Properties of the series solutions for Painlevé equations

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joint work with A.N.W. Hone and O. Ragnisco

Asymptotic and computational aspects of complex differential equations

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Introduction and motivations

P_I equation is usually written in the canonical form as

$$\frac{d^2u}{dz^2} = 6u^2 + z. \qquad (1)$$

It is known that all its solutions are non-classical, meromorphic, transcendental functions with an infinite number of poles in the complex plane. Boutroux [Ann. Ecole Norm. Sup. (3), 31, 1914] showed that, for $|z|$ large, the solutions of (1) behaves asymptotically like $u \sim \sqrt{z}\,\wp(\frac{4}{5}z^{\frac{5}{4}})$, where $\wp$ is the Weierstrass function, which satisfies the second order differential equation

$$\frac{d^2\wp}{dz^2} = 6\wp^2 - \frac{1}{2}g_2.$$

One of the purpose of our work was to highlight a direct comparison between the solutions of the P_I equation and the Weierstrass elliptic functions, at the level of series expansions rather than asymptotics.

Today I would like to convince you that P_I, P_{II} and P_{IV} possess properties generalizing, in a natural way, those of the $\wp$ function.

All these functions can be described by the solutions (entire!) of just one differential equation (see, in contrast, Hietarinta and Kruskal [Painlevé Transcendents, NATO ASI series, 1992])

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Outline

- A summary of the properties of the Weierstrass $\wp(z, g_2, g_3)$ and $\sigma(z, g_2, g_3)$ functions.
- From Weierstrass $\wp$ to Painlevé P_I.
- From Painelvé P_{IV} to P_{II} to P_I to Weierstrass $\wp$: an "all-inclusive" package.
- Conclusions and future directions.

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Classical results on the $\wp$ function

Weierstrass $\wp$ function is defined by the Mittag-Leffler expansion

$$\wp(z; g_2, g_3) = \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left(\frac{1}{(z - \Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2} \right),$$

where $\Omega_{m,n} = 2m\omega_1 + 2n\omega_2, (m,n) \in \mathbb{Z}^2$. It solves the differential equation

$$\frac{d^2\wp}{dz^2} = 6\wp^2 - \frac{g_2}{2} \quad \text{or} \quad \left(\frac{d\wp}{dz}\right)^2 = 4\wp^3 - g_2\wp - g_3$$

Apply the Painlevé analysis to the above equation [1]: if $\wp(z)$ is represented in series $\wp(z) = \sum_{j=0}^{\infty} a_j (z-p)^{j-2}$, the resonances are $r = -1$ and $r = 6$, i.e. the values of p and a_6 must be arbitrary. To match the first order ODE $g_3 = 28a_6$.

[1] To have a consistent Laurent series it is possible to replace $-\frac{g_2}{2}$ only by a linear function of z.

Classical results on the $\wp$ function

The corresponding Laurent series around the pole at $z = 0$ is written as

$$\wp(z) = \sum_{k=0} a_k z^{k-2}, \quad a_0 = 1, a_1 = a_2 = 0, \; a_k = (k-1) \sum_{m,n \neq (0,0)} \Omega_{m,n}^{-k}$$

The coefficients a_k can be rewritten in terms of Eisenstein series of weight k. By symmetry $a_{2k+1} = 0$, and

$$a_{2k} = \frac{2k-1}{(2\omega_1)^{2k}} G_{2k}(\tau), \qquad G_{2k}(\tau) \doteq \sum_{m,n \neq (0,0)} (n + m\tau)^{-2k}$$

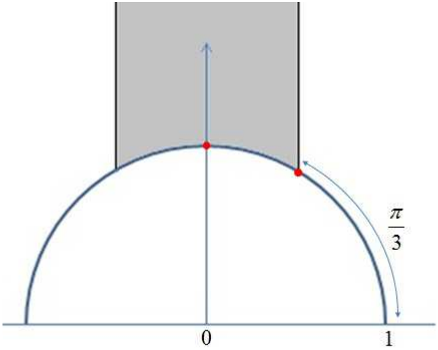
where the normalized period τ is given by $\tau = \frac{\omega_2}{\omega_1}$. Also, they solve the recurrence

$$(n+1)(n-6)a_n = 6 \sum_{j=1}^{n-1} a_j a_{n-j} - \frac{1}{2} g_2 \delta_{n,4}, \qquad n \geq 1, n \neq 6,$$

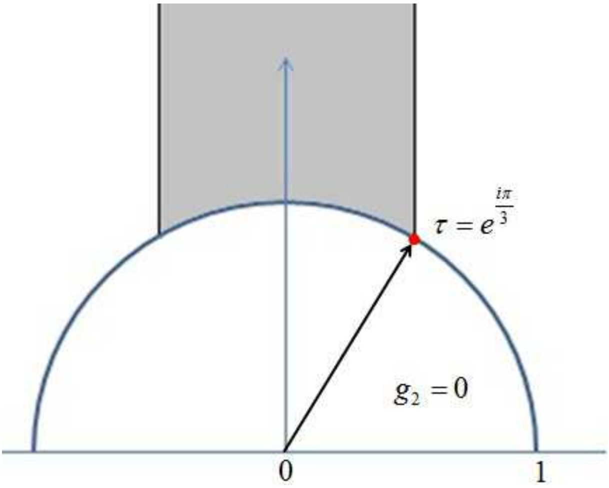
and are weighted homogeneous polynomials of degree n, i.e. $a_n(\zeta^4 g_2, \zeta^6 g_3) = \zeta^n a_n(g_2, g_3)$.

The fundamental region.

By unimodular transformations (i.e. $\tau' = \frac{a\tau+b}{c\tau+d}$, with a, b, c and d integers and $ad - bc = 1$), we can restrict the values of τ to vary in the fundamental region.

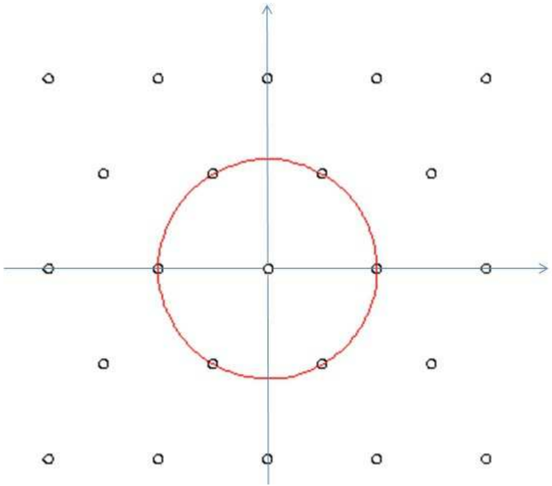


The equianharmonic case



The equianharmonic case

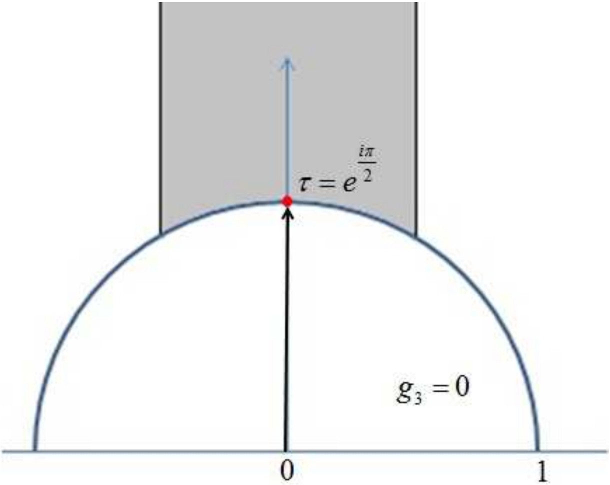
The normalized periods for $g_2 = 0$.



The oscillating converging values of $G_{6n}(e^{\frac{i\pi}{3}})$.

n	$G_{6n}(e^{\frac{i\pi}{3}})$
1	5.86303169342540159797
2	6.00963997169768048102
3	5.99971835637052593409
4	6.00001164757977973485
5	5.99999958743553301523
6	6.00000001557436652006
...	...
11	5.99999999999999892076
12	6.00000000000000003997
13	5.99999999999999999851
14	6.00000000000000000005

The lemniscatic case



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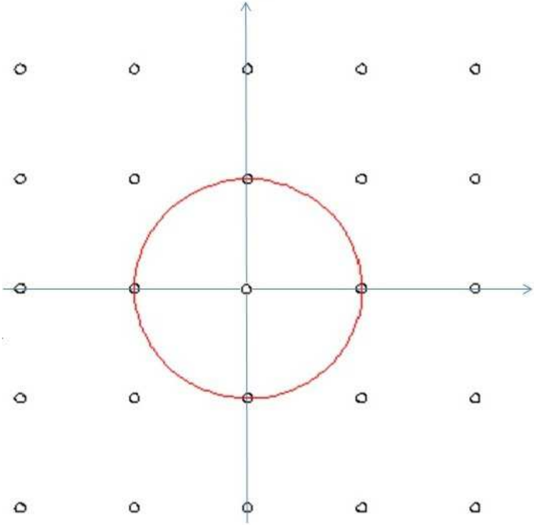

The lemniscatic case

The oscillating converging values of $G_{4n}(\mathrm{i})$.

n	$G_{4n}(\mathrm{i})$
1	3.15121200215389753821
2	4.25577303536518951844
3	3.93884901282797037475
4	4.01569503302502485587
5	3.99609675317628955957
6	4.00097680530383862810
...	...
11	3.99999904632591103400
12	4.00000023841859318284
13	3.99999994039535611558
14	4.00000001490116124950

The Weierstrass σ function.

The Weierstrass σ function associated to $\wp$ is defined by $\wp = -\ln(\sigma)''$. It is entire and possesses the product expansion

$$\sigma(z) = z \prod_{\text{m,n} \neq (0,0)} \left(1 - \frac{z}{\Omega_{m,n}}\right) e^{\frac{z}{\Omega_{m,n}} + \frac{z^2}{2\Omega_{m,n}^2}}$$

It has quasi-periodicities associated to <u>any</u> of its zeroes Ω

$$\sigma(z) = Ae^{B(z-\Omega)}\sigma(z-\Omega),$$

where $A = \sigma'(\Omega)$ and $B = \frac{1}{2}\sigma''(\Omega)/\sigma'(\Omega)$.
The σ function solves a bilinear equation (Eilbeck and Enolskii [J. Phys. A, 33, 2000])

$$D_z^4\sigma \cdot \sigma - g_2\sigma^2 = 0, \qquad D_z^n f \cdot g(z) \doteq \left(\frac{d}{dz} - \frac{d}{dz'}\right)^n f(z)g(z')|_{z'=z}$$

giving a quadratic recurrence for the Taylor series coefficients C_n

$$\sigma(z) = z + \sum_{n=2}^{\infty} C_n z^{n+1},$$

The Weierstrass σ function.

Since the coefficients C_n are weighted homogeneous polynomials of degree n, i.e. $C_n(\zeta^4 g_2, \zeta^6 g_3) = \zeta^n C_n(g_2, g_3)$, from Euler's theorem on homogeneous function it follows that

$$\left(4g_2\frac{\partial}{\partial g_2} + 6g_3\frac{\partial}{\partial g_3} - z\frac{\partial}{\partial z} + 1\right)\sigma = 0.$$

This equation gives the representation

$$\sigma(z) = \sum_{m,n\geq 0} b_{m,n}\left(\frac{1}{2}g_2\right)^m (2g_3)^n \frac{z^{4m+6n+1}}{(4m+6n+1)!}$$

Weierstrass was able to find another linear PDE for $\sigma(z)$, i.e.

$$\left(\frac{\partial^2}{\partial z^2} - 12g_3\frac{\partial}{\partial g_2} - \frac{2}{3}g_2^2\frac{\partial}{\partial g_3} + \frac{1}{12}g_2 z^2\right)\sigma = 0,$$

giving a linear recursion for the coefficients $b_{m,n}$:

$$b_{k,j} = 3(k+1)b_{k+1,j+1} + \frac{16}{3}(j+1)b_{k-2,j+1} - \frac{1}{3}(2k+3j-1)(4k+6j-1)b_{k-1,j}$$

Onishi [arXiv:1003.2927, 2010] proved that $b_{m,n} \in \mathbb{Z},\ \forall m,n \geq 0.$

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Instead of the equation P_I equation written in the canonical form

$$\frac{d^2u}{dz^2} = 6u^2 + z.$$

we consider a rescaled solution with a shift in z and a rescaling:

$$\frac{d^2u}{dz^2} = 6u^2 - 6\lambda z - \frac{g_2}{2}.$$

This form is more convenient for comparison with the Weierstrass equation for $\wp$, obtained setting $\lambda = 0$. It is clear that any solution of the P_I equation in the canonical form with a double pole at $z = p$ corresponds to a solution of the rescaled equation with a pole at $z = 0$, for a suitable choice of the constant g_2, with any $\lambda \neq 0$.

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Mittag Leffler expansion for P_I

P_I has order of growth equal to $\frac{5}{2}$ (Steinmetz[Ann. Acad. Sci. Fenn. 30, 2005]). This is also the infimum of the values μ such that the power sums over non zero poles Ω $\sum_{\Omega\neq 0}\Omega^{-\mu}$ is convergent. This means that P_I admits a Mittag Leffler expansion similar to that of $\wp$

$$u(z)=\frac{1}{z^2}+\sum_{\Omega\neq 0}\left(\frac{1}{(z-\Omega)^2}-\frac{1}{\Omega^2}\right)$$

The above observation allows to generalize the results for the Laurent expansion of $\wp$

$$u(z)=\sum_{n=0}c_n z^{n-2},\quad c_0=1, c_1=c_2=0,\ c_n=(n-1)\sum_{\Omega\neq 0}\Omega^{-n}$$

$$(n+1)(n-6)c_n=6\sum_{j=1}^{n-1}c_j c_{n-j}-\frac{1}{2}g_2\delta_{n,4}-6\lambda\delta_{n,5},\qquad n\geq 1, n\neq 6$$

Mittag Leffler expansion for P_I

In general the coefficients c_n are weighted homogeneous polynomials of order n. If $c_n \doteq \mathrm{P}_k(g_2, \lambda, g_3)$, then

$$\mathrm{P}_n(\zeta^4 g_2, \zeta^5 \lambda, \zeta^6 g_3) = \zeta^n \, \mathrm{P}_k(g_2, \lambda, g_3) \qquad \forall \zeta \in \mathbb{C}^*.$$

The invariants g_2, λ and g_3 are given by the formulae:

$$g_2 = 60 \sum_{\Omega \neq 0} \Omega^{-4}, \qquad \lambda = 4 \sum_{\Omega \neq 0} \Omega^{-5}, \qquad g_3 = 140 \sum_{\Omega \neq 0} \Omega^{-6}.$$

As for the $\wp$ function we can pick a non zero pole Ω_* such that $|\Omega_*|$ is minimal, and write:

$$c_n = \frac{n-1}{\Omega_*^n} F_n, \qquad F_n \doteq \sum_{\Omega \neq 0} \frac{\Omega_*^n}{\Omega^n}$$

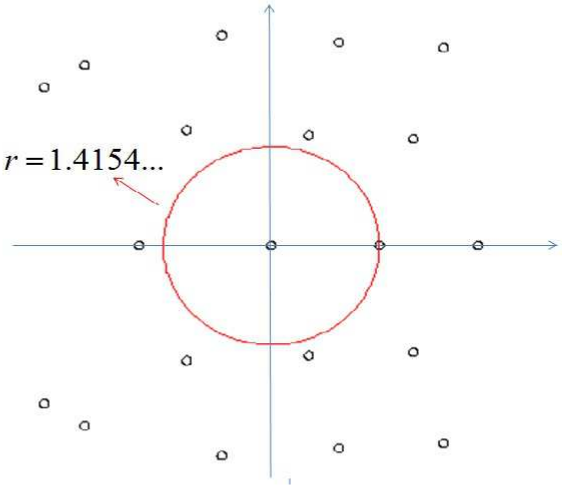
In general all the poles, except 0 and Ω_*, have modulus greater than Ω_*. Then

$$\lim_{n\to\infty} F_n = 1, \qquad \lim_{n\to\infty} \frac{n}{(n-1)} \frac{c_n}{c_{n+1}} = \Omega_*$$

With symmetries, more poles may have modulus equal to $|\Omega_*|$.

The generic case

The poles of P_I for $g_2 = 20$, $\lambda = 1$ and $g_3 = 30$.



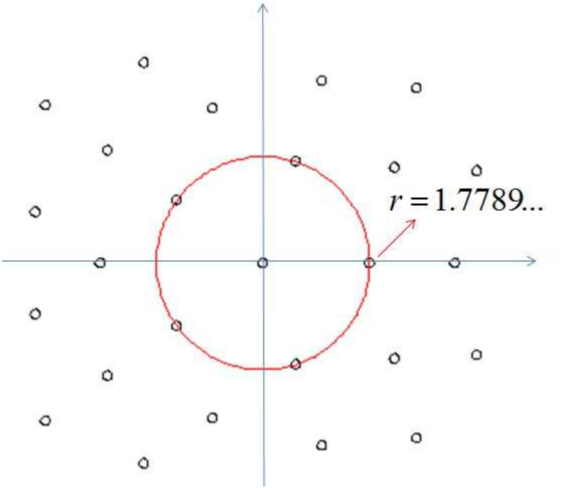
The generic case

The values of $F_n = \frac{c_n \Omega_*^n}{n-1}$.

n	F_n
10	1.5368106889286801752
14	1.1814415329441741190
18	0.9348064528870509510
...	...
60	1.0002333754335601234
68	0.9999813524476090901
...	...
174	1.0000000000111842896
175	0.9999999999973163797
...	...
263	1.0000000000000000185
264	0.9999999999999999962

The pentagonal case

The poles of P_I for $g_2 = 0$, $\lambda = 1$ and $g_3 = 0$.



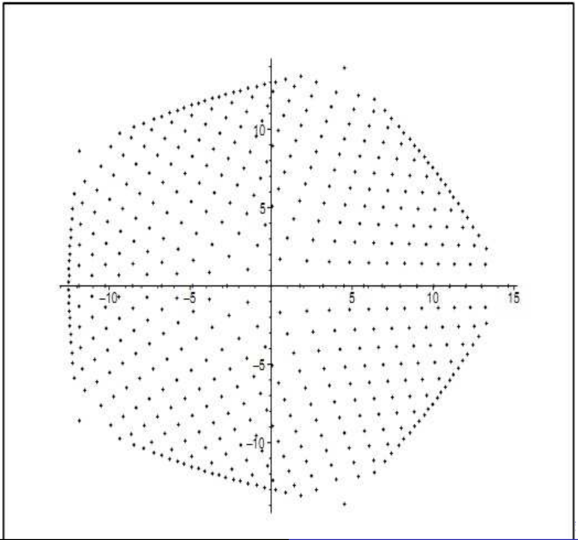
The pentagonal case

The values of $F_{5n} = \frac{c_{5n}\Omega_*^{5n}}{5n-1}$.

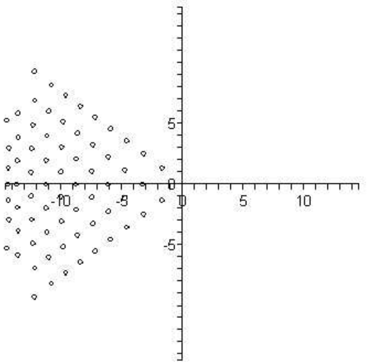
n	F_{5n}
1	4.58034567118120971780
2	5.08595550727477491733
3	4.99187877676419618478
4	5.00112762186482314743
5	4.99986996982708054870
6	5.00001616272241466830
...	...
11	4.99999999957591996469
12	5.00000000005151463070
13	4.99999999999374379485
14	5.00000000000075986461

The pentagonal case

The pentagon of poles of P_I for $g_2 = 0$, $\lambda = 1$ and $g_3 = 0$.



Poles of the tritronquée solution
($g_2 = -4.7683374.., g_3 = -1.7397996.., \lambda = 1/6$) .



The Hamiltonian and the τ function

The P_I equation is Hamiltonian, with Hamiltonian function

$$h = \frac{1}{2}v^2 - 2u^3 + \frac{1}{2}g_2 u + 6\lambda z u + \frac{1}{2}g_3.$$

The total derivative of h with respect to z is proportional to u:

$$\frac{dh}{dz} = 6\lambda u.$$

It is possible to define an entire functions having the only simple zeroes where P_I has the poles: it is the τ function associated to u, defined by:

$$u = -\frac{d^2}{dz^2}\log\tau, \qquad \text{or} \qquad h = -6\lambda\frac{d}{dz}\log\tau.$$

The τ function satisfies the Hirota bilinear equation (extending the result of Eilbeck and Enolskii)

$$D_z^4\tau\cdot\tau = (12\lambda z + g_2)\tau^2$$

The τ function

For $\lambda = 0$ the τ function reduces to the Weierstrass σ function. The generalization of the recurrence for the coefficients in the Taylor series of $\sigma(z)$, i.e. the recurrence for coefficients in the series $\tau(z) = z + \sum_{k=2} C_k z^{k+1}$ reads

$$n(n^2-1)(n-6)C_n = -\frac{1}{2}\sum_{j=1}^{n-1} b_{n,j} C_j C_{n-j} + \frac{1}{2} g_2 \sum_{j=0}^{n-4} C_j C_{n-4-j} + 6\lambda \sum_{j=0}^{n-5} C_j C_{n-5-j}$$

Notice that, due to the Hurwitz's theorem on the zeroes of converging sequences of holomorphic functions (see e.g. Titchmarsh), the zeroes of the polynomials

$$\tau_N(z) = z + \sum_{k=2}^{N} C_k z^{k+1}$$

converge to the zeroes of the τ function (i.e. to the poles of the corresponding P_I) equation.

The Taylor coefficients C_n are weighted homogeneous of degree n, i.e. $C_n(\zeta^4 g_2, \zeta^5\lambda, \zeta^6 g_3) = \zeta^n C_n(g_2, \lambda, g_3)$ and again from Euler's theorem

$$\left(4g_2\frac{\partial}{\partial g_2} + 5\lambda\frac{\partial}{\partial\lambda} + 6g_3\frac{\partial}{\partial g_3} - z\frac{\partial}{\partial z} + 1\right)\tau = 0,$$

giving the triple sum representation

$$\tau(z) = \sum_{\ell,m,n\geq 0} A_{\ell,m,n}\,(\frac{1}{2}g_2)^{\ell}(6\lambda)^m(2g_3)^n\frac{z^{4\ell+5m+6n+1}}{(4\ell+5m+6n+1)!}$$

Unlike Weierstrass, we don't have another PDE for $\tau(z)$, but, supported by numerical calculations, we conjectured that

$$A_{\ell,m,n}\in\mathbb{Z}\quad\forall\ell,m,n\geq 0.$$

Painlevé analysis can be applied directly to the equation

$$D_z^4\tau\cdot\tau-(12\lambda z+g_2)\tau^2=0,$$

obtaining the resonances $r=(-1,0,1,6)$. The values -1 and 6 corresponds to the arbitrary values of the singularity for P_I and to g_3, the values 0 and 1 corresponds to the fact that u is defined up to the gauge transformation $\tau\to\exp(az+b)\tau$. If $\tau(z)=z+\sum_{k=2}C_kz^{k+1}$, expanding around another zero at $z=\Omega\neq 0$, we obtain the formula

$$\tau(z;g_2,\lambda,g_3)=Be^{A(z-\Omega)}\tau(z-\Omega;g_2+12\lambda\Omega,\lambda,g_3+12\lambda A),$$

where $B=\tau'(\Omega)$ and $A=\frac{1}{2}\tau''(\Omega)/\tau'(\Omega)$. The quasiperiodicity of $\sigma(z)$ under shifting by a period is a special case of this.

From Painelvé P_{IV} to P_{II} to P_I to Weierstrass $\wp$: an "all-inclusive" package.

Extension to Painlevé II and IV

Painlevé equations II and IV are Hamiltonian too. The corresponding Hamiltonians h_{II} and h_{IV} are functions of the canonically conjugated variables u and v and of the " time" z. As functions of z (i.e. $s_{II}(z) = h_{II}(z, u(z), v(z))$...) they solve differential equations.
It is possible to introduce a multi-parametric differential equation encompassing all these flows.
It turns out that, as a function of z, it satisfies

$$(s'')^2 - \eta (zs' - s)^2 + 2(\gamma s' - 6\lambda)(zs' - s) + 4(s' + \mu)^3 - g_2(s' + \mu) + g_3 = 0$$

It can be shown that all the solutions of this equation are meromorphic, with simple poles with residue $+1$.

Extension to Painlevé II and IV

Proposition

By taking $\gamma = \lambda = 0$ *and* $\eta = 4$ *and setting*

$$\mu = \frac{2}{3}(\theta_0 + \theta_\infty) - \theta, \qquad g_2 = \frac{16}{3}(\theta_0 - \theta_\infty e^{\frac{i\pi}{3}})(\theta_0 - \theta_\infty e^{-\frac{i\pi}{3}}),$$

$$g_3 = \left(\frac{4}{3}\right)^3 (\theta_0 + \theta_\infty)(\theta_0 - \frac{\theta_\infty}{2})(\theta_0 - 2\theta_\infty)$$

the function $u(z, \theta_0, \theta_\infty)$ *defined by*

$$u(z, \theta_0, \theta_\infty) \doteq s(z, \theta, \theta_0, \theta_\infty + 1) - s(z, \theta, \theta_0, \theta_\infty)$$

solves the Painlevé IV equation $u'' = \frac{(u')^2}{2u} + \frac{3}{2}u^3 + 4zu^2 + 2u(z^2 - \alpha) + \frac{\beta}{u}$, *where* $\alpha \doteq 2\theta_\infty - \theta_0 + 1$ *and* $\beta \doteq -2\theta_0^2$.

Proposition

In the special case $\eta = \lambda = 0$, *the function* $-s'(z)$ *solves the Painlevé XXXIV equation.*

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$$g_3 = \left(\frac{4}{3}\right)^3 (\theta_0 + \theta_\infty)(\theta_0 - \frac{\theta_\infty}{2})(\theta_0 - 2\theta_\infty)$$

the function $u(z, \theta_0, \theta_\infty)$ *defined by*

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In the special case $\eta = \lambda = 0$, *by setting* $g_2 = 12\mu^2$ *and* $g_3 = 8\mu^3 - \frac{\gamma^2}{16}(2\alpha+1)^2$, *the function* $u(z,\gamma,\mu,\alpha)$ *defined by*

$$u(z,\gamma,\mu,\alpha) \doteq s(z,\gamma,\mu,\alpha-1) - s(z,\gamma,\mu,\alpha)$$

solves the Painlevé II equation $u'' = 2u^3 + (\gamma z + 6\mu)u + \gamma\alpha$.

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In the special case $\eta = \gamma = 0$, *the function* $u(z) = -(s'(z) + \mu)$ *solves the Painlevé I equation* $u'' - 6u^2 + 6\lambda z + \frac{g_2}{2} = 0$.

Proposition

In the special case $\eta = \gamma = \lambda = 0$, *the function* s *is written in terms of the Weierstrass zeta function as* $s(z) = \zeta(z, g_2, g_3) - \mu z$.
Equivalently, the function $u = -(s' + \mu)$ *is the Weierstrass* $\wp$ *function,* $u(z) = \wp(z, g_2, g_3)$

Extension to Painlevé II and IV

Proposition

In the special case $\eta = \lambda = 0$*, by setting* $g_2 = 12\mu^2$ *and* $g_3 = 8\mu^3 - \frac{\gamma^2}{16}(2\alpha+1)^2$*, the function* $u(z,\gamma,\mu,\alpha)$ *defined by*

$$u(z,\gamma,\mu,\alpha) \doteq s(z,\gamma,\mu,\alpha-1) - s(z,\gamma,\mu,\alpha)$$

solves the Painlevé II equation $u'' = 2u^3 + (\gamma z + 6\mu)u + \gamma\alpha.$

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Extension to Painlevé II and IV

The Painlevé analysis gives the resonances ± 1, i.e. the position of the pole p and of the coefficient of $(z-p)^0$ in the local expansion around p are arbitrary.
To every solution $s(z,\eta,\gamma,\lambda,\mu,g_2,g_3)$ of the equation there corresponds a solution $\tilde{s}(z,\eta,\gamma,\lambda,\mu,g_2,g_3)$ through the relation

$$\tilde{s}(z,\eta,\gamma,\lambda,\mu,g_2,g_3) = A + s(z-p,\eta,\tilde{\gamma},\tilde{\lambda},\tilde{\mu},\tilde{g}_2,\tilde{g}_3)$$

where A and p are arbitrary constants and

$$\tilde{\gamma} \doteq \gamma - \eta p, \quad \tilde{\lambda} \doteq \lambda - \frac{\eta A}{6}, \quad \tilde{\mu} \doteq \mu + \frac{p}{12}(2\gamma - \eta p),$$
$$\tilde{g}_2 \doteq g_2 + 12(\tilde{\mu}^2 - \mu^2) + 2A\gamma + 12\tilde{\lambda}p,$$
$$\tilde{g}_3 \doteq g_3 - 4(\tilde{\mu}^3 - \mu^3) + \tilde{g}_2\tilde{\mu} - g_2\mu - A(\eta A - 12\lambda).$$

This is a two parameter group, with translations as a group law

The order of growth of $s(z)$ is at most 4. A solution with the maximum order of growth, with a pole in $z = 0$ and with the coefficient of z^0 equal to 0, possesses the Mittag-Leffler representation

$$s(z) = \frac{1}{z} - \mu z - \gamma\frac{z^2}{8} + \frac{4\eta - g_2}{60}z^3 + \sum_{\substack{\text{poles } \Omega \\ \Omega\neq 0}} \left(\frac{1}{z-\Omega} + \frac{1}{\Omega} + \frac{z}{\Omega^2} + \frac{z^2}{\Omega^3} + \frac{z^3}{\Omega^4} \right)$$

This allows to generalize the results on the behaviour of the Laurent coefficients. Indeed the Laurent expansion reads

$$s(z) = \sum_{k=0} a_k z^{k-1}, \quad a_0 = 1,\ a_k = -\sum_{\Omega\neq 0} \Omega^{-k}; k \geq 5$$

where the coefficients a_k solve a quadratic recurrence

$$(n^2-1)(n-6)a_n = \eta(n-6)a_{n-4} + \gamma(n-3)a_{n-3} +$$
$$-6\sum_{k=1}^{n-1} a_k a_{n-k}(k-1)(n-k-1) + \frac{g_2}{2}\delta_{n,4} + (6\lambda + \gamma\mu)\delta_{n,5} - \frac{\gamma\eta}{8}\delta_{n,7}$$

The coefficients a_n are weighted homogeneous polynomials of degree n in (μ, γ, η, g_2, λ g_3). If $a_n = \mathrm{P}_n(\mu, \gamma, \eta, g_2, \lambda, g_3)$

$$\mathrm{P}_n(\xi^2\mu, \xi^3\gamma, \xi^4\eta, \xi^4 g_2, \xi^5\lambda, \xi^6 g_3) = \xi^n\, \mathrm{P}_n(\mu, \gamma, \eta, g_2, \lambda, g_3) \quad \forall \xi \in \mathbb{C}^*.$$

Again we can pick a non zero pole Ω_* such that $|\Omega_*|$ is minimal, and write:

$$a_n = -\frac{1}{\Omega_*^n} F_n, \qquad F_n \doteq \sum_{\Omega \neq 0} \frac{\Omega_*^n}{\Omega^n}$$

In general all the poles, except 0 and Ω_*, have modulus greater than Ω_*. Then

$$\lim_{n\to\infty} F_n = 1, \qquad \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \Omega_*.$$

But now we can have all the symmetries up to that of the hexagon.

Extension to Painlevé II and IV

To the function $s(z)$ it is associated a τ function having the zeros where it has poles:

$$\ln(\tau)' = s.$$

The function $\mathrm{T} = \tau e^{\mu\frac{z^2}{2}+\gamma\frac{z^3}{24}}$ solves the following bilinear equation

$$D_z^4\mathrm{T}\cdot\mathrm{T} - z(\eta z+\gamma)D_z^2\mathrm{T}\cdot\mathrm{T} + 2(\eta z-\gamma)\mathrm{T}\mathrm{T}' - (g_2 + 12\Lambda z - \gamma\frac{\eta}{4}z^3)\mathrm{T}^2 = 0.$$

where $\Lambda = \lambda + \gamma\mu/6$.
The T or τ functions are entire and possess a global Taylor series representation. Again, from the Hurwitz's root theorem, it is possible to get numerical approximations of its zeros from the zeros of the truncated series.

Extension to Painlevé II and IV

Again, we get a quadratic recurrence for the C_k in the expansion $\mathrm{T}(z) = z + \sum_{k=2} C_n z^{n+1}$

$$n(n^2-1)(n-6)C_n = -\frac{1}{2}\sum_{j=1}^{n-1} b_{n,j} C_j C_{n-j} + \frac{\eta}{2}\sum_{j=0}^{n} a^{-}_{j+1,n-j-3} C_j C_{n-4-j} +$$

$$\frac{\gamma}{2}\sum_{j=0}^{n} a^{+}_{j+1,n-j-2} C_j C_{n-3-j} + \frac{g_2}{2}\sum_{j=0}^{n-4} C_j C_{n-4-j} - \frac{\gamma\eta}{8}\sum_{j=0}^{n-7} C_j C_{n-7-j} \qquad (2)$$

$$+ 6\Lambda \sum_{j=0}^{n-5} C_j C_{n-5-j}$$

	Zeros of τ describe the poles of
$\gamma = \lambda = 0$, $\eta = 4$	P_{IV}
$\eta = \lambda = 0$	P_{XXXIV}
$\eta = \lambda = 0$	P_{II}
$\eta = \gamma = 0$	P_{I}
$\eta = \lambda = \gamma = 0$	$\wp$

Extension to Painlevé II and IV

Since the Taylor series coefficients C_n are weighted homogeneous polynomials, the tau-function can be written in the form of a multiple sum

$$\mathrm{T}(z) = \sum_{\ell,m,n,k,j\geq 0} A_{\ell,m,n,k,j} \left(\frac{1}{2}g_2\right)^{\ell} (6\Lambda)^m (2g_3)^n \left(\frac{\gamma}{4}\right)^k (2\eta)^j \frac{z^{3k+4\ell+4j+5m+6n+1}}{(3k+4\ell+4j+5m+6n+1)!}$$

for certain rational numbers $A_{\ell,m,n,k,j}$. Based on numerical evidence and on analogous results for the Weierstrass and Painlevé I cases, we conjecture that

$$A_{\ell,m,n,k,j} \in \mathbb{Z} \qquad \forall \ell, m, n, k, j \geq 0.$$

Extension to Painlevé II and IV

Denoting with $\tau(z, \eta, \gamma, \mu, g_2, \lambda, g_3)$ the function $\tau(z)$ corresponding to the parameters $\eta, \gamma, \mu, g_2, \lambda, g_3$ and with a zero at $z = 0$, by expanding around another zero at $z = p \neq 0$, we obtain quasi-periodicity formula:

$$\tau(z, \eta, \gamma, \mu, g_2, \lambda, g_3) = Be^{A(z-p)}\tau(z - p, \eta, \tilde{\gamma}, \tilde{\mu}, \tilde{g}_2, \tilde{\lambda}, \tilde{g}_3),$$

where again $B = \tau'(p)$ and $A = \frac{1}{2}\tau''(p)/\tau'(p)$. The values of the new parameters are as before, i.e.

$$\tilde{\gamma} \doteq \gamma - \eta p, \quad \tilde{\lambda} \doteq \lambda - \frac{\eta A}{6}, \quad \tilde{\mu} \doteq \mu + \frac{p}{12}(2\gamma - \eta p),$$
$$\tilde{g}_2 \doteq g_2 + 12(\tilde{\mu}^2 - \mu^2) + 2A\gamma + 12\tilde{\lambda}p,$$
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Extension to Painlevé II and IV

Let us denote with $\{\Omega(\eta, \gamma, \lambda, \mu, g_2, g_3)\}$ the set of poles of $s(z, \eta, \gamma, \lambda, \mu, g_2, g_3)$. Then it follows that

$$\{\, \Omega(\eta, \gamma, \lambda, \mu, g_2, g_3) - p \,\} = \{\, \Omega(\eta, \tilde{\gamma}, \tilde{\lambda}, \tilde{\mu}, \tilde{g_2}, \tilde{g_3}) \,\}$$

where p is *any* value in the set $\{\Omega\}$. The previous property is a direct generalization of the analogue property of the elliptic functions: the set defined by the difference among any single pole and the value of just one pole, gives the set of poles of the function $s(z)$ evaluated at different values of the parameters (the same in the Weierstrass case)

An example with Painleve IV

An example from Painleve I

A recursion for poles

It is possible to define a recursion satisfied by the poles of the function $s(z)$. It reads

$$\Omega_{k+1}(\eta,\gamma,\lambda,\mu,g_2,g_3)=\Omega_k(\eta,\gamma,\lambda,\mu,g_2,g_3)+\Omega_k(\eta,\gamma_k,\lambda_k,\mu_k,g_{2,k},g_{3,k}),$$

where $k \geq 1$ and $\gamma_k,\lambda_k,\mu_k,g_{2,k},g_{3,k}$ are explicitly given by

$$\gamma_k=\gamma-\eta\Omega_k,\quad \lambda_k=\lambda-\frac{\eta A_k}{6},\quad \mu_k=\mu+\frac{\Omega_k}{12}(2\gamma-\eta\Omega_k),$$
$$g_{2,k}=g_2+12(\mu_k^2-\mu^2)+2A_k\gamma+12\lambda_k\Omega_k,\qquad k=1\ldots,$$
$$g_{3,k}=g_3-4(\mu_k^3-\mu^3)+g_{2,k}\mu_k-g_2\mu-A_k(\eta A_k-12\lambda).$$

and the values of the constants A_k are defined by $A_k=\frac{1}{2}\tau''(\Omega_k)/\tau'(\Omega_k)$.

A recursion for poles

The values $\Omega_k(\eta, \gamma_k, \lambda_k, \mu_k, g_{2,k}, g_{3,k})$ can be obtained by the limiting value of $a_{n,k}/a_{n+1,k}$, that is

$$\Omega_k(\eta, \gamma_k, \lambda_k, \mu_k, g_{2,k}, g_{3,k}) = \lim_{n\to\infty} \frac{a_{n,k}}{a_{n+1,k}},$$

where the elements $a_{n,k}$ solve a quadratic recurrence:

$$(n^2-1)(n-6)a_{n,k} = \eta(n-6)a_{n-4,k} + \gamma_k(n-3)a_{n-3,k} +$$
$$-6\sum_{j=1}^{n-1} a_{j,k}a_{n-j,k}(j-1)(n-j-1) + +\frac{g_{2,k}}{2}\delta_{n,4} + (\lambda_+ 6\gamma_k\mu_k)\delta_{n,5} - \frac{\gamma_k\eta}{8}\delta_{n,7}$$

The recursion for poles is explicitly solved by

$$\Omega_k(\eta, \gamma, \lambda, \mu, g_2, g_3) = \Omega_1(\eta, \gamma, \lambda, \mu, g_2, g_3) + \sum_{n=1}^{k-1} \Omega_n(\eta, \gamma_n, \lambda_n, \mu_n, g_{2,n}, g_{3,n})$$

giving back the periodicity $\Omega_k = k\Omega_1$ in the Weierstrass case $\eta = \gamma = \lambda = \mu = 0$.

Conclusions

- The functions Painlevé I, II and IV are the natural extension of the elliptic functions of Weierstrass.
- It is possible to get an efficient algorithm from the pole recursion?
- The final goal would be to include them among the special functions
- What about addition or multiplication formulae? What about tabulation of values?

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Thanks!

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