$ denotes a large parameter, $a_j(x)$ is a polynomial, and ζ is a characteristic root of $(*)$, i.e.,

$$\zeta^m + a_1(x)\zeta^{m-1} + \cdots + a_m(x) = 0.$$

In the exact WKB analysis we consider the Borel sum of ψ with respect to the large parameter η, i.e.,

$$\psi_B(x,y) = \sum_n \frac{\psi_n(x)}{\Gamma(n+1/2)}(y+s(x))^{n-1/2} \quad : \text{ Borel transform}$$

$$\Psi(x,\eta) = \int_{-s(x)}^{\infty} e^{-\eta y}\psi_B(x,y)\,dy \quad : \text{ Borel sum}$$

where $s(x) = \int^x \zeta\,dx$.

"Stokes geometry"

$$\begin{cases} x = a : \text{ turning point} & \Longleftrightarrow \quad \exists\, j \neq k \text{ s.t. } \zeta_j(a) = \zeta_k(a) \\ \text{Stokes curve} & \Longleftrightarrow \quad \Im\left[\eta \int_a^x (\zeta_j(x) - \zeta_k(x))\, dx\right] = 0 \end{cases}$$

(Furthermore, we say that a Stokes curve is of type $j > k$ if $\Re\left[\eta \int_a^x (\zeta_j - \zeta_k) dx\right] > 0$ holds on it.)

▶ 2nd order case :

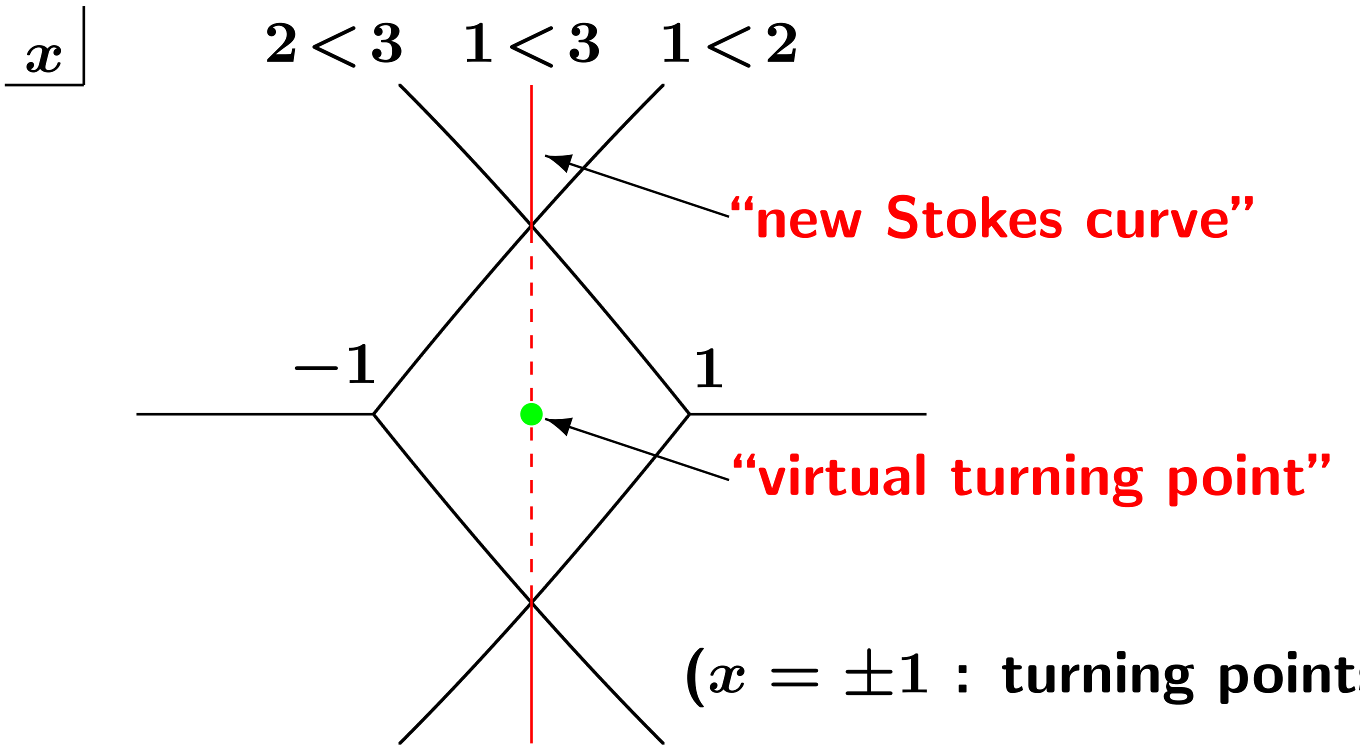
Borel summability of ψ breaks down only on Stokes curves.

▶ higher order case :

Borel summability of ψ breaks down also on "new Stokes curves".

BNR equation (Berk-Nevins-Roberts, *J. Math. Phys.* (1982))

$$\left[\left(\frac{d}{dx}\right)^3 + 3\eta^2 \frac{d}{dx} + 2ix\eta^3\right]\psi = 0$$



($x = \pm 1$: turning points)

Definition

A crossing point of a Stokes curve of type $j < k$ and a Stokes curve of type $k < l$ is called an ordered crossing point.

If we have an ordered crossing point of Stokes curves, we need to add a new Stokes curve emanating from it.

$\longrightarrow$ A recipe for obtaining a (complete) Stokes geometry.

(cf. Honda-Kawai-T. : "*Virtual Turning Points*", Springer, 2015)

However, it is not confirmed yet that the Stokes geometry thus obtained precisely describes the regions where WKB solutions are Borel summable.
(We have to discuss the effectiveness of new Stokes curves, etc.)

2 Middle convolution

Middle convolution is an operation of reduction for (systems of) differential equations.

References :

[1] Katz, "*Rigid Local Systems***", Princeton Univ. Press, 1996**

[2] Dettweiler-Reiter, *J. Algebra* **(2007)**

They introduced middle convolution to study rigid local systems.

[3] Oshima, "*Fractional Calculus of Weyl Algebra and Fuchsian Differential Equations***", Math. Soc. Japan, 2012**

He developed a systematic study of ODEs with polynomial coefficients by using middle convolutions.

Definition (“middle convolution with a large parameter”)

Let $\mu \in \mathbb{C} \setminus \{0\}$. Then for a differential operator P of the form $(*)$ we define its middle convolution $mc_{\mu\eta}P$ by

$$
\begin{aligned}
mc_{\mu\eta}P &= (\eta^{-1}\partial_x)^l \circ \mathrm{Ad}(\partial_x^{-\mu\eta})P \\
&= (\eta^{-1}\partial_x)^l \circ \partial_x^{-\mu\eta} \circ P \circ \partial_x^{\mu\eta}
\end{aligned}
$$

where $\partial_x = \partial/\partial x$ and $l = \max\{\deg a_j + j - m\,; 1 \leq j \leq m\}$.

For example,

$$\mathrm{Ad}(\partial_x^{-\mu\eta})x^k = (x - \mu\eta\partial_x^{-1})^k,$$

$$\mathrm{Ad}(\partial_x^{-\mu\eta})(\eta^{-1}\partial_x)^j = (\eta^{-1}\partial_x)^j,$$

$$mc_{\mu\eta}[(\eta^{-1}\partial_x)^2 - x] = (\eta^{-1}\partial_x)^3 - x(\eta^{-1}\partial_x) + \mu - \eta^{-1}.$$

In what follows we denote $mc_{\mu\eta}P$ simply by $\tilde{P}$.

Note that $\tilde{P} = mc_{\mu\eta}P$ is of order $m + l$.

Furthermore, if ψ is a solution of $P\psi = 0$, then a solution $\tilde{\psi}$ of $\tilde{P}\tilde{\psi} = 0$ is provided by the Euler transform

$$\tilde{\psi}(x, \eta) = \frac{1}{\Gamma(\mu\eta)} \int_C (x - z)^{\mu\eta - 1} \psi(z, \eta)\, dz$$

for a suitably chosen integration path C.

3 Problem and main result

Problem

If a higher order ODE $\tilde{P}\tilde{\psi} = 0$ of the form $(*)$ is obtained from $P\psi = 0$ via middle convolution, then what can we say about the Stokes geometry of $\tilde{P}\tilde{\psi} = 0$ or, equivalently, the Borel summability of a WKB solution $\tilde{\psi}$ of $\tilde{P}\tilde{\psi} = 0$?

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Basic idea

To employ the "exact steepest descent method" proposed by Aoki-Kawai-T. (*J. Math. Phys.* (2001)).

To be more specific, we take a WKB solution

$$\psi_k = \exp\left(\eta \int^x \zeta_k dx\right) \sum_n \eta^{-(n+1/2)} \psi_{k,n}(x)$$

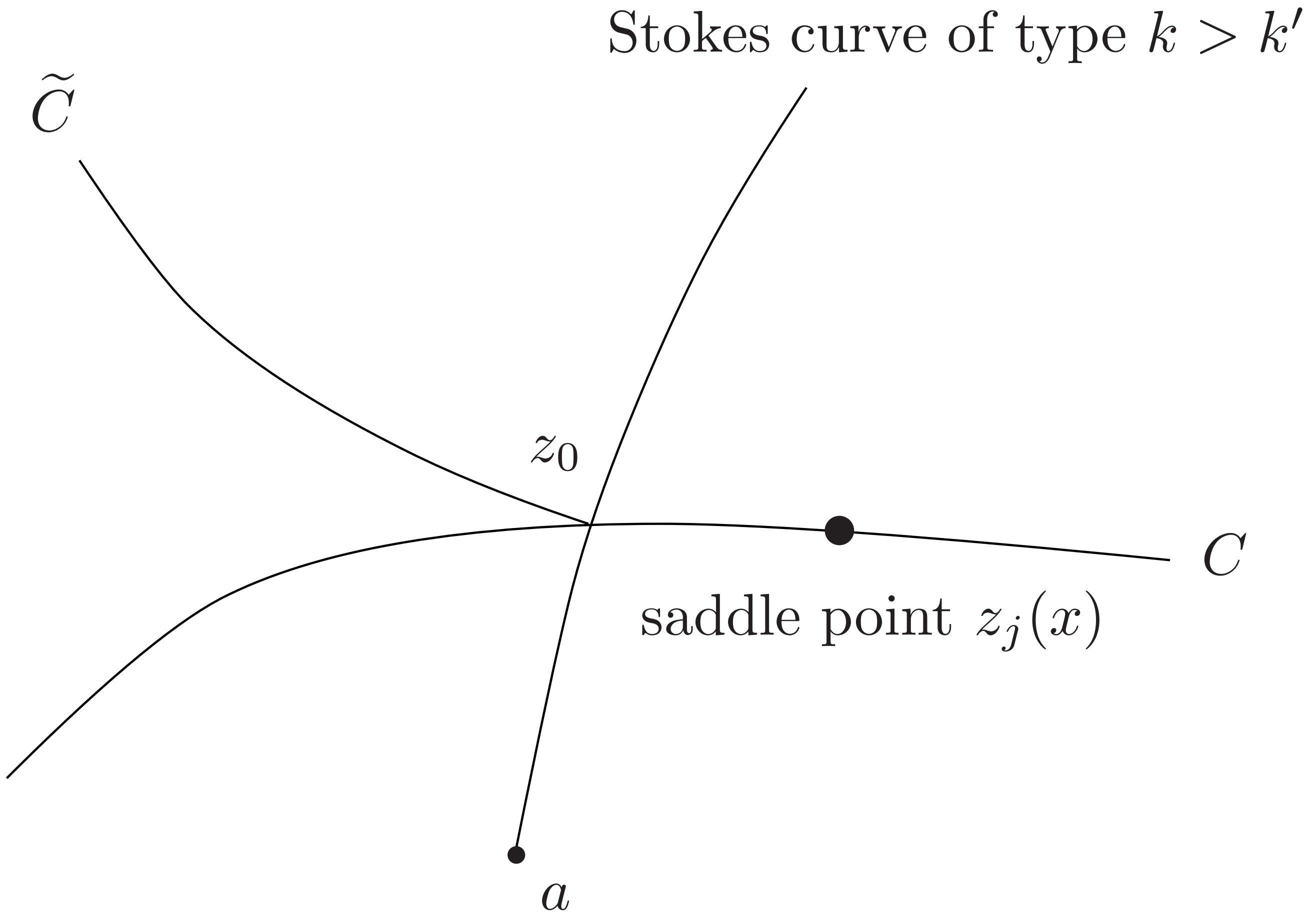
of $P\psi = 0$, where ζ_k $(k = 1, \ldots, m)$ is a characteristic root of $P\psi = 0$, and consider

$$\begin{aligned}\tilde{\psi} &= \int (x-z)^{\mu\eta-1} \psi_k(z,\eta)\, dz \\ &= \int \exp(\eta f_k) \sum_n \eta^{-(n+1/2)} \psi_{k,n}(z)\, dz\end{aligned}$$

with

$$f_k = f_k(x,z) = \mu \log(x-z) + \int^z \zeta_k(z) dz.$$

In particular, we pick up a saddle point $z_j(x)$ $(j = 1, \ldots, J)$ of f_k and consider the integral along a steepest descent path C of $\Re f_k$ passing through $z_j(x)$.



Assume that C crosses a Stokes curve of type $k > k'$ of $P\psi = 0$ at $z = z_0$.

$\longrightarrow$ We bifurcate another steepest descent path $\widetilde{C}$ of $\Re f_{k'}$ from z_0.

We repeat this bifurcation process of steepest descent paths until no further new crossing points appear.

"exact steepest descent path C^{exact} passing through $z_j(x)$"

$$= \text{ totality of such steepest descent paths}$$

$$= C \cup \widetilde{C} \cup \cdots$$

Main Theorem

Let $\tilde{P}\tilde{\psi} = 0$ be obtained from $P\psi = 0$ via middle convolution. Assume

- P is of second order,
- all turning points of $P\psi = 0$ are simple,
- no Stokes curve of $P\psi = 0$ connects two turning points,

and further assume that the above bifurcation process to define an exact steepest descent path terminates in finite steps. Then for a given x, if the exact steepest descent path passing through $z_j(x)$ does not hit any other saddle point, the integral

$$\eta^{-1/2}\tilde{\psi} = \eta^{-1/2} \int (x - z)^{\mu\eta - 1} \psi_k(z, \eta)\, dz \qquad (**)$$

along C^{exact} defines a WKB solution of $\tilde{P}\tilde{\psi} = 0$ and it is Borel summable.

Conjecture

x **is located on an effective portion of a Stokes curve of** $\tilde{P}\tilde{\psi} = 0$**.**

$\Longleftrightarrow$ **two saddle points are connected by an exact steepest descent path.**

4 Outline of the proof

Proposition 1

$(\partial f_k/\partial z)(x, z_j(x)) = 0$, i.e., $z = z_j(x)$ is a saddle point of f_k.

$\iff$ $\frac{\mu}{x - z_j(x)}$ is a characteristic root of $\tilde{P}\tilde{\psi} = 0$.

In particular, there exist $m+l$ saddle points of f_k (i.e., $J = m+l$).

Proposition 2

$$\frac{d}{dx} f_k(x, z_j(x)) = \frac{\mu}{x - z_j(x)}.$$

Prop. 1 & 2 suggest that the integral $(**)$ gives a WKB solution

$$\tilde{\psi}_j = \exp\left(\eta \int^x \frac{\mu}{x - z_j(x)}\, dx\right) \sum_n \eta^{-(n+1/2)} \tilde{\psi}_{j,n}(x)$$

of $\tilde{P}\tilde{\psi} = 0$.

Analytically speaking, we should discuss

$$\int_C (x-z)^{\mu\eta-1}\Psi_k(z,\eta)\,,$$

where Ψ_k is the Borel sum of ψ_k. Then we have

$$\begin{aligned}
&\int_C (x-z)^{\mu\eta-1}\Psi_k(z,\eta)\,dz \\
&\quad = \int_C (x-z)^{\mu\eta-1}\int_{\substack{\tilde{y}=-\int\zeta_k dz+u \\ u\geq 0}} e^{-\eta\tilde{y}}\psi_{k,B}(z,\tilde{y})d\tilde{y}dz \\
&\quad = \int_{\substack{y=-f_{k,0}+v \\ v\geq 0}} e^{-\eta y}\int_{C_v} \psi_{k,B}(z,y+\mu\log(x-z))(x-z)^{-1}dzdy
\end{aligned}$$

where $f_{k,0}=f_k(x,z_j(x))$ and C_v is a compact portion of C determined by v. (Here we have used $y=\tilde{y}-\mu\log(x-z)$.)

Let us define

$$\chi(x,y)=\int_{C_v}\psi_{k,B}(z,y+\mu\log(x-z))(x-z)^{-1}dz.$$

Proposition 3

$\chi(x,y)$ coincides with the Borel transform of $\tilde{\psi}_j$ near $y=-\int^x \frac{\mu}{x-z_j(x)}\,dx$.

Prop. 3 is verified through the analysis near a saddle point $z=z_j(x)$.

Global behavior of $\chi(x, y)$

Case 1 : C does not cross a Stokes curve of $P\psi = 0$.

In this case it follows from the assumption that the integration path C_v does not meet a singularity of $\psi_{k,B}$ and hence $\chi(x, y)$ is well-defined for all $v \geq 0$.

Case 2 : C crosses a Stokes curve of $P\psi = 0$ once.

A singularity of $\psi_{k,B}$ hits C_v at the crossing point z_0.

$\longrightarrow$ We need to deform the integration path.

$$\longrightarrow \quad \chi(x, y) = \int_{C_v} \psi_{k,B}(z, y + \mu \log(x - z))(x - z)^{-1} dz + \int_{\widetilde{C_v}} \psi_{k,B}(z, y + \mu \log(x - z))(x - z)^{-1} dz$$

where $\widetilde{C_v}$ is a compact portion (determined by v) of a bifurcated steepest descent path $\widetilde{C}$.

Furthermore, using the connection formula for the Stokes phenomenon which occurs with ψ_k at z_0, we find that the second term can be expressed in terms of $\psi_{k',B}$ as follows:

$$\int_{\widetilde{C}_v} (V_B * \psi_{k',B})\Big(z, y + \mu \log(x-z) + \int_a^{z_0} (\zeta_k - \zeta_{k'}) dz\Big)(x-z)^{-1} dz$$

where V_B denotes the Borel transform of the Stokes coefficient V that appears in the connection formula.

Then the assumption again entails that $\chi(x, y)$ is well-defined for all $v \geq 0$.

Case 3 : C (and/or $\widetilde{C}$) crosses further Stokes curves of $P\psi = 0$.

We repeat the above argument.

Corollary

The Borel sum of $\eta^{-1/2}\tilde{\psi}_j$ is expressed as

$$\int_C (x-z)^{\mu\eta-1}\Psi_k(z,\eta)\,dz + V\int_{\widetilde{C}} (x-z)^{\mu\eta-1}\Psi_{k'}(z,\eta)\,dz + \cdots$$

5 Examples

Example 1

$$P_1 = 3(\eta^{-1}\partial_x)^2 + 2c(\eta^{-1}\partial_x) + x$$

$$\tilde{P}_1 = mc_{\mu\eta}P_1 = 3(\eta^{-1}\partial_x)^3 + 2c(\eta^{-1}\partial_x)^2 + x(\eta^{-1}\partial_x) - \mu + \eta^{-1}$$

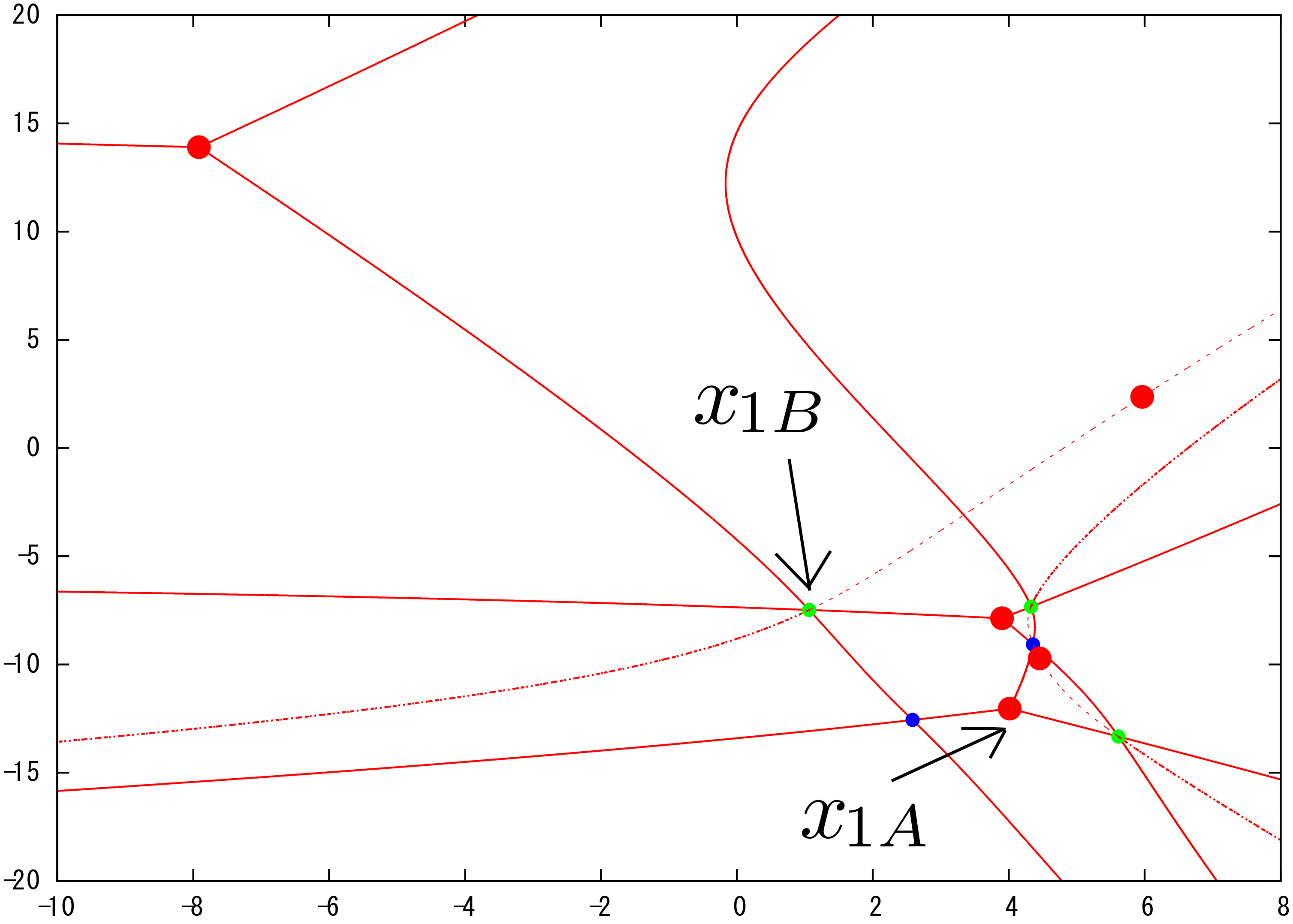
with $c = -3 + 3i$, $\mu = 1 - 6i$.

We investigate the exact steepest descent paths of

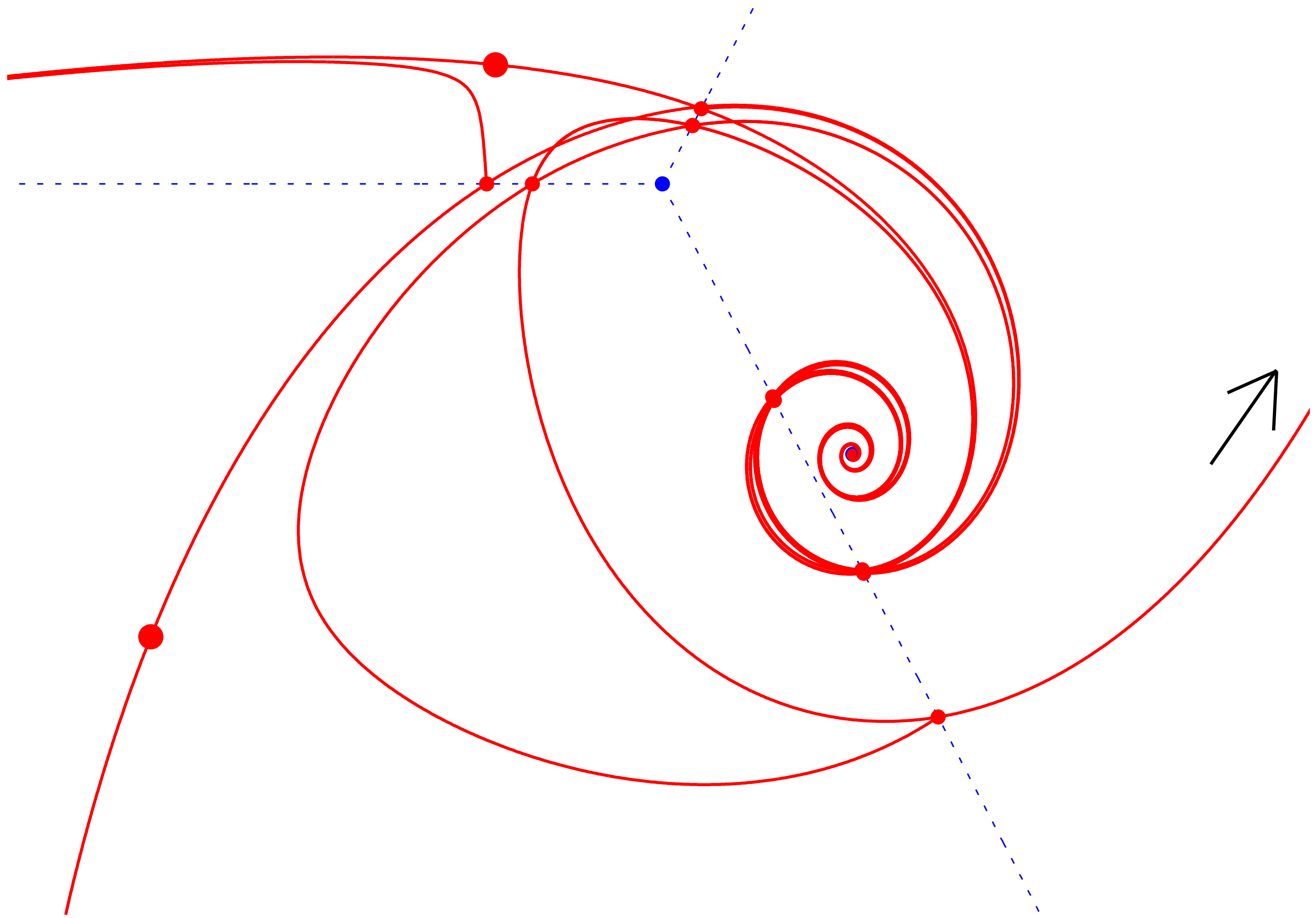
$$\int (x - z)^{\mu\eta - 1}\Psi_k(z, \eta)\, dz$$

near the point x_{1B} specified in the following figure, that is, at $x = x_{1B} + 0.1\exp(k\pi i/9)$ $(0 \le k \le 17)$.

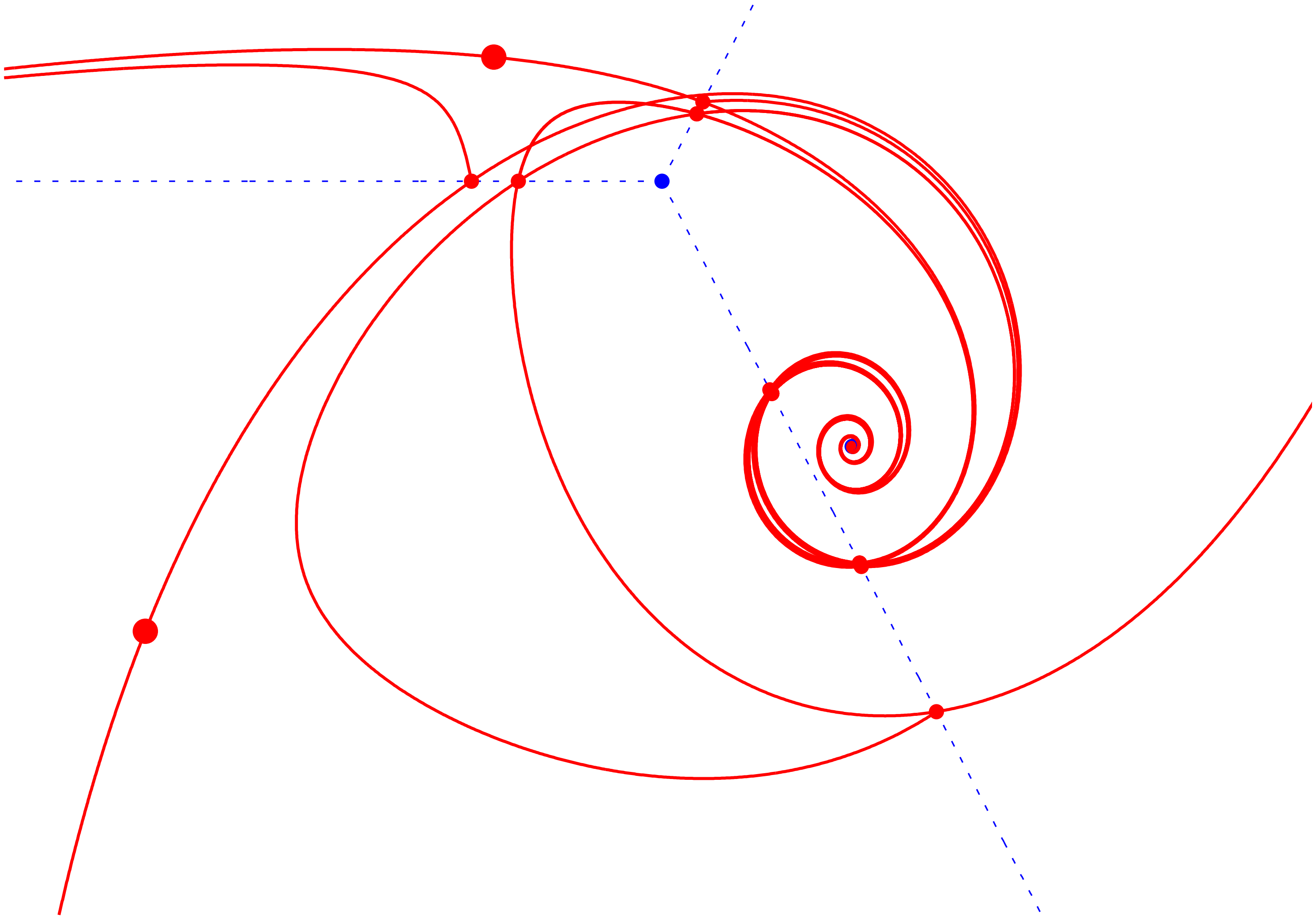
Stokes geometry of $\tilde{P}_1\tilde{\psi} = 0$



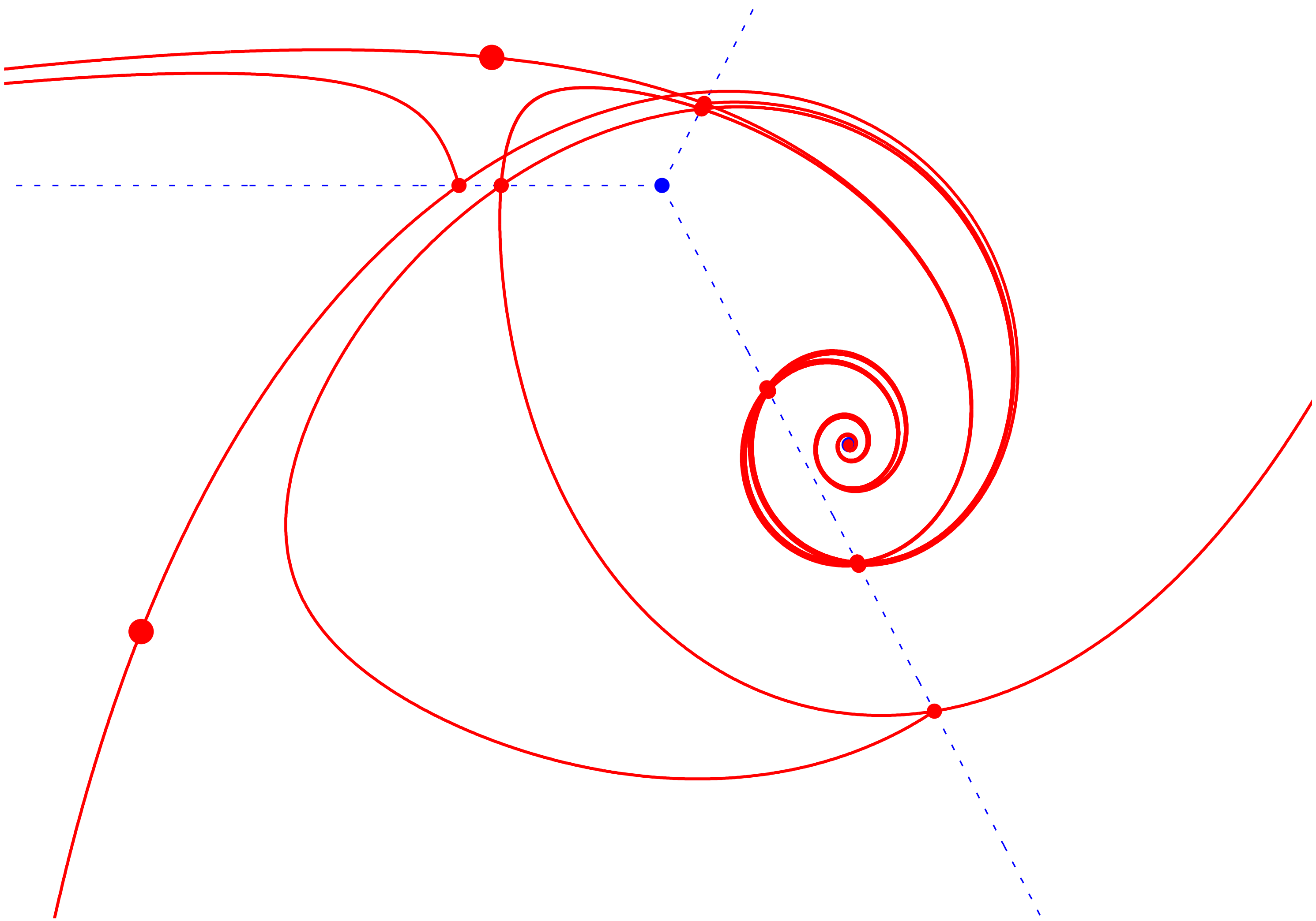
Configuration of exact steepest descent paths



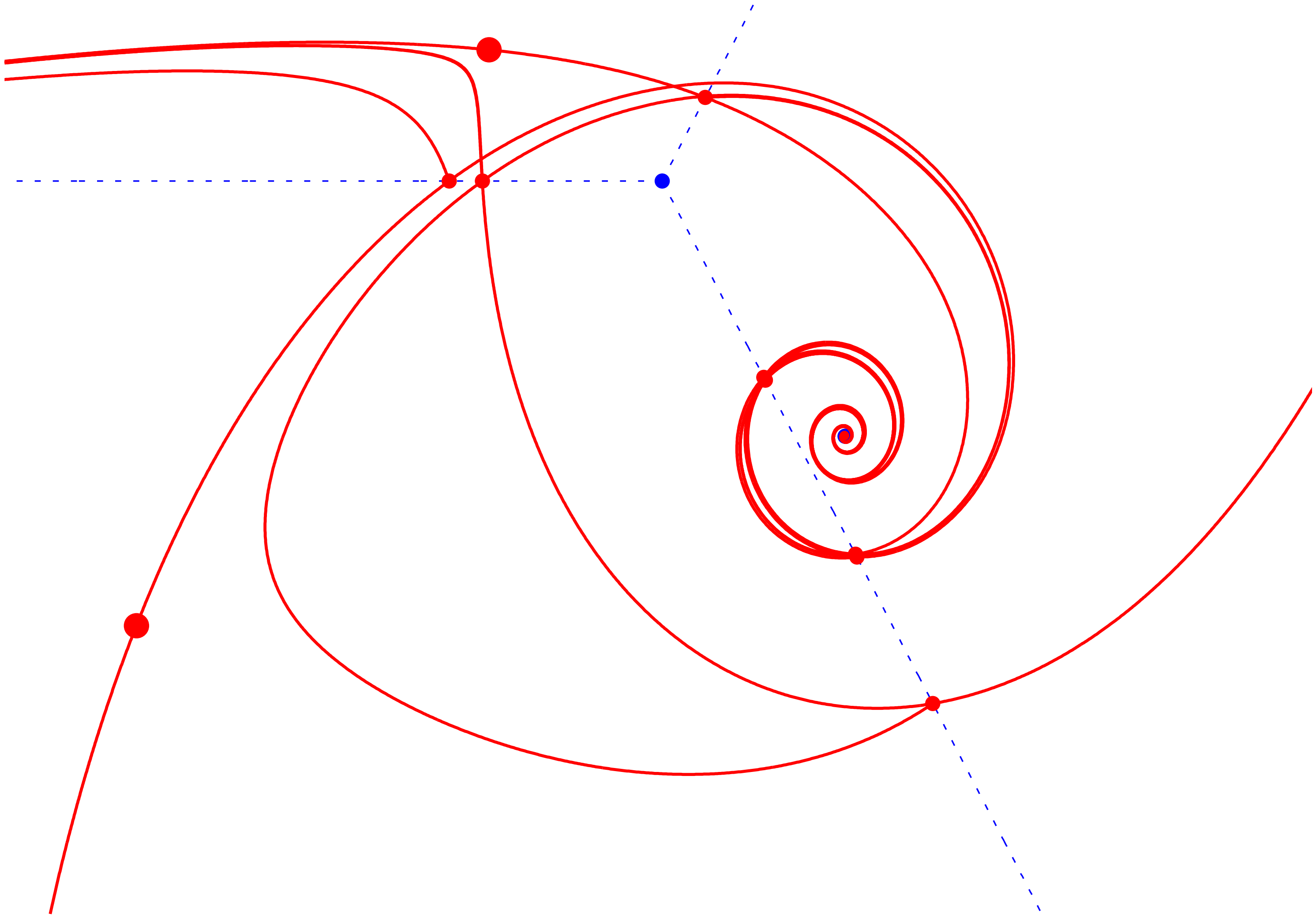
$k = 0$



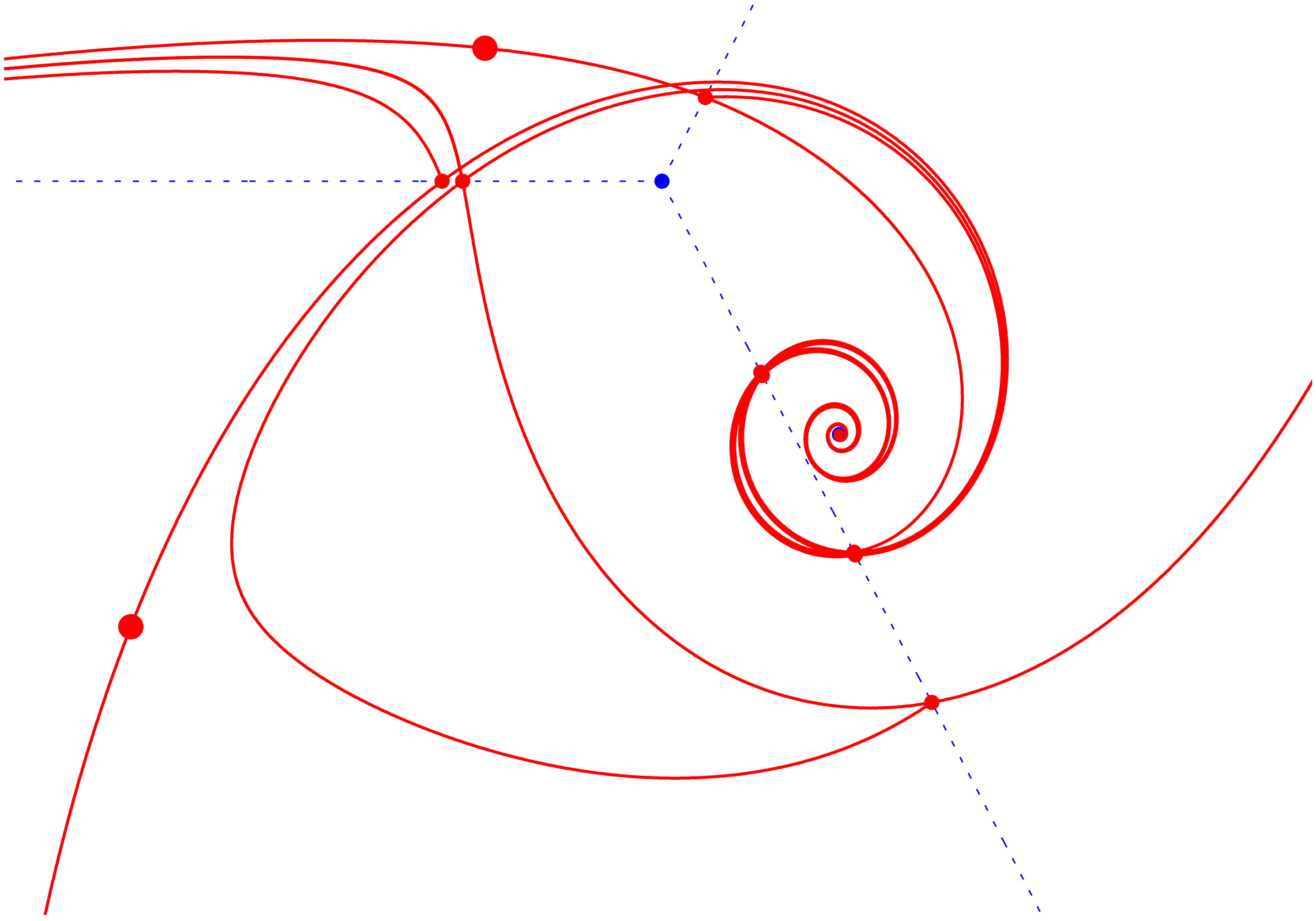
$$\boldsymbol{k = 1}$$



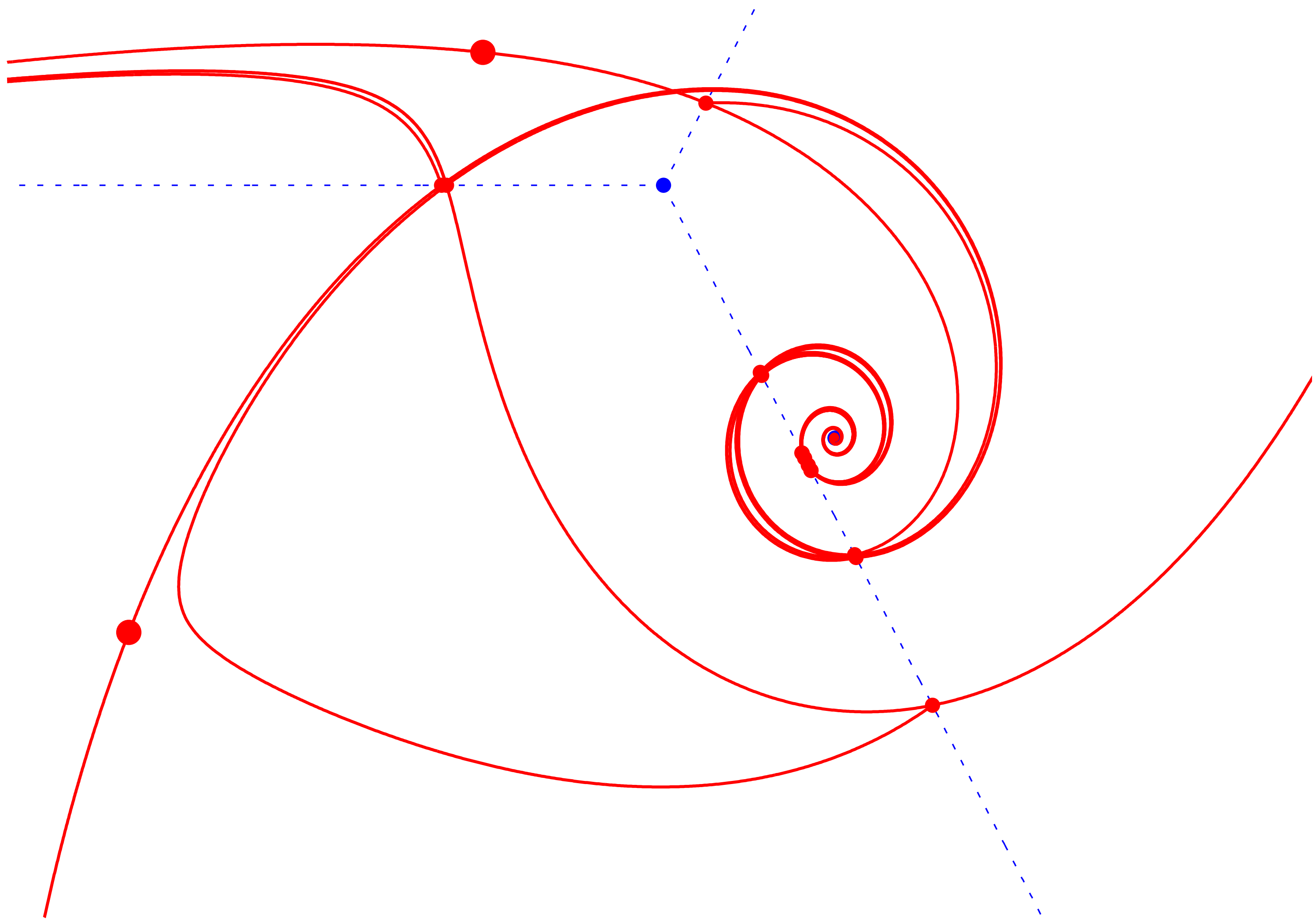
$$\boldsymbol{k = 2}$$



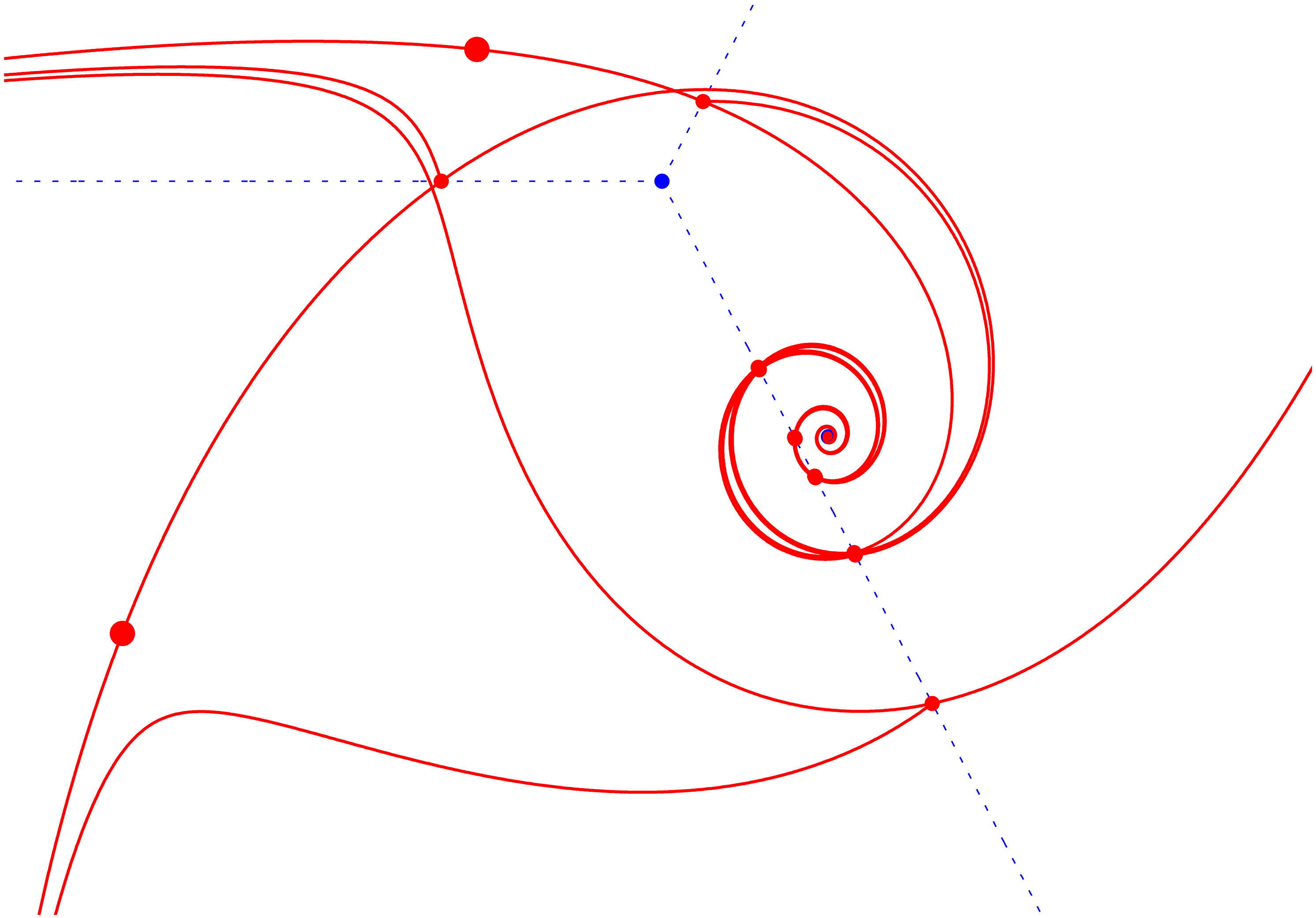
$k = 3$



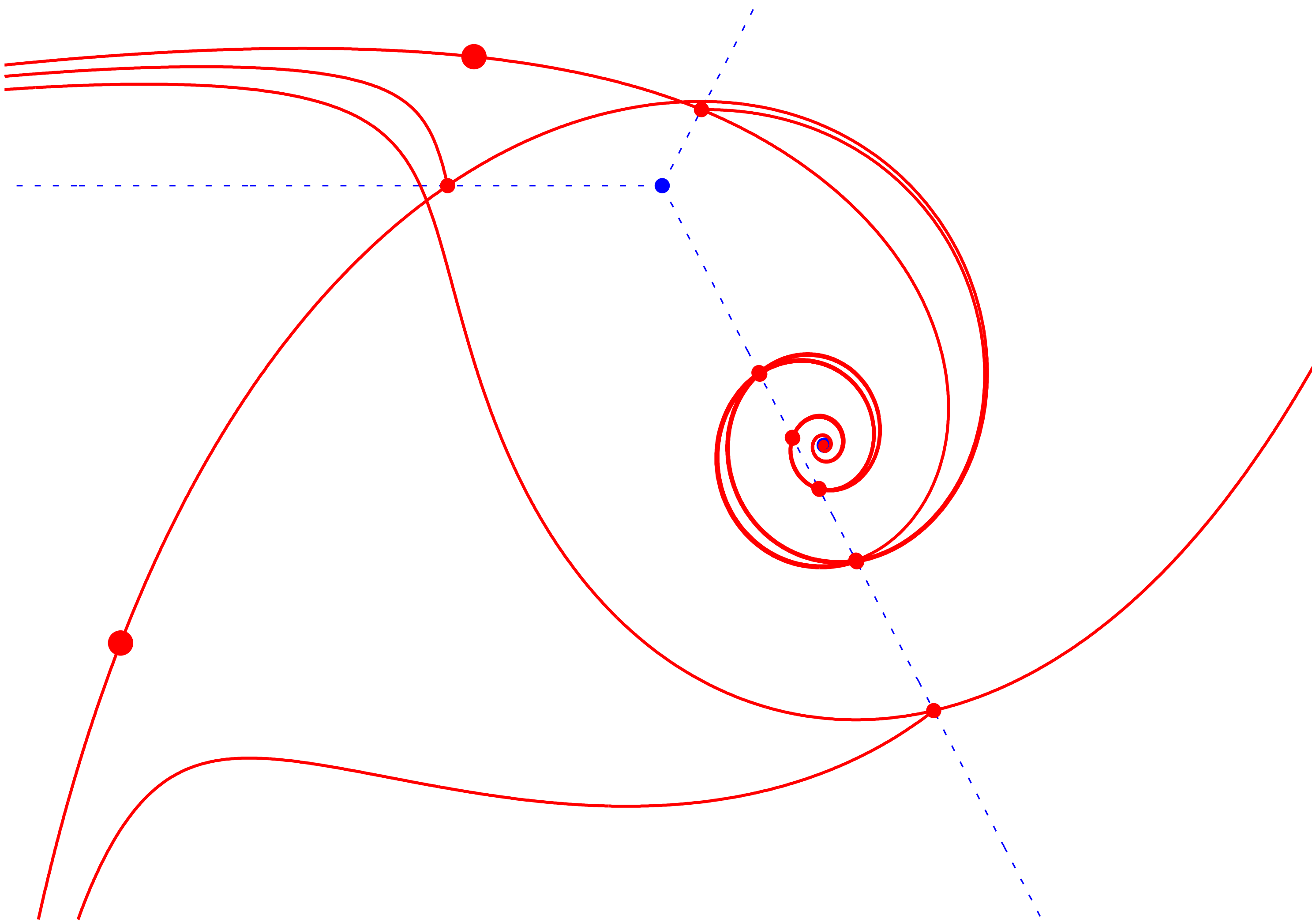
$\boldsymbol{k = 4}$



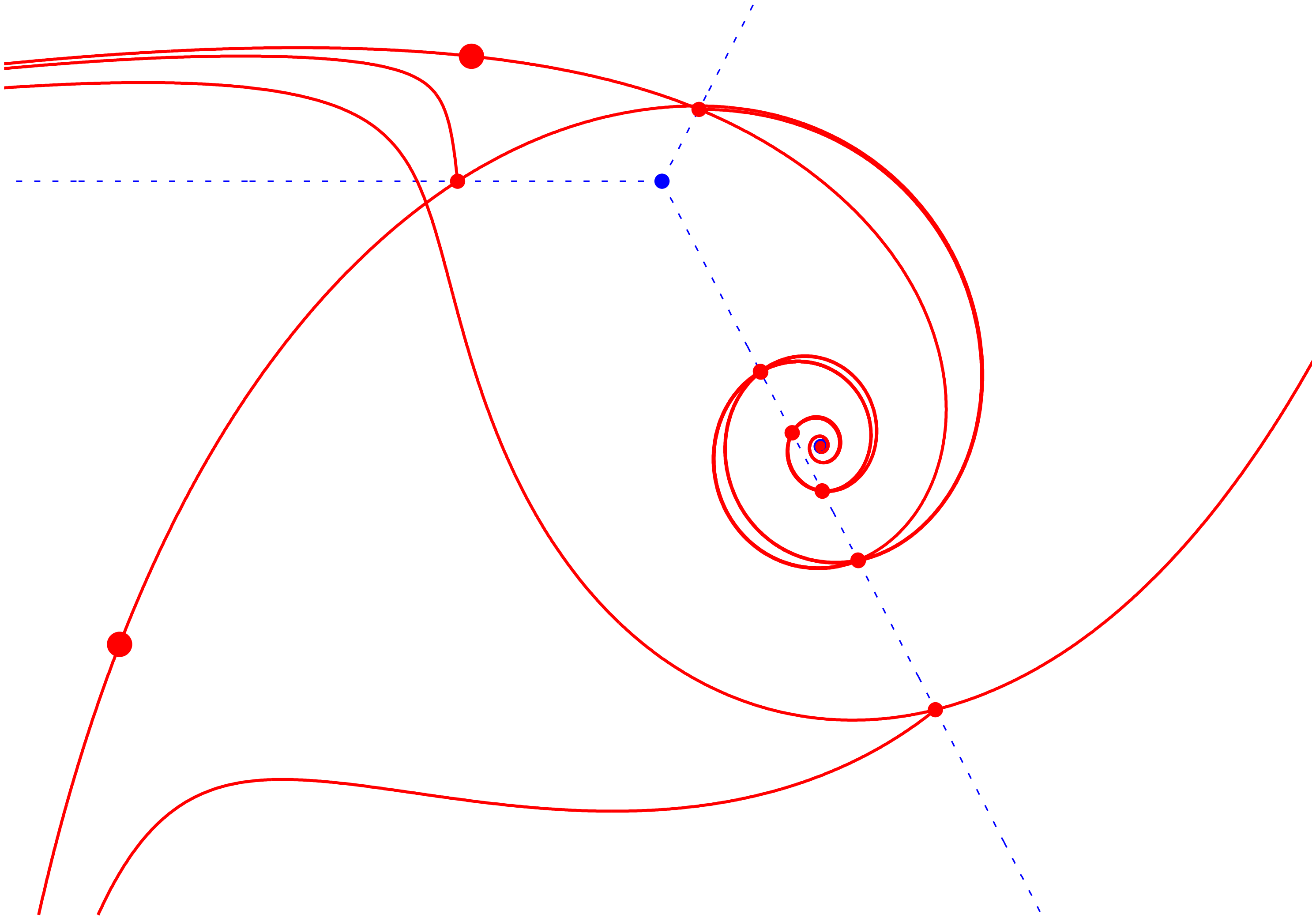
$\boldsymbol{k = 5}$



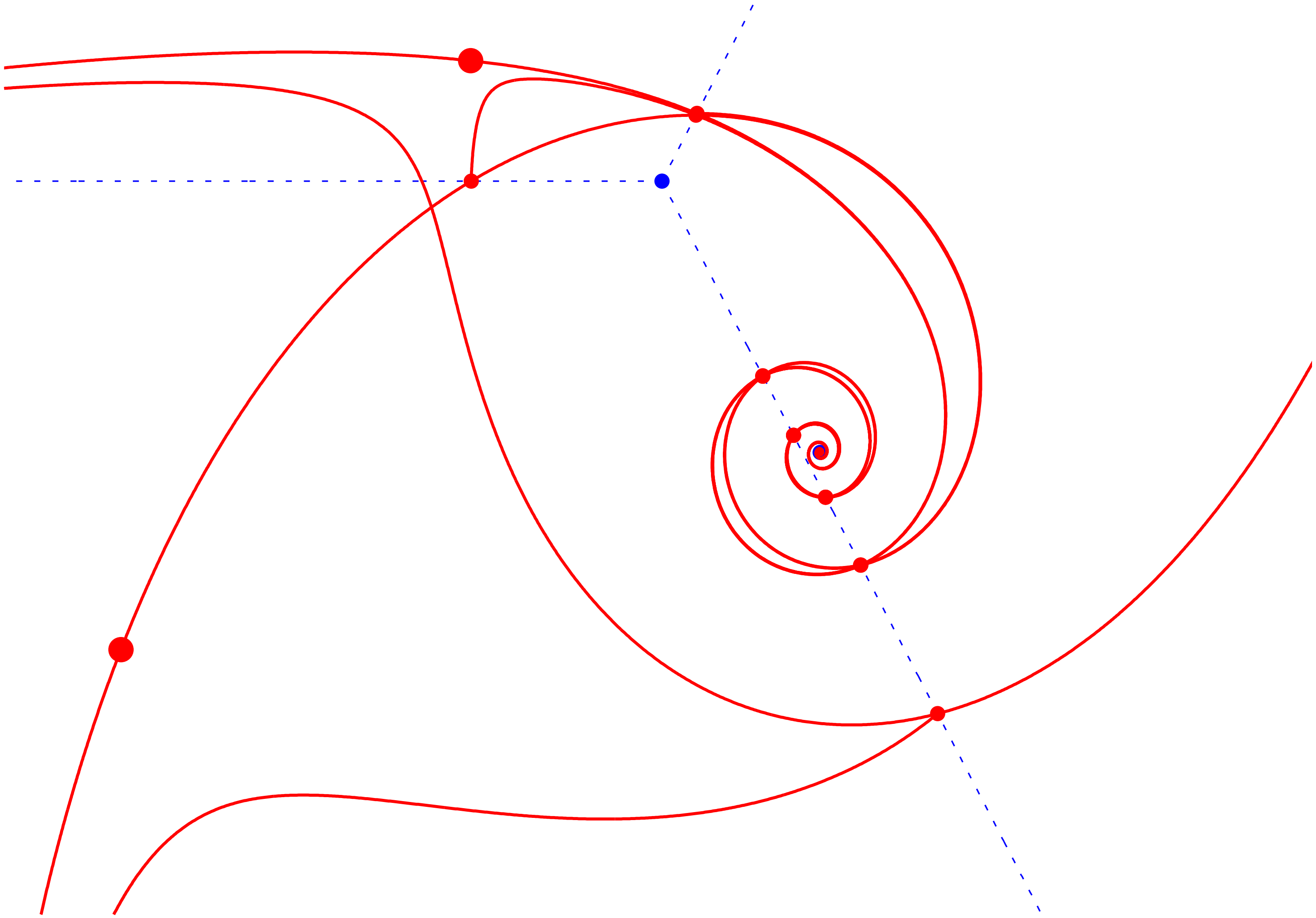
$k = 6$



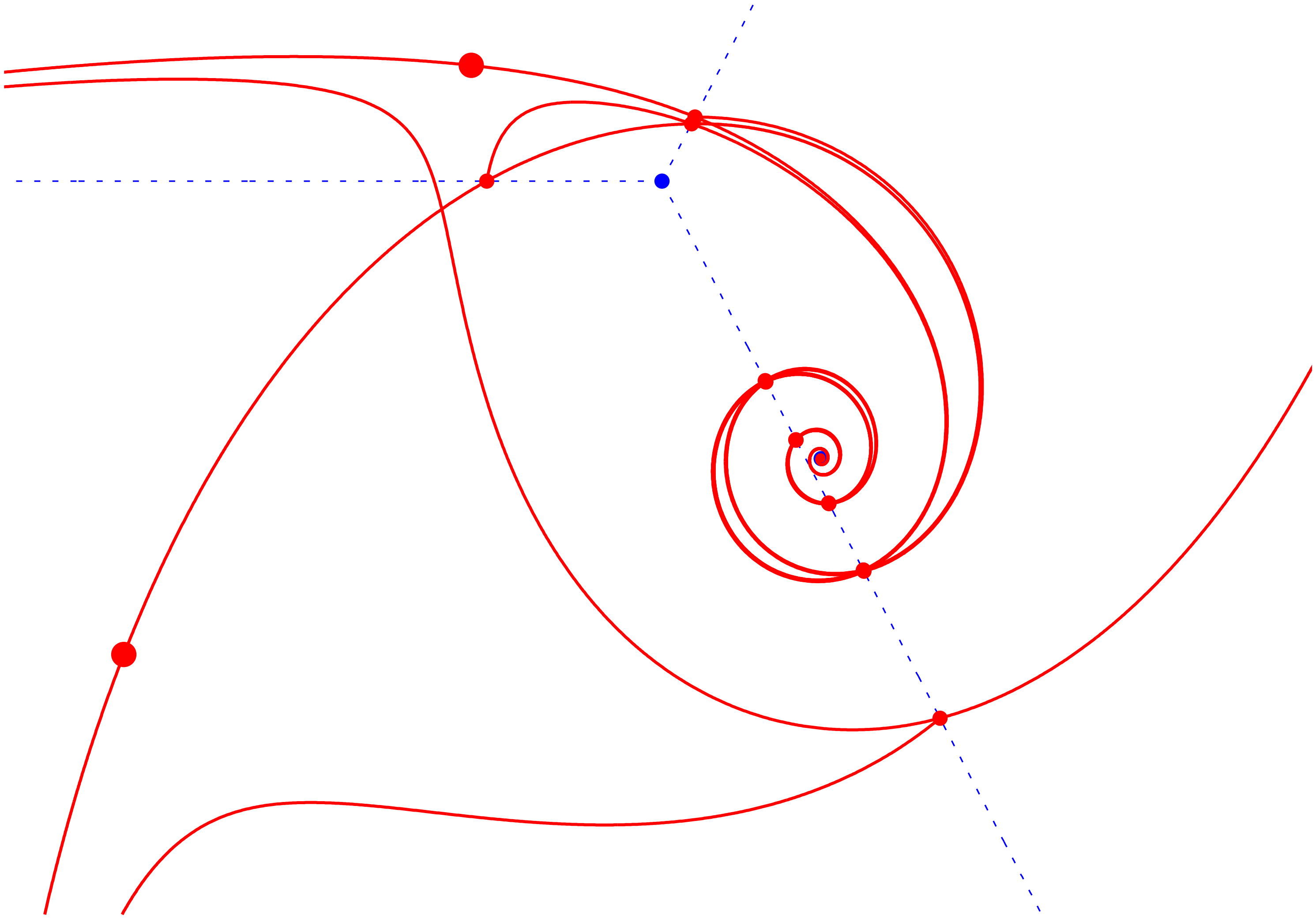
$\boldsymbol{k = 7}$



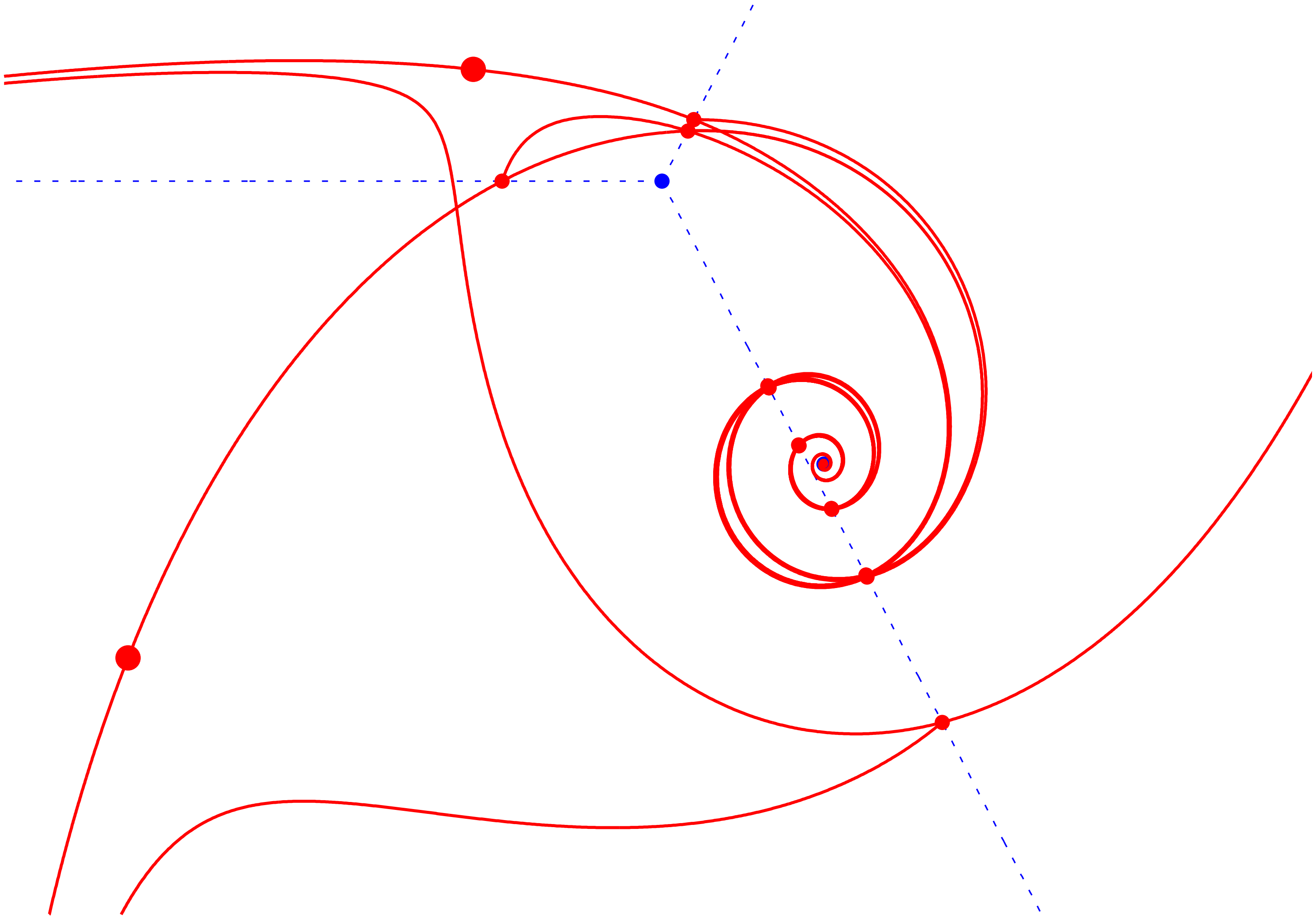
$k = 8$



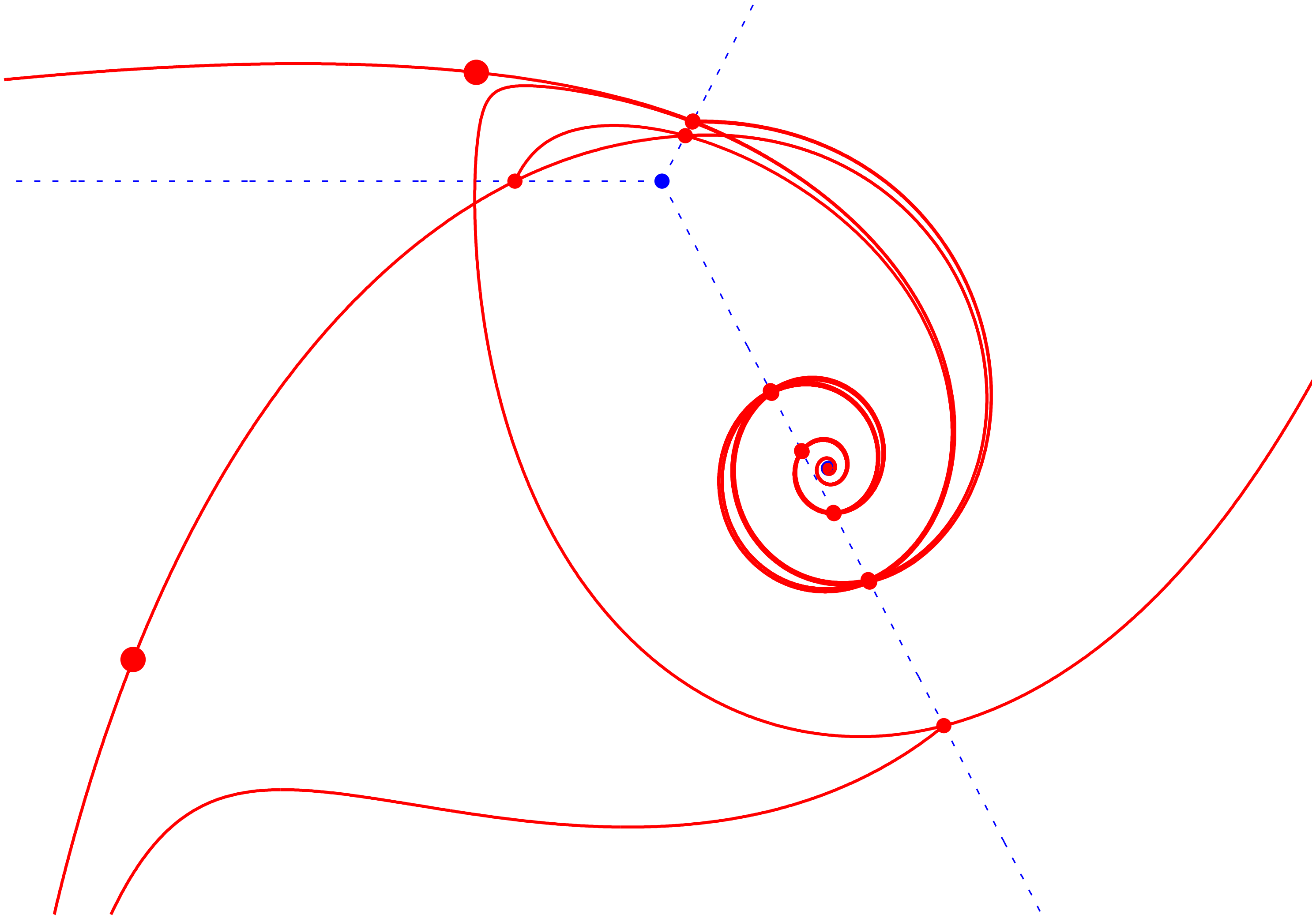
$\boldsymbol{k = 9}$



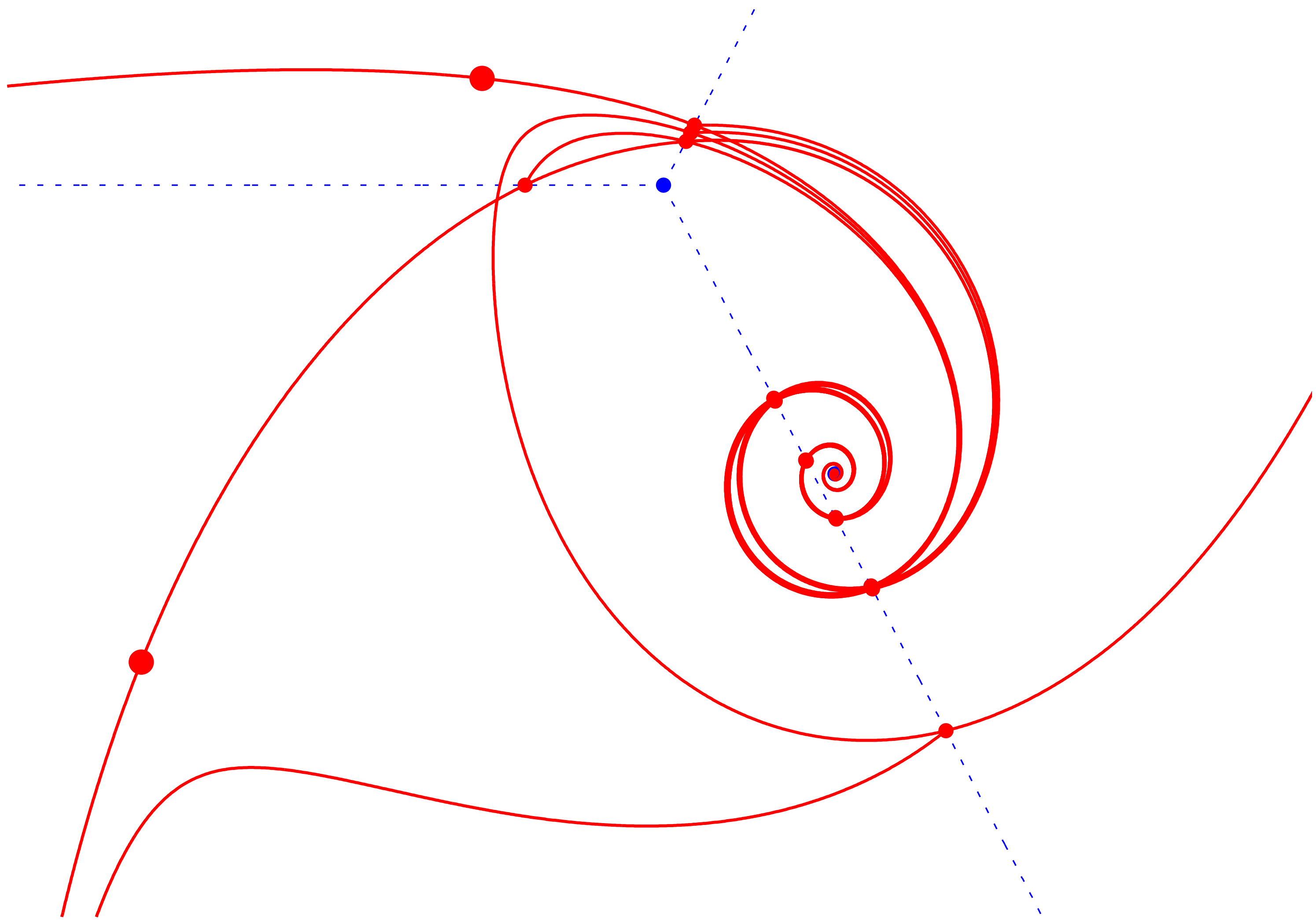
$k = 10$



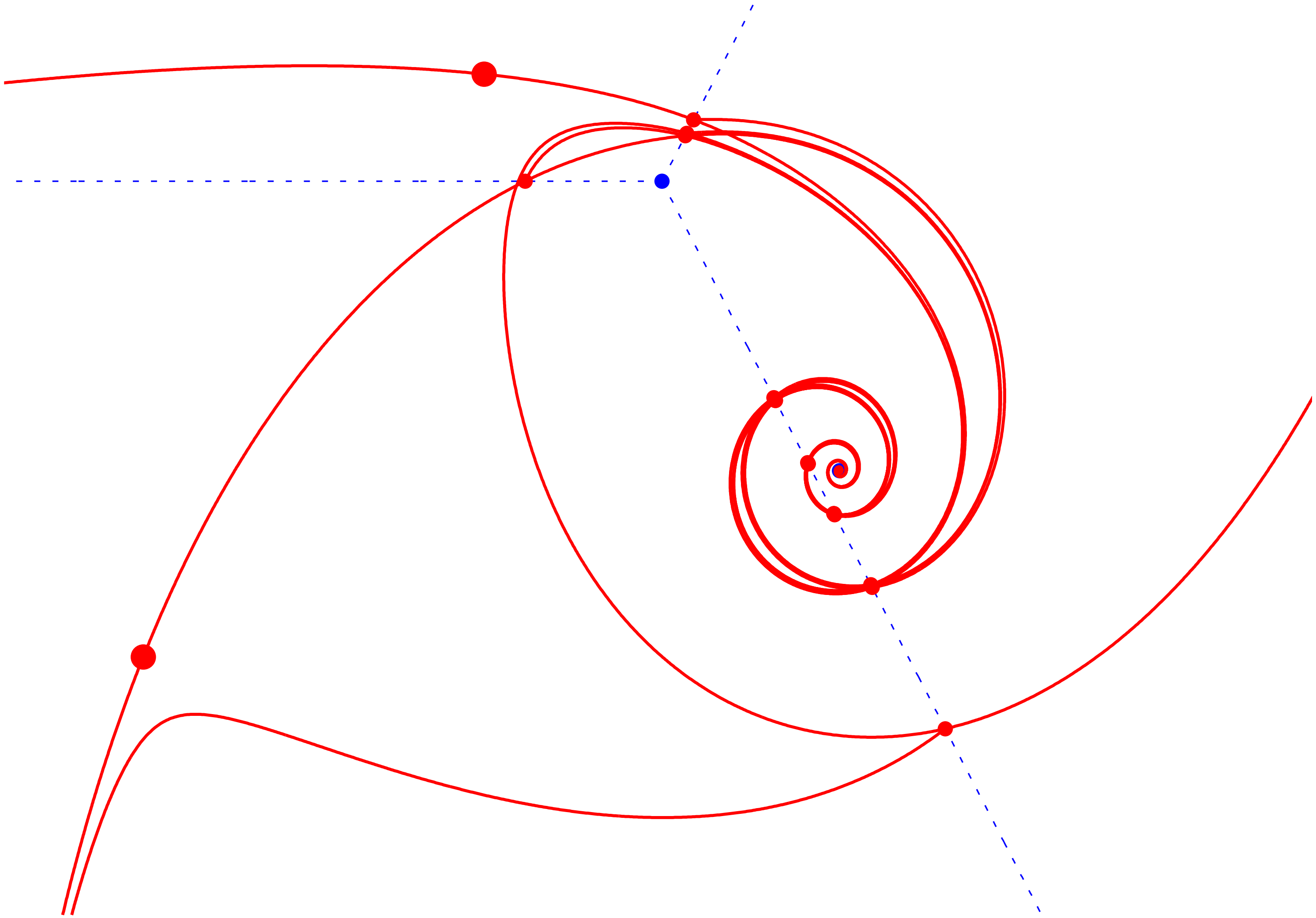
$\boldsymbol{k = 11}$



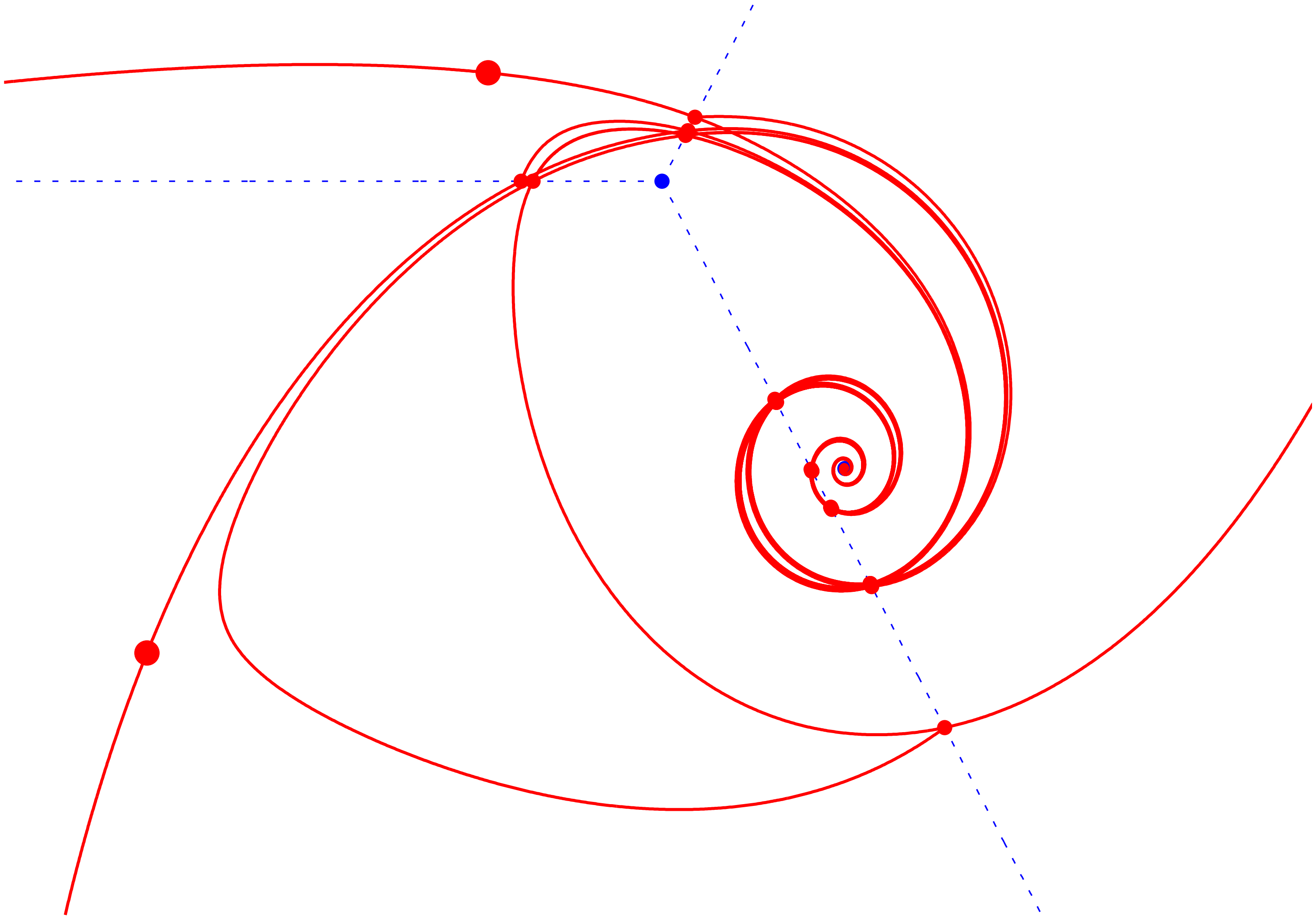
$\boldsymbol{k = 12}$



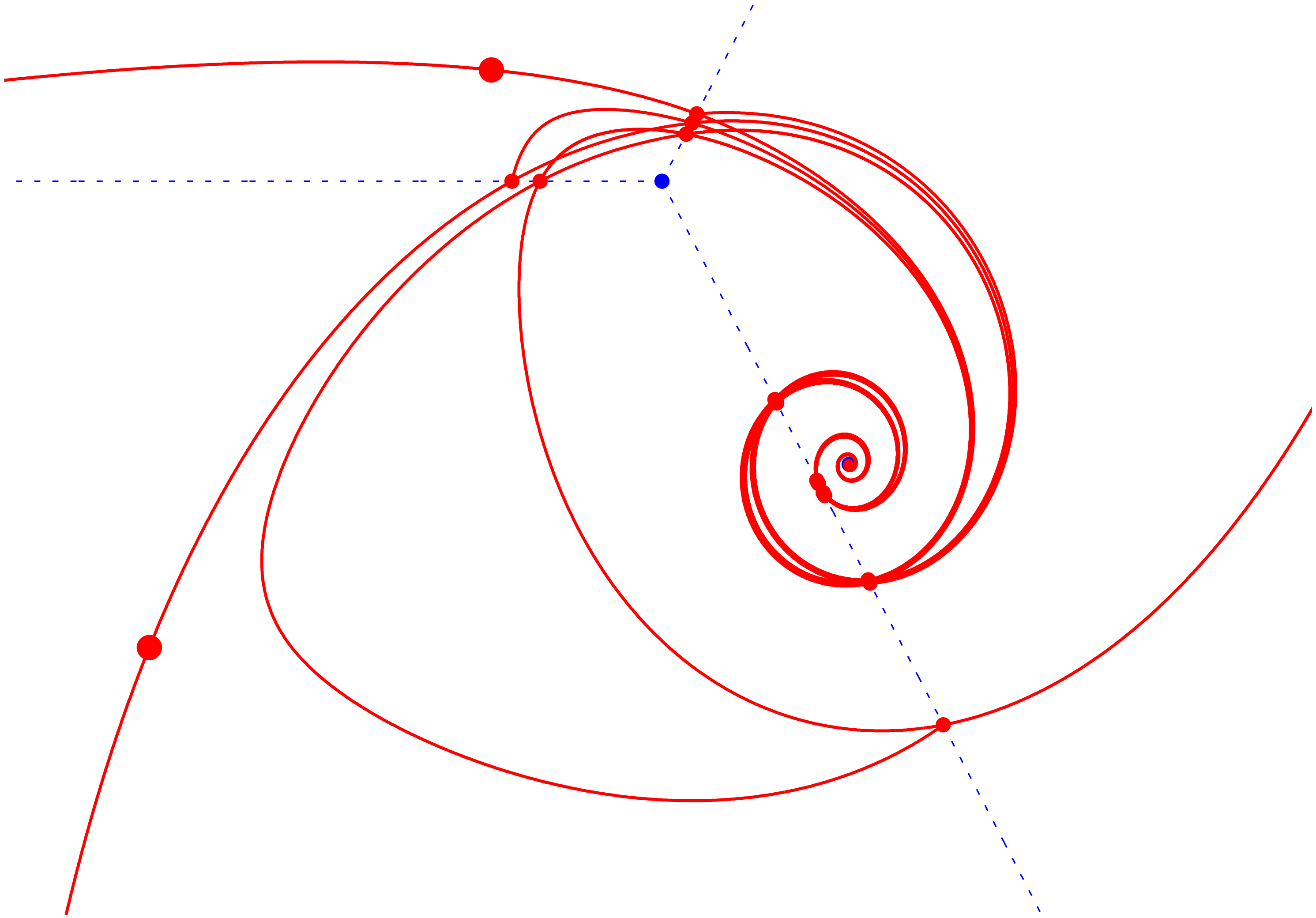
$\boldsymbol{k = 13}$



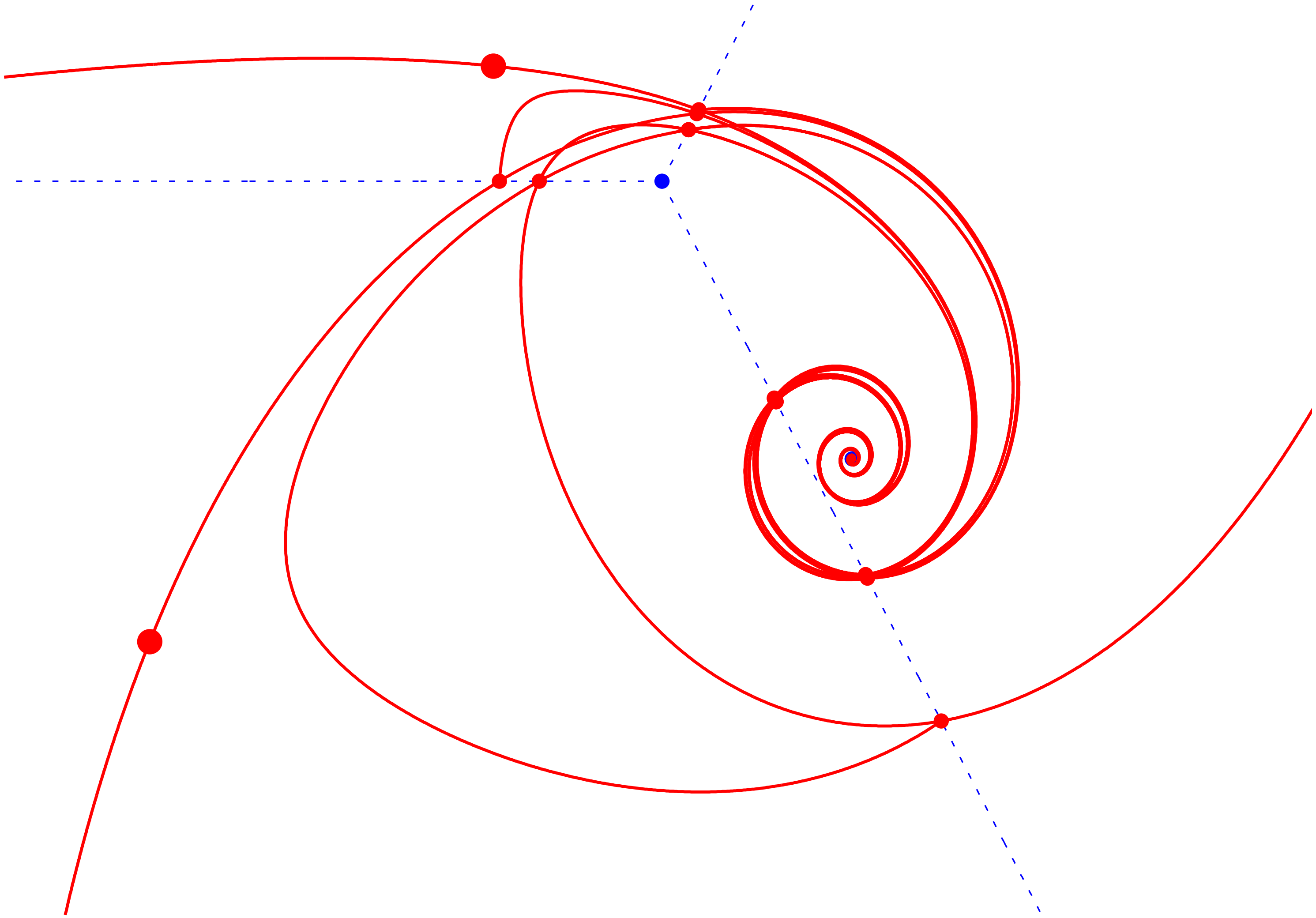
$\boldsymbol{k = 14}$



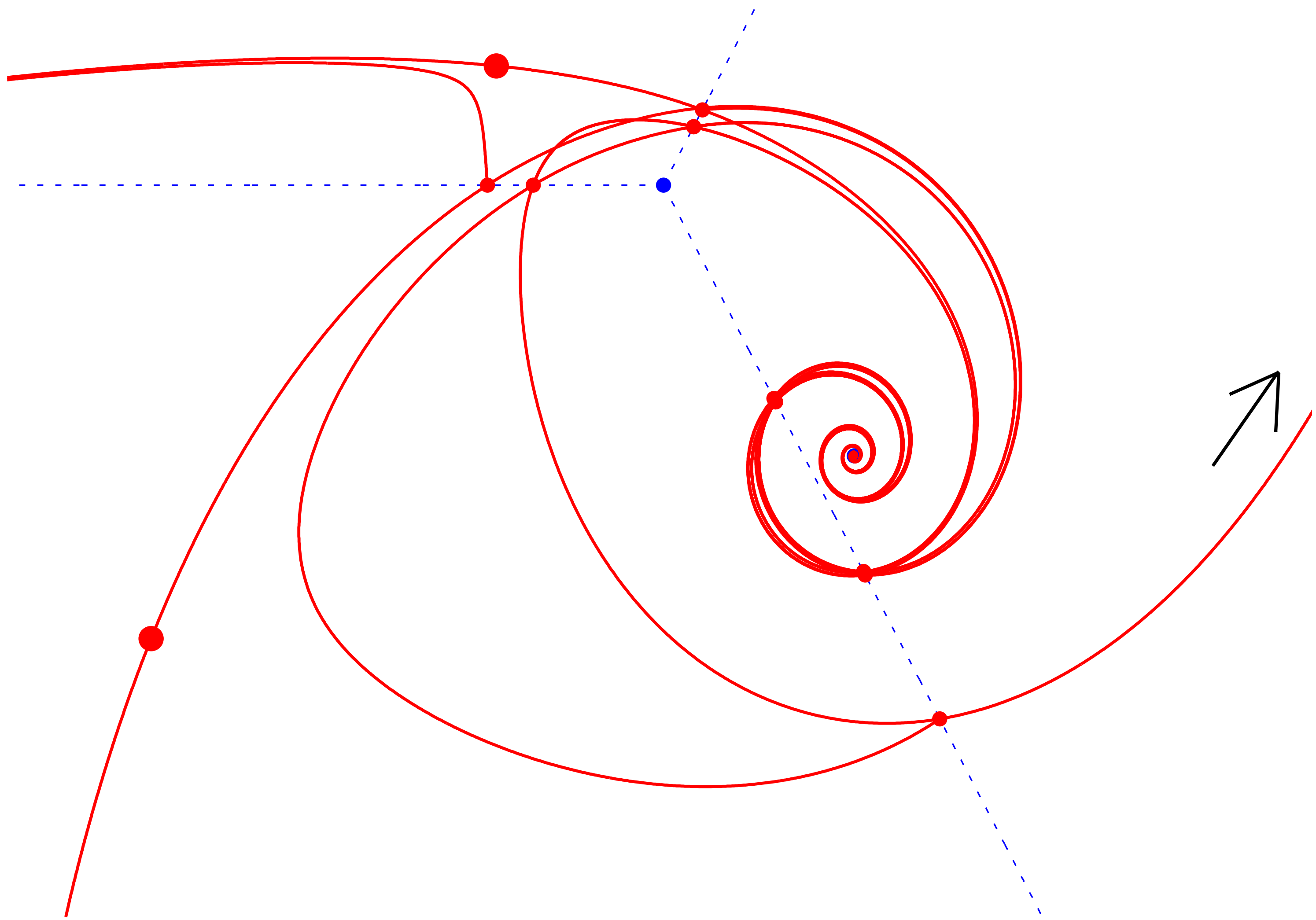
$\boldsymbol{k = 15}$



$$k = 16$$



$\boldsymbol{k = 17}$

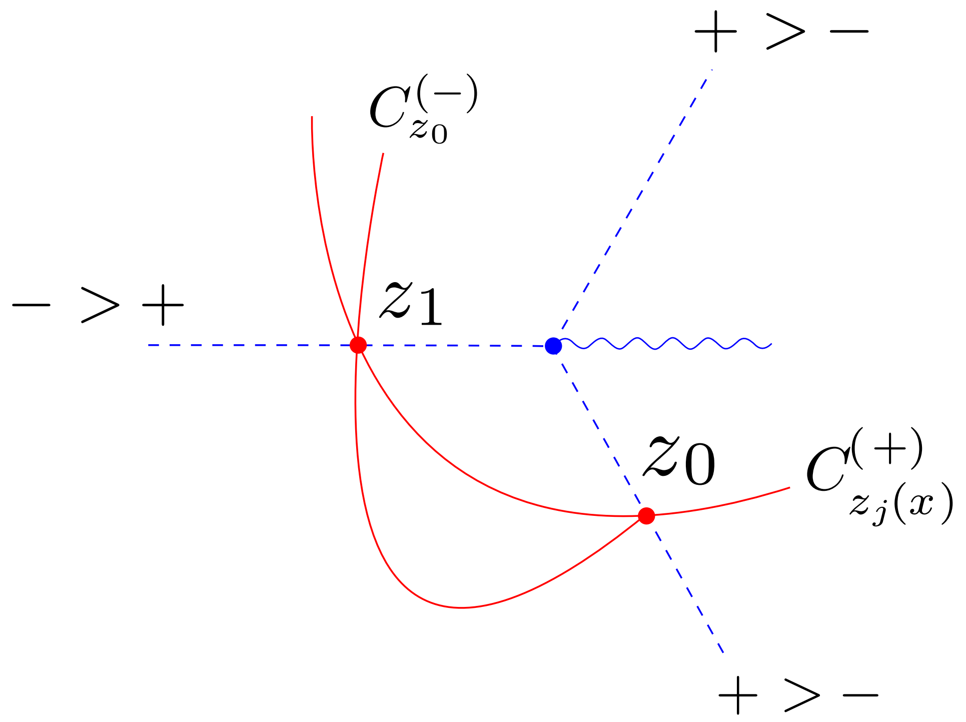


$k = 0$

We find configuration changes 6 times:

between $k = 2, 3$; between $k = 5, 6$; between $k = 8, 9$;
between $k = 11, 12$; between $k = 14, 15$; between $k = 17, 0$.

Among them a change between $k = 2, 3$ is superfluous.



As a matter of fact, steepest descent paths overlap on the portion in question and a cancellation occurs.

Example 2

$$P_2 = (\eta^{-1}\partial_x)^2 + x^2 + c$$

$$\tilde{P}_2 = (\eta^{-1}\partial_x)^4 + (x^2+c)(\eta^{-1}\partial_x)^2 + (-2\mu x+4x\eta^{-1})(\eta^{-1}\partial_x) + \mu^2 - 3\mu\eta^{-1} + 2\eta^{-2}$$

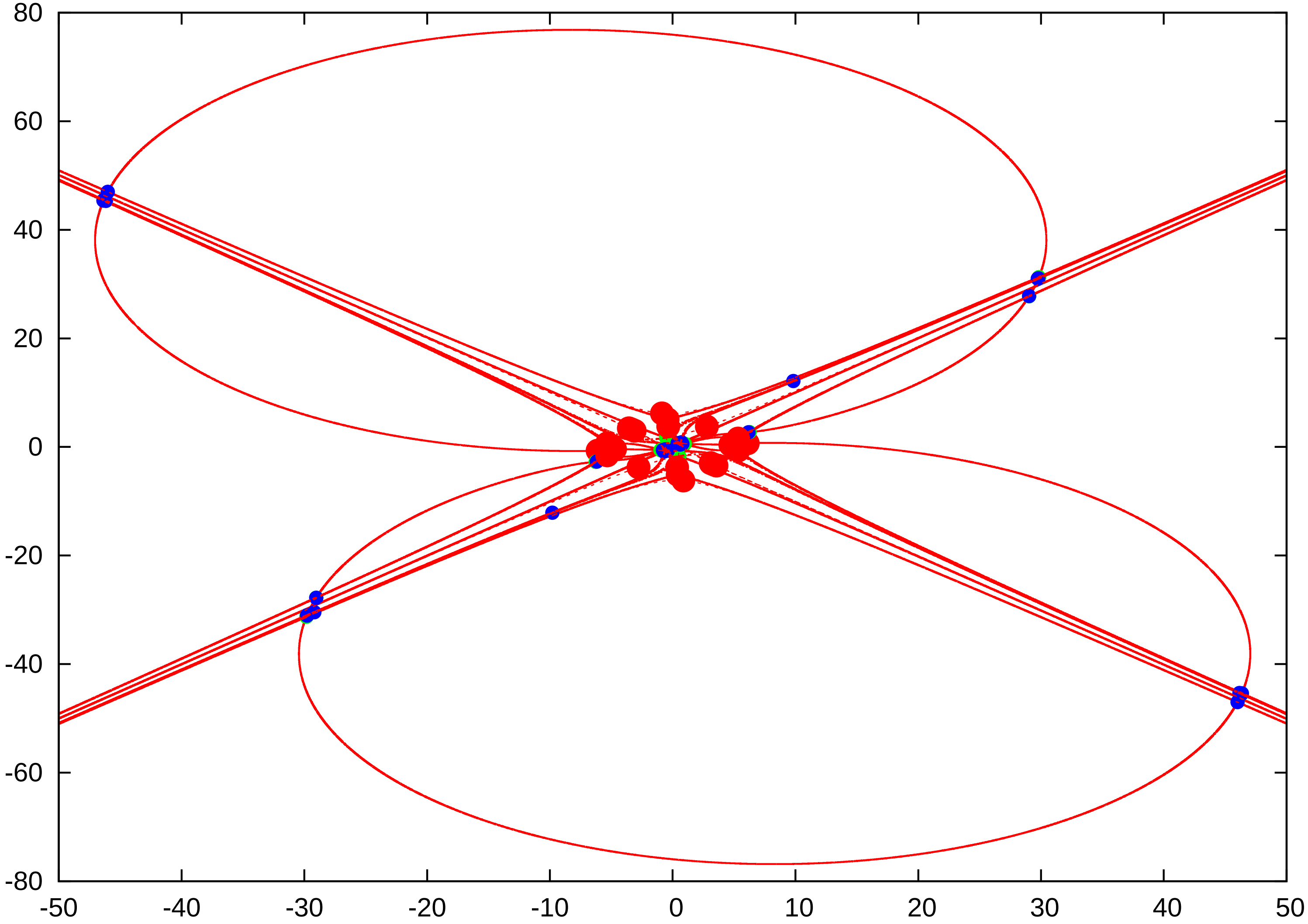
with $c = 1 + 0.1i$, $\mu = 1 - 6i$.

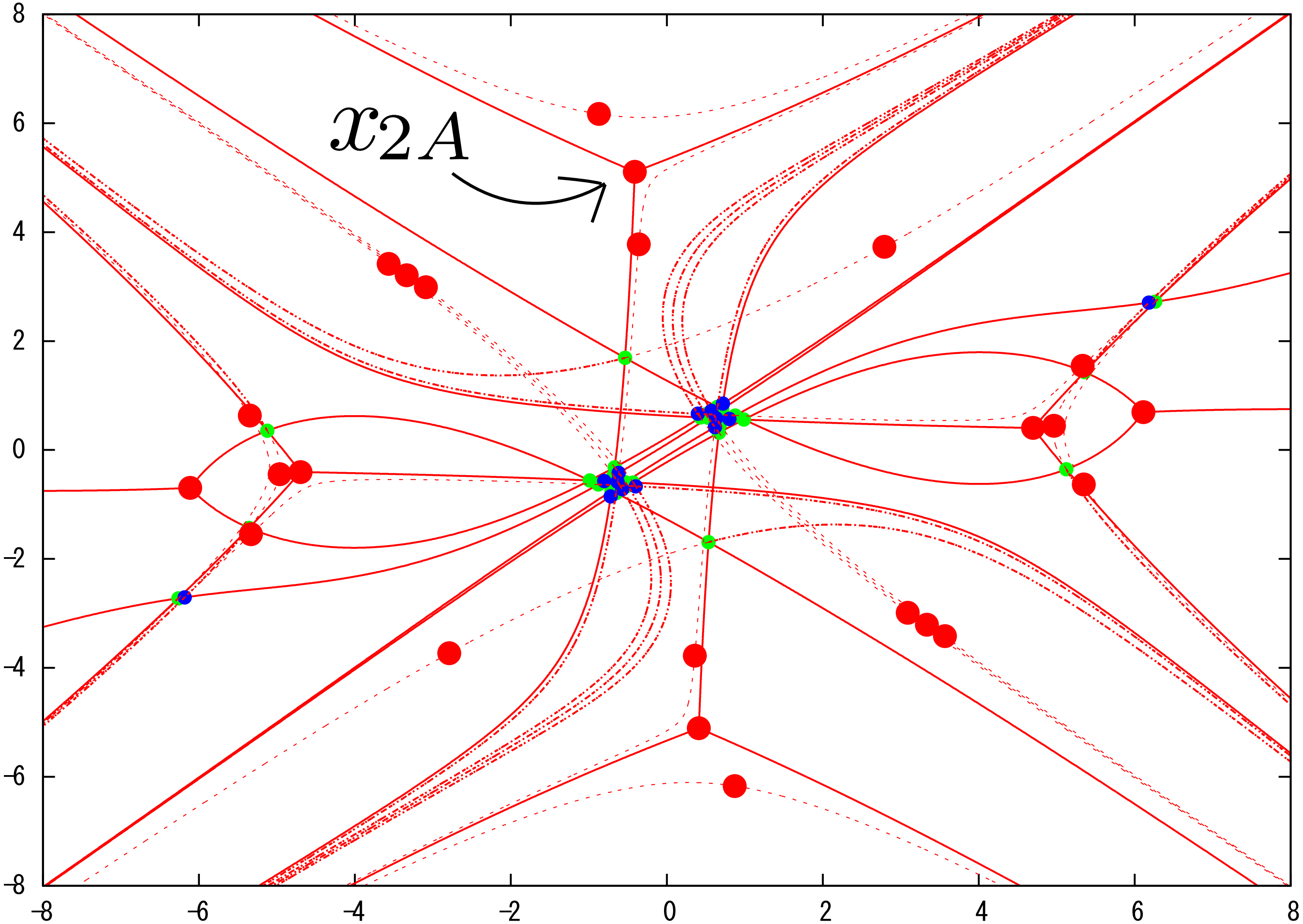
We investigate the exact steepest descent paths of

$$\int (x - z)^{\mu\eta^{-1}} \Psi_k(z, \eta)\, dz$$

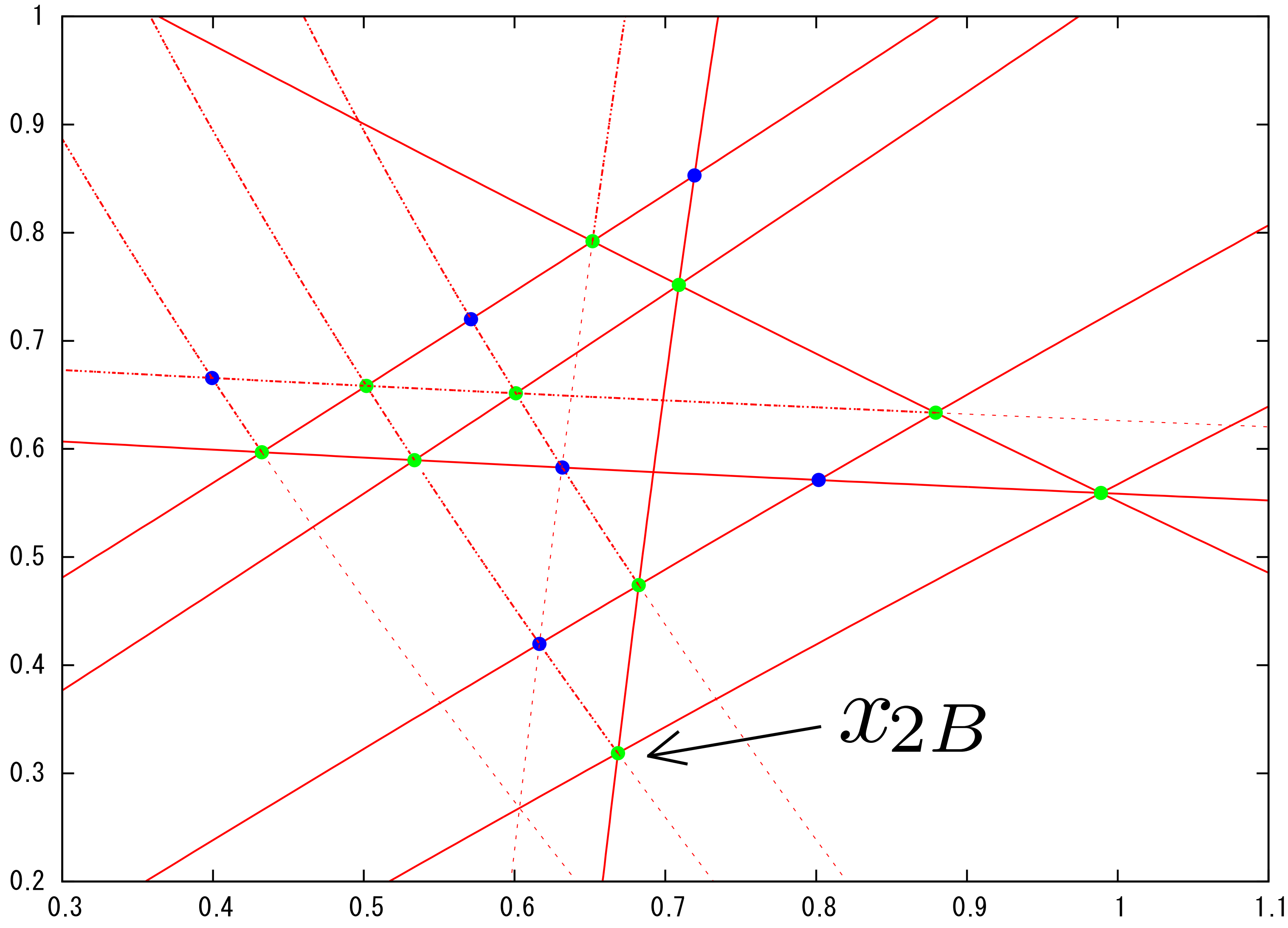
near the point x_{2B} specified in the following figure, that is, at $x = x_{2B} + 0.01\exp(k\pi i/9)$ ($0 \le k \le 17$).

Stokes geometry of $\tilde{P}_2\tilde{\psi} = 0$



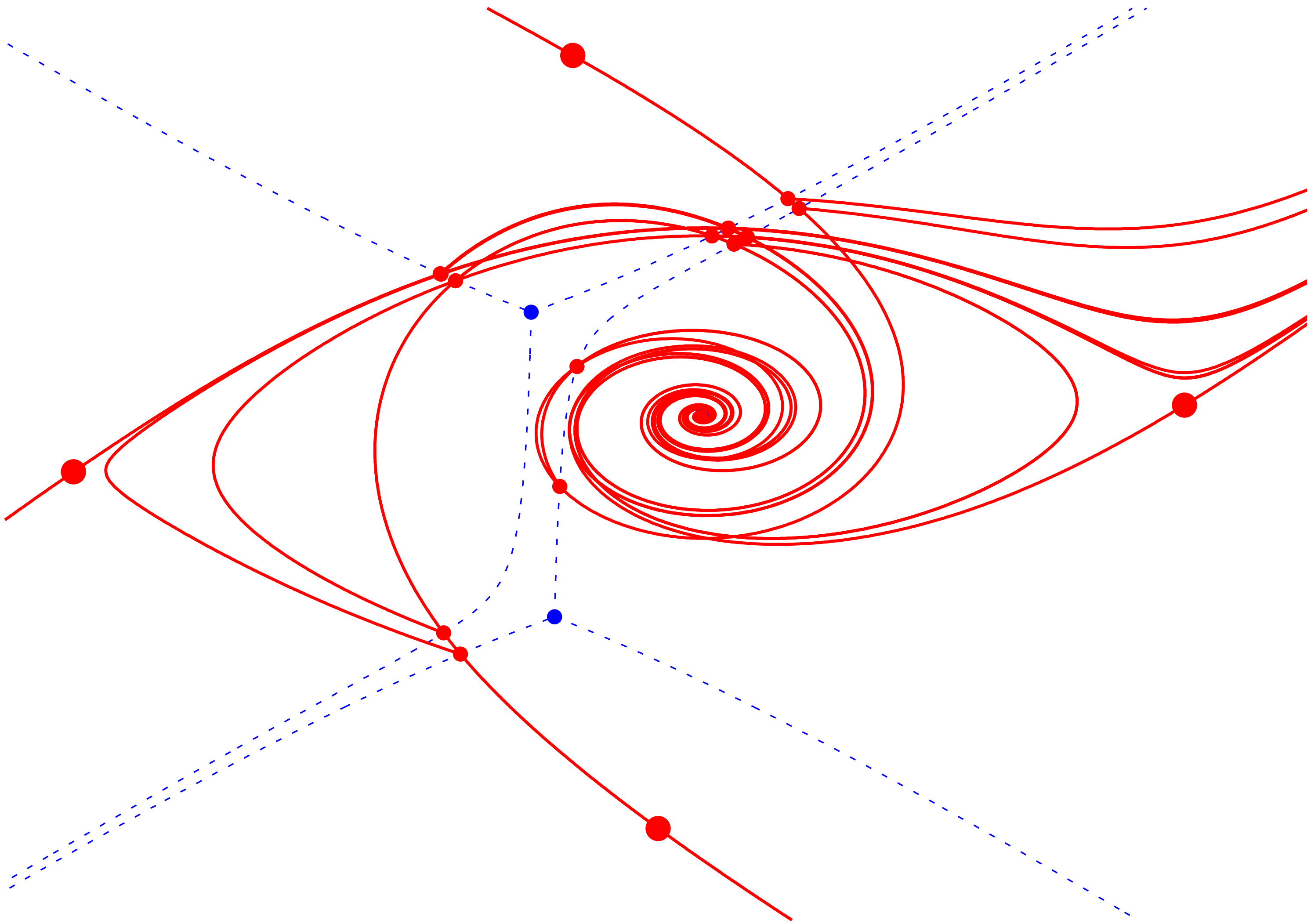


(Enlarged near the center)

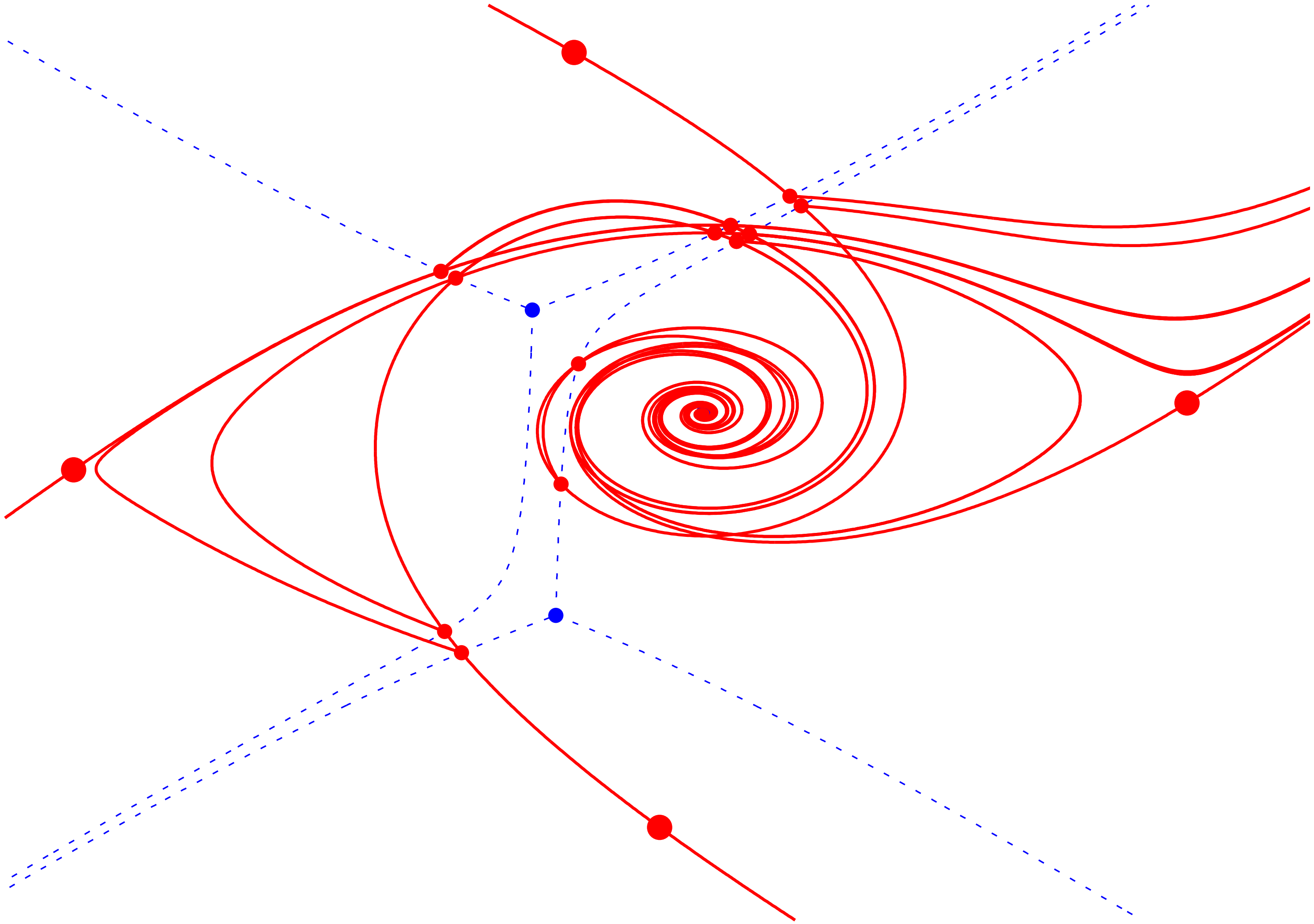


(More enlarged in $[0, 2] \times [0, 2]$)

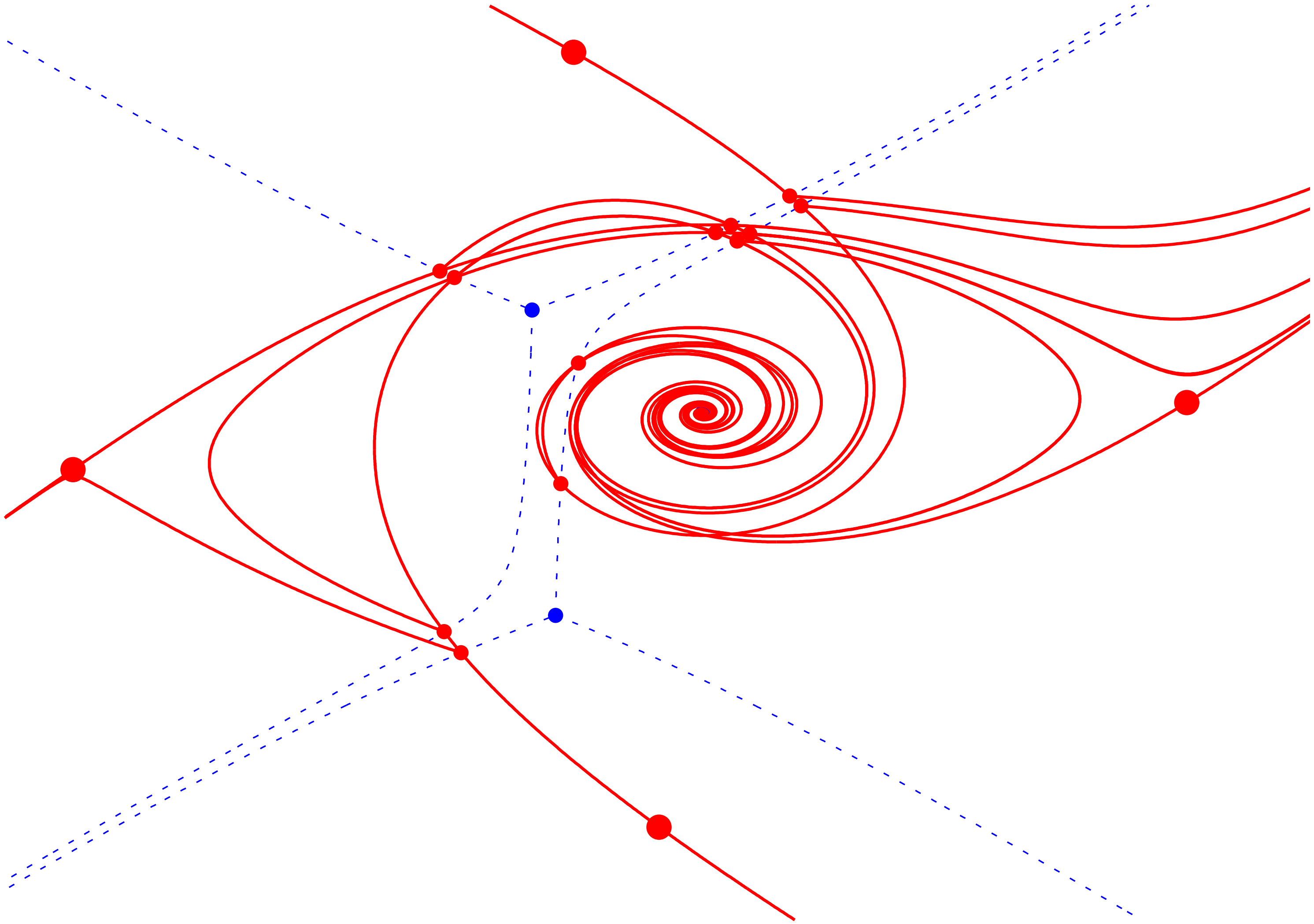
Configuration of exact steepest descent paths



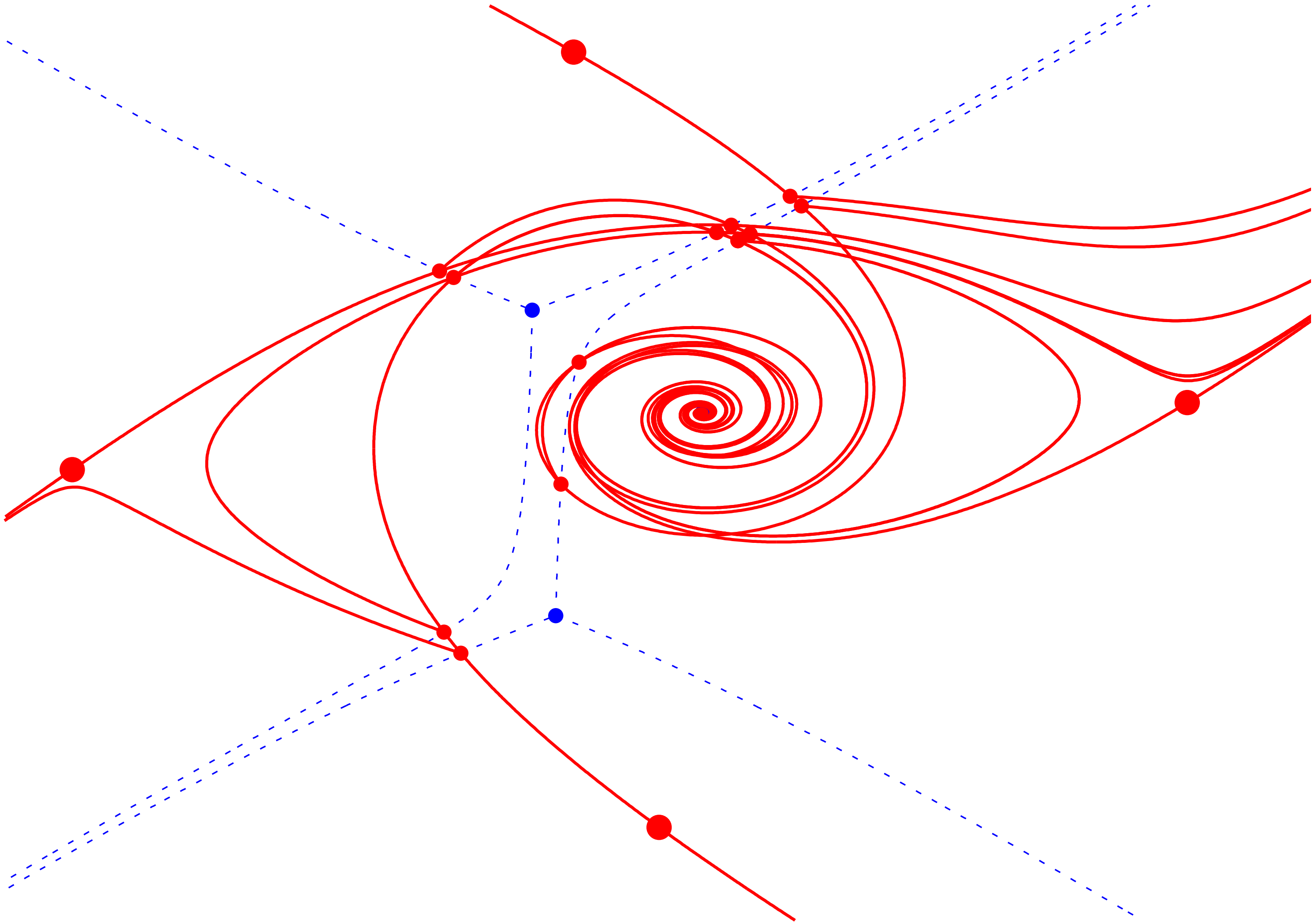
$\boldsymbol{k = 0}$



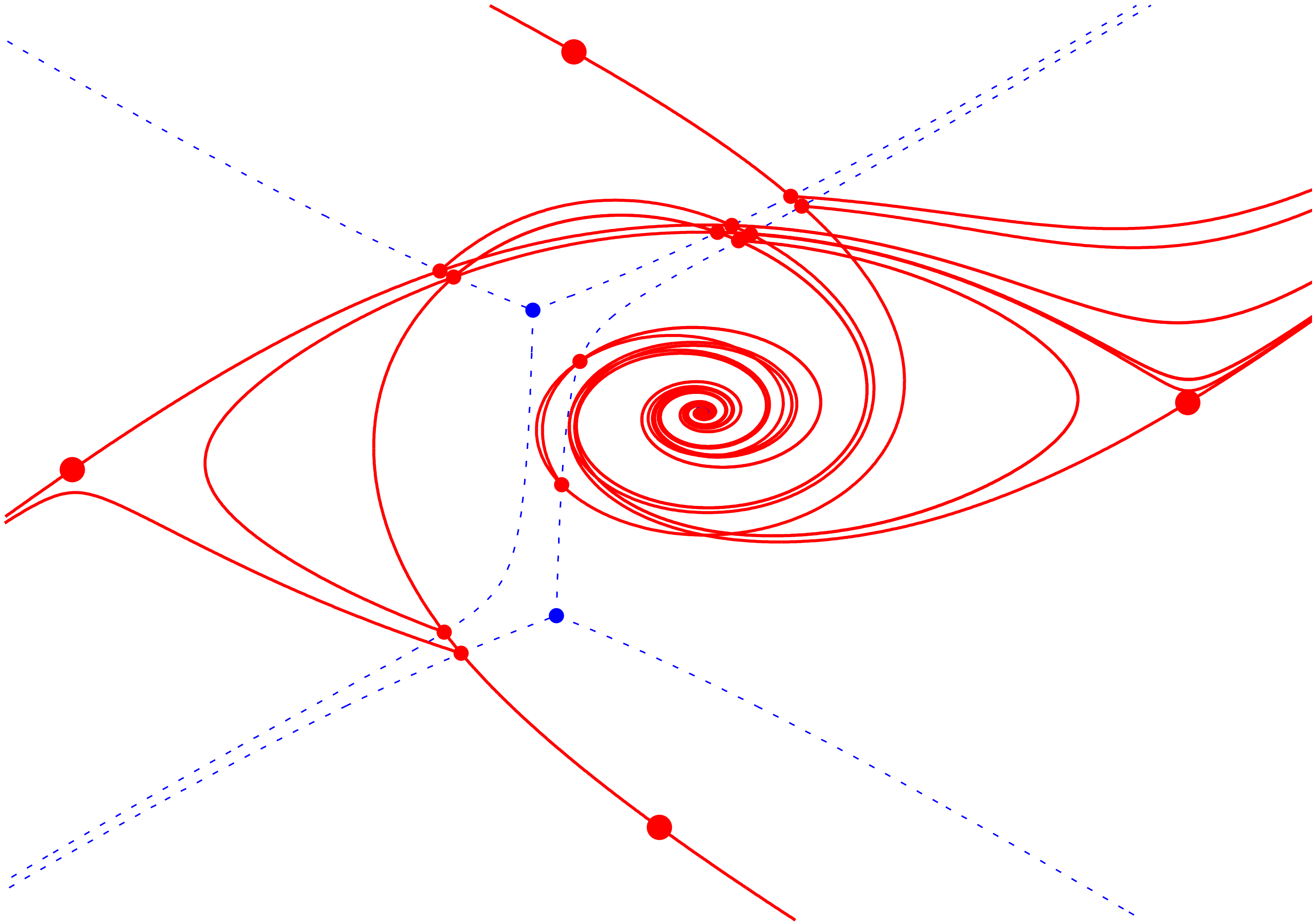
$k = 1$



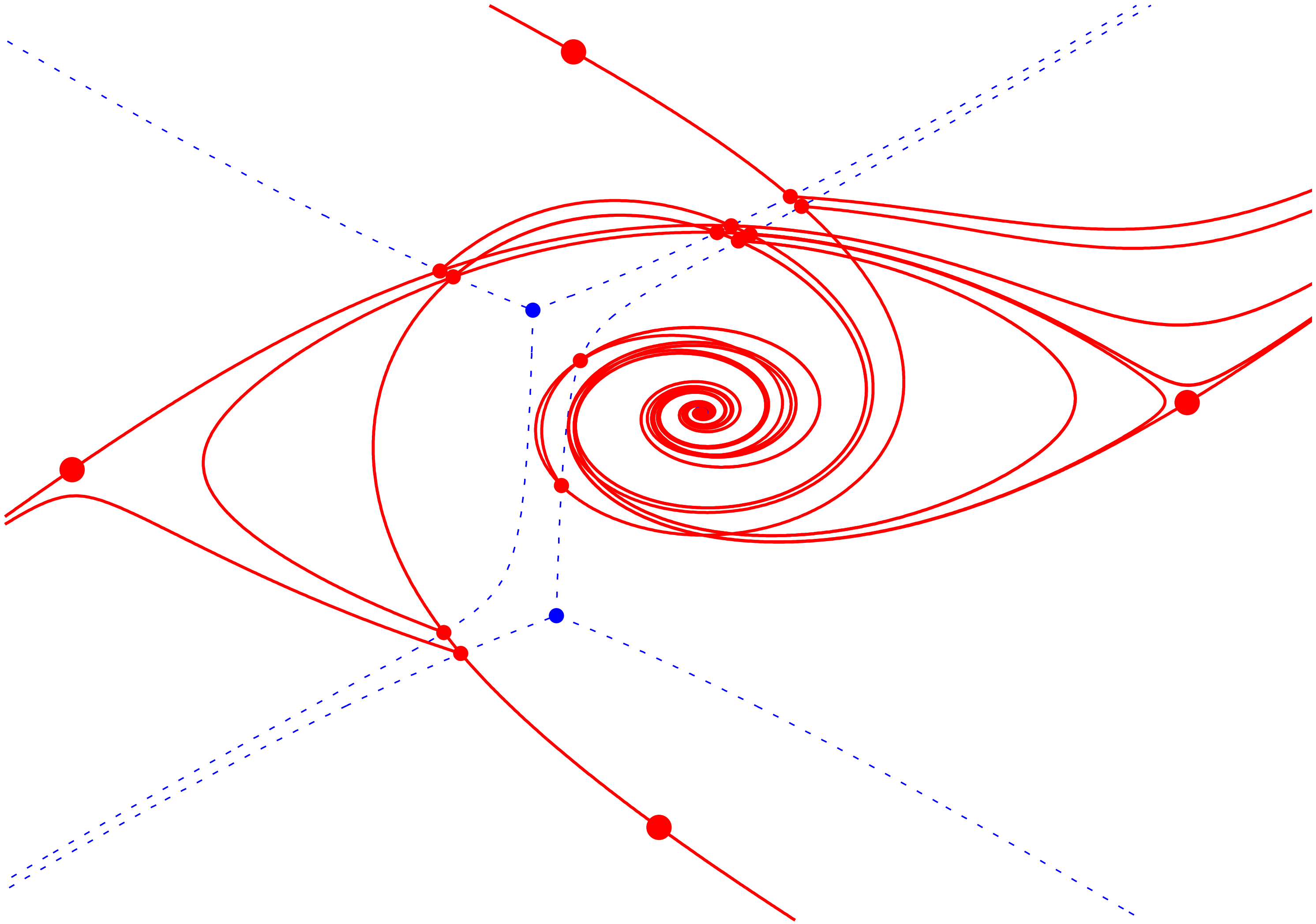
$k = 2$



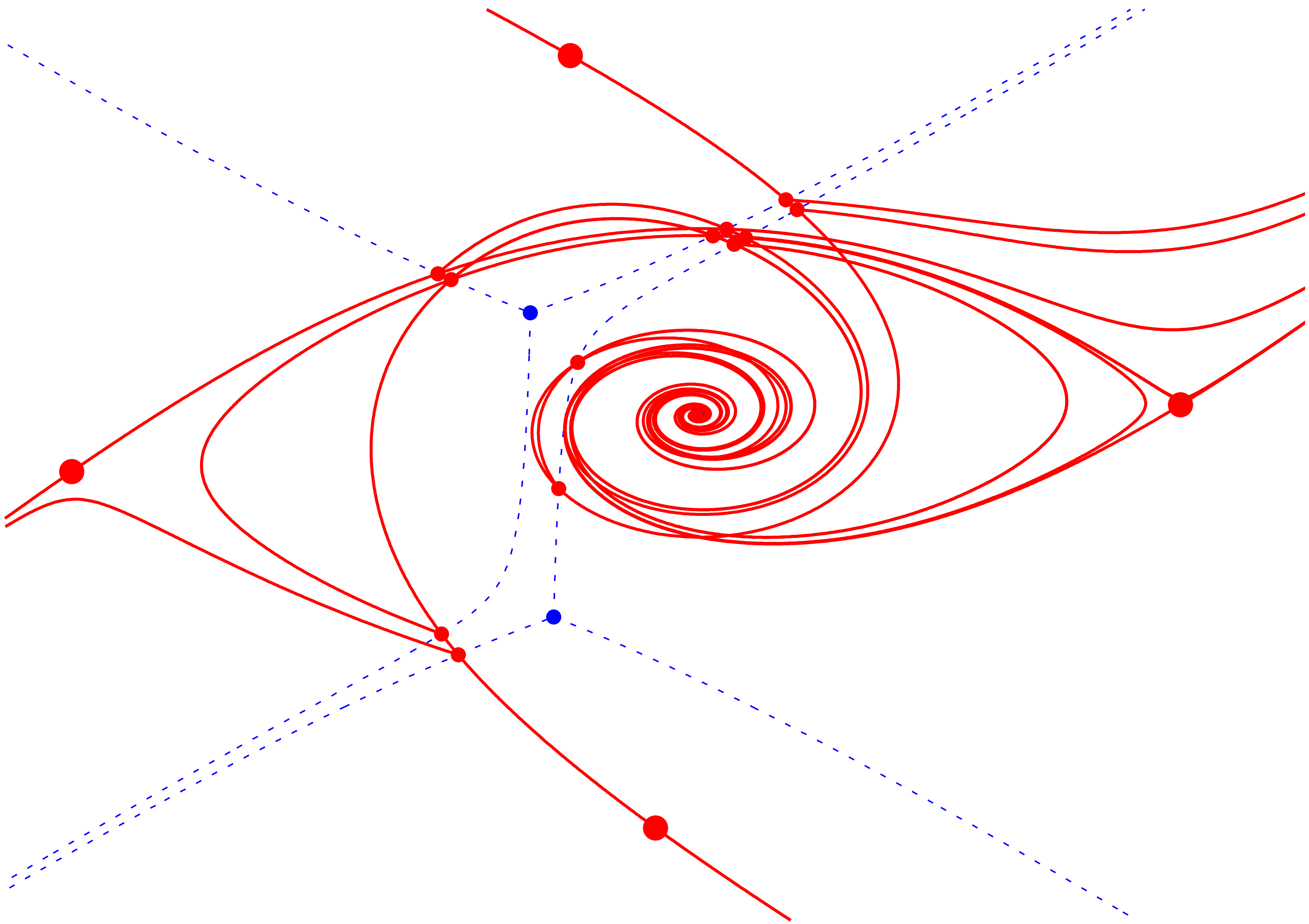
$\boldsymbol{k = 3}$



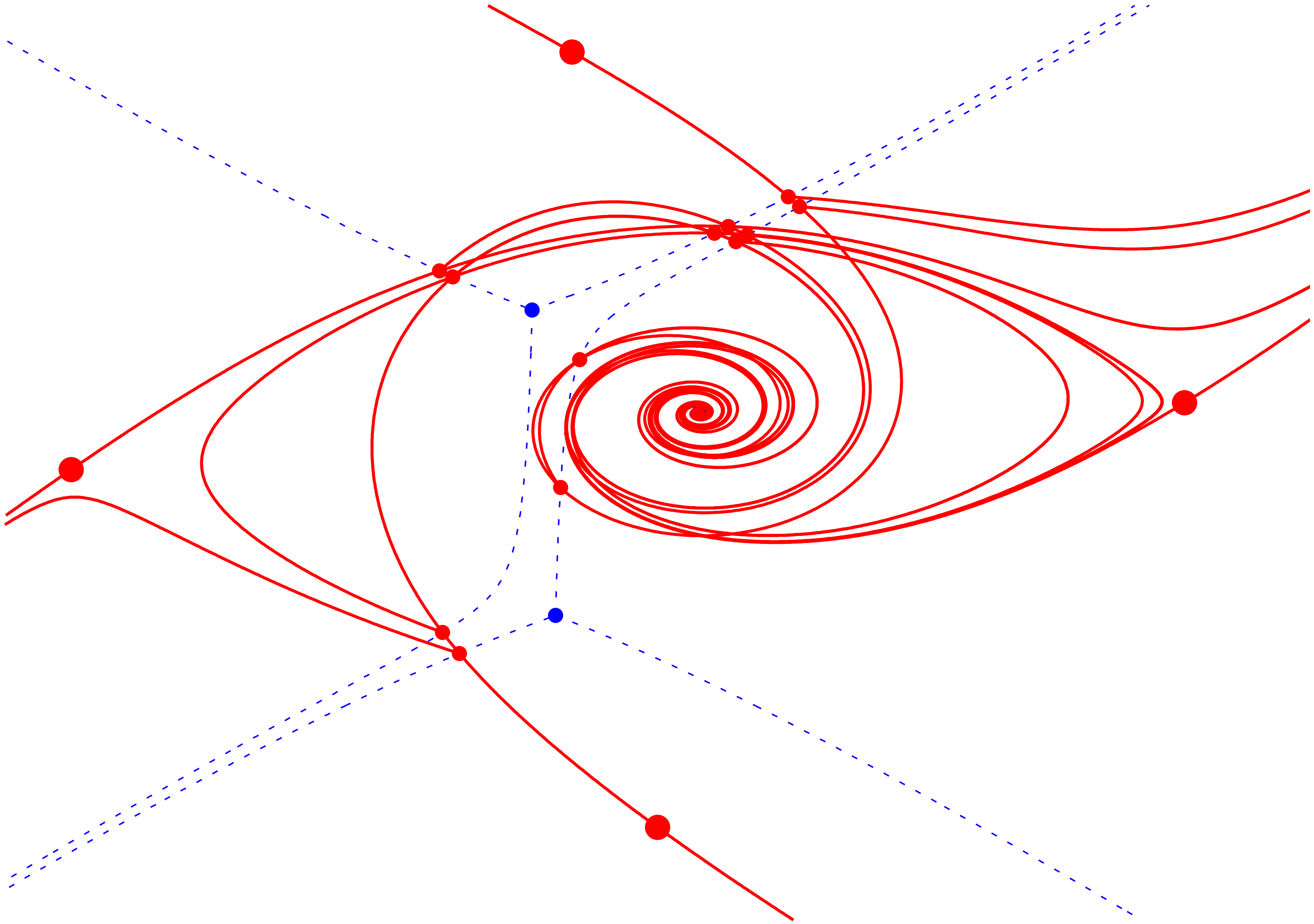
$k = 4$



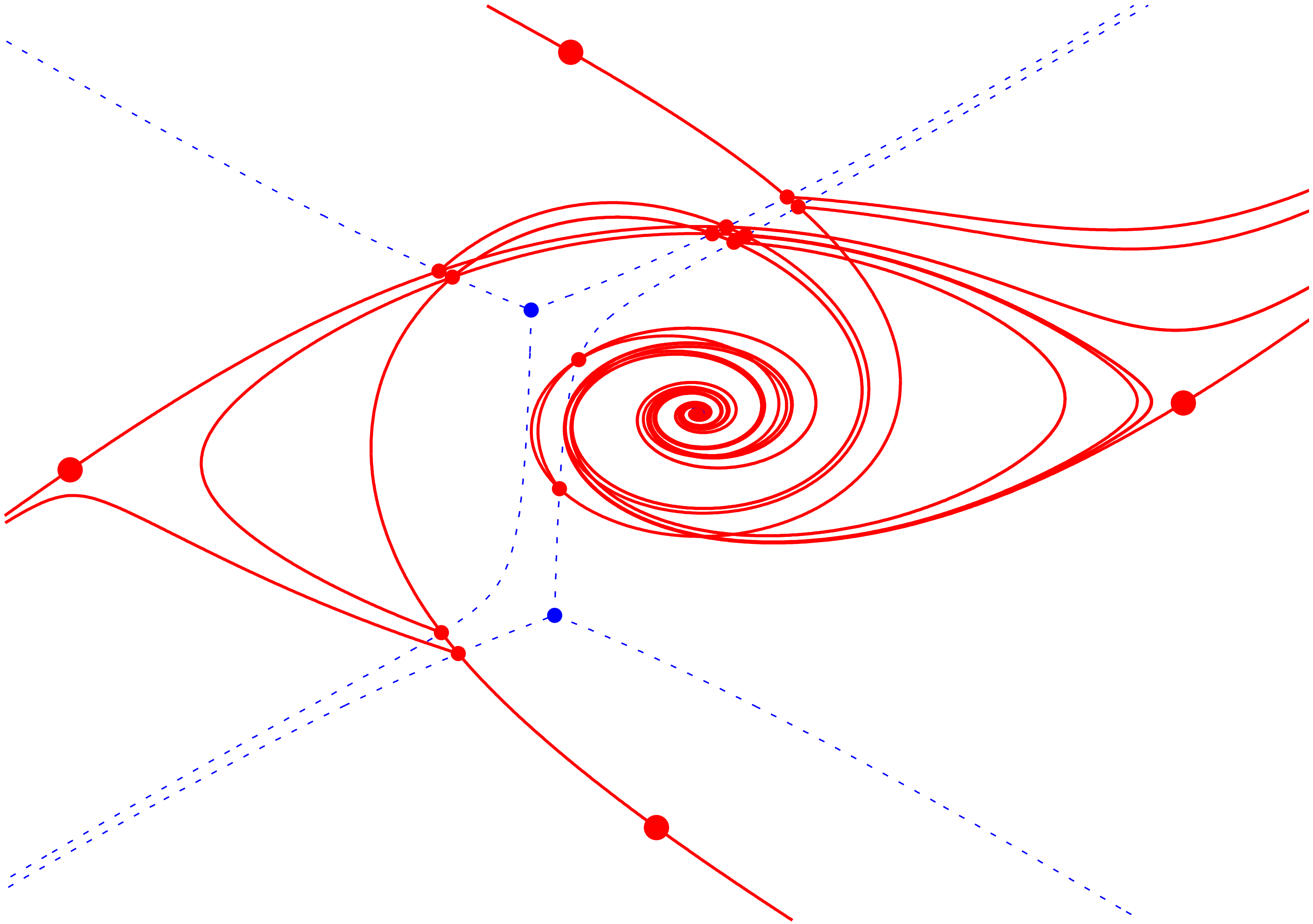
$\boldsymbol{k = 5}$



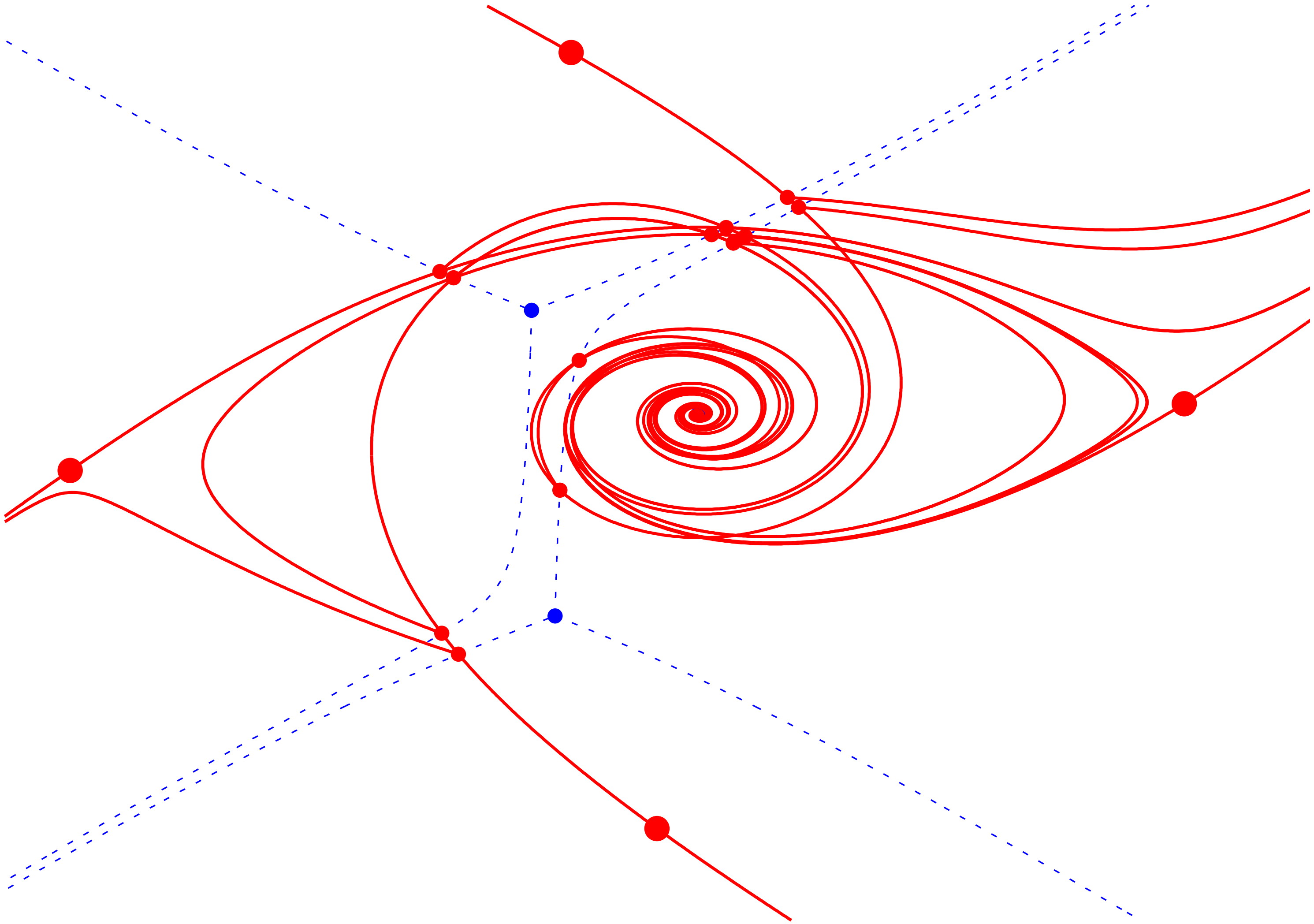
$k = 6$



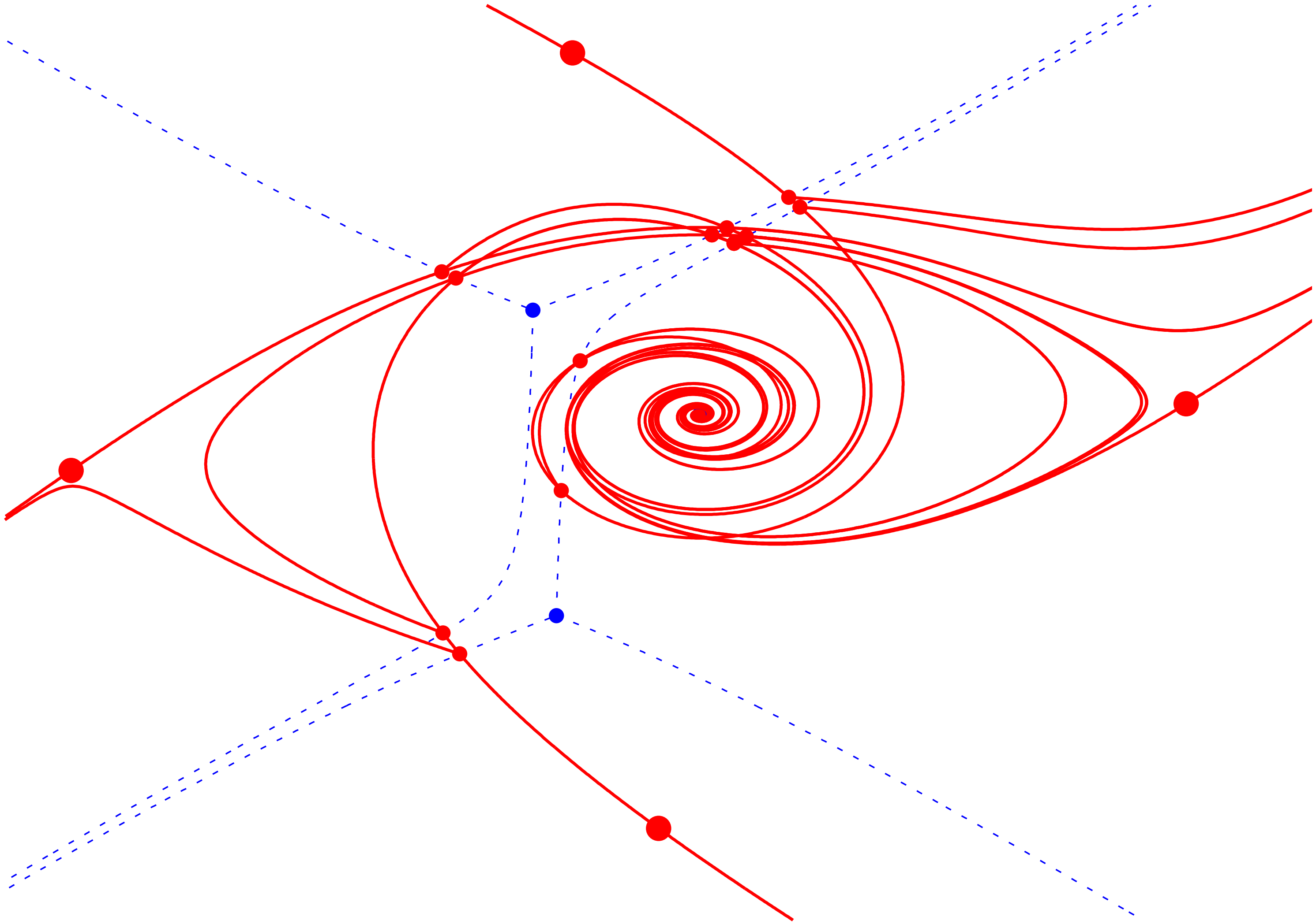
$\boldsymbol{k = 7}$



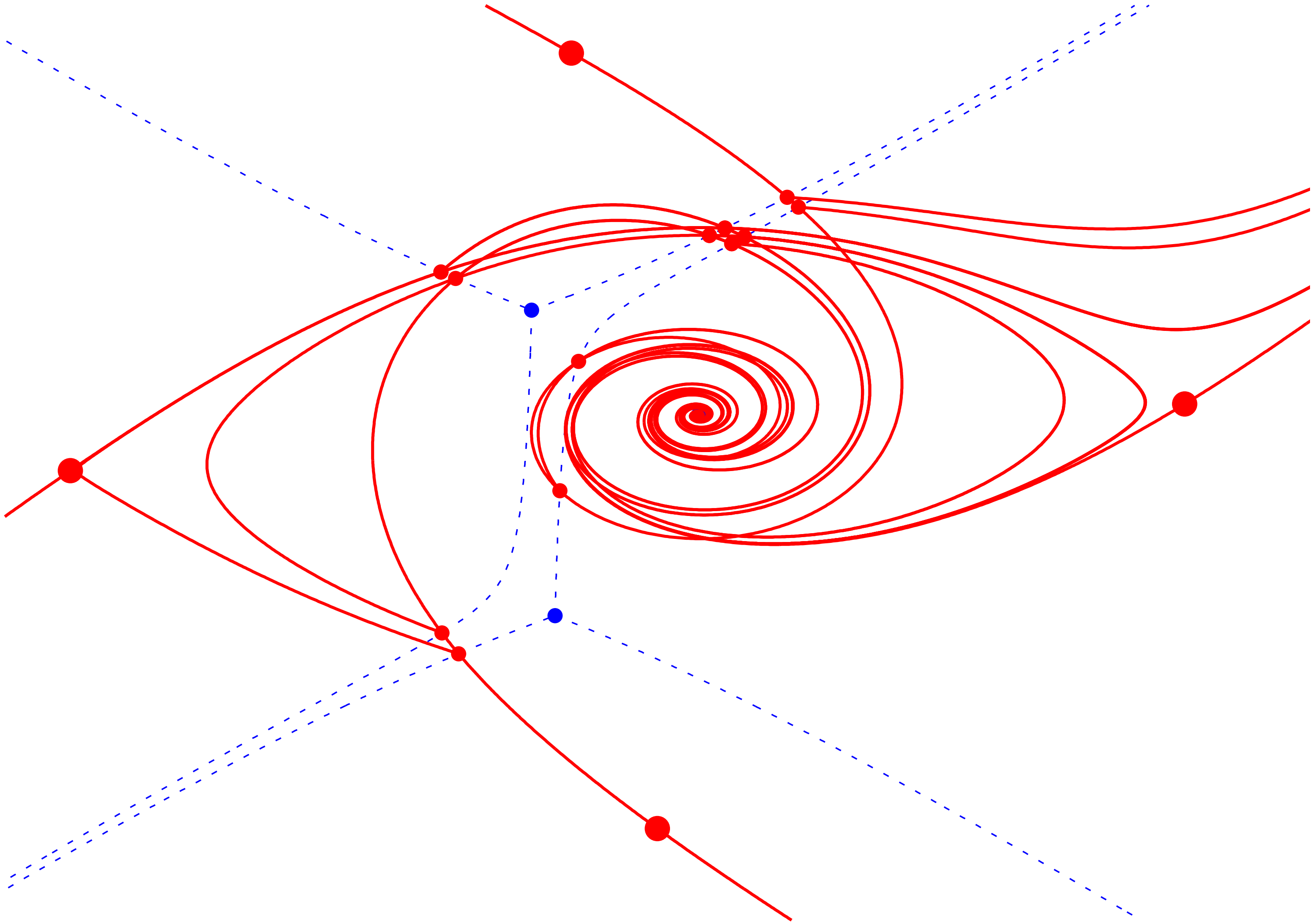
$k = 8$



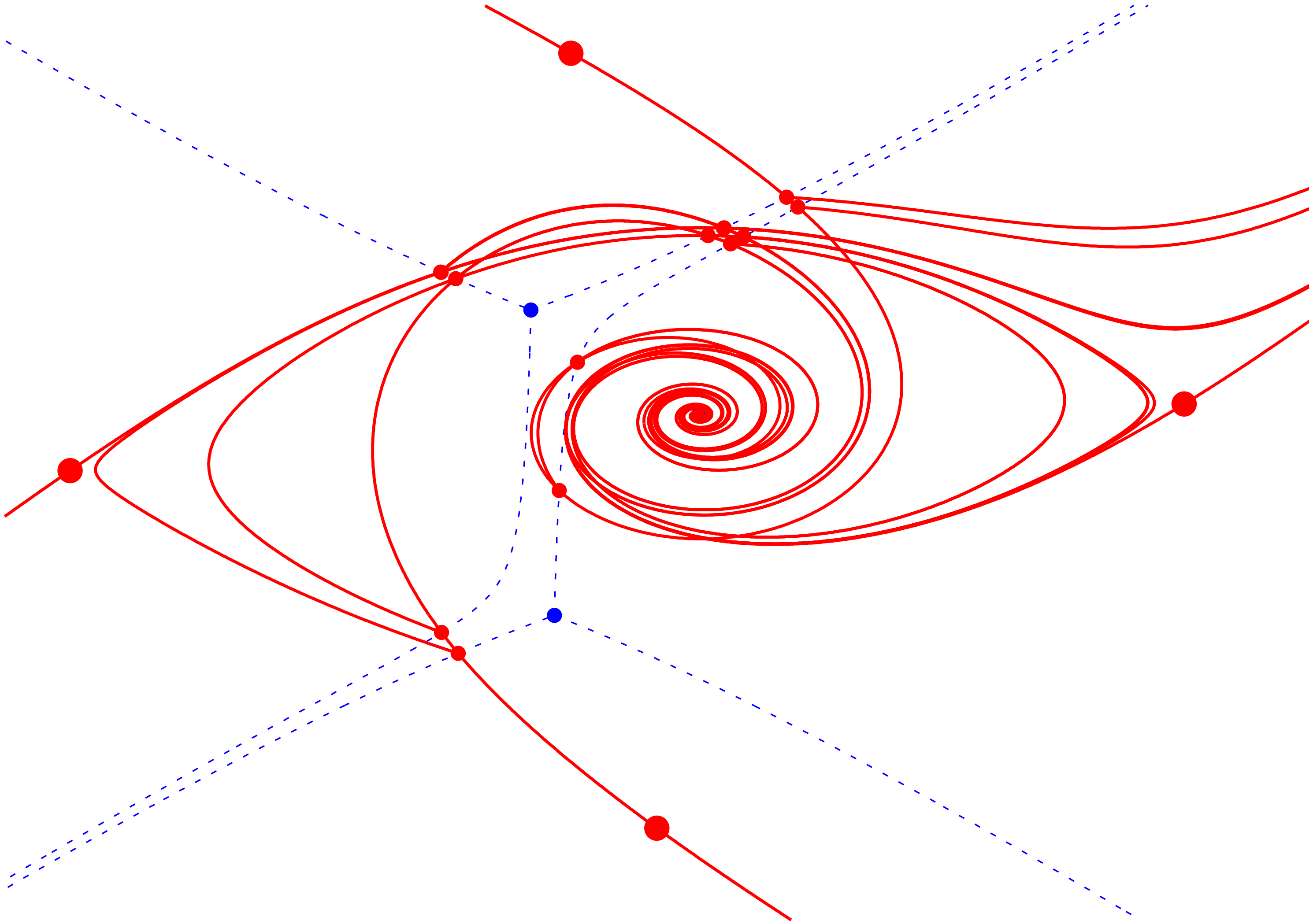
$k = 9$



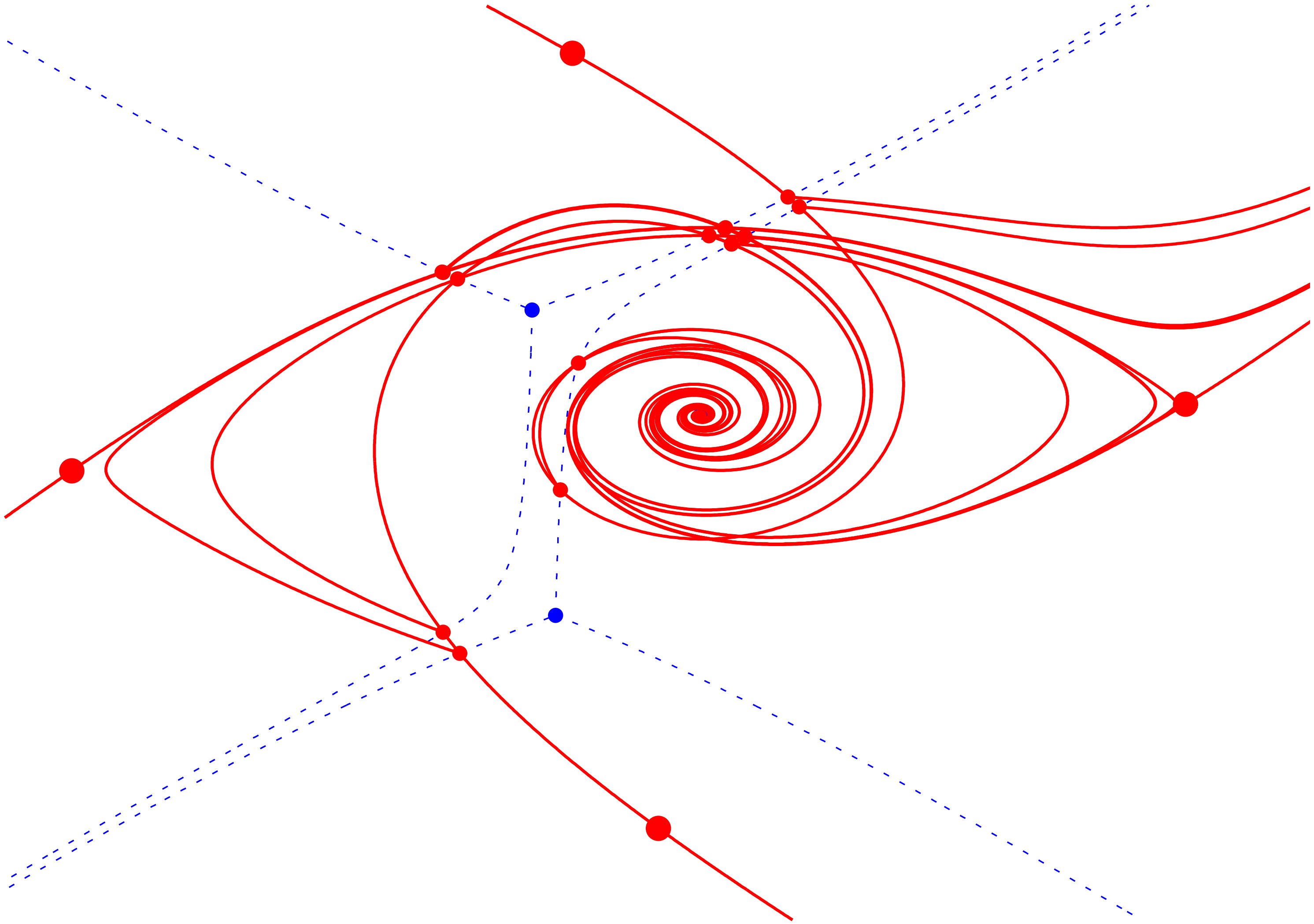
$k = 10$



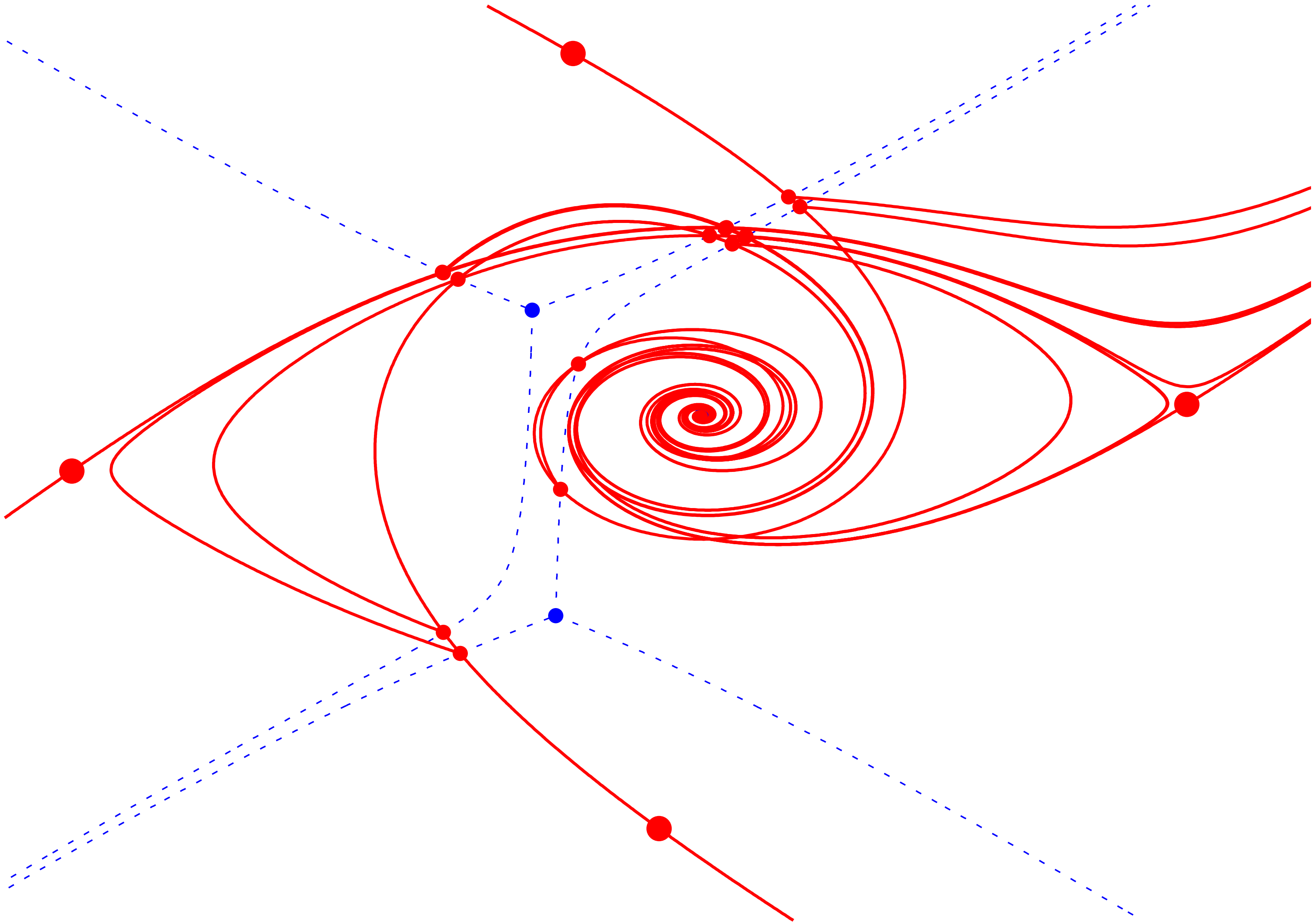
$\boldsymbol{k = 11}$



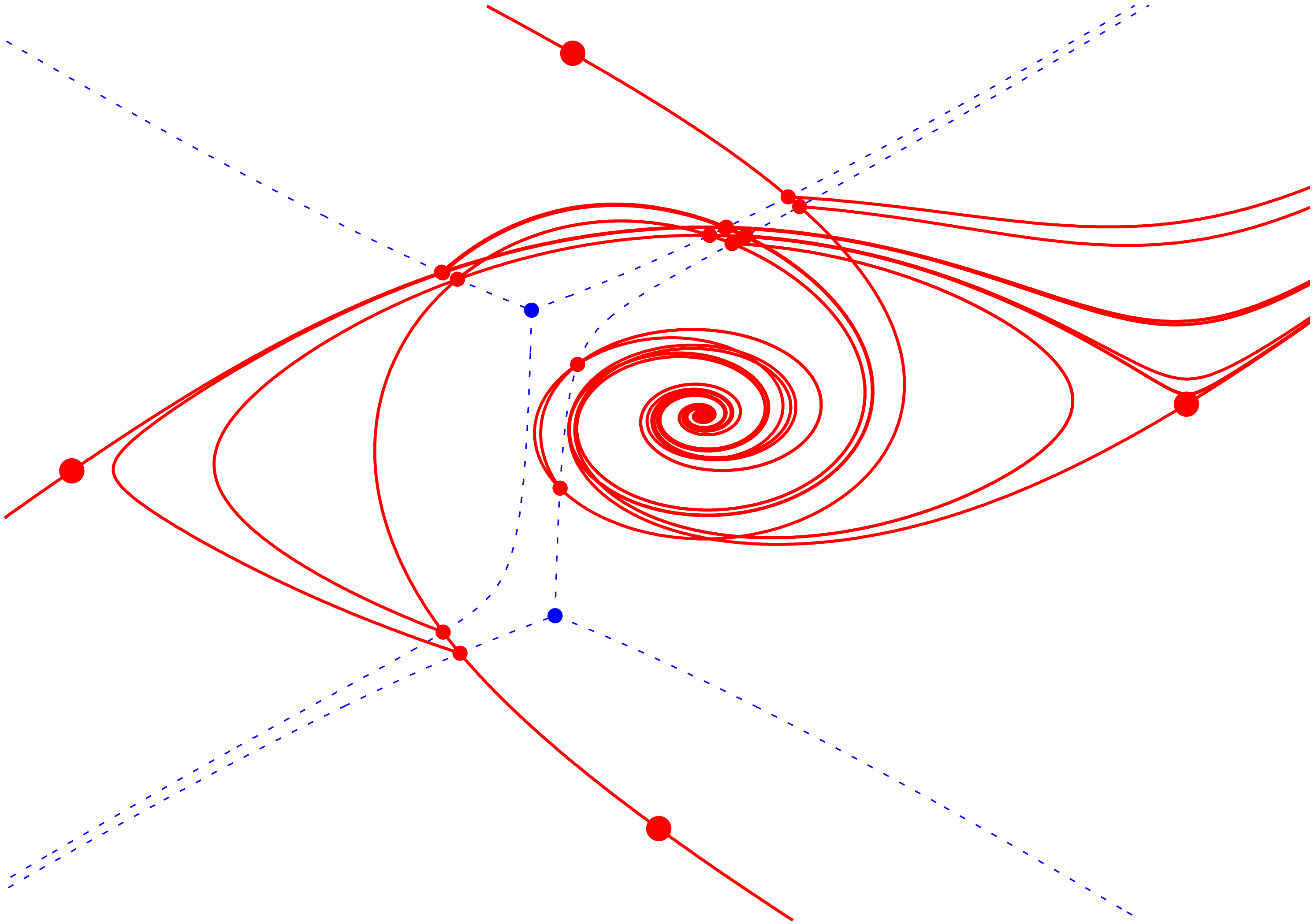
$k = 12$



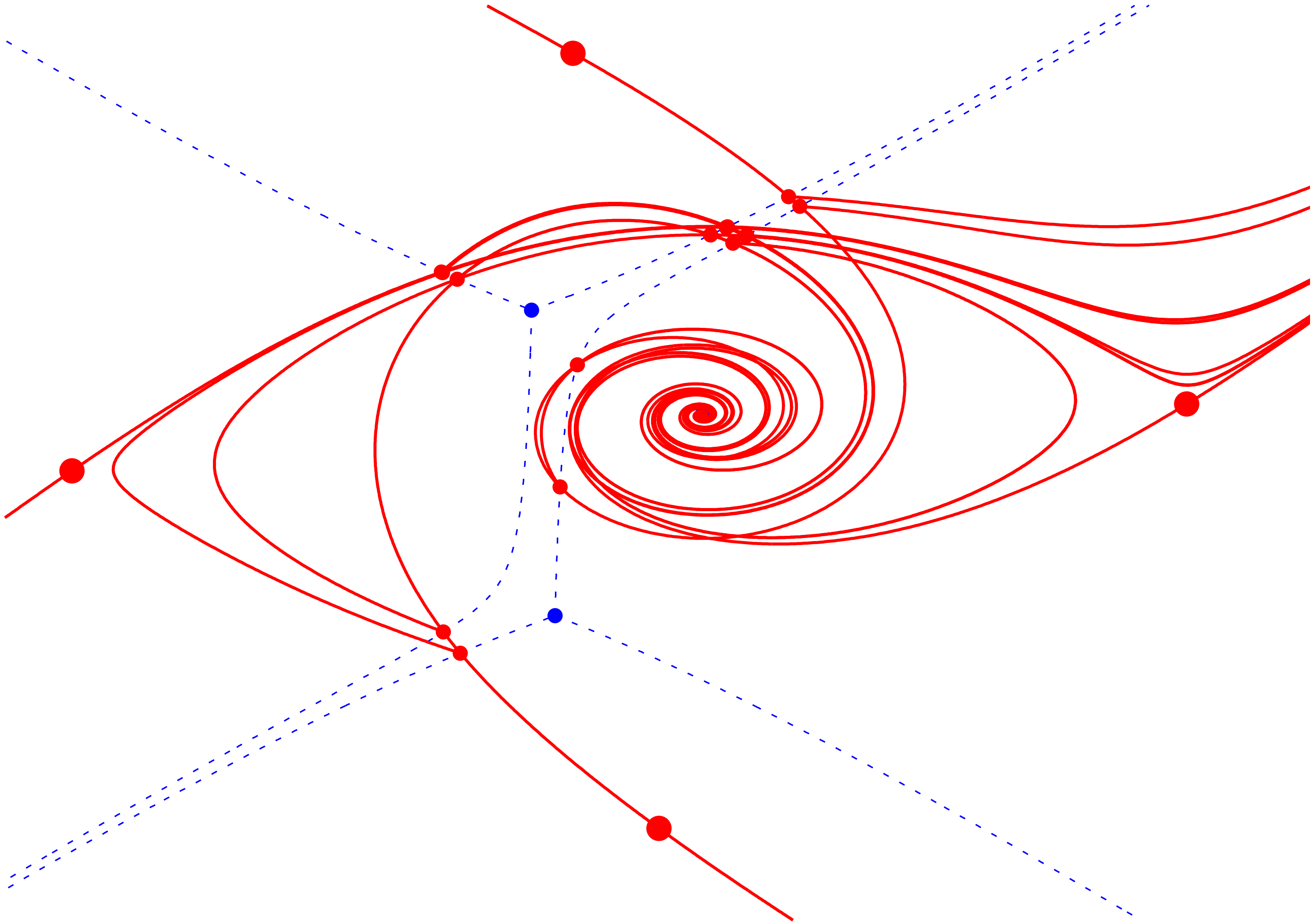
$\boldsymbol{k = 13}$



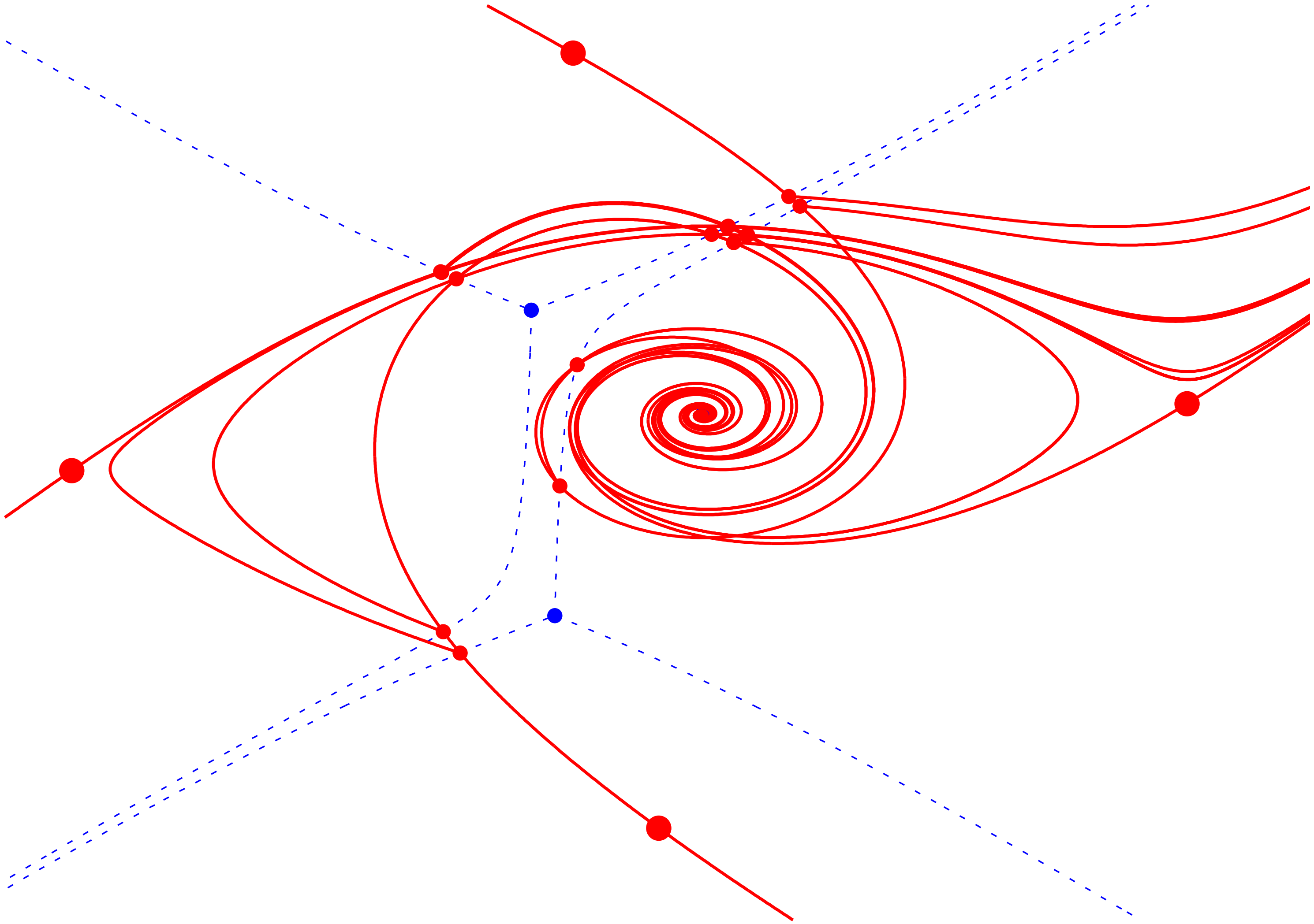
$k = 14$



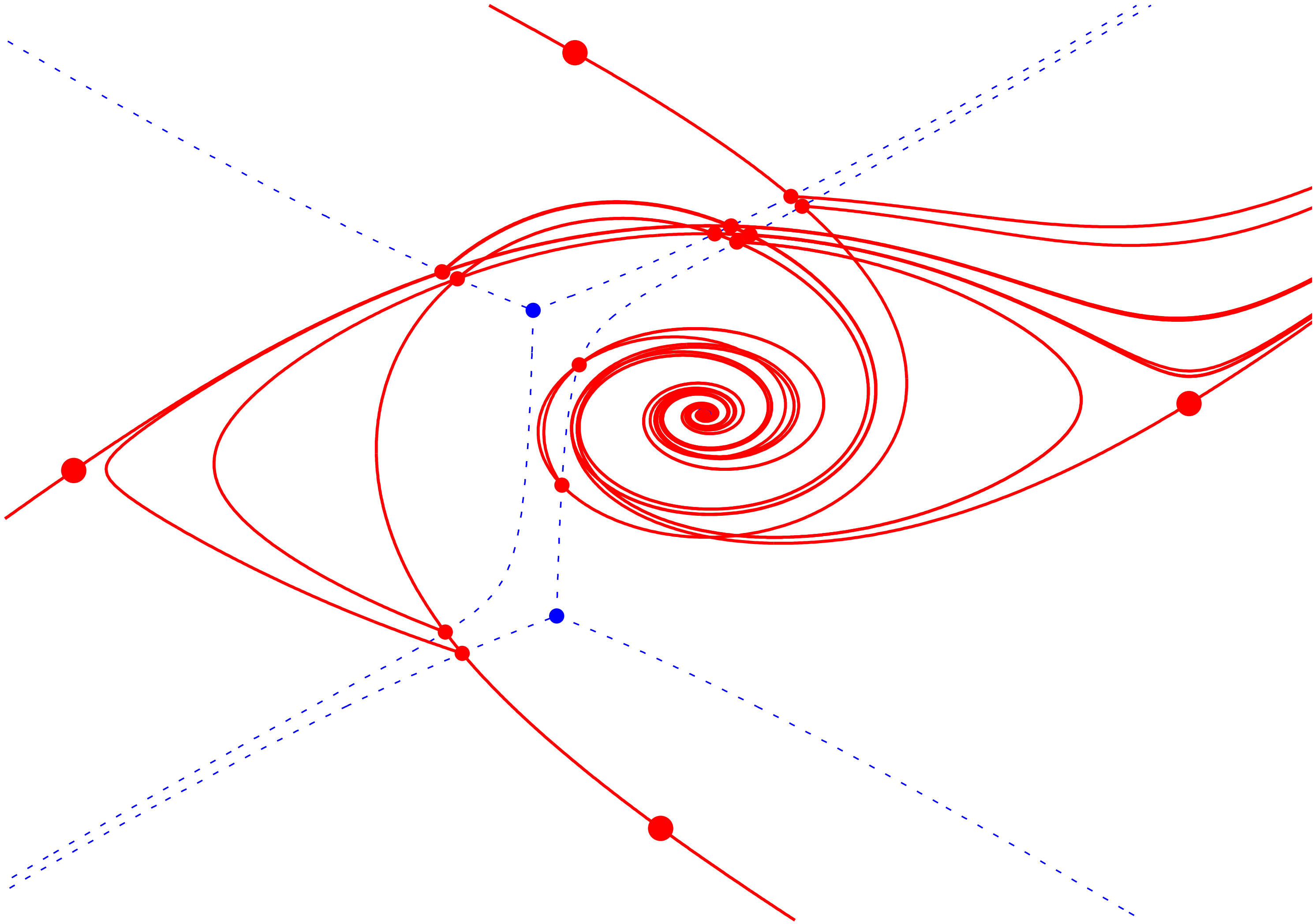
$\boldsymbol{k = 15}$



$\boldsymbol{k = 16}$



$\boldsymbol{k = 17}$

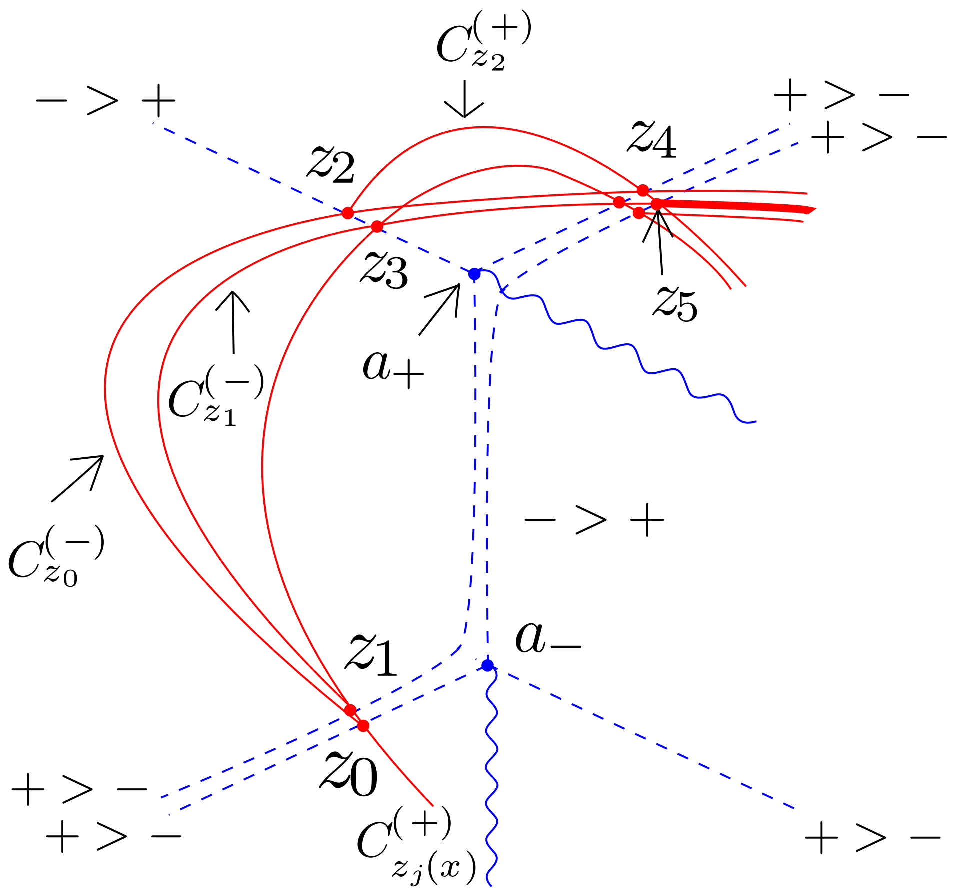


$\boldsymbol{k = 0}$

We find configuration changes 6 times:

between $k = 1, 2$;	**between** $k = 4, 5$;	**between** $k = 6, 7$;
between $k = 10, 11$;	**between** $k = 13, 14$;	**between** $k = 14, 15$.

Among them a change between $k = 14, 15$ is superfluous since an overlap of steepest descent paths and a cancellation again occur.



Conclusion

If a higher order ODE is obtained from a second order ODE via middle convolution, the Borel summability of its WKB solutions can be examined by the exact steepest descent method.